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# Elastodynamic homogenization of 2D periodic microstructured media based on the Christoffel tensor

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## Abstract

*The properties of elastic waves propagating within architected media have been the subject of recent investigations. The study of wave propagation and its interaction with the periodic microstructure of architected materials for very long wavelengths has shown that a homogeneous continuum of Cauchy type can correctly describe the behaviour of such materials. However, as the frequency or the wavenumber increases, the behaviour becomes more complex and hence Classical elasticity is not rich enough to capture specific physical phenomena, such as dispersivity for instance. The elastodynamic homogenization method based on the Floquet-Bloch analysis allows to characterise the effective properties through the computation of the acoustic tensor, which depends on the directions of propagation and the symmetry class of the material. It is worth noting that, assuming a specific symmetry class, some information concerning the material might be lost. However, the identification of effective properties becomes difficult since no particular assumptions are involved with respect to the symmetry class of the effective elasticity tensor. In this article, we propose a parameterisation of the acoustic tensor using the harmonic decomposition of a generic elasticity tensor. The acoustic tensor's invariants will be expressed as a function of this decomposition's elements, for a given direction of propagation.*

**Keywords : Architected Materials, 2D Elasticity, Elastodynamics, Homogenization, Harmonic Decomposition, Acoustic tensor.**

## 1 Introduction

Periodic architected materials are those that possess an inner geometry, they result from a combination of at least two phases, where one phase can be either solid or void, assembled in unit cells to form a new material. Their effective properties are controlled by the internal geometry rather than the specific nature of phases and depends strongly on the scale separation between the wavelength of the loading and the size of the unit cell. The resulting local kinematics can be rather complex and lead to an extreme

mechanical behaviour such as negative Poisson's ratio [1], negative compressibility [2], negative thermal expansion [3], etc. The study of elastic waves propagating in architected materials is of particular interest. In fact, the microstructure and the topology of the medium strongly affects the propagation of elastic waves and unusual effects can be observed at the macroscale, such as an anisotropic elastodynamic behaviour at high frequencies in honeycombs [4]. In order to study these phenomena, it is convenient to substitute the original heterogeneous continuum with a homogeneous one that has an equivalent behaviour for a given range of parameters. Such equivalent continuum will be denoted "Equivalent Homogeneous Medium" (EHM). For Long Wave-Lengths/Low-Frequencies (LW-LF), the EHM is generally considered as a classical Cauchy continuum, the effective properties of which can be obtained by periodic elastostatic homogenization.

To illustrate the identification method, we propose a comparison between the EHM and the real medium behaviour in the LW-LF regime for some architected materials belonging to different symmetry classes. The unit cells of the studied lattices are shown in the figure (1)

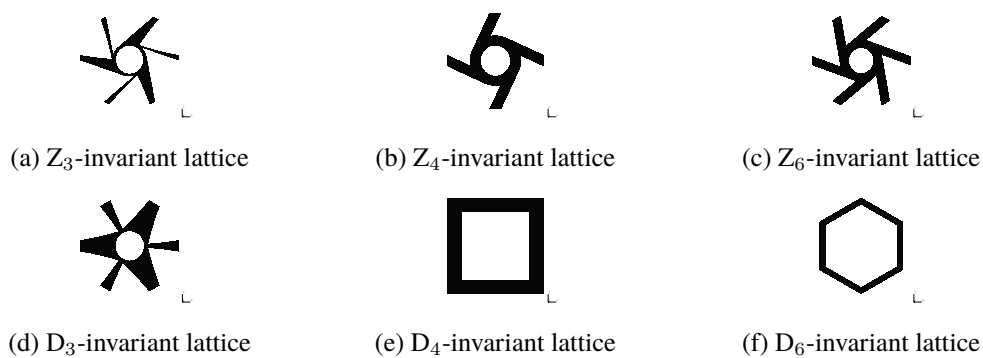


Figure 1: Unit cells of the studied architected materials

In this figure,  $D_k$  refers to the dihedral group, which contains rotation symmetry of angle  $2\pi/k$  around a center point and  $k$  mirror lines,  $Z_k$  refers to the cyclic group which contains only the rotation symmetry but no mirror line.

## 2 Elastodynamic homogenization : LW-LF approx.

Bloch-wave homogenization approach allows to characterise the effective medium in the LW-LF approximation through the analysis of wave propagation. The main idea of the proposed approach is to set up an identification method using waves velocities and the Christoffel tensor polynomial invariants in order to identify the effective properties. The global steps of the method are summarized as follows

1. Floquet-Bloch analysis  $\rightarrow$  Computing wave velocities;
2. parameterisation of the Christoffel tensor field using harmonic decomposition elements  $\mathbf{C}^H$ ;
3. identification algorithm of the effective elasticity tensor  $\mathbf{C}^H$ .

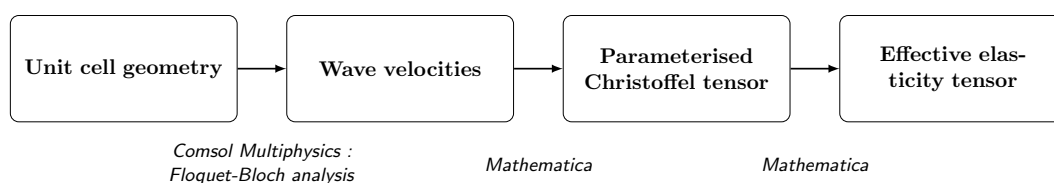


Figure 2: Schematic of elastodynamic homogenization method and softwares used

## 2.1 Floquet-Bloch analysis

The Floquet-Bloch analysis allows to compute the dispersion diagram which is the fingerprint of the architected material [5]. All the information needed to characterise the medium can be extracted from this diagram, in particular phase velocities. Let consider a wave propagation problem in a periodic heterogeneous medium  $\Omega$  generated by a unit cell  $\mathcal{T}$ , the equation of motion can be expressed in terms of the displacement field  $\mathbf{u}(\mathbf{x}, t)$

$$\nabla \cdot [\mathbf{C}(\mathbf{x}) : (\mathbf{u}(\mathbf{x}, t) \otimes^s \nabla)] = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t), \quad \forall \mathbf{x} \in \Omega, \quad (2.1)$$

where  $\mathbf{C}$  and  $\rho$  are  $\mathcal{T}$ -periodic fields.

In order to analyze the elastodynamic behaviour of the medium, we apply the Floquet-Bloch theorem over its unit cell to obtain the elementary solutions of (2.1) in the form of Bloch-waves :

$$\mathbf{u}_{\mathbf{k}}(\mathbf{x}, t) = \mathcal{R}e \left( \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}) e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \right), \quad (2.2)$$

in which  $\tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})$  is a complex  $\mathcal{T}$ -periodic polarization field. Introducing the ansatz (2.2) in (2.1), we apply the Floquet-Bloch transform [6] to the motion equation which becomes

$$(\nabla - i\mathbf{k}) \cdot \{\mathbf{C}(\mathbf{x}) : [\tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}) \otimes^s (\nabla - i\mathbf{k})]\} = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{T}. \quad (2.3)$$

The latter can be viewed as a family of eigenvalues problems of wave propagation at the angular frequency  $\omega$ , parametrized by the wave vector  $\mathbf{k}$ . The existence of free-waves is subjected to the following condition

$$\det [\mathbf{Z}_{\mathbf{k}}(\omega)] = \det [(\nabla - i\mathbf{k}) \cdot \mathbf{C}(\mathbf{x}) \cdot (\nabla - i\mathbf{k}) - \omega^2 \rho(\mathbf{x}) \mathbf{I}] = 0, \quad (2.4)$$

which is called dispersion relation. The graphical representation of the set of all couples  $(\mathbf{k}, \omega)$  satisfying the dispersion relation represents the dispersion diagram.

## 2.2 Christoffel tensor revisited

In this study, we analyze the wave propagation for very Long-Wavelength/Low-Frequency. Hence, the dispersion relation is linear and can be obtained by solving the the Christoffel equation [7]

$$(\Gamma(\mathbf{n}) - \rho^H v_\phi^2 \mathbf{I}) \tilde{\mathbf{u}} = \mathbf{0} \quad \text{with} \quad \Gamma(\mathbf{n}) = \mathbf{n} \cdot \mathbf{C}^H \cdot \mathbf{n}. \quad (2.5)$$

In (2.5),  $\Gamma$  is the Christoffel tensor in the direction  $\mathbf{n}$ ,  $\rho^H$  is the effective mass density and  $\mathbf{C}^H$  is the elasticity tensor of the effective medium in the LW-LF approximation. Since the wave velocities derive from the eigenvalues of  $\Gamma$ , we will use this information to determine the constants of  $\mathbf{C}^H$ . In order to well identifying the effective medium, in particular its symmetry class, we propose a parameterisation of the Christoffel tensor in terms of the direction of propagation  $\mathbf{n}_\theta$ <sup>1</sup>, and the harmonic decomposition<sup>2</sup> elements of  $\mathbf{C}^H = (\mathbf{H}, \mathbf{h}, \alpha_2, \alpha_0)$ , which provides a physical interpretation of the anisotropic part of  $\mathbf{C}^H$ . More precisely,  $\mathbf{C}^H$  can be decomposed into irreducible sum of 4<sup>th</sup> order harmonic tensor  $\mathbf{H}$ <sup>2</sup>, a 2<sup>nd</sup>

<sup>1</sup>the direction of propagation is now parametrized by the angle of propagation  $\theta$  with respect to the reference frame.

<sup>2</sup>A  $n^{\text{th}}$ -order harmonic tensor is a completely symmetric and traceless  $n^{\text{th}}$ -order tensor

order harmonic tensor  $\mathbf{h}$  and two scalars  $\alpha_2$  and  $\alpha_0$  [8]

$$\mathbf{C}^H = \underbrace{\mathbf{H} + \frac{1}{2}(\mathbf{h} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{h})}_{\text{Anisotropic part}} + \underbrace{\frac{\alpha_2}{2}\mathbf{J} + 2\alpha_0\mathbf{K}}_{\text{Isotropic part}}, \quad (2.6)$$

in which  $\mathbf{K}$  and  $\mathbf{J}$  are respectively the 4<sup>th</sup> order spheric and deviatoric projectors. Hence, the Christoffel tensor can be re-written using the compact notation

$$\Gamma(\mathbf{n}_\theta; \mathbf{C}^H) = \Gamma(\mathbf{n}_\theta; \mathbf{H}, \mathbf{h}, \alpha_2, \alpha_0). \quad (2.7)$$

The set of propagation angles  $\theta$  defines a Christoffel tensor field, carrying all the information about the effective medium. The Christoffel tensor being real and symmetric, wave velocities are related to the components of  $\mathbf{C}^H$  through the Christoffel's polynomial invariants

$$\mathcal{I}_1(\theta) = \text{tr}[\Gamma(\mathbf{n}_\theta; \mathbf{C}^H)], \quad \mathcal{I}_2(\theta) = \det[\Gamma(\mathbf{n}_\theta; \mathbf{C}^H)]. \quad (2.8)$$

More precisely, we have the following relationships

$$\mathcal{I}_1(\theta) = \rho^H (v_P^2(\theta) + v_S^2(\theta)), \quad \mathcal{I}_2(\theta) = (\rho^H)^2 v_P^2(\theta)v_S^2(\theta), \quad (2.9)$$

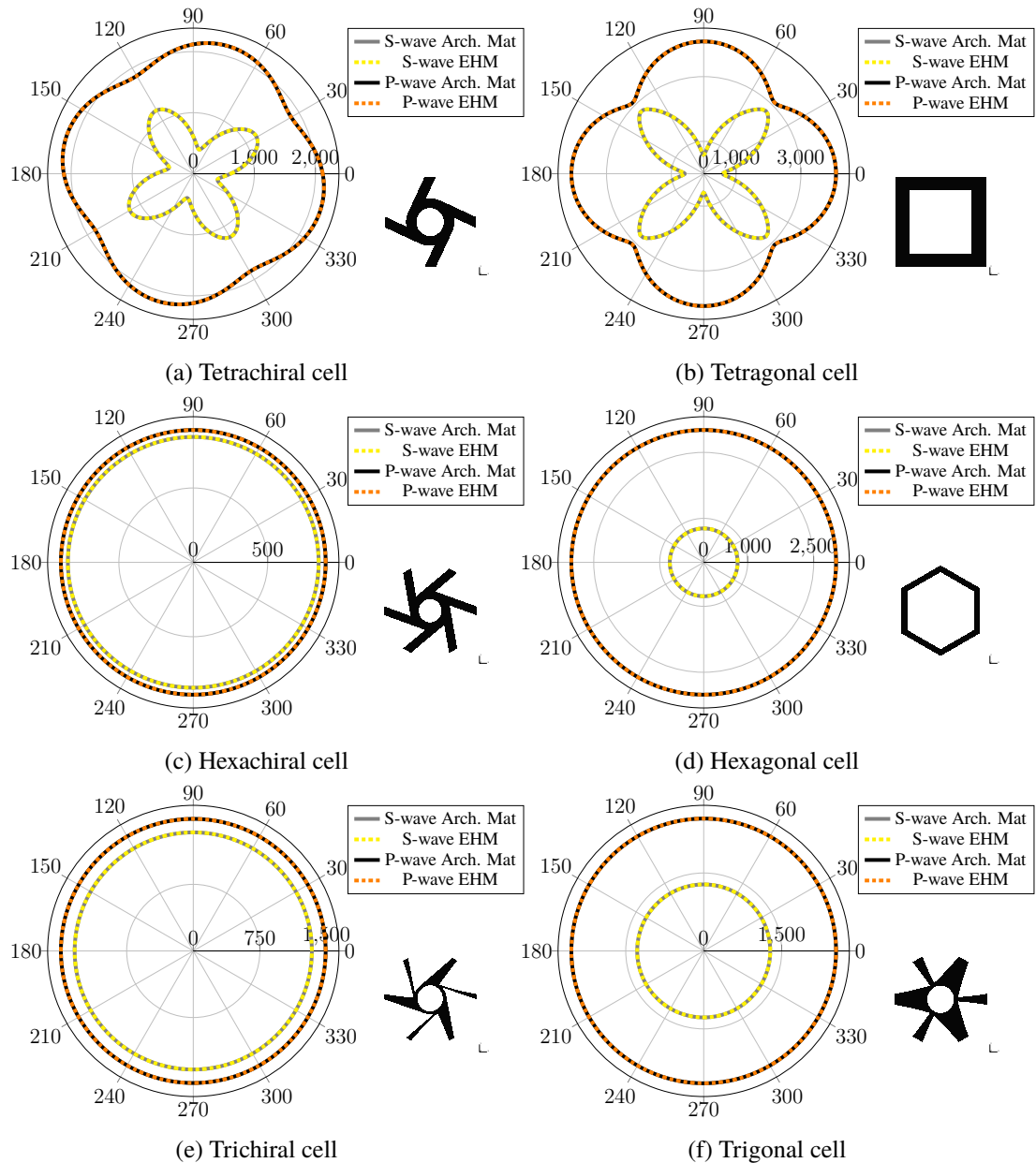
where  $v_P(\theta)$  and  $v_S(\theta)$  are respectively the longitudinal and the transverse wave velocities in the direction of angle  $\theta$ .

## 2.3 Elastodynamic homogenization approach

The main idea of elastodynamic homogenization approach in the LW-LF regime consists of solving the wave propagation problem over a unit cell using the Floquet-Bloch analysis, and subsequently identifying the harmonic decomposition elements of the effective elasticity tensor. Given some directions of propagation, the relationships (2.9) are numerically computed, and related to its analytical expressions (2.8) which provides a set of equations taking the form of a waterfall system. The resolution of the inverse problem proceed in three steps (3 subsystems): the first step gives the isotropic parameters ( $\alpha_2, \alpha_0$ ), the second and the third steps provide respectively the components of  $\mathbf{h}$  and those of  $\mathbf{H}$ . Once the harmonic decomposition components are identified, the effective elasticity tensor can be reconstructed according to the formula (2.6).

## 3 Numerical results

We solved the wave propagation problem for  $\theta = \{0^\circ, 30^\circ, 45^\circ, 90^\circ\}$  using COMSOL MULTIPHYSICS 5.4 software to compute waves velocities and the Christoffel tensor's invariants, which are the input parameters of the identification algorithm. In order to validate our method, we calculated the effective Christoffel tensors from the identified elasticity tensors  $\mathbf{C}^H$  and the corresponding phase velocities (2.9), then we compared them to phase velocities in the architected materials obtained via the Floquet-Bloch analysis. The numerical results of the wave velocities within different architected materials are shown in the figures below.



## 4 Conclusion

In this study, we proposed a new method of homogenization of architected materials using Bloch-waves in the LW-LF approximation. The study of wave propagation provides useful informations to characterise the effective properties of the medium, by exploiting the Christoffel tensor. The proposed parameterisation of the Christoffel tensor gives a clear physical interpretation of the effective properties without making any assumptions with respect to the symmetry class. Results obtained from this approach agree well with the LW-LF behaviour of the real architected material for different geometries:

- tetrachiral and tetragonal anisotropy are well captured by the Cauchy model;
- isotropic behaviour for hexagonal and hexachiral honeycombs;
- trichiral and trigonal lattices behave isotropically.

Despite the fact that Cauchy model is not rich enough to capture certain physical phenomena related to the mesostructure effects [9, 4], the elastodynamic homogenization approach studied in this article offers simple and efficient approximation in the LW-LF regime, and can be enriched and extended to predict the behaviour outside the LW-LF regime [10, 11].

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