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# Existence of minimizers in the variational approach to brittle fracture

Antonin Chambolle<sup>[0000-0002-9465-4659]</sup>

**Abstract** We review the variational approach to crack growth of Francfort and Marigo, based on incremental *free discontinuity problems* in linearized elasticity, and the successive techniques used to establish the existence of minimizers for such problems. In particular, we mention the crucial Korn-Poincaré inequality for functions with small jump set (proved with S. Conti and G. Francfort) which is at the root of many subsequent important results. In the last section, we give, in the much simpler scalar case, an overview of the proofs of a Poincaré inequality with small jump set.

## 1 Introduction: the variational approach to brittle fracture

The classical theory of crack growth in linear-elastic brittle materials assumes that the bulk (elastic) energy dissipated to increase a crack by a certain amount is proportional to the length (in 2D) or surface (in 3D) of the new crack. This was formalized by Griffith [65, 66] in the early XXth century. In a modern formulation of that theory, the following situation is considered:

- One is given a reference configuration  $\Omega \subset \mathbb{R}^2$  of a linear-elastic material;
- On some part of the boundary  $\partial^D \Omega \subset \partial \Omega$ , a boundary load  $U^0$ , possibly depending on time, is prescribed;
- One is also given a crack path  $\Gamma(\ell)$ ,  $\ell \geq 0$ , which is a curve parameterized by its length.

For any  $\ell \geq 0$ , one can compute the energy given the crack  $\Gamma([0, \ell])$  of length  $\ell$ :

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$$\mathcal{W}(\ell) = \min \left\{ \frac{1}{2} \int_{\Omega \setminus \Gamma([0, \ell])} \mathbb{C}e(u) : e(u) \, dx \ : \ u = U^0 \text{ on } \partial^D \Omega \right\}. \quad (1)$$

Here, one assumes  $u \in H_{\text{loc}}^1(\Omega \setminus \Gamma([0, \ell]); \mathbb{R}^2)$ ,  $e(u) = (Du + Du^T)/2$  is its symmetrized gradient, and  $\mathbb{C}$  is a symmetric tensor defining a coercive quadratic form of  $e(u)$  (yet which does not control the skew-symmetric part of the gradient). Denoting  $\sigma$  the internal stress, the relationship  $\sigma = \mathbb{C}e(u)$  is the *Hooke's law*. The modeling behind this form is very classical [36, Sec. 6.2]: if the configuration  $\Omega$  is deformed by some smooth diffeomorphism into  $\Phi(\Omega)$ , then typically a first order, frame invariant energy will depend only on the singular values of  $D\Phi$ , hence be a unitary invariant function of  $(D\Phi)^T D\Phi$ . Hence, if the deformation is given by an infinitesimal displacement of the form  $\Phi(x) = x + \varepsilon u(x)$ , using that  $(D\Phi)^T D\Phi = I + \varepsilon(Du^T + Du) + \varepsilon^2 Du^T Du$ , at the first order the elastic energy is a function of  $e(u)$ . The specific form (1) is rigorously derived by  $\Gamma$ -convergence arguments in [44]. A typical Hooke tensor  $\mathbb{C}$ , for an isotropic linear-elastic material, is defined by

$$\mathbb{C}e(u) = \lambda \text{Tr } e(u) I + 2\mu e(u)$$

with  $\lambda, \mu$  the *Lamé constants*, see [36, Sec. 3.8].

It is clear that if the crack length increases (freezing the boundary load at the current time), then the value of  $\mathcal{W}(\ell)$  will decrease since the minimization in (1) is performed on a larger set. Then, a fundamental quantity is the *energy release rate*  $-\mathcal{W}'(\ell) \geq 0$ , which expresses the amount of elastic energy is released by the growth of the crack of one unit of length. The idea behind Griffith's theory is that the amount of energy dissipated to increase the crack by a given length should be proportional, with some factor  $\gamma > 0$  (the *toughness*), to this additional length, so that the crack will actually grow as soon as this release rate reaches this threshold.

At time  $t \geq 0$ , assuming the crack has reached a length  $\ell = \ell(t)$  (and is thus given by  $\Gamma([0, \ell(t)])$ ), Griffith's idea postulates therefore that at equilibrium:

- either  $-\mathcal{W}'(\ell(t)) < \gamma$  and  $\ell$  remains constant;
- or  $-\mathcal{W}'(\ell(t)) = \gamma$  and  $\ell$  may increase.

This is summarized by the following rules:

$$\begin{cases} -\mathcal{W}'(\ell) \leq \gamma; \\ (\gamma + \mathcal{W}'(\ell))\dot{\ell} = 0. \end{cases} \quad (2)$$

In general, to consider a situation where a crack should actually grow, one may consider an increasing boundary load. In practice, in this short note, we will consider as in [55] a load of the form  $t \mapsto tU_0$  on  $\partial^D \Omega$ . More general assumptions require that the boundary load be absolutely continuous (for instance, [46] and subsequent works).

This theory of Griffith is widely considered as valid (in a very simplified framework) by the mechanics community, has a good predicting power, and is the basis for many finite elements (mostly "XFEM" elements, which incorporate discontinuities)

implementations in the mechanical engineering community [83, 1, 86], yet it has some limitation:

1. crack initiation is theoretically impossible;
2. the crack path should be known in advance;
3. it is hard to extend properly to the more realistic 3D setting.

For the first point, to be more precise, it is known that in a smooth enough domain, if the minimizer of (1) for  $\ell = 0$  has no strong singularity (that is, a point  $x$  such that  $\int_{B_r(x)} |e(u)|^2 dx \sim r$  as  $r \rightarrow 0$ ), then the derivative  $\mathcal{W}'(0) = 0$ . In particular, in the proposed setting where the load is of the form  $tU^0$  on  $\partial^D\Omega$ , the solution will be at each “time”  $tu^0$ , where  $u^0$  solves (1) for  $\ell = 0$ , and the energy release rate at any possible “crack tip” (for no crack, though) will remain 0 whatever  $t$  and the load. This is of course absurd for a brittle material. On the other hand, if  $\ell(0) > 0$ , then it is known [67, 40, 14] that at the tip  $x = \Gamma(\ell(0))$ , one will have  $\int_{B_r(x)} |e(u^0)|^2 dx \sim r$  as  $r \rightarrow 0$ , and in particular  $\mathcal{W}'(\ell(0)) < 0$ . Then for the load  $t \mapsto tU^0$ , the energy release rate will grow quadratically with  $t$  and certainly reach  $\gamma$  at some point, where the crack will start to grow (continuously).

For the second point, there have been many attempts to understand the possible direction of an increase of a crack at a tip, when the load increases. Typically, one should evaluate the energy release rate as a function of the direction of the crack starting from the tip. Then, one selects a rule to choose the direction. This is done at the cost of complicated calculation (see for instance the non exhaustive list [78, 11, 13, 50]), and the rule seems not entirely clear [11]: whether one should favor stability (and hence ask that the new crack grows in a direction where the crack opening is in “mode I”, that is, without shearing further the tip and provoking immediately a new turn), or an energetic criterion (and hence ask that the crack grows in the first direction where the release rate reaches the toughness). Both criteria seem logical yet they also seem incompatible, and one way to resolve this paradox could be to admit that, as well as for crack initiation, in case the crack has to kink and change direction it can only do so in a “brutal” way, by increasing of a finite (nonzero) amount of additional length [33, 34]. Of course, we reach here a situation where the model is too simplified to describe what really should happen at small scales for a material in such a physical situation. In addition, these models which are purely quasi-static cannot capture the dynamics of brutal changes in the material configuration. For extensions to 3D linearized elasticity, the mathematical analysis is much harder and more technical, see for instance [88, 79, 80, 76, 77].

In the paper [55], the authors considered Griffith’s theory with a new point of view. Instead of fixing the crack path and viewing  $\ell(t)$  as the unknown, they proposed to view also the path as a global variable. This was inspired by previous models of damage [54] and the emerging interest for *free-discontinuity problems* [48, 8], and in particular the study of the *Mumford-Shah functional* [84] for image segmentation. These new models led to the introduction of a whole lot of analytic tools (based on geometric measure theory and *BV* spaces [52, 51, 95, 8]) and it was early proposed to use them in the modeling of fracture [56, 5]. In [56, 55], the authors pushed the idea a bit further by introducing an incremental model for computing crack evolutions

under a growing load, and a limiting “time”<sup>1</sup>-continuous model which generalizes Griffith’s evolutions. The setting is as before (in what follows,  $\Omega$  may be a subset of  $\mathbb{R}^d$  for any  $d \geq 1$ , the dimensions  $d = 2, 3$  being the physical dimensions), except that now  $\Gamma \subset \Omega \cup \partial^D \Omega$  is part of the unknown. In the incremental model, one will start with  $t_0 = 0$ ,  $\Gamma^0$  possibly empty, and find for any  $k \geq 1$ , given  $t_k \geq t_{k-1}$ , a new configuration  $(u^k, \Gamma^k)$  which solves:

$$\min \left\{ \frac{1}{2} \int_{\Omega \setminus \Gamma} \mathbb{C}e(u) : e(u) \, dx + \gamma \mathcal{H}^{d-1}(\Gamma) : u = t_k U^0 \text{ on } \partial^D \Omega \setminus \Gamma, \Gamma \supseteq \Gamma^{k-1} \right\}. \quad (3)$$

Here,  $\mathcal{H}^{d-1}$  is the  $(d-1)$ -dimensional Hausdorff measure (imposing thus that  $\Gamma$  be an at most  $(d-1)$ -dimensional set, and in particular with vanishing Lebesgue measure), see for instance [52, 51, 8]. The last constraint accounts for the non-reversibility of the crack, which may only grow. We will denote by  $\mathcal{E}(u, \Gamma)$  the energy appearing in (3).

A time-continuous evolution is obtained in the limit  $\sup_k (t_{k+1} - t_k) \rightarrow 0$ , and one hopes to converge to a configuration  $u(t), \Gamma(t)$  which satisfies

- (*irreversibility*)  $\Gamma^0 \subseteq \Gamma(t) \subseteq \Gamma(s)$  for any  $0 \leq t < s$ ,
- (*unilateral minimality*) for each  $t > 0$ ,  $u(t) = tU^0$  on  $\partial^D \Omega \setminus \Gamma(t)$  and  $(u(t), \Gamma(t))$  is a *unilateral minimizer*:

$$\mathcal{E}(u(t), \Gamma(t)) \leq \mathcal{E}(v, K)$$

for each  $K \supseteq \bigcup_{s < t} \Gamma(s)$  and any  $v \in H_{\text{loc}}^1(\Omega \setminus K; \mathbb{R}^d)$  with  $v = tU^0$  on  $\partial^D \Omega \setminus K$ .

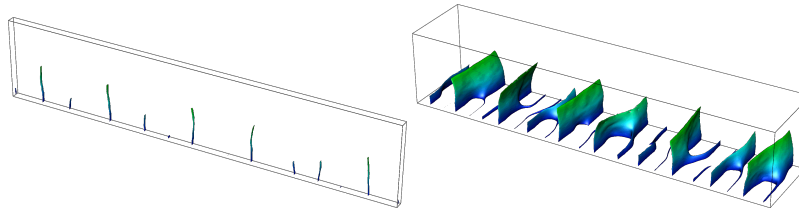
- (*continuity and energy conservation*)  $t \mapsto \mathcal{E}(u(t), \Gamma(t))$  is absolutely continuous and one has  $d\mathcal{E}(u(t), \Gamma(t))/dt = \int_{\partial^D \Omega \setminus \Gamma(t)} \mathbb{C}e(u(t)) : (U^0 \odot \nu_\Omega) d\mathcal{H}^{d-1}$  for a.e.  $t \geq 0$ .

(Here, for  $a, b \in \mathbb{R}^d$ ,  $a \odot b = \frac{1}{2}(a_i b_j + a_j b_i)_{i,j=1}^d$ .) The last condition is written in a weaker sense in the original work [55], and is derived in this form first in [46] in dimension 2, for a simplified scalar setting. The linearized-elasticity counterpart of the results of [46] can be found in [24]. A fundamental result which links the variational theory to the classical one is [55, Prop. 4.8]. This result shows that if  $\Gamma(t)$  is growing along a rectifiable curve and satisfies the properties above, then it is a classical evolution in the sense of Griffith’s theory.

The great improvement of this approach upon the classical one is that it solves all three issues (1)-(2)-(3) mentioned before, that is, initiation (yet, most probably brutal, that is, not satisfying the continuity axiom above at least at  $t = 0$  [35, 33]) becomes possible (and will always occur), the crack path is determined by the process, and the formulation is the same in two or three dimensions (or any dimension). This comes at several costs. One important issue, for instance, is the fact that the *global* minimization introduced in the model (or in the axiom of unilateral minimality) is nonphysical (in particular taking into account internal or boundary forces would be

<sup>1</sup> In the quasistatic models described in this note, the time is just a parameter accounting for non-reversibility but should not be considered as a precise physical time.

impossible, since one may reach arbitrarily low energies by breaking the material into small pieces sent to infinity, corresponding to an infinite work of the forces). Some progress towards considering local minimization was developed in [45, 74, 75]. An analysis of the differences between local and global minimizations is proposed in [85]. The authors provide an example where the variational solution is different from the classical one.



**Fig. 1** Numerical simulation: an almost 2D hot ceramic plate (left) and a thicker plate (right) plunged into cold water at the bottom (experimental setup described in [21], simulation by Blaise Bourdin)

Yet, the fact that evolutions computed with this approach are compatible with Griffith’s theory, and even more the link with the phase-field approach independently developed in the physics community [70, 68, 87] (this link is rigorously established through  $\Gamma$ -convergence arguments for the Ambrosio-Tortorelli approximation of (3) see [10, 20, 69, 28, 57]), have turned this theory more and more into a “standard tool” for fracture modeling and computation (see for instance [82, 90, 71]). The quality of the crack growth computed by this approach and their fidelity to the physical experiments are sometimes remarkable: for instance, Figure 1, computed by Blaise Bourdin [21], shows numerical simulations in accordance with the experiments from [91] (a hot ceramic sample is brutally cooled and exhibits oscillating crack patterns starting from the stressed bottom boundary). See also [21, Fig. 3].

On the other hand, this theory has raised numerous mathematical difficulties. While existence of minimizers for the Mumford-Shah functional was proved only a few years after its introduction in the late 80s [84, 43, 49], it was not until about 2010 that Gianni Dal Maso introduced in [42] an appropriate energy space for Griffith’s problem (3), the space “ $GSBD_2(\Omega)$ ”. Before this, although many results were already available in simplified settings (with additional bounds in the space  $SBD$  [6, 17], or in dimension 2 with a connectedness assumption on the crack [46, 24]), it was difficult to tackle the problem in its full generality and reproduce the programme developed for Mumford-Shah. We explain this in this note: in the next Section, we show how existence for the so-called “weak” formulation has been established in a series of papers after 2016. Then in Section 2.2 we sketch an explanation of how the weak solutions are shown to be also strong minimizers of (3). Some open issues are raised in Section 3. In Section 4, we describe in the scalar setting some of the techniques for proving the rigidity estimate of [25], stated below. This result is the starting point for most of the recent progress in this field.

## 2 Existence for Griffith's variational problem

### 2.1 Poincaré-Korn inequality with small jump set and compactness

There are two main issues for showing existence of the minimizers of (3). First, one needs to get a control on  $u$  (or on a minimizing sequence). This is not natural as a displacement  $u$  which has a very large value in a small ball  $B$  with  $\partial B \subset K$  may have a very low energy. Then, one has to show that an almost minimizer (or the limit of a minimizing sequence) has a crack set  $K$  which can be properly described as a closed  $(d - 1)$ -dimensional set with bounded surface, and does not spread out everywhere in the domain. In the scalar setting, the first issue corresponds to the compactness and lower-semicontinuity result shown in [4] for the Mumford-Shah energy, the second to the density estimates proved in [49] and which allow to show that weak minimizers are strong solutions of the Mumford-Shah problem (see also [43] for an alternative approach). In the setting of linearized elasticity, the situation is far more complex. The main issue for showing compactness is the lack of control of the maximum of  $|u|$ , indeed, if one adds an artificial box constraint on the displacement, then the compactness and lower-semicontinuity result proved in [17] is sufficient to show existence of weak minimizers to (3), in the framework of the space  $SBD(\Omega)$ . This latter space is a subset of  $BD(\Omega)$ , the space of displacements  $u \in L^1(\Omega; \mathbb{R}^d)$  such that the distributional symmetrized gradient  $Eu$  is a bounded Radon measure [92, 93]. Yet, without such a box constraint, there is no result which ensures that a minimizing sequence should remain in that space.

A first progress was accomplished in [25] where the following *Poincaré-Korn inequality for functions with small jump set* was proved. We quote [25, Prop. 3.1]:

**Proposition 1** *Let  $Q = (-r, r)^d$ ,  $Q' = (-r/2, r/2)^d$ ,  $u \in SBD(Q)$  and  $p \in [1, \infty)$ . We have the following:*

- (i) *There exist a set  $\omega \subset Q'$  and an affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $e(a) = 0$  such that*

$$|\omega| \leq cr \mathcal{H}^{d-1}(J_u)$$

and

$$\int_{Q' \setminus \omega} |u - a|^{dp/(d-1)} dx \leq cr^{d(p-1)/(d-1)} \left( \int_Q |e(u)|^p \right)^{d/(d-1)}.$$

- (ii) *If, additionally,  $p > 1$ , then there is  $q > 0$  (depending on  $p$  and  $n$ ) such that, for a given mollifier  $\varphi_r \in C_c^\infty(B_{r/4})$ ,  $\varphi_r(x) = r^{-d}\varphi_1(x/r)$ , the function  $v = u\chi_{Q' \setminus \omega} + a\chi_\omega$  obeys*

$$\int_{Q''} |e(v * \varphi_r) - e(u) * \varphi_r|^p dx \leq c \left( \frac{\mathcal{H}^{d-1}(J_u)}{r^{d-1}} \right)^q \int_Q |e(u)|^p dx,$$

where  $Q'' = (-r/4, r/4)^d$ .

The constant in (i) depends only on  $p$  and  $d$ , the one in (ii) also on  $\varphi_1$ .

In this statement,  $SBD(Q)$  stands for the space of “special functions with bounded deformation”. These functions  $u$  are such that their symmetrized distributional gradient  $Eu = \frac{1}{2}(Du + Du^T)$  is a bounded Radon measure decomposed as

$$Eu = e(u)dx + (u^+ - u^-) \odot \nu_u \mathcal{H}^{d-1} \llcorner J_u.$$

In this expression,  $e(u)$  is the absolutely continuous part of  $Eu$  with respect to the Lebesgue measure  $dx$ , and  $J_u$  is the set of jump points  $x$ , that is, where the blow-ups  $y \mapsto u(x + ry)$  converge, as  $r \rightarrow 0$  and in  $L^1(B_1; \mathbb{R}^d)$ , to a function taking two values  $u^\pm$  on both sides of a hyperplane  $\nu_u^\pm$ . This intrinsic jump set is known to be a  $(d - 1)$ -countably rectifiable set, that is, it may be covered by countably many  $C^1$  hypersurfaces up to a  $\mathcal{H}^{d-1}$ -negligible set with normal vector  $\nu_u$   $\mathcal{H}^{d-1}$ -almost everywhere. We refer to [6, 17] for details.

We mention that roughly at the same time, a series of similar Poincaré-Korn-type inequalities were proved by Manuel Friedrich. In a first paper [58], the result above (point (i)) was proved in dimension 2 with different techniques, leading in particular (for  $p = 2$ ) to a control on the *perimeter* of the exceptional set  $\omega$ , and an actual *Korn* inequality, that is, a control on  $|\nabla u|$ , outside of  $\omega$ . This was remarkable as in particular, the techniques of [25], described in Section 4, certainly cannot give any control on the perimeter of  $\omega$ . A following paper [59] was stating a much more precise inequality in dimension 2 (splitting  $\Omega$  into a partition of sets where  $u$  was controlled up to a different rigid motion in each set), and showing, in arbitrary dimensions, a variant of our main result [25, Theorem 1.1], valid for  $p = 2$  only but also with an estimate of the perimeter of the exceptional set, and a more precise control of  $u$  (in particular,  $u\chi_{\omega^c}$  is shown to be in  $BV$ ): see [59, Theorem 2.10]. Similar results were proved afterwards with the techniques of [25], first in dimension 2 [38], then in general [22]. In both latter references, the inequality is for any  $p > 1$ , the perimeter of  $\omega$  is controlled by the size of the jump, and it is shown that out of  $\omega$ ,  $u$  coincides with a function  $v \in W^{1,p}(Q; \mathbb{R}^d)$  so that in particular also  $\nabla u$  is controlled.

While the proofs in [25] are written in the so-called space  $SBD(Q)$  (as that work started actually before [42] was announced), the proof applies to the more general space  $GSBD(Q)$  introduced in [42] as the correct energy space for (3). Indeed, the definition of this space is based only on 1D slicing properties of the functions, upon which the proof of Proposition 1 mostly relies: see Section 4 where we explain the main mechanisms in the simpler scalar case.

Proposition 1 turned out to be the starting point for many developments: first, after some time, we realized with Vito Crismale that this was sufficient to obtain compactness for minimizing sequences for (3). The idea, developed in [30], is quite simple. One introduces a scale  $\delta > 0$ , covers the domain with cubes  $Q_{z,\delta} = \delta z + [0, \delta]^d$ ,  $z \in \mathbb{Z}^d$  (considering only the cubes inside  $\Omega$ ), and for each function of the minimizing sequence  $(u_n, \Gamma_n)_n$  one introduces  $\omega_n^{z,\delta}$  and  $a_n^{z,\delta}$  obtained by invoking Proposition 1 in each cube  $Q_{z,\delta}$  (or actually, in slightly larger cubes). We obtain piecewise affine functions  $a_n = \sum_z a_n^{z,\delta} \chi_{Q_{z,\delta}}$  with



$$\int_{\Omega \setminus \omega_n} |u_n - a_n|^2 dx \lesssim \delta^2 \int_{\Omega} |e(u_n)|^2 dx,$$

where  $\omega_n = \bigcup_z \omega_n^{z, \delta}$  (plus a small set near  $\partial\Omega$ ) satisfies

$$|\omega_n| \lesssim \delta \mathcal{H}^{d-1}(\Gamma_n).$$

(In practice, one works with the “weak” formulation, where  $\Gamma_n$  is to be replaced with the intrinsic jump set  $J_{u_n}$  of  $u_n$ , which might not be closed.)

Thanks to the fact that  $(a_n)_n$  is finite-dimensional, one can extract a subsequence which converges (up to pieces which may go to infinity), and as  $\delta$  gets smaller, we end up with showing that (a subsequence of)  $(u_n)_n$  is a Cauchy sequence (in some sense), converging either to infinity or to a limiting  $u$ .

The set (denoted by  $A$  in [30]) where  $|u_n| \rightarrow \infty$  is not really an issue, as replacing  $u$  with 0 in that set, one obtains (reproducing almost verbatim the arguments in [17]) lower semicontinuity of the energy and that the limit is actually a minimizer of the weak formulation.

A natural question which emerges is “what happens in the set  $A$ ”? This seems of little importance (as one eventually replaces  $u$  with 0 in  $A$ ), yet in more general problems where the elastic energy is not homogeneous and may not be minimal for  $e(u) = 0$  (consider for instance the case of a pre-stress as in [91, 21], cf Fig. 1), then this trick will not work. This issue has been addressed in [32, 31]. The main result in [31], inspired from previous results in *GBD* [3] and *GSBV* [60], shows that if  $(u_n)_n$  is a bounded sequence for the energy (3), then there exists a subsequence, a (Cacciopoli, that is with bounded perimeter) partition  $(E_i)_i$  of  $\Omega$ , and sequences of infinitesimal rigid motions  $(a_n^i)_n$  (for each  $E_i$ ), such that

$$u_n - \sum_i a_n^i \chi_{E_i} \rightarrow u \quad \text{a.e.}$$

and the energy of the limit  $u$  is bounded by the energies of  $(u_n)_n$ . See [31] for details.

## 2.2 Existence of strong solutions

Once weak minimizers to (3) have been found, one should show that they are “strong”, that is, their discontinuity set  $J_u$  (which, in general, might be not closed or might even be dense) is concentrated rather than spread out, so that its closure  $\Gamma = \overline{J_u}$  will have the same surface (Hausdorff measure) and  $(u, \Gamma)$  is a minimizer of (3). This is where Proposition 1 (ii) played a major role.

It is well-known since this issue was solved in [49] for the Mumford-Shah functional, that this follows from a *lower density estimate* of the form:

*There exists  $r_0 > 0$  and  $\beta > 0$  such that for any  $x \in J_u$  and any ball  $B_r(x) \subset \Omega$  with  $r \leq r_0$ , then*

$$\mathcal{H}^{d-1}(J_u \cap B_r(x)) \geq \beta r^{d-1}.$$

Indeed, it follows easily that the same holds for  $\overline{J_u}$ , then that the latter set has finite surface (see for instance [8, Theorem 2.56]), and that this surface is the same as the surface of  $J_u$  (at least inside  $\Omega$ ).

For (3), this was first shown in dimension 2 in [37, 39]. Then, this was extended to arbitrary dimensions in [27], while in the paper [29], with Vito Crismale, we showed the full existence to minimizers of (3) by establishing the density estimate up to the Dirichlet boundary (when the crack is along this boundary), see [29, Corollary 5.8].

Proposition 1 (ii) is crucial for establishing the lower density in [27, 29]. Indeed, it allows to build, in a ball  $B_r$  of radius  $r$  where  $\mathcal{H}^{d-1}(J_u \cap B_r) \ll r^{d-1}$ , a competitor  $v$  which coincides with  $u$  near  $\partial B_r$  but is smooth inside a smaller ball and has much smaller jump. This is the basic brick of a ‘‘Decay Lemma’’, following the main idea of [49]: one shows that for  $\tau < 1$ , there exist  $r_0, \varepsilon$ , such that for a minimizer, if  $\mathcal{H}^{d-1}(J_u \cap B_r(x)) < \varepsilon r^{d-1}$ ,  $r \leq r_0$  then the energy in the ball  $B_{\tau r}(x)$  is controlled by  $\tau^{d-1+s}$  times the energy in the ball  $B_r(x)$  for some  $s > 0$ . This is shown by contradiction: assuming this is not true, one finds a sequence of (quasi) minimizers  $(u_r)$  in balls with vanishing radius  $r$ , with vanishing jump and for which the energy does not decay at the given rate. Scaling everything to the ball of radius 1 and using the previous construction, one can replace these functions by smooth competitors in most of the ball, showing that in the limit the energy should behave like the energy of smooth functions without jump, and decay like the volume of the ball (that is,  $\tau^d$ ), which gives a contradiction.

Once a Decay Lemma is proven, the conclusion follows: indeed, at rectifiability points  $x$  of  $J_u$  ( $\mathcal{H}^{d-1}$ -almost all points), one has  $\lim_{r \rightarrow 0} \mathcal{H}^{d-1}(J_u \cap B_r(x))/r^{d-1} = \omega_{d-1}$ , the volume of the  $(d-1)$ -ball. But if at larger scales  $r \sim r_0$  one had  $\mathcal{H}^{d-1}(J_u \cap B_r(x)) \leq \varepsilon r$ , the Decay Lemma would imply that the energy  $\mathcal{E}(u, B_r(x)) = \frac{1}{2} \int_{B_r(x)} \mathbb{C}e(u) : e(u) dx + \gamma \mathcal{H}^{d-1}(J_u \cap B_r(x))$  decays too fast, with  $\mathcal{E}(u, B_r(x))/r^{d-1} \rightarrow 0$  as  $r \rightarrow 0$ : a contradiction since this limit is at least  $\gamma \omega_{d-1}$ .

### 2.3 Regularity of the jump set

One natural question, once the existence of minimizers to (3) has been established, is the regularity one can hope to show for the crack set  $\Gamma$ . This is a tough question already in the Mumford-Shah case, the full ‘‘Mumford-Shah conjecture’’ [84] (the jump set in 2D is a finite union of smooth curves with possible endpoints, meeting only at triple point with three angles of  $120^\circ$ , and at the boundary with an angle of  $90^\circ$ ) being still only proved for isolated connected components of  $\Gamma$  [18, 19]. In higher dimensions, [9, 7] have first shown that  $\Gamma$  is regular up to a set of dimension smaller than  $d-1$  (see also [8]).

For the Griffith energy, very important progress has been accomplished in the past few years, towards showing similar results. First in [16], it is proved (in 2D) that up to a  $\mathcal{H}^1$ -null set, any isolated connected component of  $\Gamma$  is  $C^{1,\alpha}$  for some  $\alpha > 0$ . This is extended to arbitrary dimensions in [72], yet under the assumption that locally, the jump set is separating a ball into two ‘‘big’’ pieces (and possibly

smaller ones). Finally, the authors of [62] have considerably improved in 2D the results of [16], showing that  $K$  is in fact, almost everywhere locally a  $C^{1,1/2}$  curve, up to a set of Hausdorff dimension strictly below 1, yet, as for the Mumford-Shah functional, it is not known whether the singular set is finite. In higher dimensions, removing the separating assumption is also still a challenge.

### 3 Extensions and open problems

Among the other difficult issues which remain open, the most important from the modeling point of view is the existence of generalized Griffith evolutions with continuously growing crack: to complete the theory introduced in [55], one would like to be able to send the “time-steps”  $t_{k+1} - t_k$  to 0 in the most general setting. In the scalar setting, there are a few situations where this is known, since the work [46] (in 2D, with the assumption that the crack is connected), extended then in the strong setting to the general 2D case in [15]. In the weak formulation, the general scalar case was first addressed in [53] where the celebrated “jump transfer lemma” was proven. This lemma shows that under some assumptions, a sequence of minimizers of Mumford-Shah type problems converges to a minimizer, and, unfortunately, no such results exists for the Griffith energy in the general setting. This lemma was proved in dimension 2 in [64], whose authors could therefore show existence of continuous evolutions in the weak setting. (Until then, the only clear situation was the extension of [46] to linearized elasticity in [24], assuming connectedness of the 1D crack.) Establishing the existence of continuous-time evolution in higher dimension, or simply of strong evolutions (with a closed crack  $\Gamma$ ) in 2D as in [15], still seems very difficult.

For similar reasons, homogenization (most of the time discussed in the scalar setting [47, 94] or with scalings which exclude (3) as the limit problem [89], see [23] for a general study) or linearization (of non-linear elasticity models with cracks) remain challenging. A generalization of [44] in the Griffith setting has been proved in 2D by Manuel Friedrich [61] and extended to incorporate a non-interpenetration condition in [2]. Important progress has been made very recently concerning the convergence and integral representation [63, 41] of Griffith-type functionals in arbitrary dimensions and one may expect more results in the next years.

Many other issues remain open in the framework of the phase-field approximations of Griffith’s problem, based on [10] and widely adopted for simulations. Incorporating constraints, such as non-interpenetration [12, 73], is tricky to rigorously justify and well-understood only in 2D [26]. One technical reason behind this issue is that in 2D, one can easily approximate a displacement with very small jump set by smooth functions without adding additional jump, as first observed in [38] (as given a ball  $B_r$  such that  $\mathcal{H}^1(B_r \cap J_u) \ll r$ , there will be many circles  $\partial B_s$ ,  $s < r$ , with  $J_u \cap \partial B_s = \emptyset$  and one can select radii  $s$  where it is possible to wipe out the jump inside  $B_s$  without changing too much the energy), while this is in general impossible in higher dimensions without adding some spurious (small) crack set

where the constraint may be impossible to check (as a very tiny jump set could cross all spheres  $\partial B_s$ ,  $s < r$ , in the same setting as before). We observe that in this setting, existence of weak minimizers can be shown using the same techniques as described in this note, yet it remains unclear whether these are also strong minimizers.

## 4 Ideas of proofs (simpler scalar case)

### 4.1 Poincaré inequality with a small jump set (the scalar case)

We give here the main ideas of the proofs in [25], in the (much) simpler case where the functions are real-valued with integrable gradient (and where many alternative results may be derived, such as [49, Thm 3.1]). As for most proofs of Poincaré or Sobolev-type inequalities, the results rely on integrating along lines to get control on a value at some point with values at other points. The same proofs for vectorial fields, involving only the symmetrized gradients, are much more involved, since such integrals can control only the difference of the components along the direction between the two points. Hence, each time such an argument is invoked below, it has to be replaced with a (quite) more complicated construction involving  $d$  different directions in order to gain control on all the components of the field. Throughout all this section, except otherwise stated, we work at scale 1 (and, even, in  $(0, 1)^d$ ), the corresponding statements in a cube  $(-r, r)^d$  are recovered without effort by appropriately scaling the functions and sets.

Consider  $Q = (0, 1)^d$  and  $u \in W^{1,p}(Q \setminus K)$  ( $p \in [1, +\infty)$ ), for  $K \subset Q$  a closed, rectifiable set. We assume  $\mathcal{H}^{d-1}(K)$  is small. The setting could be as well  $SBV(Q)$  or  $GSBV(Q)$  [8], with  $J_u$  replacing  $K$ , and it would not change anything to what follows. In a first step, we prove:

**Proposition 2** *There exists  $C$  depending only on  $d$  and  $p$  such that, given  $u$  and  $K$  as above, there exists  $c \in \mathbb{R}$  and  $\omega \subset Q$  with:*

$$|\omega| \leq C \mathcal{H}^{d-1}(K), \quad (4)$$

$$\int_{Q \setminus \omega} |u - c|^p dy \leq C \int_Q |\nabla u|^p dx. \quad (5)$$

*Proof.* We follow closely [25]. Given  $x, y \in Q$ , we let  $v(x, y) := \mathcal{H}^0(K \cap [x, y])$ , and we observe that if  $v(x, y) = 0$ , then (for a.e.  $x, y$ , or assuming first  $u$  is smooth away from  $K$  and reasoning then by density):

$$u(y) = u(x) + \int_0^1 \nabla u(x + s(y-x)) \cdot (y-x) ds.$$

Then (extending all functions by 0 out of  $Q$ ),

$$\begin{aligned}
& \int_Q \int_Q (1 - v(x, y))^+ |u(x) - u(y)|^p dy dx \\
& \leq \int_Q dx \int_Q dy (1 - v(x, y))^+ \int_0^1 |\nabla u(x + s(y - x))|^p |y - x|^p ds \\
& \leq \int_0^1 ds \int_Q dx \int_{[-1, 1]^d} |\nabla u(x + s\xi)|^p |\xi|^p d\xi \\
& = \int_0^1 ds \int_{[-1, 1]^d} |\xi|^p d\xi \int_{Q - s\xi} |\nabla u(x + s\xi)|^p dx \leq 2^d \sqrt{d}^p \int_Q |\nabla u|^p dx,
\end{aligned}$$

while, similarly,

$$\int_Q \int_Q v(x, y) dx dy \leq \int_{[-1, 1]^d} \int_Q \mathcal{H}^0((x + \mathbb{R}\xi) \cap K) dx d\xi \leq 2^d \sqrt{d} \mathcal{H}^{d-1}(K).$$

Denoting  $\omega_x = \{y : v(x, y) \geq 1\}$ , we deduce that for any  $\eta > 0$ ,

$$\begin{aligned}
& \left| \left\{ x \in Q : |\omega_x| \geq \frac{2^d \sqrt{d}}{\eta} \mathcal{H}^{d-1}(K) \right\} \right| \leq \eta, \\
& \left| \left\{ x \in Q : \int_{Q \setminus \omega_x} |u - u(x)|^p dy \geq \frac{2^d \sqrt{d}^p}{\eta} \int_Q |\nabla u|^p dy \right\} \right| \leq \eta.
\end{aligned}$$

Hence, if  $\eta < |Q|/2 = 1/2$ , there is a set of positive measure of points  $x$  such that

$$|\omega_x| \leq \frac{2^d \sqrt{d}}{\eta} \mathcal{H}^{d-1}(K), \quad \int_{Q \setminus \omega_x} |u - u(x)|^p dy \leq \frac{2^d \sqrt{d}^p}{\eta} \int_Q |\nabla u|^p dy.$$

This shows the result.  $\square$

We now show how to improve the estimate on  $\omega$  to the right dimensional scaling.

## 4.2 Improvement: scaling of the volume estimate

Let now  $u, K$  and  $\omega, c$  as in Proposition 2. Consider the set  $\omega_1^* = \omega_1 \cup \tilde{\omega}_1$ , where

$$\omega_1 = \{x \in Q : (x + \mathbb{R}e_1) \cap K \neq \emptyset\}, \quad \tilde{\omega}_1 = \left\{ x \in Q : \mathcal{H}^1((x + \mathbb{R}e_1) \cap \omega) \geq \frac{1}{2} \right\}. \quad (6)$$

Here,  $(e_1, \dots, e_d)$  is the canonical basis of  $\mathbb{R}^d$ . Clearly,  $\omega_1^* = (0, 1) \times \Pi_1(\omega_1^*)$  is invariant by translation (in  $Q$ ) in the direction  $e_1$ , where  $\Pi_1$  is the projection onto  $e_1^\perp$ . In addition,  $|\omega_1| \leq \mathcal{H}^{d-1}(K)$ , while we have, letting  $Q' = (0, 1)^{d-1}$ :

$$|\omega| = \int_{Q'} \int_0^1 \chi_\omega(x_1, x') dx_1 dx' = \int_{Q'} |\omega \cap ((0, x') + \mathbb{R}e_1)| dx' \geq \frac{|\tilde{\omega}_1|}{2},$$

so that (from (4)):  $|\tilde{\omega}_1| \leq 2C \mathcal{H}^{d-1}(K)$  and  $|\omega_1^*| \leq (2C + 1) \mathcal{H}^{d-1}(K)$ .

Now, if  $x \in Q \setminus \omega_1^*$ , for any  $t$  with  $x + te_1 \in Q$ ,

$$u(x + te_1) = u(x) + \int_0^t \partial_1 u(x + se_1) ds,$$

and, averaging over  $\{t \in \mathbb{R} : x + te_1 \in Q \setminus \omega\}$  we get:

$$u(x) = \int_{(x+\mathbb{R}e_1) \cap Q \setminus \omega} u d\mathcal{H}^1 - \int_{-x_1}^{1-x_1} \frac{\chi_{Q \setminus \omega}(x + te_1)}{\mathcal{H}^1((x + \mathbb{R}e_1) \cap Q \setminus \omega)} \int_0^t \partial_1 u(x + se_1) ds dt.$$

It follows (for any  $c \in \mathbb{R}$  but we consider here the constant from Proposition 2):

$$\begin{aligned} |u(x) - c| &\leq \left| \int_{(x+\mathbb{R}e_1) \cap Q \setminus \omega} u d\mathcal{H}^1 - c \right| + \int_{(x+\mathbb{R}e_1) \cap Q} |\partial_1 u| d\mathcal{H}^1 \\ &\leq 2 \int_{(x+\mathbb{R}e_1) \cap Q \setminus \omega} |u - c| d\mathcal{H}^1 + \int_{(x+\mathbb{R}e_1) \cap Q} |\partial_1 u| d\mathcal{H}^1. \end{aligned} \quad (7)$$

In the last estimate we have used that  $x \notin \tilde{\omega}_1$  so that  $|(x + \mathbb{R}e_1) \cap Q \setminus \omega| \geq 1/2$ . We deduce from Jensen's lemma (or Hölder's inequality):

$$|u(x) - c|^p \leq 2^{p-1} \left( 2^p \int_{(x+\mathbb{R}e_1) \cap Q \setminus \omega} |u - c|^p d\mathcal{H}^1 + \int_{(x+\mathbb{R}e_1) \cap Q} |\partial_1 u|^p d\mathcal{H}^1 \right). \quad (8)$$

Integrating (8) over  $x \in Q \setminus \omega_1^*$ , we get

$$\int_{Q \setminus \omega_1^*} |u - c|^p dx \leq 2^{2p-1} \int_{Q \setminus \omega} |u - c|^p dx + 2^{p-1} \int_Q |\partial_1 u|^p d\mathcal{H}^1, \quad (9)$$

then thanks to (5) we get that for some constant, still denoted  $C$  (depending on  $d$  and  $p$ ),

$$\int_{Q \setminus \omega_1^*} |u - c|^p dx \leq C \int_Q |\nabla u|^p dx. \quad (10)$$

Letting  $\omega^* = \bigcap_{i=1}^d \omega_i^*$  (where  $\omega_i^*$  is defined as  $\omega_1^* = \omega_1 \cup \tilde{\omega}_1$  in (6), but in the direction  $e_i$  instead of  $e_1$ ), we find (changing again  $C$ , which is multiplied by  $d$ ):

$$\int_{Q \setminus \omega^*} |u - c|^p dx \leq C \int_Q |\nabla u|^p dx.$$

Using now that  $|\Pi_i(\omega^*)| \leq C\mathcal{H}^{d-1}(K)$  for  $i = 1, \dots, d$ , we deduce  $|\omega^*| \leq C\mathcal{H}^{d-1}(K)^{d/(d-1)}$  (From Loomis-Whitney's inequality [81], see also [25, Lemma 4.1], or follow the proof in Section 4.3 below, applied to the functions  $f_i = \chi_{\omega_i^*}$ , which are such that  $\chi_{\omega^*} = \prod_{i=1}^d f_i$ ). We have shown:

**Proposition 3** *In Prop. 2, one may replace (4) with the estimate*

$$|\omega| \leq C\mathcal{H}^{d-1}(K)^{\frac{d}{d-1}}. \quad (11)$$

### 4.3 Higher integrability

Now, we show how a better control on  $u - c$  is obtained, using the standard approach to Sobolev's inequalities (see for instance [51, §4.5]). We consider (8) for  $i = 1, \dots, d$ :

$$\chi_{Q \setminus \omega_i^*} |u(x) - c|^p \leq 2^{p-1} \chi_{Q \setminus \omega_i^*} \int_{(x + \mathbb{R}e_i) \cap Q} \left( 2^p (1 - \chi_\omega) |u - c|^p + |\partial_i u|^p \right) d\mathcal{H}^1.$$

Denoting by  $f_i(x)$  the right-hand side, we stress that it does not depend on  $x_i$ . We multiply these inequalities for  $i = 1, \dots, d$  and obtain that for all  $x \in Q$ :

$$f(x) := \chi_{Q \setminus \omega^{**}} |u(x) - c|^{\frac{pd}{d-1}} \leq \prod_{i=1}^d f_i(x)^{\frac{1}{d-1}},$$

where we have denoted  $\omega^{**} = \bigcup_{i=1}^d \omega_i^*$ . Then, we pursue as in [51, p. 139] or [81]:

$$\begin{aligned} \int_0^1 f(x) dx_1 &\leq f_1(x)^{\frac{1}{d-1}} \int_0^1 \prod_{i=2}^d f_i(x)^{\frac{1}{d-1}} dx_1 \leq f_1(x)^{\frac{1}{d-1}} \left( \prod_{i=2}^d \int_0^1 f_i(x) dx_1 \right)^{\frac{1}{d-1}} \\ &= \left( \prod_{i=1}^d \int_0^1 f_i(x) dx_1 \right)^{\frac{1}{d-1}} \end{aligned}$$

(using Hölder's inequality);

$$\begin{aligned} \int_0^1 \int_0^1 f(x) dx_1 dx_2 &\leq \left( \int_0^1 f_2(x) dx_1 \right)^{\frac{1}{d-1}} \int_0^1 \left( \prod_{i \neq 2} \int_0^1 f_i(x) dx_1 \right)^{\frac{1}{d-1}} dx_2 \leq \\ &\left( \int_0^1 f_2(x) dx_1 \right)^{\frac{1}{d-1}} \left( \prod_{i \neq 2} \int_0^1 \int_0^1 f_i(x) dx_1 dx_2 \right)^{\frac{1}{d-1}} = \left( \prod_{i=1}^d \int_0^1 \int_0^1 f_i(x) dx_1 dx_2 \right)^{\frac{1}{d-1}}, \end{aligned}$$

and the same argument for  $i = 3, \dots, d$ . Eventually, we find

$$\int_Q f(x) dx \leq \left( \prod_{i=1}^d \int_Q f_i(x) dx \right)^{\frac{1}{d-1}},$$

and it follows:

$$\begin{aligned} \int_{Q \setminus \omega^{**}} |u(x) - c|^{\frac{pd}{d-1}} dx &\leq 2^{\frac{(p-1)d}{d-1}} \prod_{i=1}^d \left( 2^p \int_{Q \setminus \omega} |u - c|^p dx + \int_Q |\partial_i u|^p dx \right)^{\frac{1}{d-1}} \\ &\leq C \left( \int_Q |\nabla u|^p dx \right)^{\frac{d}{d-1}}, \end{aligned}$$

for some constant depending only on  $d, p$ . By construction, one has here  $|\omega^{**}| \leq C\mathcal{H}^{d-1}(K)$ . By simple scaling arguments, we have eventually proven:

**Proposition 4** *Let  $Q = (-r, r)^d$ ,  $K \subset Q$  with  $\mathcal{H}^{d-1}(K) < +\infty$  and  $u \in W_{\text{loc}}^{1,p}(Q \setminus K)$  with  $\int_{Q \setminus K} |\nabla u|^p dx < +\infty$ ,  $p \in [1, +\infty)$ . Then there exists  $\omega \subset Q$  and  $c \in \mathbb{R}$  such that*

$$|\omega| \leq Cr\mathcal{H}^{d-1}(K)$$

and

$$\int_{Q \setminus \omega} |u - c|^{\frac{pd}{d-1}} dx \leq Cr^{\frac{d(p-1)}{d-1}} \left( \int_Q |\nabla u|^p dx \right)^{\frac{d}{d-1}},$$

where the constant  $C$  depends only on  $d$  and  $p$ .

This is the scalar counterpart of Proposition 1, (i), except that in this setting it is easier to obtain the estimate in the whole cube  $Q$  and not a smaller one. Although we worked with compact sets  $K$  and  $u \in W^{1,p}(Q \setminus K)$ , the experts will notice that the result holds in  $GSBV(Q)$  with the same proof and  $K$  replaced with  $J_u$ . Note that, similarly to the vectorial setting, one cannot easily obtain here the Sobolev exponent  $pd/(d-p)$ . In classical proofs of the Sobolev inequality, this is derived from the case  $p = 1$ , applied to the function  $|u|^q \in W_{\text{loc}}^{1,1}(Q \setminus K)$  (for  $q = p(d-1)/(d-p)$ , and assuming in a first stage that  $|u|$  is bounded). Yet, doing so, one will obtain on the right-hand side an integral of  $|u|^{q-1}|\nabla u|$  on the whole set  $Q$ , which cannot be compared to the left-hand side where  $|u|^{qd/(d-1)}$  is integrated only on  $Q \setminus \omega$ . To get such a result, much more involved proofs have to be developed, still based on Proposition 1, yet using both (i) and (ii)), see [22] for details (in  $GSBD$ ).

Here again the estimate on  $|\omega|$  could be improved as in Proposition 3 at the expense of a more complicated construction, involving intersecting sets in various directions, see [25, §4].

#### 4.4 Proof of Proposition 1, (ii) in the scalar case

Here, there is a priori not too much to simplify with respect to [25]. Yet we found some simplification (also possible for the vectorial case) which we detail below.

We consider as before  $r = 1$ , and as in the statement, a smooth mollifier  $\varphi$  with support in  $B_{1/4}$ . One will obtain a result which is valid in  $(1/4, 3/4)^d \subset\subset Q$ : one could of course, if needed, obtain similar results in larger cubes, by changing/scaling the function  $\varphi$ , yet then the constants depend on  $\varphi$  and will degenerate to  $+\infty$  as the inner cube tends to  $Q$ .

In this section, we need to assume  $p > 1$ . We consider  $\omega$  such that (4)-(5) hold<sup>2</sup>, and assume without loss of generality that  $c = 0$ . We define, as in the statement,  $v = u\chi_{Q \setminus \omega}$ , and we start as in Section 4.3, with the set  $\omega_1^*$  defined as before, and recalling (10).

<sup>2</sup> Actually, it could be the original set in Prop. 2, as well as any of the sets built later on in Prop. 3 or Prop. 4.



We let  $w_1 = u\chi_{Q \setminus \omega_1^*}$  and since the set  $\omega_1^* = (0, 1) \times \Pi_1(\omega_1^*)$ , one may check that the distributional derivative  $\partial_1 w_1$  is in fact  $(\partial_1 u)\chi_{Q \setminus \omega_1^*} \in L^p(Q)$ . In particular,  $\partial_1(\varphi * w_1) = \varphi * \partial_1 w_1 = \varphi * ((\partial_1 u)\chi_{Q \setminus \omega_1^*})$  in  $Q' := (1/4, 3/4)^d$ . We decompose:

$$\partial_1(\varphi * v) - \varphi * (\partial_1 u) = \partial_1 \varphi * (v - w_1) + \varphi * (\partial_1 w_1 - \partial_1 u)$$

and we estimate both parts. We write

$$\|\varphi * (\partial_1 w_1 - \partial_1 u)\|_p \leq \|\varphi\|_p \int_{\omega_1^*} |\partial_1 u| dx \leq C|\omega_1^*|^{1-\frac{1}{p}} \|\nabla u\|_p$$

(where now the constant  $C$  depends on  $d, p$  and  $\|\varphi\|_p$ ). Similarly,

$$\begin{aligned} \|\partial_1 \varphi * (v - w_1)\|_p &\leq \|\partial_1 \varphi\|_p \int_Q |u| |\chi_{Q \setminus \omega} - \chi_{Q \setminus \omega_1^*}| dx \\ &= \|\partial_1 \varphi\|_p \int_Q |u| (\chi_{\omega_1^* \setminus \omega} + \chi_{\omega \setminus \omega_1^*}) dx. \end{aligned}$$

Then,

$$\int_Q |u| \chi_{\omega_1^* \setminus \omega} dx \leq |\omega_1^*|^{1-\frac{1}{p}} \|u\|_{L^p(Q \setminus \omega)} \leq C|\omega_1^*|^{1-\frac{1}{p}} \|\nabla u\|_p,$$

while thanks to (10),

$$\int_Q |u| \chi_{\omega \setminus \omega_1^*} \leq C|\omega|^{1-\frac{1}{p}} \|\nabla u\|_p.$$

We deduce:

$$\|\partial_1(\varphi * v) - \varphi * (\partial_1 u)\|_p \leq C(|\omega_1^*|^{1-\frac{1}{p}} + |\omega|^{1-\frac{1}{p}}) \|\nabla u\|_p \leq C\mathcal{H}^{d-1}(K)^{1-\frac{1}{p}} \|\nabla u\|_p \quad (12)$$

for a constant  $C$  which now depends on  $d, p, \|\varphi\|_p$  and  $\|\nabla \varphi\|_p$ . We have shown:

**Proposition 5** *In the setting of proposition 4, in addition, if  $p > 1$  and  $\varphi \in C_c^\infty(B_{1/4})$  is a smooth mollifier with  $\varphi_r(x) = r^{-d}\varphi(x/r)$ , if we let  $v = u\chi_{Q \setminus \omega} + c\chi_\omega$  then the smooth function  $\varphi_r * v$  obeys:*

$$\int_{Q'} |\nabla(\varphi_r * v) - \varphi_r * \nabla u|^p dx \leq C \left( \frac{\mathcal{H}^{d-1}(K)}{r^{d-1}} \right)^{p-1} \int_Q |\nabla u|^p dx$$

where  $Q' = (-3r/4, 3r/4)^d$ ; the constant  $C$  depends only on  $p, d, \varphi$ .

*Remark 1 (Set  $\omega$  from Proposition 3)* If we perform the previous construction starting from the set  $\omega$  built in Section 4.2, then the exponent  $p - 1$  above is to be replaced with  $d(p - 1)/(d - 1)$ .

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## References

1. Murad S. Abuaisha, David Eaton, Jeffrey Priest, and Ron Wong. Hydraulic fracture simulation using the GEOS code. In *Microseismic Industry Consortium: Annual Research Report, Volume 5*. February 2015.
2. Stefano Almi, Elisa Davoli, and Manuel Friedrich. Non-interpenetration conditions in the passage from nonlinear to linearized Griffith fracture. *J. Math. Pures Appl. (9)*, 175:1–36, 2023.
3. Stefano Almi and Emanuele Tasso. A new proof of compactness in G(S)BD. *Adv. Calc. Var.*, 16(3):637–650, 2023.
4. Luigi Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital. B (7)*, 3(4):857–881, 1989.
5. Luigi Ambrosio and Andrea Braides. Energies in *SBV* and variational models in fracture mechanics. In *Homogenization and applications to material sciences (Nice, 1995)*, volume 9 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, pages 1–22. Gakkōtoshō, Tokyo, 1995.
6. Luigi Ambrosio, Alessandra Coscia, and Gianni Dal Maso. Fine properties of functions with bounded deformation. *Arch. Rational Mech. Anal.*, 139(3):201–238, 1997.
7. Luigi Ambrosio, Nicola Fusco, and Diego Pallara. Partial regularity of free discontinuity sets. II. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(1):39–62, 1997.
8. Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
9. Luigi Ambrosio and Diego Pallara. Partial regularity of free discontinuity sets. I. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(1):1–38, 1997.
10. Luigi Ambrosio and Vincenzo Maria Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via  $\Gamma$ -convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.
11. Michel Amestoy and Jean-Baptiste Leblond. Crack paths in plane situations. II. Detailed form of the expansion of the stress intensity factors. *Internat. J. Solids Structures*, 29(4):465–501, 1992.
12. Hanen Amor, Jean-Jacques Marigo, and Corrado Maurini. Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments. *Journal of the Mechanics and Physics of Solids*, 57(8):1209–1229, 2009.
13. Ivan I. Argatov and Sergei A. Nazarov. Energy release in the kinking of a crack in a plane anisotropic body. *Prikl. Mat. Mekh.*, 66(3):502–514, 2002.
14. Jean-François Babadjian, Antonin Chambolle, and Antoine Lemenant. Energy release rate for non-smooth cracks in planar elasticity. *J. Éc. polytech. Math.*, 2:117–152, 2015.
15. Jean-François Babadjian and Alessandro Giacomini. Existence of strong solutions for quasi-static evolution in brittle fracture. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(4):925–974, 2014.
16. Jean-François Babadjian, Flaviana Iurlano, and Antoine Lemenant. Partial regularity for the crack set minimizing the two-dimensional Griffith energy. *J. Eur. Math. Soc. (JEMS)*, 24(7):2443–2492, 2022.

17. Giovanni Bellettini, Alessandra Coscia, and Gianni Dal Maso. Compactness and lower semi-continuity properties in  $SBD(\Omega)$ . *Math. Z.*, 228(2):337–351, 1998.
18. A. Bonnet. On the regularity of edges in image segmentation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 13(4):485–528, 1996.
19. Alexis Bonnet and Guy David. Cracktip is a global Mumford-Shah minimizer. *Astérisque*, (274):vi+259, 2001.
20. Blaise Bourdin, Gilles A. Francfort, and Jean-Jacques Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids*, 48(4):797–826, 2000.
21. Blaise Bourdin, Jean-Jacques Marigo, Corrado Maurini, and Paul Sicsic. Morphogenesis and propagation of complex cracks induced by thermal shocks. *Phys. Rev. Lett.*, 112:014301, Jan 2014.
22. Filippo Cagnetti, Antonin Chambolle, and Lucia Scardia. Korn and Poincaré-Korn inequalities for functions with a small jump set. *Math. Ann.*, 383(3-4):1179–1216, 2022.
23. Filippo Cagnetti, Gianni Dal Maso, Lucia Scardia, and Caterina Ida Zeppieri.  $\Gamma$ -convergence of free-discontinuity problems. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(4):1035–1079, 2019.
24. Antonin Chambolle. A density result in two-dimensional linearized elasticity, and applications. *Arch. Ration. Mech. Anal.*, 167(3):211–233, 2003.
25. Antonin Chambolle, Sergio Conti, and Gilles Francfort. Korn-Poincaré inequalities for functions with a small jump set. *Indiana Univ. Math. J.*, 65(4):1373–1399, 2016.
26. Antonin Chambolle, Sergio Conti, and Gilles A. Francfort. Approximation of a brittle fracture energy with a constraint of non-interpenetration. *Arch. Ration. Mech. Anal.*, 228(3):867–889, 2018.
27. Antonin Chambolle, Sergio Conti, and Flaviana Iurlano. Approximation of functions with small jump sets and existence of strong minimizers of Griffith’s energy. *J. Math. Pures Appl.* (9), 128:119–139, 2019.
28. Antonin Chambolle and Vito Crismale. A density result in  $GSBD^p$  with applications to the approximation of brittle fracture energies. *Arch. Ration. Mech. Anal.*, 232(3):1329–1378, 2019.
29. Antonin Chambolle and Vito Crismale. Existence of strong solutions to the Dirichlet problem for the Griffith energy. *Calc. Var. Partial Differential Equations*, 58(4):Paper No. 136, 27, 2019.
30. Antonin Chambolle and Vito Crismale. Compactness and lower semicontinuity in  $GSBD$ . *J. Eur. Math. Soc. (JEMS)*, 23(3):701–719, 2021.
31. Antonin Chambolle and Vito Crismale. A general compactness theorem in  $G(S)BD$ . Preprint HAL-03807668, to appear in *Indiana Univ. Math. J.*, October 2022.
32. Antonin Chambolle and Vito Crismale. Equilibrium Configurations For Nonhomogeneous Linearly Elastic Materials With Surface Discontinuities. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, XXIV(3):1575–1610, September 2023.
33. Antonin Chambolle, Gilles A. Francfort, and Jean-Jacques Marigo. When and how do cracks propagate? *J. Mech. Phys. Solids*, 57(9):1614–1622, 2009.
34. Antonin Chambolle, Gilles A. Francfort, and Jean-Jacques Marigo. Revisiting energy release rates in brittle fracture. *J. Nonlinear Sci.*, 20(4):395–424, 2010.
35. Antonin Chambolle, Alessandro Giacomini, and Marcello Ponsiglione. Crack initiation in brittle materials. *Arch. Ration. Mech. Anal.*, 188(2):309–349, 2008.
36. Philippe G. Ciarlet. *Mathematical elasticity. Vol. I*, volume 20 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1988. Three-dimensional elasticity.
37. Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Existence of minimizers for the 2D stationary Griffith fracture model. *C. R. Math. Acad. Sci. Paris*, 354(11):1055–1059, 2016.
38. Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Integral representation for functionals defined on  $SBD^p$  in dimension two. *Arch. Ration. Mech. Anal.*, 223(3):1337–1374, 2017.
39. Sergio Conti, Matteo Focardi, and Flaviana Iurlano. Existence of strong minimizers for the Griffith static fracture model in dimension two. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(2):455–474, 2019.

40. Martin Costabel and Monique Dauge. Crack singularities for general elliptic systems. *Math. Nachr.*, 235:29–49, 2002.
41. Vito Crismale, Manuel Friedrich, and Francesco Solombrino. Integral representation for energies in linear elasticity with surface discontinuities. *Adv. Calc. Var.*, 15(4):705–733, 2022.
42. Gianni Dal Maso. Generalised functions of bounded deformation. *J. Eur. Math. Soc. (JEMS)*, 15(5):1943–1997, 2013.
43. Gianni Dal Maso, Jean-Michel Morel, and Sergio Solimini. A variational method in image segmentation: existence and approximation results. *Acta Math.*, 168(1-2):89–151, 1992.
44. Gianni Dal Maso, Matteo Negri, and Danilo Percivale. Linearized elasticity as  $\Gamma$ -limit of finite elasticity. *Set-Valued Anal.*, 10(2-3):165–183, 2002. Calculus of variations, nonsmooth analysis and related topics.
45. Gianni Dal Maso and Rodica Toader. A model for the quasi-static growth of brittle fractures based on local minimization. *Math. Models Methods Appl. Sci.*, 12(12):1773–1799, 2002.
46. Gianni Dal Maso and Rodica Toader. A model for the quasi-static growth of brittle fractures: existence and approximation results. *Arch. Ration. Mech. Anal.*, 162(2):101–135, 2002.
47. Gianni Dal Maso and Caterina Ida Zeppieri. Homogenization of fiber reinforced brittle materials: the intermediate case. *Adv. Calc. Var.*, 3(4):345–370, 2010.
48. Ennio De Giorgi and Luigi Ambrosio. New functionals in the calculus of variations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 82(2):199–210 (1989), 1988.
49. Ennio De Giorgi, Michele Carriero, and Antonio Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.
50. Philippe Destuynder and Mohamed Djaoua. Sur une interprétation mathématique de l’intégrale de Rice en théorie de la rupture fragile. *Math. Methods Appl. Sci.*, 3(1):70–87, 1981.
51. Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
52. Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
53. Gilles A. Francfort and Christopher J. Larsen. Existence and convergence for quasi-static evolution in brittle fracture. *Comm. Pure Appl. Math.*, 56(10):1465–1500, 2003.
54. Gilles A. Francfort and Jean-Jacques Marigo. Stable damage evolution in a brittle continuous medium. *European J. Mech. A Solids*, 12(2):149–189, 1993.
55. Gilles A. Francfort and Jean-Jacques Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46(8):1319–1342, 1998.
56. Gilles A. Francfort and Jean-Jacques Marigo. Cracks in fracture mechanics : A time indexed family of energy minimizers. In P. Argoul, M. Frémond, and Q. S. Nguyen, editors, *IUTAM Symposium on Variations of Domain and Free-Boundary Problems in Solid Mechanics, Paris, April 1997*, pages 197–202, Dordrecht, 1999. Springer Netherlands.
57. Francesco Freddi and Gianni Royer-Carfagni. Regularized variational theories of fracture: a unified approach. *J. Mech. Phys. Solids*, 58(8):1154–1174, 2010.
58. Manuel Friedrich. A Korn-type inequality in SBD for functions with small jump sets. *Math. Models Methods Appl. Sci.*, 27(13):2461–2484, 2017.
59. Manuel Friedrich. A piecewise Korn inequality in *SBD* and applications to embedding and density results. *SIAM J. Math. Anal.*, 50(4):3842–3918, 2018.
60. Manuel Friedrich. A compactness result in *GSBVP* and applications to  $\Gamma$ -convergence for free discontinuity problems. *Calc. Var. Partial Differential Equations*, 58(3):Paper No. 86, 31, 2019.
61. Manuel Friedrich. Griffith energies as small strain limit of nonlinear models for nonsimple brittle materials. *Math. Eng.*, 2(1):75–100, 2020.
62. Manuel Friedrich, Camille Labourie, and Kerrek Stinson. On regularity for griffith almost-minimizers in the plane. preprint arXiv:2310.07670, 2023.
63. Manuel Friedrich, Matteo Perugini, and Francesco Solombrino.  $\Gamma$ -convergence for free-discontinuity problems in linear elasticity: homogenization and relaxation. *Indiana Univ. Math. J.*, 72:1949–2023, 2023.
64. Manuel Friedrich and Francesco Solombrino. Quasistatic crack growth in 2D-linearized elasticity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 35(1):27–64, 2018.

65. Alan A. Griffith. VI. The phenomena of rupture and flow in solids. *Phil. Trans. Roy. Soc. A*, 221(586):163–198, 1920.
66. Alan A. Griffith. The theory of rupture. In C.B. Biereno and J.M. Burgers, editors, *Proc. 1st Intern. Congr. Appl. Mech., Delft*, pages 54–63, 1924.
67. Pierre Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
68. Vincent Hakim and Alain Karma. Laws of crack motion and phase-field models of fracture. *Journal of the Mechanics and Physics of Solids*, 57(2):342–368, 2009.
69. Flaviana Iurlano. A density result for *GSD* and its application to the approximation of brittle fracture energies. *Calc. Var. Partial Differential Equations*, 51(1-2):315–342, 2014.
70. Alain Karma, David A. Kessler, and Herbert Levine. Phase-field model of mode III dynamic fracture. *Phys. Rev. Lett.*, 87:045501, Jul 2001.
71. Philip K. Kristensen, Christian F. Niordson, and Emilio Martínez-Pañeda. An assessment of phase field fracture: crack initiation and growth. *Philos. Trans. Roy. Soc. A*, 379(2203):20210021, 2021.
72. Camille Labourie and Antoine Lemenant. Epsilon-Regularity for Griffith Almost-Minimizers in Any Dimension Under a Separating Condition. *Arch. Ration. Mech. Anal.*, 247(6):105, 2023.
73. Giovanni Lancioni and Gianni Royer-Carfagni. The variational approach to fracture mechanics. A practical application to the French *Panthéon* in Paris. *J. Elasticity*, 95(1-2):1–30, 2009.
74. Christopher J. Larsen. Epsilon-stable quasi-static brittle fracture evolution. *Comm. Pure Appl. Math.*, 63(5):630–654, 2010.
75. Christopher J. Larsen. Local minimality and crack prediction in quasi-static Griffith fracture evolution. *Discrete Contin. Dyn. Syst. Ser. S*, 6(1):121–129, 2013.
76. Véronique Lazarus and Jean-Baptiste Leblond. Three-dimensional crack-face weight functions for the semi-infinite interface crack. I. Variation of the stress intensity factors due to some small perturbation of the crack front. *J. Mech. Phys. Solids*, 46(3):489–511, 1998.
77. Véronique Lazarus and Jean-Baptiste Leblond. Three-dimensional crack-face weight functions for the semi-infinite interface crack. II. Integrodifferential equations on the weight functions and resolution. *J. Mech. Phys. Solids*, 46(3):513–536, 1998.
78. Jean-Baptiste Leblond. Crack paths in plane situation – I, General form of the expansion of the stress intensity factors. *Int. J. Solids Struct.*, 25:1311–1325, 1989.
79. Jean-Baptiste Leblond. Crack paths in three-dimensional elastic solids. I. Two-term expansion of the stress intensity factors—application to crack path stability in hydraulic fracturing. *Internat. J. Solids Structures*, 36(1):79–103, 1999.
80. Jean-Baptiste Leblond, Véronique Lazarus, and Salah Mouchrif. Crack paths in three-dimensional elastic solids. II. Three-term expansion of the stress intensity factors—applications and perspectives. *Internat. J. Solids Structures*, 36(1):105–142, 1999.
81. Lynn H. Loomis and Hassler Whitney. An inequality related to the isoperimetric inequality. *Bull. Amer. Math. Soc.*, 55:961–962, 1949.
82. Christian Miehe, Martina Hofacker, and Fabian Welschinger. A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering*, 199(45):2765–2778, 2010.
83. Nicolas Moës, John Dolbow, and Ted Belytschko. A finite element method for crack growth without remeshing. *Internat. J. Numer. Methods Engrg.*, 46(1):131–150, 1999.
84. David Mumford and Jayant Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42(5):577–685, 1989.
85. Matteo Negri and Christoph Ortner. Quasi-static crack propagation by Griffith’s criterion. *Math. Models Methods Appl. Sci.*, 18(11):1895–1925, 2008.
86. Jerzy Podgórski and Jakub Gontarz. Simulation of the Griffith’s crack using own method of predicting the crack propagation. *Advances in Science and Technology Research Journal*, 15(4):1–13, 2021.
87. Efim Brener Robert Spatschek and Alain Karma. Phase field modeling of crack propagation. *Philosophical Magazine*, 91(1):75–95, 2011.

88. Robert Arno Sack. Extension of griffith's theory of rupture to three dimensions. *Proceedings of the Physical Society*, 58(6):729, nov 1946.
89. Lucia Scardia. Damage as the  $\Gamma$ -limit of microfractures in linearized elasticity under the non-interpenetration constraint. *Adv. Calc. Var.*, 3(4):423–458, 2010.
90. Christoph Schreiber, Tim Etrich, Charlotte Kuhn, and Ralf Müller. A phase field modeling approach of crack growth in materials with anisotropic fracture toughness. In *2nd International Conference of the DFG International Research Training Group 2057–Physical Modeling for Virtual Manufacturing (iPMVM 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.
91. Yingfeng Shao, Xianghong Xu, Songhe Meng, Guanghui Bai, Chiping Jiang, and Fan Song. Crack patterns in ceramic plates after quenching. *Journal of the American Ceramic Society*, 93(10):3006–3008, 2010.
92. Pierre-Marie Suquet. Sur un nouveau cadre fonctionnel pour les équations de la plasticité. *C. R. Acad. Sci. Paris Sér. A-B*, 286(23):A1129–A1132, 1978.
93. Roger Temam and Gilbert Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75(1):7–21, 1980/81.
94. Caterina Ida Zeppieri. Homogenisation of high-contrast brittle materials. *Math. Eng.*, 2(1):174–202, 2020.
95. William P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.