Global and Local Duality in Grassmann Algebras

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Abstract

The concept of *dual* vector spaces was first introduced about 100 years ago by Hahn (1927), Schauder (1930), and Banach (1932). The word *dual* itself was first introduced by Bourbaki (1938). When the concept of a vector space over a field is broadened to a Grassmann algebra of null vectors over a field, a new concept of *local duality* arises. This paper explores the relationships between *global* and *local* duality in Grassmann algebras of null vectors, and the Clifford geometric algebras they become.

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0 Introduction

The concept of *dual* vector spaces was first introduced about 100 years ago by Hahn (1927), Schauder (1930), and Banach (1932). The word *dual* itself was first introduced by Bourbaki (1938) (from Wikipedia). When the concept of a vector space over a field is broadened to a Grassmann algebra of null vectors over a field, a new concept of *local duality* arises.

Section 1, sets down the basic definitions and results from previous papers that are needed here, [4, 5, 6, 7]. The *spectral basis* of a *mother geometric algebra* $\mathbb{G}_{n,n}$ of neutral signature, consists of nilpotents and idempotents. Matrices, with their usual rules of addition and multiplication, become faithful *coordinate matrices* of geometric numbers [8]. Whereas the outcome of global duality in a pair of real Grassmann algebras is called the *mother* geometric algebra $\mathbb{G}_{n,n}$, the outcome of local duality in a single Grassmann algebra might appropriately be called the *father geometric algebra* $\mathbb{G}_{1,n}$. The most famous such geometric algebra is $\mathbb{G}_{1,3}$, isomorphic to Dirac algebra, or spacetime algebra of relativity and quantum mechanics [1, 3, 7].

Section 2, explores fundamental relationships that exist between the global and local null vector bases of the real geometric algebras $\mathbb{G}_{n,n}$ and $\mathbb{G}_{1,2n-1}$, respectively. Whereas the geometric algebra $\mathbb{G}_{n,n}$ is algebraically isomorphic

to it's faithful *real* coordinate matrix algebra, the Lorentz geometric algebra $\mathbb{G}_{1,2n-1}$ is algebraically isomorphic to its faithful *complex* coordinate matrix algebra. The relationship between the global and local dual bases is reflected in the definition of a new $\star$-Hermitian transpose of the matrix relating these different bases.

Section 3, proposes a Program for Future Research developing the ideas that have been revealed by the concept of local duality. This includes in quantum mechanics, Fourier and Wavelet transforms, and perhaps even providing new insight into the nagging problem of the unification of general relativity and quantum mechanics [12, 19, 20].

1 Basic definitions

A *null vector* or *nilpotent* is a vector whose square is zero. Denote by $\mathcal{N}_{\mathcal{F}}$ the *universal algebra* generated by taking all sums and products of null vectors, and obeying all of the familiar rules of matrix addition and multiplication over a field $\mathcal{F}$, [9].

A set of n linearly independent null vectors over a field of characteristic $\neq 2$ generate a 2^n-dimensional Grassmann algebra. Two 2^n-dimensional Grassmann algebras $\mathcal{A}_n, \mathcal{B}_n \subset \mathcal{N}$ are *globally dual* if for every $a_i \in \mathcal{A}_n$ and $b_j \in \mathcal{B}_n$,

$$2a_i \cdot b_j := a_i b_j + b_j a_i = \delta_{ij}. \tag{1}$$

It is well-known that $2n$ null vectors $\mathcal{A}_n \cup \mathcal{B}_n \subset \mathcal{N}$, taken with the inner product (1), generate a 2^{2n}-dimensional geometric algebra $\mathbb{G}_{n,n}$ of neutral signature, [8].

The concept of *local duality* in a 2^{n+1}-dimensional $\wedge$-Grassmann algebra $\mathcal{C}_{n+1} \subset \mathcal{N}$, generated by $n+1$ null vectors is defined differently. Two null vectors $c_i, c_j \in \mathcal{C}_{n+1}$ are said to be *locally dual* if

$$2c_i \cdot c_j := c_i c_j + c_j c_i = 1 - \delta_{ij}. \tag{2}$$

If *every pair* of null vectors $c_i, c_j \in \mathcal{C}_{n+1}$ satisfies (2), then the $\wedge$-Grassmann algebra $\mathcal{C}_{n+1}$, taken with the inner product (2), generates the geometric algebra $\mathbb{G}_{1,n}$ of Lorentz signature $p + q = 1 + n$, [5].

2 Global and local duality

Note that for $n = 1$, the concepts of global and local duality are identical in the geometric algebra $\mathbb{G}_{1,1}$. In this case, the Grassmann algebra $\mathcal{A}_1 \cup \mathcal{B}_1$, where $\mathcal{A}_1 = \{a_1\}$, $\mathcal{B}_1 := \{b_1\}$, and $\mathcal{C}_2 := \{c_1, c_2\}$. Choosing null vectors $c_1 = a_1$, and $c_2 = b_1$, it is easily checked that

$$a_1 \cdot b_1 = c_1 \cdot c_2 = \frac{1}{2},$$

satisfying both conditions (1) and (2).

For the globally dual Grassmann algebra $\mathcal{A}_2 \cup \mathcal{B}_2$, where $\mathcal{A}_2 = \{a_1, a_2\}$ and $\mathcal{B}_2 = \{b_1, b_2\}$, satisfying the additional property (1),

$$\{\mathcal{A}_2, \mathcal{B}_2\}^T := \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}$$

defines the *spectral basis* of null vectors for the geometric algebra $\mathbb{G}_{2,2}$. In terms of the spectral basis, the *standard basis* of the geometric algebra $\mathbb{G}_{1,3}$ is defined by

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & j & j \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_1 - b_1 \\ j(a_2 + b_2) \\ a_2 - b_2) \end{pmatrix}, \tag{3}$$

where $j := \sqrt{-1}$, [7, 11]. Defining

$$\Omega_{2^\rho} := \begin{pmatrix} \rho & \rho \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \Omega_{2_\rho} := \begin{pmatrix} 1 & 1 \\ \rho & -\rho \end{pmatrix},$$

where $\rho \in \mathbb{C}$, equation (3) takes the block matrix form

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \Omega_2 & [0]_2 \\ [0]_2 & \Omega_{2^j} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}. \tag{4}$$

Noting

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} j & j \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} -j & 1 \\ -j & -1 \end{pmatrix},$$

it follows that

$$\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -j & 1 \\ 0 & 0 & -j & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{5}$$

or equivalently,

$$\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega_2 & [0]_2 \\ [0]_2 & \Omega_{2^j} \end{pmatrix}^\star \begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{6}$$

where the $\star$-*Hermitian transpose* is used, [7].

In the local dual null vector basis $\mathcal{C}_4 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$, the geometric algebra $\mathbb{G}_{1,3}$ is defined by

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \tag{7}$$

or equivalently,

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Omega_{2\rho} & \Omega^-_{2\rho} \\ \Omega_2 & -\Omega^-_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{8}$$

for $\rho = \frac{1}{\sqrt{3}}$.

Noting that

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ \sqrt{3} & -1 & 1 & -1 \\ \sqrt{3} & -1 & -1 & 1 \\ \sqrt{3} & 1 & -1 & -1 \end{pmatrix} =: \frac{1}{4} \begin{pmatrix} \Omega_{2\rho} & \Omega^-_{2\rho} \\ \Omega_2 & -\Omega^-_2 \end{pmatrix}^\star$$

for $\rho = \sqrt{3}$, it follows that

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \Omega_{2\rho} & \Omega^-_{2\rho} \\ \Omega_2 & -\Omega^-_2 \end{pmatrix}^\star \begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{9}$$

We say that the matrix in (7), relating the standard and local bases of $\mathbb{G}_{1,3}$, is the *local basis $\star$-Hermitian transpose* of the matrix in (9). Thus, by definition, under the $\star$-Hermitian transpose in $\mathbb{G}_{1,3}$,

$$\frac{1}{\sqrt{3}} \longleftrightarrow \sqrt{3}.$$

Clearly, the local basis $\star$-Hermitian transpose is an involution.

The relations (3) and (8) imply

$$\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \left\{ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -j & -j & j & j \\ -j & -j & j & j \end{pmatrix} \right\} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

and

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \left\{ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}^T + \begin{pmatrix} \sqrt{3} & \sqrt{3} & j & j \\ \sqrt{3} & \sqrt{3} & j & j \\ \sqrt{3} & \sqrt{3} & -j & -j \\ \sqrt{3} & \sqrt{3} & -j & -j \end{pmatrix} \right\} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix},$$

or, equivalently, in the forms

$$\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \Omega_2 & [0]_2 \\ [0]_2 & \Omega_{2^j} \end{pmatrix}^\star \begin{pmatrix} \Omega_{2^\rho} & \Omega_{2^\rho}^- \\ \Omega_2 & -\Omega_2^- \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \tag{10}$$

where $\rho = \frac{1}{\sqrt{3}}$, and

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \Omega_{2^\rho} & \Omega_{2^\rho}^- \\ \Omega_2 & -\Omega_2^- \end{pmatrix}^\star \begin{pmatrix} \Omega_2 & [0]_2 \\ [0]_2 & \Omega_{2^j} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}, \tag{11}$$

where in this case $\rho = \sqrt{3}$ as we have already seen in (9).

The relationships (10) and (11), between the *global spectral basis* $\{a_1, b_1, a_2, b_2\}$ and the *local dual basis* $\{c_1, c_2, c_3, c_4\}$, is expressed in terms of the *local* $\star$*-Hermitian involution*. The global spectral basis in (10) is the $\star$*-Hermitian transpose* of the local dual basis (11) because there exists constants $\alpha, \beta \in \mathbb{R}$ such (11) is true. In this case, equation (11) shows that $\alpha = \frac{1}{2\sqrt{2}}$ and $\beta = \sqrt{3}$, and for (10), $\alpha = \frac{1}{2\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{3}}$.

The global spectral basis

$$\{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$$

is similarly related to the local dual basis

$$\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}.$$

In this case,

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & j & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j & j & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j & j \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ a_4 \\ b_4 \end{pmatrix},$$

or in iterated form

$$\begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega_{4^j} & [0]_4 \\ [0]_4 & \Omega_{4^j} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ \cdot \\ \cdot \\ a_4 \\ b_4 \end{pmatrix}. \tag{12}$$

Using what we learned in the 4-dimensional case, inverting (12) gives

$$\begin{pmatrix} a_1 \\ b_1 \\ \cdot \\ \cdot \\ a_4 \\ b_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega_{4^j} & [0]_4 \\ [0]_4 & \Omega_{4^j} \end{pmatrix}^\star \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix}. \tag{13}$$

Similarly

$$\begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix},$$

or in iterated form

$$\begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega_{4^\rho} & \Omega_{4^\rho}^- \\ \Omega_4 & -\Omega_4^- \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_8 \end{pmatrix}, \tag{14}$$

where $\rho = \frac{1}{\sqrt{7}}$. Inverting this last relation,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{7} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \sqrt{7} & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \sqrt{7} & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \sqrt{7} & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ \sqrt{7} & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ \sqrt{7} & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ \sqrt{7} & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ \sqrt{7} & -1 & 1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix},$$

or in iterated form

$$\begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_8 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega_{4^\rho} & \Omega_{4^\rho}^- \\ \Omega_4 & -\Omega_4^- \end{pmatrix}^\star \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix}, \tag{15}$$

where $\rho = \sqrt{7}$. Together, (12) and (14) imply the global spectral basis of $\mathbb{G}_{1,7}$ is the $\star$-Hermitian transpose of the local dual basis of $\mathbb{G}_{1,7}$, with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{\sqrt{7}}$.

Using (14) and (15), we can directly relate the global and local bases vectors of $\mathbb{G}_{1,7}$, getting $(\, a_1 \quad b_1 \quad a_2 \quad b_2 \quad a_3 \quad b_3 \quad a_4 \quad b_4 \,)^T =$

$$\frac{1}{4}\begin{pmatrix} 1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} \\ -1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & -1+\frac{1}{\sqrt{7}} & 1+\frac{1}{\sqrt{7}} \\ 1-j & 1+j & -1-j & -1+j & -1+j & -1-j & 1+j & 1-j \\ -1-j & -1+j & 1-j & 1+j & 1+j & 1-j & -1+j & -1-j \\ 1-j & -1-j & -1-j & 1-j & 1+j & -1+j & -1+j & 1+j \\ -1-j & 1-j & 1-j & -1-j & -1+j & 1+j & 1+j & -1+j \\ 1-j & 1+j & -1-j & -1+j & 1-j & 1+j & -1-j & -1+j \\ -1-j & -1+j & 1-j & 1+j & -1-j & -1+j & 1-j & 1+j \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}.$$

Analogous to (10) and (11), relating spectral basis of $\mathbb{G}_{2,2}$ to local dual basis of $\mathbb{G}_{1,3}$, for $\mathbb{G}_{4,4}$ and $\mathbb{G}_{1,7}$,

$$\begin{pmatrix} a_1 \\ b_1 \\ \cdot \\ \cdot \\ a_4 \\ b_4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \Omega_4 & [0]_4 \\ [0]_4 & \Omega_{4^j} \end{pmatrix}^\star \begin{pmatrix} \Omega_{4^\rho} & \Omega^-_{4^\rho} \\ \Omega_4 & -\Omega^-_4 \end{pmatrix} \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ c_8 \end{pmatrix}, \tag{16}$$

and

$$\begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ c_8 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \Omega_{4^\rho} & \Omega^-_{4^\rho} \\ \Omega_4 & -\Omega^-_4 \end{pmatrix}^\star \begin{pmatrix} \Omega_4 & [0]_4 \\ [0]_4 & \Omega_{4^j} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ \cdot \\ \cdot \\ a_4 \\ b_4 \end{pmatrix}, \tag{17}$$

with $\alpha = \frac{1}{4}$ and $\beta = \sqrt{7}$. Block matrices greatly simplify the calculations using the iterated $(2^k \times 2^k)$-block matrices $\Omega_{2^k}, \Omega^-_{2^k}$, their inverses and $\star$-Hermitian involution.

Because of the periodicity modulo(8) of Clifford geometric algebras, higher order geometric algebras $\mathbb{G}_{p+1,q+1}$, for $p+q = 8$, can be broken down into blocks of 2×2 matrix algebras with entries in $\mathbb{G}_{p,q}$, [8, p.71]. The power of a 2×2 block matrix structure has been richly exploited in the analysis of the Möbius transformations of the conformal group [9, 10], [16, p.244].

3 Program for Future Research

In [4], I began the study of the Lorentz geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ in terms of what I called in that paper positively and negatively correlated null vectors. In particular, I was struck by the close relationship to many ideas in graph theory, and began the development of a geometric calculus on those algebras, similar to the now more widely known geometric calculus on Euclidean space. Clearly, there is much left to explore along these lines, [2, 11, 15].

In a second paper, *Calculus of Compatible Nilpotents* the vector derivative, and basic differential formulas for elementary functions, can be developed in the basis of compatible null vectors of a in geometric algebra with a Lorentz metric signature, [5].

In the third paper of this series, *Itinerant Quantum Integers: The Language of Quantum Computers*, the exploration of basic ideas of quantum mechanics is begun. In particular, it seems natural to identify a *qubit* as a pair of compatible null vectors, the *up state* being their sum and *down state* being their difference. In this paper, two new representations of the symmetric group are given, [6].

The fourth paper in this series, *Global and Local nilpotent Bases of Matrices*, gives up the idea of negatively correlated null vectors, and instead considers positively correlated null vectors over the complex number field. This brings the idea of local and global duality into much sharper focus. A surprising connection is exhibited between discrete Fourier and Wavelet transforms by local duality in a single Grassmann algebra, and perhaps provides a new route on the road to unification of general relativity and quantum mechanics, [7]. Clearly, further research can be done on the 2×2 block matrix structure and the Vahlen matrices of the conformal group, [9].

Since the *father* geometric algebras $\mathbb{G}_{1,n}$ have now been recognized, alongside the well known *mother* geometric algebras $\mathbb{G}_{n,n}$, one may wonder if there are other unknown siblings to be discovered?

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References

[1] W.E. Baylis, G. Sobczyk, *Relativity in Clifford's Geometic Algebras of Space and Spacetime*, International Journal of Theoretical Physics 43, 10 (Oct.2004), 2061-2079.

[2] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, 2nd edition, Kluwer 1992.

[3] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, NY 1966.

[4] G. Sobczyk, *Geometric Algebras of Light Cone Projective Graph Geometries*, https://arxiv.org/pdf/2303.03452.pdf

[5] G. Sobczyk, *Calculus of Compatible Nilpotents*, preprint, https://hal.science/hal-04108375

[6] G. Sobczyk, *Itinerant Quantum Integers: The Language of Quantum Computers*, http://dx.doi.org/10.13140/RG.2.2.16627.50720

[7] G. Sobczyk, *Global and Local Nilpotent Bases of Matrices*, (Sept 2023) http://arxiv.org/2309.13084

[8] G. Sobczyk, *Matrix Gateway to Geometric Algebra, Spacetime and Spinors,* Independent Publisher November 2019. https://www.garretstar.com

[9] G. Sobczyk, *Geometric Algebras of Compatible Null Vectors*, Advanced Computational Applications of Geometric Algebra Springer Volume 13771, Lecture Notes in Computer Science series, (to appear)

[10] G. Sobczyk, *Conformal Mappings in Geometric Algebra*, Notices of the AMS, Volume 59, Number 2, p.264-273, 2012.

[11] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, New York 2013.

[12] G. Sobczyk, T. Yarman, Unification of space-time-matter-energy. Appl. & Computat. Math. 7(2), 255–68. http://acmij.az/view.php?lang=az&menu=journal&id=219

[13] G. Sobczyk, *What's in a Pauli Matrix?* https://arxiv.org/abs/1905.08102v1

[14] G. Sobczyk, *Spinors in Spacetime Algebra and Euclidean 4-Space* https://arxiv.org/abs/1703.01244v1

[15] Khovanova, T., *Clifford Algebras and Graphs*, Geombinatories, 20(2): 56-76, 2010, https://0810.3322

[16] Lounesto, P.: Crumeyrolle's Bivectors and Spinors, in R. Ablamowicz, P. Lounesto (eds.), Clifford Algebras and Spinor Structures, pp. 137-166, Kluwer (1995).

[17] G. Sobczyk, *Spacetime Vector Analysis*, Physics Letters A, Vol 84A, p.45-49, 1981.

[18] Garret Sobczyk, *Special relativity in complex vector algebra* https://arxiv.org/abs/0710.0084v1

[19] Discrete fourier transform - Wikipedia en.wikipedia.org

[20] Discrete wavelet transform - Wikipedia en.wikipedia.org

[21] Cruz Guzman, J., Page, B., "Zbigniew Oziewicz Seminar on Fundamental Problems in Physics", FESC-Cuautitlan Izcalli UNAM, Mexico. https://www.youtube.com/channel/UCBcXAdMO3q6JBNyvBBVLmQg

[22] Oziewicz, Z., *Science Must Never be Objective*, 16-th International Conference, The Science and Quality of Life At: Vilnius, Lithuania, Vol. Studium Vilnense A 10 (2013) pages 3-8. ISSN 1648-7907.