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# The Frisch-Parisi conjecture II: Besov spaces in multifractal environment, and a full solution

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## Abstract

Frisch-Parisi conjecture claims the existence of Baire function spaces in which Baire typical functions share the same multifractal behavior, prescribed in advance, and obey a multifractal formalism. In this paper, we introduce a family  $\mathbf{B}$  of *heterogeneous* Besov spaces, which generalize the standard Besov spaces - they are obtained by replacing the Lebesgue measure (which plays a key role in the definition of the standard Besov spaces) by multifractal Radon measures belonging to some class constructed in the companion paper [1]. We find a characterization of the elements of  $\mathbf{B}$  in terms of wavelet coefficients, and then describe the multifractal properties (singularity spectrum, validity of the multifractal formalism) of their Baire typical functions. This allows us to fully solve the Frisch-Parisi conjecture inside  $\mathbf{B}$ .

## Résumé.

La conjecture de Frisch-Parisi affirme l'existence d'espaces fonctionnels de Baire dans lesquels les fonctions génériques partagent le même comportement multifractal, prescrit à l'avance, et satisfont un formalisme multifractal. Dans cet article, nous introduisons une famille  $\mathbf{B}$  d'espaces de Besov *hétérogènes*, qui généralisent les espaces de Besov standard - ils sont obtenus en remplaçant la mesure de Lebesgue (qui joue un rôle clé dans la définition des espaces de Besov standard) par les éléments d'une classe de mesures de Radon multifractales construite dans l'article associé [1]. Nous trouvons une caractérisation des éléments de  $\mathbf{B}$  en termes de coefficients d'ondelettes, puis nous décrivons les propriétés multifractales (spectre de singularité, validité du formalisme multifractal) de leurs fonctions génériques. Ceci nous permet de résoudre complètement la conjecture de Frisch-Parisi dans  $\mathbf{B}$ .

*Keywords:* Hausdorff dimension, multifractal formalism, Besov spaces, Fréchet spaces, wavelets.  
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## 1. Introduction

The so-called Frisch-Parisi conjecture is the inverse problem, raised by S. Jaffard, consisting in seeking for Baire function spaces in which typical elements share the same prescribed multifractal behavior and obey a multifractal formalism. To be more specific, recall that for a real valued function  $f \in L_{\text{loc}}^{\infty}(\mathbb{R}^d)$ , the pointwise Hölder exponent function  $h_f$  is defined as follows. Given  $x_0 \in \mathbb{R}^d$ , and  $H \in \mathbb{R}_+$ ,  $f$  is said to belong to  $\mathcal{C}^H(x_0)$  if there exist a polynomial  $P$  of degree at most  $\lfloor H \rfloor$ , a constant  $C > 0$ , and a neighborhood  $V$  of  $x_0$  such that

$$\forall x \in V, \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^H.$$

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The *pointwise Hölder exponent* of  $f \in L_{\text{loc}}^\infty(\mathbb{R}^d)$  at  $x_0$  is

$$h_f(x_0) = \sup \{H \in \mathbb{R}_+ : f \in \mathcal{C}^H(x_0)\}, \quad (1.1)$$

and  $f$  is said to have a Hölder singularity of order  $h_f(x_0)$  at  $x_0$ .

Then the multifractal spectrum, or singularity spectrum of  $f$  is the mapping

$$\sigma_f : \begin{cases} \mathbb{R} \rightarrow [0, d] \cup \{-\infty\} \\ H \mapsto \dim E_f(H), \end{cases}$$

where

$$E_f(H) = \{x \in \mathbb{R}^d : h_f(x) = H\},$$

$\dim$  stands for the Hausdorff dimension, and  $\dim E = -\infty$  if and only if  $E = \emptyset$ .

**Conjecture 1** (Frisch-Parisi conjecture). *Let  $\mathcal{S}_d$  be the set of functions  $\sigma : \mathbb{R} \rightarrow [0, d] \cup \{-\infty\}$  such that  $\sigma$  is concave, continuous, with compact support included in  $(0, +\infty)$  and whose maximum equals  $d$ . For every  $\sigma \in \mathcal{S}_d$ , there exists a Baire functional space of functions defined on  $\mathbb{R}^d$  in which any typical element  $f$  obeys some multifractal formalism and satisfies  $\sigma_f = \sigma$ .*

We refer to the companion paper [1] for an introduction to multifractals and this conjecture. In [1], we constructed a set  $\mathcal{M}_d$  of  $\mathbb{Z}^d$ -invariant Radon measures on  $\mathbb{R}^d$ , exhausting the possible multifractal behaviors of fully supported measures obeying a multifractal formalism, and with a prescribed singularity spectrum compactly supported in  $(0, \infty)$ . Considering the set of capacities  $\mathcal{E}_d = \{\nu^s : \nu \in \mathcal{M}_d, s > 0\}$ , the prescription part of the conjecture followed thanks to a family of Baire spaces  $\{\mathbf{B}^\mu(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d}$  extending naturally the Hölder-Zygmund spaces and defined using wavelet expansion of uniformly bounded Hölder functions (see [1, Theorems 2 and 7]); in particular typical elements in  $\mathbf{B}^\mu(\mathbb{R}^d)$  inherit the singularity spectrum of  $\mu$ . In this paper, we introduce *heterogeneous* Besov spaces denoted by  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  (depending on a capacity  $\mu$  and two indices  $p, q \in (0, +\infty]$ ), which generalise in a natural direction the standard Besov spaces defined through  $L^p$  moduli of smoothness. In particular, one will see that  $\mathbf{B}_\infty^{\mu,\infty}(\mathbb{R}^d) = \mathbf{B}^\mu(\mathbb{R}^d)$ . Roughly speaking, the central role played by the Lebesgue measure in the structure of Besov spaces is now played by (possibly) multifractal measures and more generally capacities. We first characterize these new spaces using wavelet coefficients. Then, we consider the family  $\mathbf{B} = \{\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)\}_{\mu \in \mathcal{E}_d, (p,q) \in [1, +\infty]^2}$  of such spaces where  $\mu$  belongs to set of capacities  $\mathcal{E}_d$ . Through a delicate study we identify the multifractal behavior of the typical elements in any function space  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ ; this behavior depends in a non trivial way on that of  $\mu$  and on  $p$ . Then, we show that these typical functions obey a multifractal formalism. Finally, we solve the inverse problem exhaustively inside  $\mathbf{B}$ .

Sections 1.1 and 1.2 respectively introduce the heterogeneous Besov spaces (called Besov spaces in multifractal environment) considered in this paper, and provide the characterisation of these spaces using wavelets (Theorem 2). Then, basic multifractal properties of the elements of  $\mathcal{E}_d$  are gathered in Section 1.3 (Theorem 3). The typical multifractal behavior in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  is presented in Section 1.4 (Theorem 4), while the multifractal formalism used in this paper and its typical validity in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  are the subject of Section 1.5 (Theorem 5). The full solution to the conjecture is given in Section 1.6 (Theorems 6 and 7).

### 1.1. Definitions of heterogeneous Besov spaces

Standard Besov spaces can be defined by using  $L^p$  moduli of smoothness, and are characterized using decay rate of wavelet coefficients. To define Besov spaces in multifractal environment, the classical definition of  $L^p$  moduli of smoothness is extended using Hölder capacities. For  $x \in \mathbb{R}^d$ ,  $r \in \mathbb{R}_+$ ,  $B(x, r)$  denotes the closed Euclidean ball with center  $x$  and radius  $r$ .

**Definition 1.1.** *The set of Hölder set functions on  $\mathcal{B}(\mathbb{R}^d)$  is defined as*

$$\mathcal{H}(\mathbb{R}^d) = \{\mu : \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}_+ \cup \{\infty\} : \exists C, s > 0, \forall E \subset \mathbb{R}^d, \mu(E) \leq C|E|^s\}. \quad (1.2)$$

Then, the set of Hölder capacities is defined as

$$\mathcal{C}(\mathbb{R}^d) = \{\mu \in \mathcal{H}(\mathbb{R}^d) : \forall E, F \in \mathcal{B}(\mathbb{R}^d), E \subset F \Rightarrow \mu(E) \leq \mu(F)\}. \quad (1.3)$$

The topological support  $\text{supp}(\mu)$  of  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is the set of points  $x \in \mathbb{R}^d$  for which  $\mu(B(x, r)) > 0$  for every  $r > 0$ . A capacity  $\mu$  is fully supported when  $\text{supp}(\mu) = \mathbb{R}^d$ .

40 Similarly,  $\mathcal{H}([0, 1]^d)$  and  $\mathcal{C}([0, 1]^d)$  are defined by replacing above  $\mathbb{R}^d$  by  $[0, 1]^d$ .

**Definition 1.2.** For  $h \in \mathbb{R}^d$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , consider the finite difference operator  $\Delta_h f : x \in \mathbb{R}^d \mapsto f(x+h) - f(x)$ . Then, for  $n \geq 2$ , set  $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$ .

For every fully supported set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$ , for every  $n \in \mathbb{N}^*$ ,  $h \in \mathbb{R}^d \setminus \{0\}$  and  $x \in \mathbb{R}^d$ , set

$$\Delta_h^{\mu, n} f(x) = \frac{\Delta_h^n f(x)}{\mu(B[x, x+nh])},$$

where  $B[x, y]$  stands for the Euclidean ball of diameter  $[x, y]$ .

For  $p \in [1, +\infty]$ , the  $\mu$ -adapted  $n$ -th order  $L^p$  modulus of smoothness of  $f$  is defined at any  $t > 0$  by

$$\omega_n^\mu(f, t, \mathbb{R}^d)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu, n} f\|_{L^p(\mathbb{R}^d)}. \quad (1.4)$$

Observe that when  $\mu(E) = 1$  for every set  $E$ , then  $\omega_n^\mu(f, t, \mathbb{R}^d)_p$  is a modification of the standard  $n$ -th order  $L^p$  modulus of smoothness of  $f$  defined by

$$\omega_n(f, t, \mathbb{R}^d)_p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\mathbb{R}^d)}. \quad (1.5)$$

Recall that when  $s > 0$ , and  $p, q \in [1, +\infty]$ , the Besov space  $B_q^{s,p}(\mathbb{R}^d)$  is the set of those functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\|f\|_{B_q^{s,p}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{s,p}(\mathbb{R}^d)} < \infty, \quad (1.6)$$

where

$$|f|_{B_q^{s,p}(\mathbb{R}^d)} = \|(2^{js}(\omega_n(f, 2^{-j}, \mathbb{R}^d)_p))_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} < +\infty,$$

and  $n$  is an integer  $\geq s$ . The dependence on  $n$  in  $|f|_{B_q^{s,p}(\mathbb{R}^d)}$  is voluntarily omitted. Indeed, the norm  $\|\cdot\|_{B_q^{s,p}(\mathbb{R}^d)}$  makes  $B_q^{s,p}(\mathbb{R}^d)$  a Banach space, and different values of  $n > s$  yield equivalent norms (see [2, Remark 3.2.2]).

Let us recall some notations used in [1]. For every  $j \in \mathbb{Z}$ , let  $\mathcal{D}_j$  stand for the collection of (closed) dyadic cubes of generation  $j$ , i.e.  $\lambda_{j,k} = 2^{-j}k + 2^{-j}[0, 1]^d$ , where  $k \in \mathbb{Z}^d$ . Let us also set  $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j$ , and if  $\lambda = \lambda_{j,k} \in \mathcal{D}_j$  we set  $x_\lambda = 2^{-j}k$ .

50 For  $x \in \mathbb{R}^d$ ,  $\lambda_j(x)$  is the closure of the unique dyadic cube of generation  $j$ , product of semi-open to the right dyadic intervals, which contains  $x$ .

For  $j \in \mathbb{Z}$ ,  $\lambda \in \mathcal{D}_j$ , and  $N \in \mathbb{N}^*$ ,  $N\lambda$  denotes the cube with same center as  $\lambda$  and radius equal to  $N \cdot 2^{-j-1}$  in  $(\mathbb{R}^d, \|\cdot\|_\infty)$ . For instance,  $3\lambda$  is the union of those  $\lambda' \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\lambda' \neq \emptyset$  ( $\partial\lambda$  stands for the frontier of the cube  $\lambda$ ).

55 The capacities considered in this paper and in [1] satisfy additional properties.

**Definition 1.3.** Let  $\Phi$  be the set of non decreasing functions  $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$  such that  $\lim_{j \rightarrow +\infty} \frac{\phi(j)}{j} = 0$ . A capacity  $\mu \in \mathcal{C}(\mathbb{R}^d)$  is almost doubling when there exists  $\phi \in \Phi$  such that

$$\text{for all } x \in \text{supp}(\mu) \text{ and } j \in \mathbb{N}, \quad \mu(3\lambda_j(x)) \leq e^{\phi(j)} \mu(\lambda_j(x)). \quad (1.7)$$

**Definition 1.4.** A set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfies property (P) if there exist  $C, s_1, s_2 > 0$  and  $\phi \in \Phi$  such that:

(P<sub>1</sub>) for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ,

$$C^{-1}2^{-js_2} \leq \mu(\lambda) \leq C2^{-js_1}. \quad (1.8)$$

(P<sub>2</sub>) for all  $j, j' \in \mathbb{N}$  with  $j' \geq j$ , for all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ :

$$C^{-1}2^{-\phi(j)}2^{(j'-j)s_1}\mu(\lambda') \leq \mu(\tilde{\lambda}) \leq C2^{\phi(j)}2^{(j'-j)s_2}\mu(\lambda'). \quad (1.9)$$

In particular,  $\mu$  is almost-doubling, and doubling if  $\phi = 0$ .

**Definition 1.5.** For  $s > 0$ , a set function  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is  $s$ -Hölder when there exists  $C > 0$  such that  $\mu(E) \leq C|E|^s$  for all  $E \in \mathcal{B}(\mathbb{R}^d)$ .

Then, for  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $s > 0$ , and  $E \in \mathbb{R}^d$ , define

$$\mu^s(E) = \mu(E)^s \quad \text{and} \quad \mu^{(+s)}(E) = \mu(E)|E|^s,$$

and if  $\mu$  is  $s_0$ -Hölder, then for all  $s \in (0, s_0)$ , define

$$\mu^{(-s)}(E) = \begin{cases} 0 & \text{if } |E| = 0, \\ \mu(E)|E|^{-s} & \text{if } 0 < |E| < +\infty, \\ \infty & \text{otherwise.} \end{cases}$$

Starting from  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $\mu^s$ ,  $\mu^{(+s)}$  and  $\mu^{(-s)}$  as defined above still belong to  $\mathcal{H}(\mathbb{R}^d)$  (with  $s$  small enough in the case of  $\mu^{(-s)}$ ).

We are now ready to introduce heterogenous Besov spaces in  $\mu$ -environment.

**Definition 1.6** (Besov spaces in  $\mu$ -environment). Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfy property (P<sub>1</sub>) of Definition 1.4 with exponents  $0 < s_1 \leq s_2$ , and consider an integer  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$ .

For  $1 \leq p, q \leq \infty$ , let

$$B_q^{\mu,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : |f|_{B_q^{\mu,p}(\mathbb{R}^d)} < +\infty\},$$

where

$$|f|_{B_q^{\mu,p}(\mathbb{R}^d)} = \left\| 2^{jd/p}(\omega_n^\mu(f, 2^{-j}, \mathbb{R}^d)_p)_{j \in \mathbb{N}} \right\|_{\ell^q(\mathbb{N})}. \quad (1.10)$$

Also, let

$$\mathbf{B}_q^{\mu,p}(\mathbb{R}^d) = \bigcap_{0 < \varepsilon < \min(s_1, 1)} B_q^{\mu^{(-\varepsilon)}, p}(\mathbb{R}^d). \quad (1.11)$$

Note that  $B_q^{\mu,p}(\mathbb{R}^d) \subset \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ . The spaces  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  will be referred to as Besov spaces in  $\mu$ -environment.

At this stage, both  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  depend a priori on the choice of  $n$ . However, the dependence in  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$  can be dropped for  $B_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu$  is a doubling capacity, and also for  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , under the (rather weak) extra property (P<sub>2</sub>) of Definition 1.4 (see Theorem 2 for a precise statement). Moreover, endowed with the norm  $\|\cdot\|_{L^p(\mathbb{R}^d)} + \|\cdot\|_{B_q^{\mu,p}(\mathbb{R}^d)}$ ,  $B_q^{\mu,p}(\mathbb{R}^d)$  is a Banach space. Hence,  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  is naturally endowed with a Fréchet space structure, as the intersection of a nested family of such spaces. The Fréchet spaces  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  will be used to solve the Frisch-Parisi conjecture.

Let  $\mathcal{L}^d$  stand for the  $d$ -dimensional Lebesgue measure. For  $\mu = (\mathcal{L}^d)^{\frac{s}{d} - \frac{1}{p}}$ , it is quite direct to see that  $B_q^{\mu,p}(\mathbb{R}^d) = B_q^{s,p}(\mathbb{R}^d)$  when  $s > d/p$ . When  $\mu \in \mathcal{H}(\mathbb{R}^d)$  is multifractal (typically  $\mu = \nu^s$  where  $\nu$  is a multifractal measure and  $s > 0$ ) the heterogeneity associated to the distribution of the values of  $\mu$  at small scales makes natural to see such a capacity as defining an *heterogeneous* environment imposing local distortions in the computation of the moduli of smoothness in comparison to positive powers of  $\mathcal{L}^d$ , which are *homogeneous* in space. Like for  $B_q^{s,p}(\mathbb{R}^d)$ , to study the typical multifractal behavior in  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , it is essential to establish a wavelet characterization of these spaces. Such a characterization exists for  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu$  is almost doubling, and for  $B_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu$  is doubling (see Theorem 2).

## 1.2. Wavelet characterization of Besov spaces in almost doubling environments

85 It is standard that classical Besov spaces are characterized in terms of wavelet coefficients decay. We investigate the situation for the spaces  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

Let  $\{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\}$  be a family of wavelets defining a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$  (see [3, Ch. 2 and 3] for a general construction).

Let  $\Lambda = \bigcup_{j \in \mathbb{Z}} \Lambda_j$ , where for  $j \in \mathbb{Z}$

$$\Lambda_j = \{(i, j, k) : i \in \{1, \dots, 2^d - 1\}, k \in \mathbb{Z}^d\}.$$

For every  $\lambda = (i, j, k) \in \Lambda$ , denote by  $\psi_\lambda$  the function  $x \mapsto \psi^{(i)}(2^j x - k)$ . The functions  $2^{dj/2} \psi_\lambda$ ,  $j \in \mathbb{Z}$ ,  $\lambda \in \Lambda_j$ , form an orthonormal basis of  $L^2(\mathbb{R}^d)$ , and every  $f \in L^2(\mathbb{R}^d)$  can be expanded, in two equivalent manners, as

$$f = \sum_{k \in \mathbb{Z}^d} \beta(k) \phi(\cdot - k) + \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda = \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda,$$

where

$$\beta(k) = \int_{\mathbb{R}^d} f(x) \phi(x - k) dx \quad \text{and} \quad c_\lambda = \int_{\mathbb{R}^d} 2^{dj} \psi_\lambda(x) f(x) dx \quad (k \in \mathbb{Z}^d, \lambda \in \Lambda). \quad (1.12)$$

90 Recall that a mapping  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  has  $r$  vanishing moments when for every multi-index  $\alpha \in \mathbb{N}^d$  of length smaller than or equal to  $r$ ,  $\int_{\mathbb{R}^d} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \psi(x) dx = 0$ .

**Definition 1.7.** For every  $r \in \mathbb{N}$ , call  $\mathcal{F}_r$  the set of those  $\{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\}$  which define a multi-resolution analysis with reconstruction in  $L^2(\mathbb{R}^d)$ , and such that  $\phi$  and the  $\psi^{(i)}$  are compactly supported,  $r$  times continuously differentiable functions, and every  $\psi^{(i)}$  has  $r$  vanishing moments.

The set  $\mathcal{F}_r$  is not empty for all  $r \in \mathbb{N}$  (see [3, Prop. 4, section 3.7]).

Fix  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ . For any  $f \in L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ , define the sequences  $(\beta(k))_{k \in \mathbb{Z}^d}$  and  $(c_\lambda)_{\lambda \in \Lambda}$  as in (1.12). Besov spaces are characterized by their wavelet coefficients as follows (see [3, Ch. 6], [4], or [2, Corollary 3.6.2]): For  $r > s > d/p$ ,

$$f \in B_q^{s,p}(\mathbb{R}^d) \iff \begin{cases} \beta \in \ell^p(\mathbb{Z}^d), \\ (\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N}), \text{ where } \varepsilon_j = \left\| \left( 2^{j(s-d/p)} c_\lambda \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}, \end{cases} \quad (1.13)$$

95 and the decomposition  $f = \sum_{k \in \mathbb{Z}^d} \beta(k) \phi(\cdot - k) + \sum_{j \in \mathbb{N}} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$  holds. Moreover,  $\|\beta\|_{\ell^p(\mathbb{Z}^d)} + \|(\varepsilon_j)\|_{\ell^q(\mathbb{N})}$  is a norm equivalent to the norm  $\|f\|_{B_q^{s,p}(\mathbb{R}^d)}$  defined in (1.6). Note that  $\psi^{(i)} \in B_q^{s,p}(\mathbb{R}^d)$ , and  $B_q^{s,p}(\mathbb{R}^d) \hookrightarrow B_{+\infty}^{s-\frac{d}{p}, +\infty}(\mathbb{R}^d) = \mathcal{C}^{s-\frac{d}{p}}(\mathbb{R}^d)$ , where for all  $\alpha > 0$ ,  $\mathcal{C}^\alpha(\mathbb{R}^d) = B_\infty^{\alpha, \infty}(\mathbb{R}^d)$  is the Hölder-Zygmund space of order  $\alpha$ .

100 **Definition 1.8.** Let  $\mu \in \mathcal{H}(\mathbb{R}^d)$  satisfy property  $(P_1)$  of Definition 1.4 with exponents  $0 < s_1 \leq s_2$ , and consider an integer  $n \geq \lfloor s_2 + \frac{d}{p} \rfloor + 1$ .

Fix a wavelet  $\Psi \in \mathcal{F}_n$ , and consider for a function  $f \in L^p(\mathbb{R}^d)$  the quantity

$$|f|_{\mu,p,q,\Psi} = \|(\varepsilon_j^\mu)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})}, \text{ where } \varepsilon_j^\mu = \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}. \quad (1.14)$$

Then define

$$B_{q,\Psi}^{\mu,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : |f|_{\mu,p,q,\Psi} < +\infty\}. \quad (1.15)$$

The space  $(B_{q,\Psi}^{\mu,p}(\mathbb{R}^d), \|\cdot\|_{L^p(\mathbb{R}^d)} + |\cdot|_{\mu,p,q,\Psi})$  is complete.

**Theorem 2.** *Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  be an almost doubling capacity. Let  $0 < s_1 \leq s_2$  and  $r = \lfloor s_2 + \frac{d}{p} \rfloor + 1$ . Suppose that property (P) holds for  $\mu$  with the exponents  $(s_1, s_2)$  and that  $B_q^{\mu,p}(\mathbb{R}^d)$  has been constructed by using the  $\mu$ -adapted  $n$ -th order  $L^p$  moduli of smoothness, for some integer  $n \geq r$ .*

105 *Let  $\Psi \in \mathcal{F}_r$ .*

*For every  $\varepsilon \in (0, 1)$ , there exists a constant  $C_{\varepsilon,\Psi} > 1$  such that for all  $f \in L^p(\mathbb{R}^d)$ ,*

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu,p,q,\Psi} \leq C_{\varepsilon,\Psi} (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)}), \quad (1.16)$$

$$\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu,p}(\mathbb{R}^d)} \leq C_{\varepsilon,\Psi} (\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon),p,q,\Psi}}). \quad (1.17)$$

*In particular, if  $0 < \varepsilon < \min(1, s_1)$ ,  $B_q^{\mu,p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu^{(-\varepsilon)},p}(\mathbb{R}^d)$  and  $B_q^{\mu,p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu^{(-\varepsilon)},p}(\mathbb{R}^d)$ . Moreover, if  $\mu$  is doubling and satisfies property (P) with  $\phi = 0$ , then  $B_q^{\mu,p}(\mathbb{R}^d) = B_q^{\mu,p}(\mathbb{R}^d)$  and the norms  $\|\cdot\|_{L^p} + |\cdot|_{\mu,p,q,\Psi}$  and  $\|\cdot\|_{L^p} + |\cdot|_{B_q^{\mu,p}(\mathbb{R}^d)}$  are equivalent.*

Thus, when  $\mu$  is doubling and satisfies (P) with  $\phi = 0$ , the space  $B_q^{\mu,p}(\mathbb{R}^d)$  coincides with  $B_{q,\Psi}^{\mu,p}(\mathbb{R}^d)$  and possesses two equivalent definitions based either on  $L^p$  moduli of smoothness or on wavelet coefficients, and this definition is independent of the choice of  $n \geq r$  and  $\Psi \in \mathcal{F}_r$ . Moreover, when  $\mu$  satisfies property (P), combining (1.16) and (1.17) shows that  $f \in B_q^{\mu,p}(\mathbb{R}^d)$  if and only if  $f \in B_q^{\mu^{(-\varepsilon)},p}(\mathbb{R}^d)$  for every  $\varepsilon \in (0, \min(1, s_1))$ , hence one also gets a wavelet characterization of  $B_q^{\mu,p}(\mathbb{R}^d)$ . And since by construction the family of Banach spaces  $\{B_\varepsilon := B_{q,\Psi}^{\mu^{(-\varepsilon)},p}(\mathbb{R}^d)\}_{0 < \varepsilon < \min(s_1, 1)}$  satisfies  $B_\varepsilon \hookrightarrow B_{\varepsilon'}$  for all  $0 < \varepsilon \leq \varepsilon' < \min(s_1, 1)$ , one obtains another way to see that the space  $B_q^{\mu,p}(\mathbb{R}^d)$  can be endowed with a Fréchet space structure, of which a countable basis of neighborhoods of the origin is given by

$$\left\{ \mathcal{N}_m = \left\{ f \in B_q^{\mu,p}(\mathbb{R}^d) : \|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(-\frac{1}{m}),p,q,\Psi}} < \frac{1}{m} \right\} \right\}_{\substack{m \in \mathbb{N}, \\ m > \max(1, s_1^{-1})}}. \quad (1.18)$$

**Remark 1.9.** (1) *The embeddings  $B_q^{s_2 + \frac{d}{p},p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu,p}(\mathbb{R}^d) \hookrightarrow B_q^{s_1 + \frac{d}{p},p}(\mathbb{R}^d)$  and  $B_q^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d) \hookrightarrow B_q^{\mu,p}(\mathbb{R}^d)$  hold under (P<sub>1</sub>).*

(2) *It is direct from the proof of Theorem 2 that under the weaker assumption that (P) holds for all  $(s'_1, s'_2)$  such that  $0 < s'_1 < s_1 \leq s_2 < s'_2$ , the statement remains true.*

(3) *Fundamental examples of doubling capacities satisfying property (P) with  $\phi = 0$ , namely Gibbs capacities, will be given in Remark 1.15.*

115 **1.3. Recalls about the class of multifractal environments constructed in [1]**

In this section, we resume some of the results proved in the companion paper [1], concerning the construction of capacities  $\mu \in \mathcal{H}([0, 1]^d)$  having a prescribed multifractal spectrum and satisfying a multifractal formalism.

**Definition 1.10.** *Let  $\mu \in \mathcal{H}([0, 1]^d)$ . For  $x \in \text{supp}(\mu)$ , the lower and upper pointwise Hölder exponents of  $\mu$  at  $x$  are respectively defined by*

$$\underline{h}_\mu(x) = \liminf_{j \rightarrow +\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \quad \text{and} \quad \bar{h}_\mu(x) = \limsup_{j \rightarrow \infty} \frac{\log_2 \mu(\lambda_j(x))}{-j}.$$

*Whenever  $\underline{h}_\mu(x) = \bar{h}_\mu(x)$ , the common limit is called  $h_\mu(x)$ . Then, for  $\alpha \in \mathbb{R}$ ,*

$$\begin{aligned} \underline{E}_\mu(\alpha) &= \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) = \alpha\} & \bar{E}_\mu(\alpha) &= \{x \in \text{supp}(\mu) : \bar{h}_\mu(x) = \alpha\}, \\ \text{and} \quad E_\mu(\alpha) &= \underline{E}_\mu(\alpha) \cap \bar{E}_\mu(\alpha). \end{aligned}$$

The singularity (or multifractal) spectrum of  $\mu$  is then the mapping

$$\sigma_\mu : \alpha \in \mathbb{R} \mapsto \dim E_\mu(\alpha).$$

**Definition 1.11.** The  $L^q$ -spectrum of  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$  is defined by

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{j \rightarrow +\infty} -\frac{1}{j} \log_2 \sum_{\substack{\lambda \in \mathcal{D}_j, \lambda \subset [0, 1]^d, \\ \mu(\lambda) > 0}} \mu(\lambda)^q.$$

Recall the definition of the Legendre transform  $g^*$  of a mapping  $g : \mathbb{R} \rightarrow \mathbb{R}$ : for every  $\alpha \in \mathbb{R}$ ,  $g^*(\alpha) = \inf_{q \in \mathbb{R}} q\alpha - g(q)$ . One always has (see [5, 6])

$$\sigma_\mu(\alpha) \leq \tau_\mu^*(\alpha) := \inf_{q \in \mathbb{R}} q\alpha - \tau_\mu(q). \quad (1.19)$$

**Definition 1.12.** A set function  $\mu \in \mathcal{H}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$  is said to obey the multifractal formalism (MF) over an interval  $I \subset \mathbb{R}$  when for all  $\alpha \in I$ ,

$$\sigma_\mu(\alpha) = \tau_\mu^*(\alpha). \quad (1.20)$$

It is said to obey the strong multifractal formalism (SMF) over  $I$  if for all  $\alpha \in I$ , in addition to (1.20) one has  $\dim E_\mu(\alpha) = \tau_\mu^*(\alpha)$ .

When  $I = \mathbb{R}$ , one simply says that the MF or the SMF holds for  $\mu$ .

The following result is proved in [1] ( $\mathcal{S}_d$  is defined in Conjecture 1).

**Theorem 3** ([1], Corollary 5). *There exists a family of environments  $\mathcal{E}_d \subset \mathcal{C}(\mathbb{R}^d)$  such that :*

1. Every  $\mu \in \mathcal{E}_d$  is  $\mathbb{Z}^d$ -invariant, fully supported on  $\mathbb{R}^d$ , almost doubling and satisfies property (P), and the SMF holds for  $\mu|_{[0, 1]^d}$ .
2. For every  $\sigma \in \mathcal{S}_d$ , there exists  $\mu \in \mathcal{E}_d$  such that  $\sigma = \sigma_\mu$ .

By Remark 1.9 (2), when  $\mu \in \mathcal{E}_d$ , since property (P) holds with any  $(s_1, s_2)$  such that  $0 < s_1 < \tau'_\mu(+\infty) \leq \tau'_\mu(-\infty) < s_2$ ,  $\mathbf{B}_q^{\mu, p}(\mathbb{R}^d)$  is well defined by (1.11) independently of the integer  $n \geq s_\mu$ , where

$$s_\mu = \left\lfloor \tau'_\mu(-\infty) + \frac{d}{p} \right\rfloor + 1, \quad (1.21)$$

and the wavelet characterization of  $\mathbf{B}_q^{\mu, p}(\mathbb{R}^d)$  holds with any  $\Psi \in \mathcal{F}_{s_\mu}$ .

#### 1.4. Typical singularity spectrum in Besov spaces in multifractal environment

Our result on the multifractal nature of the elements of  $\mathbf{B}_q^{\mu, p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  is the following. The multifractal formalism's validity is dealt with in the next section.

**Theorem 4.** *Let  $\mu \in \mathcal{E}_d$ , let  $p, q \in [1, +\infty]$ , and consider the mapping*

$$\zeta_{\mu, p}(t) = \begin{cases} \frac{p-t}{p} \tau_\mu \left( \frac{p}{p-t} t \right) & \text{if } t \in (-\infty, p) \\ \tau'_\mu(+\infty) t & \text{if } t \in [p, +\infty). \end{cases} \quad (1.22)$$

1. For all  $f \in \mathbf{B}_q^{\mu, p}(\mathbb{R}^d)$ ,

$$\sigma_f(H) \leq \begin{cases} \zeta_{\mu, p}^*(H) & \text{if } H \leq \zeta'_{\mu, p}(0^+) \\ d & \text{if } H > \zeta'_{\mu, p}(0^+). \end{cases} \quad (1.23)$$



2. For typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , one has  $\sigma_f = \zeta_{\mu,p}^*$ .

The possible shapes of  $\sigma_f$  when  $f$  is typical in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  are investigated in detail in Section 4 (see Proposition 4.2): depending on the values of  $p$  and  $\sigma_\mu^*(\alpha_{\min})$ , various phenomena may occur. See for instance Figures 1 and 3 for a representation of typical singularity spectrum in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , according to whether  $\sigma_\mu(\alpha_{\min}) = 0$  or  $\sigma_\mu(\alpha_{\min}) > 0$ . Next remark gathers key information, proved in Proposition 4.2.

**Remark 1.13.** 1. The map  $\zeta_{\mu,p}$  is always concave. Also, it is immediate that  $\zeta_{\mu,+\infty} = \tau_\mu$ , so typical functions in  $\mathbf{B}_q^{\mu,+\infty}(\mathbb{R}^d)$  satisfy  $\sigma_f = \tau_\mu^*$ .

2. The support of  $\zeta_{\mu,p}^*$  is the compact subinterval  $[\zeta_{\mu,p}(+\infty), \zeta'_{\mu,p}(-\infty)] \subset (0, +\infty)$ . Moreover, since  $\zeta_{\mu,p}(0) = \tau_\mu(0) = -d$ , the maximum of  $\zeta_{\mu,p}^*$  is  $d$ , and it is reached at  $H$  if and only if  $H \in [\zeta'_{\mu,p}(0^+), \zeta_{\mu,p}(0^-)]$ .

3. One has  $\zeta'_{\mu,p}(-\infty) \leq \tau'_\mu(-\infty) + \frac{d}{p}$  (see the first item of Section 4.2).

Two examples of capacities are given in the next remarks: the first ones are the Lebesgue measure and its powers, this case is included both in Jaffard's results and as a particular case of capacities belonging to  $\mathcal{E}_d$ . The second ones are the Gibbs capacities: although they do not belong to  $\mathcal{E}_d$ , they share the same multifractal properties as the capacities of  $\mathcal{E}_d$ , and the conclusions of Theorems 4 and 5 remain true with them.

**Remark 1.14.** The set of environments  $\mathcal{E}_d$  includes all the positive powers of the Lebesgue measure  $\mathcal{L}^d$  (see [1]). Taking  $s > d/p$  and  $\mu = (\mathcal{L}^d)^{s/d-1/p}$ , Theorem 4 coincides with the celebrated Jaffard's theorem [7], which can be stated as follows:

1. For all  $f \in B_q^{s,p}(\mathbb{R}^d)$ ,  $\sigma_f(H) \leq \begin{cases} \min\{p(H - (s - \frac{d}{p})), d\} & \text{if } H \geq s - d/p, \\ -\infty & \text{if } H < s - d/p. \end{cases}$

2. Typical  $f \in B_q^{s,p}(\mathbb{R}^d)$  satisfy  $\sigma_f(H) = \begin{cases} p(H - (s - \frac{d}{p})) & \text{if } H \in [s - d/p, s], \\ -\infty & \text{otherwise.} \end{cases}$

In this case,  $\tau_\mu(t) = (s - d/p)t - d$  so  $\tau'_\mu(-\infty) = \tau'_\mu(+\infty) = s - d/p$ ,  $\tau_\mu^*(H) = d$  if  $H = s - d/p$  and  $-\infty$  otherwise. Hence,  $\zeta_{\mu,p}(t) = st - d$  for  $t < p$  and  $\zeta_{\mu,p}(t) = (s - d/p)t$  for  $t \geq p$ , whose Legendre transform is the typical spectrum in  $B_q^{s,p}(\mathbb{R}^d)$ .

**Remark 1.15.** Gibbs capacities are a fundamental class of multifractal doubling capacities obeying property (P) with  $\phi = 0$ , and for which Theorem 4, as well as Theorem 5 below, hold not only for  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , but also for  $B_q^{\mu,p}(\mathbb{R}^d)$ . Such a capacity is of the form  $\mu = \nu^s$ , where  $s > 0$  and  $\nu$  is a Gibbs measure defined as follows: let  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathbb{Z}^d$ -invariant real valued Hölder continuous function. The sequence of Radon measures

$$\nu_n(dx) = \frac{\exp(S_n\varphi(x))}{\int_{[0,1]^d} \exp(S_n\varphi(t)) \mathcal{L}^d(dt)} \mathcal{L}^d(dx), \quad \text{where } S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(2^k x),$$

converges vaguely to a  $\mathbb{Z}^d$ -invariant Radon measure  $\nu$  fully supported on  $\mathbb{R}^d$ , called Gibbs measure associated with  $\varphi$ .

Also, the so-called topological pressure of  $\varphi$ ,  $P(\varphi) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{[0,1]^d} 2^n \exp(S_n\varphi(x)) \mathcal{L}^d(dx)$  exists, and  $\tau_{\nu|_{[0,1]^d}}(t) = tP(\varphi) - P(t\varphi)$ . Moreover,  $\tau_{\nu|_{[0,1]^d}}$  is analytic (see [8, 9]).

One can check that when  $p = +\infty$ , or  $\tau_{\nu|_{[0,1]^d}}^*(\tau'_{\nu|_{[0,1]^d}}(+\infty)) = 0$ , or the potential  $\varphi$  reaches its minimum at 0, the proofs of Theorems 4 and 5 when  $\mu \in \mathcal{E}_d$  remain true (up to slight modifications) for  $\mu = \nu^s$ . The general case requires additional efforts.

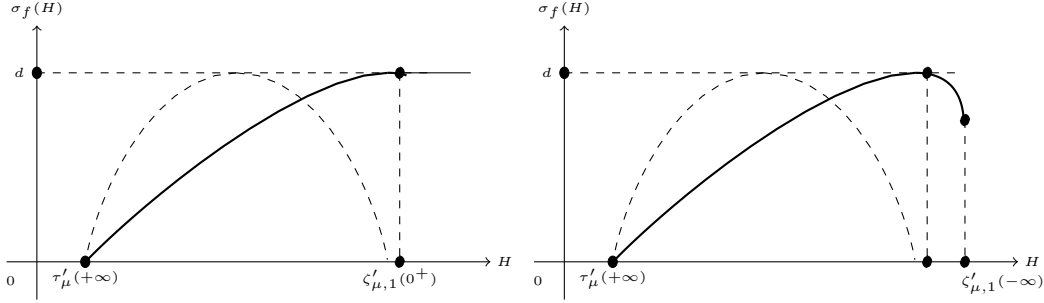


Figure 1: **Left:** Upper bound for the singularity spectrum of every  $f \in \mathbf{B}_q^{\mu,1}(\mathbb{R}^d)$ . **Right:** Singularity spectrum of a typical  $f \in \mathbf{B}_q^{\mu,1}(\mathbb{R}^d)$ . The dashed graph represents the (initial) singularity spectrum of  $\mu$ . When  $p = +\infty$  and  $f$  is typical in  $\mathbf{B}_q^{\mu,+\infty}(\mathbb{R}^d)$ ,  $\sigma_f = \sigma_\mu$ .

### 1.5. Multifractal formalism for functions in $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$

The formalism used in this paper is based on the one developed by Jaffard in [10]. Let us begin  
165 with the definition of wavelet leaders.

**Definition 1.16** (Wavelet leaders). *Given  $\Psi \in \bigcup_{r \in \mathbb{N}} \mathcal{F}_r$  and  $f \in L_{\text{loc}}^p(\mathbb{R}^d)$  for  $p \in [1, +\infty]$ , denoting the wavelet coefficients of  $f$  associates with  $\Psi$  by  $(c_\lambda)_{\lambda \in \Lambda}$ , the wavelet leader of  $f$  associated with  $\lambda \in \mathcal{D}$  is defined as:*

$$L_\lambda^f = \sup\{|c_{\lambda'}| : \lambda' = (i, j, k) \in \Lambda, \lambda'_{j,k} \subset 3\lambda\}. \quad (1.24)$$

Pointwise Hölder exponents of Hölder continuous functions (recall (1.1)) are related to the wavelet leaders as follows (see [10, Corollary 1]).

**Proposition 1.17.** *Let  $r \in \mathbb{N}^*$  and  $\Psi \in \mathcal{F}_r$ . If  $f \in \mathcal{C}^\varepsilon(\mathbb{R}^d)$  for some  $\varepsilon > 0$ , then for every  $x_0 \in \mathbb{R}^d$ ,  $h_f(x_0) < r$  if and only  $\liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j}^f(x_0)}{\log(2^{-j})} < r$ , and in this case*

$$h_f(x_0) = \liminf_{j \rightarrow \infty} \frac{\log L_{\lambda_j}^f(x_0)}{\log(2^{-j})}. \quad (1.25)$$

In order to estimate from above the singularity spectrum  $\sigma_f$  of  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , it is then natural to consider, exactly as it was done for the elements of  $\mathcal{H}([0, 1]^d)$ , the  $L^q$ -spectrum of  $f$  relative to  $\Psi$  defined as follows: For any  $N \in \mathbb{N}^*$ , set

$$\zeta_f^{N,\Psi} = \liminf_{j \rightarrow +\infty} \zeta_{f,j}^{N,\Psi}, \text{ where } \zeta_{f,j}^{N,\Psi} : t \in \mathbb{R} \mapsto -\frac{1}{j} \log_2 \left( \sum_{\lambda \in \mathcal{D}_j, \lambda \subset N[0,1]^d, L_\lambda^f > 0} (L_\lambda^f)^t \right). \quad (1.26)$$

Recall that  $(N[0, 1]^d)_{N \in \mathbb{N}^*}$  is the increasing sequence of boxes  $[-(N-1)/2, (N+1)/2]^d$ , whose union covers  $\mathbb{R}^d$ .

**Definition 1.18.** *The  $L^q$ -spectrum of  $f$  relative to  $\Psi$  is the concave function*

$$\zeta_f^\Psi = \inf\{\zeta_f^{N,\Psi} : N \in \mathbb{N}^*\} = \lim_{N \rightarrow +\infty} \zeta_f^{N,\Psi}. \quad (1.27)$$

The concavity of  $\zeta_f^\Psi$  follows from the fact that  $(\zeta_f^{N,\Psi})_{N \geq 1}$  is a non-increasing sequence of functions,  
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It is remarkable that  $\zeta_f^\Psi|_{\mathbb{R}_+}$  does not depend on  $\Psi$  [10, Th. 3]. This would be the case over  $\mathbb{R}$  if  $\Psi$  belonged to the Schwarz class [10, Th. 4]. However, the wavelet characterization of  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$

175 makes it necessary to use compactly supported wavelets, which never belong to  $\mathcal{C}^\infty(\mathbb{R}^d)$  [11]. For simplicity,  $\zeta_f^\Psi|_{\mathbb{R}_+}$  is simply denoted by  $\zeta_f|_{\mathbb{R}_+}$ .

Let us now define the multifractal formalism for functions. First, recall that by [10], when  $H < r$ , the Legendre transform of  $\zeta_f^\Psi$  at  $H$  (recall formula (1.19) of the Legendre transform) provides an upper bound for  $\dim E_f(H)$ , i.e. one has

$$\sigma_f(H) \leq (\zeta_f^\Psi)^*(H). \quad (1.28)$$

The formalism we use will combines Jaffard's formalism (based on wavelet leaders) with a variant of it (used to control the decreasing part of  $\sigma_f$ ). This variant is necessary since when  $\mu \in \mathcal{E}_d$ ,  $q < +\infty$  and the elements of  $\Psi$  are smooth, it is generic in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  that  $\zeta_f^\Psi|_{\mathbb{R}_+}$  equals  $-\infty$  (see Theorem 5(3)). Hence, for  $H \geq (\zeta_f)^\Psi(0^+)$ ,  $(\zeta_f^\Psi)^*(H)$  only provides the trivial upper bound  $\sigma_f(H) \leq d$ .  
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**Definition 1.19.** Let  $r \in \mathbb{N}^*$  and  $f \in \bigcup_{s>0} \mathcal{C}^s(\mathbb{R}^d)$ . Suppose that  $\sigma_f$  has a compact domain included in  $(0, r)$ . Let  $I \subset \text{dom}(\sigma_f)$  be a compact interval.

1. The wavelet leaders multifractal formalism (WMF) holds for  $f$  on  $I$  when there exists  $\tilde{r} \geq r$  such that for all  $H \in I$  and all  $\Psi \in \mathcal{F}_{\tilde{r}}$ ,  $\sigma_f(H) = (\zeta_f^\Psi)^*(H)$ .
- 185 2. The weak wavelet leaders multifractal formalism (WWMF) holds for  $f$  on  $I$  relatively to  $\Psi \in \mathcal{F}_r$  when the following property holds: there exists an increasing sequence  $(j_k)_{k \in \mathbb{N}}$  such that for all  $N \in \mathbb{N}$ ,  $\lim_{k \rightarrow \infty} \zeta_{f, j_k}^{N, \Psi} = \zeta_{f, w}^{(N), \Psi}$  exists, and setting  $\zeta_{f, w}^\Psi = \lim_{N \rightarrow +\infty} \zeta_{f, w}^{(N), \Psi}$ , one has  $\sigma_f(H) = (\zeta_{f, w}^\Psi)^*(H)$  for all  $H \in I$ .

**Remark 1.20.** Contrarily to (1.28), in general, even if there exists such a subsequence  $(j_k)_{k \in \mathbb{N}}$ , one cannot get the a priori inequality  $\sigma_f \leq (\zeta_{f, w}^\Psi)^*$ . This justifies the terminology "weak". Nevertheless, the existence of  $\zeta_{f, w}^\Psi$  emphasizes that the sequences  $(\zeta_{f, j}^{N, \Psi}(t))_{j \in \mathbb{N}}$  converge along the same subsequence for all  $N$  and  $t$ . This property is typical in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , and holds simultaneously for countably many  $\Psi$ 's.  
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Theorem 4 can now be completed by the following result on the validity of the multifractal formalism. Recall (1.22) and (1.21) for the definitions of  $\zeta_{\mu,p}$  and  $s_\mu$  respectively, as well as Remarks 1.9 (2) and 1.13.  
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**Theorem 5** (Validity of the multifractal formalism). Let  $\mu \in \mathcal{E}_d$ .

1. For all  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , one has  $\zeta_f|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ .
2. Typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the WMF on  $[\zeta_{\mu,p}'(+\infty), \zeta_{\mu,p}'(0^+)]$  (i.e. in the increasing part of  $\sigma_f$ ), and  $\zeta_f|_{\mathbb{R}_+} = \zeta_{\mu,p}|_{\mathbb{R}_+}$ .  
200
3. (i) Let  $\Psi \in \mathcal{F}_{s_\mu}$ . Typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the WWMF on  $\text{dom}(\sigma_f) = [\zeta_{\mu,p}'(+\infty), \zeta_{\mu,p}'(-\infty)]$  relatively to  $\Psi$ , with  $\zeta_{f, w}^\Psi = \sigma_f^* = \zeta_{\mu,p}$ . Moreover, if  $q < +\infty$ , the property  $\zeta_f^\Psi|_{\mathbb{R}_+^*} = -\infty$  is typical as well.  
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- (ii) Given a countable subset  $\mathcal{F}$  of  $\mathcal{F}_{s_\mu}$ , typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the WWMF on the interval  $\text{dom}(\sigma_f)$  relatively to any  $\Psi \in \mathcal{F}$ , with  $\zeta_{f, w}^\Psi = \sigma_f^* = \zeta_{\mu,p}$ , and  $\zeta_f^\Psi|_{\mathbb{R}_+^*} = -\infty$  if  $q < +\infty$ .

Let us mention that, although typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  are multifractal and satisfy a multifractal formalism, they do not possess any self-similar structure, consolidating the idea that being multifractal is far from being exceptional.

210 *1.6. Solutions to the Frisch-Parisi conjecture*

It is worth recalling the results obtained by Jaffard in [7]. Consider an increasing continuous and concave function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}^+$ , with positive slope  $\eta'(+\infty)$  at  $\infty$ , such that  $\eta(0) \in [0, d]$ , and  $\eta^*$  takes values in  $[-d, 0]$  over its domain. Setting  $\zeta = \eta - d$ , Jaffard seeks for a Baire space in which the increasing part of the typical singularity spectrum is given by  $\zeta^*$ . He works with the so-called homogeneous Besov spaces  $\dot{B}_q^{s,p}(\mathbb{R}^d)$ , introduced the Baire space  $V = \bigcap_{\epsilon > 0} \bigcap_{t > 0} \dot{B}_{t,\text{loc}}^{(\eta(t)-\epsilon)/t,t}(\mathbb{R}^d)$  [7] and proved that for typical functions  $f \in V$ ,  $\sigma_f = \tilde{\zeta}^*$ , where

$$\tilde{\zeta}(t) = \begin{cases} d(t/t_c - 1) & \text{if } t < t_c \\ \zeta(t) & \text{if } t \geq t_c \end{cases},$$

$t_c$  being the unique solution of  $\zeta(t_c) = 0$ . In particular,  $\sigma_f$  is necessarily increasing, with domain  $[\zeta'(+\infty), d/t_c]$ , and with an affine part over the interval  $[\zeta'(t_c+), d/t_c]$ . Also,  $\sigma_f$  coincides with  $\zeta^*$  over  $[\zeta'(+\infty), \zeta'(t_c+)]$ .

In addition, in the multifractal formalism used in [7], the scaling function  $\zeta_f(t)$  is defined as  $\sup\{s \geq 0 : f \in \dot{B}_{\infty,\text{loc}}^{s/t,t}(\mathbb{R}^d)\} - d$  for  $t > 0$ , and with this definition typical functions in  $V$  satisfy  $\zeta_f = \zeta$ . Thus the associated multifractal formalism holds on  $[\zeta'(+\infty), \zeta'(t_c+)]$  only. However, it can be checked that the WMF does hold for  $f$  with  $\zeta_f = \tilde{\zeta}$  on  $[\zeta'(+\infty), d/t_c]$ .

Hence, although this approach was a substantial progress, it allowed to reach only increasing singularity spectra, necessarily composed by an affine part followed by a concave part. Up to now, no better solution to Conjecture 1 has been proposed.

Combining the previously stated results (Theorems 3, 4 and 5), we can now state our main theorems. The first one is a direct corollary of Theorems 4 and 5.

**Theorem 6.** *The Frisch-Parisi conjecture 1 is true. Given  $\sigma \in \mathcal{S}_s$  and  $\mu \in \mathcal{E}_d$  such that  $\sigma_\mu = \sigma$ , the associated inverse problem is solved by  $\mathbf{B}_q^{\mu,+\infty}(\mathbb{R}^d)$ , for any  $q \in [1, +\infty]$ .*

Our second statement provides solutions of the form  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  with  $1 \leq p < +\infty$ . Its proof is given in Section 8.

**Theorem 7 (Solutions of the form  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  with  $p < +\infty$ ).** *Let  $\sigma \in \mathcal{S}_d$  and denote its domain by  $[H_{\min}, H_{\max}]$ .*

1. *If  $\sigma$  is the typical singularity spectrum in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  for some  $1 \leq p < +\infty$ ,  $q \in [1, +\infty]$  and  $\mu \in \mathcal{E}_d$ , then  $\sigma(H_{\min}) = 0$  and  $\sigma'(H_{\min}^+) \leq p$ .*
2. *Suppose  $\sigma(H_{\min}) = 0$  and  $\sigma'(H_{\min}^+) < +\infty$ . For all  $p \in [\max(1, \sigma'(H_{\min}^+)), +\infty)$ , there exists  $\mu \in \mathcal{E}_d$  such that for all  $q \in [1, +\infty]$ ,  $\sigma$  is the singularity spectrum of the typical elements of  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ ; also, typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  satisfy the WMF in the increasing part of  $\sigma$  and the WWMF over  $[H_{\min}, H_{\max}]$  relatively to any  $\Psi$  in a countable family of elements of  $\mathcal{F}_{s_\mu}$ .*

Let us make a final remark. Like for Besov spaces, one can let  $p$  or  $q$  take values in  $(0, +\infty]$  in the definition of Besov spaces in multifractal environment, and all our results remain valid, the only change to make being to take  $p \in [\sigma'(H_{\min}^+), +\infty)$  in Theorem 7(2). This provides a larger set of solutions to the inverse problem 1.

240 *1.7. Organization of the rest of the paper*

In Section 2, the wavelet characterization of the space  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  is established when  $\mu$  is an almost doubling capacity satisfying property (P) (Theorem 2).

In Section 3 are gathered the main properties proved in [1] to be satisfied by the capacities  $\mu \in \mathcal{E}_d$ .

245 The various shapes of  $\zeta_{\mu,p}$  and  $\zeta_{\mu,p}^*$  are investigated in Section 4, where  $\zeta_{\mu,p}^*$  is explicitly expressed in terms of  $\tau_\mu^*$ ; this expression turns out to be very useful in the proof of the WMF's validity for typical functions.

Next, in section 5, the upper bound for the singularity spectrum of all functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  is obtained (part (1) of Theorem 4), as a consequence of part (1) of Theorem 5 which is also proved there. Part (2) of Theorem 4 is shown in Section 6. It consists first in building a specific function whose singularity spectrum is typical, and then in building a dense  $G_\delta$ -set included in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  in which all functions share the same multifractal spectrum. Parts (2) and (3) of Theorem 5 are established in Section 7. Finally, the proof of Theorem 7 is given in Section 8.

## 2. Wavelet characterization of $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$

255 After some definitions and two basic lemmas in Section 2.1, Theorem 2 is proved when  $p \in [1, +\infty)$  in Section 2.2. The much simpler case  $p = +\infty$  is left to the reader who can easily adapt the lines used to treat the case  $p < +\infty$ .

### 2.1. Preliminary definitions and observations

260 We start by extending the definition of the moduli of smoothness (1.4) and (1.5) to all Borel sets  $\Omega \subset \mathbb{R}^d$ .

**Definition 2.1.** Let  $\Omega \subset \mathbb{R}^d$ . For  $h \in \mathbb{R}^d$ , let

$$\Omega_{h,n} = \{x \in \Omega : x + kh \in \Omega, k = 1, \dots, n\}. \quad (2.1)$$

Then, for  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\mu \in \mathcal{H}(\mathbb{R}^d)$ ,  $t > 0$  and  $n \geq 1$  set

$$\omega_n^\mu(f, t, \Omega)_p = \sup_{t/2 \leq |h| \leq t} \|\Delta_h^{\mu,n} f\|_{L^p(\Omega_{h,n})} \quad (2.2)$$

$$\text{and } \omega_n(f, t, \Omega)_p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\Omega_{h,n})}. \quad (2.3)$$

Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$  be an almost doubling capacity such that property (P) holds with exponents  $0 < s_1 \leq s_2$ . Let  $n \geq r = \lfloor s_2 + \frac{d}{p} \rfloor + 1$  and  $\Psi = \{\phi, \{\psi^{(i)}\}_{i=1, \dots, 2^d-1}\} \in \mathcal{F}_r$  (see Definition 1.7).

265 Also, recall that for  $\lambda = (i, j, k) \in \Lambda_j$ ,  $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ . It follows from the construction of  $\Psi$  (see [3, Section 3.8]) that there exists an integer  $N_\Psi \in \mathbb{N}^*$  such that  $\text{supp}(\phi)$  and  $\text{supp}(\psi^{(i)})$  are included in  $N_\Psi[0, 1]^d$ . Our proofs will use some estimates established in [2]. These estimates require to associate with each  $\lambda = (i, j, k) \in \Lambda_j$  a larger cube  $\tilde{\lambda}$  described in the following definition.

**Definition 2.2.** For each  $\lambda = (i, j, k) \in \Lambda_j$ , set

$$\tilde{\lambda} = \lambda_{j,k} + 2^{-j}(\text{supp}(\phi) - \text{supp}(\phi)).$$

Note that  $\lambda_{j,k} \subset \text{supp}(\psi_\lambda) \subset \tilde{\lambda} \subset 3N_\Psi \lambda_{j,k}$ , the second embedding coming from the construction of compactly supported wavelets (see [3, Section 3.8]).

For every  $j \in \mathbb{N}$ , the cubes  $(\tilde{\lambda})_{\lambda \in \Lambda_j}$  do not overlap too much, in the sense that

$$K_\Psi := \sup_{j \in \mathbb{N}} \sup_{\lambda \in \Lambda_j} \#\{\lambda' \in \Lambda_j : \tilde{\lambda} \cap \tilde{\lambda}' \neq \emptyset\} < +\infty. \quad (2.4)$$

**Lemma 2.3.** Let  $p \in [1, +\infty)$  and  $n \in \mathbb{N}^*$ . There exists a constant  $C_{d,n,p}$  (depending on  $p$ ,  $n$ , and  $d$  only) such that for all  $f \in L_{\text{loc}}^p(\mathbb{R}^d)$ ,  $t > 0$  and  $\lambda \in \Lambda$ , the following inequality holds:

$$\omega_n(f, t, \tilde{\lambda})_p^p \leq C_{d,n,p} t^{-d} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda} + B(0, 2nt)} |\Delta_y^n f(x)|^p dx dy.$$

270 *Proof.* The approach follows the lines of the proof of [2, inequality (3.3.17)], where a similar inequality is proved.

Fix  $f$ ,  $t$  and  $\lambda$  as in the statement. For any  $h, y \in \mathbb{R}^d$ , recall the following equality (see (3.3.19) in [2]):

$$\Delta_h^n f(x) = \sum_{k=1}^n (-1)^k \binom{n}{k} [\Delta_{ky}^n f(x + kh) - \Delta_{h+ky}^n f(x)].$$

Integrating  $|\Delta_h^n f|^p$  over  $\tilde{\lambda}_{h,n}$  (recall definition (2.1)), one sees that for some constant  $C_{n,p} > 0$ , when  $|h| \leq t$ ,

$$\begin{aligned} \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p &\leq C_{n,p} \sum_{k=1}^n \|\Delta_{ky}^n f(\cdot + kh)\|_{L^p(\tilde{\lambda}_{h,n})}^p + \|\Delta_{h+ky}^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \\ &\leq C_{n,p} \sum_{k=1}^n \|\Delta_{ky}^n f\|_{L^p(\tilde{\lambda}+B(0,2nt))}^p + \|\Delta_{h+ky}^n f\|_{L^p(\tilde{\lambda}+B(0,2nt))}^p. \end{aligned}$$

Then, defining  $C_d = \mathcal{L}^d(B(0,3) \setminus B(0,2))$ , an integration with respect to  $y$  over  $B(0,3t) \setminus B(0,2t)$  yields

$$C_d t^d \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \leq C_{n,p} \sum_{k=1}^n \int_{2t \leq |y| \leq 3t} \int_{\tilde{\lambda}+B(0,2nt)} |\Delta_{ky}^n f(x)|^p + |\Delta_{h+ky}^n f(x)|^p dx dy.$$

Further, operating the change of variable  $y' = ky$  in each term of the sum yields

$$\begin{aligned} t^d \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p &\leq C_d^{-1} C_{n,p} \sum_{k=1}^n \int_{2kt \leq |y| \leq 3kt} \int_{\tilde{\lambda}+B(0,2nt)} |\Delta_y^n f(x)|^p + |\Delta_{h+y}^n f(x)|^p dx dy \\ &\leq 2n C_d^{-1} C_{n,p} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda}+B(0,2nt)} |\Delta_y^n f(x)|^p dx dy. \end{aligned}$$

where one used that  $t \leq |h+y| \leq 4nt$  when  $|h| \leq t$  and  $|y| \geq 2t$ . The previous upper bound being independent of  $h \in B(0,t)$ , one concludes that

$$\omega_n(f, t, \tilde{\lambda})_p^p = \sup_{0 \leq |h| \leq t} \|\Delta_h^n f\|_{L^p(\tilde{\lambda}_{h,n})}^p \leq \frac{2n C_d^{-1} C_{n,p}}{t^d} \int_{t \leq |y| \leq 4nt} \int_{\tilde{\lambda}+B(0,2nt)} |\Delta_y^n f(x)|^p dx dy,$$

as desired.  $\square$

**Lemma 2.4.** *Let  $\varepsilon > 0$  and  $\mu \in \mathcal{C}(\mathbb{R}^d)$  that satisfies Property (P) with exponents  $s_1$  and  $s_2$ .*

*There exists a constant  $C = C(\varepsilon, n, \mu) \geq 1$  such that for every  $j \in \mathbb{N}$  and  $\lambda \in \Lambda_j$ , for every  $x \in \tilde{\lambda} + B(0, 2n2^{-j})$  and  $y \in \mathbb{R}^d$  such that  $2^{-j} \leq |h| \leq 4n2^{-j}$ , for every  $f : \tilde{\lambda} \rightarrow \mathbb{R}$ , one has*

$$\frac{|\Delta_h^n f(x)|}{\mu(\lambda)} \leq C \frac{|\Delta_h^n f(x)|}{\mu^{(+\varepsilon)}(B[x, x + nh])}.$$

*Proof.* Observe first that under the assumptions of the Lemma, the inequality

$$\frac{\mu(B[x, x + ny])}{\mu(\lambda)} \leq C(n|y|)^{-\varepsilon},$$

follows easily from the definition of the almost doubling property (1.7). Then, Lemma 2.4 is deduced from last inequality and the definition of  $\mu^{(+\varepsilon)}$ .  $\square$

275 *2.2. Proof of Theorem 2 when  $1 \leq p < +\infty$*

Let us now explain our approach to get Theorem 2 when  $p \in [1, +\infty)$ . Recall that  $B_q^{\mu,p}(\mathbb{R}^d)$  is defined via  $L^p$  moduli of smoothness of order  $n \geq r = \lfloor s_2 + d/p + 1 \rfloor$ , and that  $\Psi$  belongs to  $\mathcal{F}_r$ . The purpose of this theorem is to establish relations between this definition (1.10) and the wavelet-based one (1.14).

280 In Section 2.2.1, it is shown that, for any  $\varepsilon \in (0, 1)$ , when  $B_q^{\mu^{(+\varepsilon)},p}(\mathbb{R}^d)$  is defined via the  $L^p$  modulus of smoothness of order  $n$ , then (1.16) holds for any  $\Psi \in \mathcal{F}_n$ . It is only a partial proof of the statement, since one wants to obtain (1.16) for any  $\Psi \in \mathcal{F}_r$ .

Then, in Section 2.2.2, (1.17) is completely proved to hold for any  $\varepsilon \in (0, 1)$  and any  $\Psi \in \mathcal{F}_r$  when  $B_q^{\mu,p}(\mathbb{R}^d)$  is defined via the  $L^p$  modulus of smoothness of order exactly equal to  $r$ . Since 285  $\mathcal{F}_n \subset \mathcal{F}_r$ , the statement also holds for  $\Psi \in \mathcal{F}_n$ .

Finally, from the two preceding observations, we conclude that (1.16) holds for any  $\varepsilon \in (0, 1)$  and any  $\Psi \in \mathcal{F}_r$ , by applying:

- first (1.16) with the environment  $\mu$ , the  $n$ -th order difference operator,  $\varepsilon/3$  and any wavelet  $\tilde{\Psi} \in \mathcal{F}_n$ ,
- 290 • then (1.17) with the environment  $\mu^{(+\varepsilon/3)}$ , the  $r$ -th order difference operator,  $\varepsilon/3$  and the same  $\tilde{\Psi} \in \mathcal{F}_n$ ,
- finally (1.16) with the environment  $\mu^{(+2\varepsilon/3)}$ , the  $r$ -th order difference operator,  $\varepsilon/3$  and  $\Psi \in \mathcal{F}_r$ .

2.2.1. Proof of inequality (1.16) in Theorem 2

295 Assume that  $\Psi \in \mathcal{F}_n \subset \mathcal{F}_r$ . Fix  $\varepsilon > 0$ ,  $f \in L^p(\mathbb{R}^d)$  and  $j \in \mathbb{N}$ .

For every  $\lambda = (i, j, k) \in \Lambda_j$ , since  $\psi_\lambda$  is orthogonal to any polynomial  $P$  of degree  $\leq n$ , the wavelet coefficient  $c_\lambda$  can be written

$$c_\lambda = 2^{jd} \int_{\mathbb{R}^d} (f(x) - P(x))\psi_\lambda(x)dx.$$

Due to the local approximation of  $f$  by polynomials (equation (3.3.13) in [2]), there exists a polynomial  $P_\lambda$  of degree  $\leq n$  such that

$$\|f - P_\lambda\|_{L^p(\tilde{\lambda})} \leq C\omega_n(f, 2^{-j}, \tilde{\lambda})_p,$$

where  $C$  depends on  $n$  and  $p$  only. Recall that  $\text{supp}(\psi_\lambda) \subset \tilde{\lambda}$ .

The last inequalities, together with Hölder's inequality, yield

$$\begin{aligned} \frac{|c_\lambda|}{\mu(\lambda)} &\leq 2^{jd} \frac{\|\psi_\lambda\|_{L^{p'}(\mathbb{R}^d)} \|f - P_\lambda\|_{L^p(\tilde{\lambda})}}{\mu(\lambda)} \leq C 2^{jd} \frac{2^{-jd/p'} \|\psi^{(i)}\|_{L^{p'}(\mathbb{R}^d)} \omega_n(f, 2^{-j}, \tilde{\lambda})_p}{\mu(\lambda)} \\ &\leq \tilde{C} 2^{jd/p} \frac{\omega_n(f, 2^{-j}, \tilde{\lambda})_p}{\mu(\lambda)}, \end{aligned} \quad (2.5)$$

where  $\tilde{C} = C \sup \{ \|\psi^{(i)}\|_{L^{p'}(\mathbb{R}^d)} : 1 \leq i \leq 2^d - 1 \}$ .

Then, Lemma 2.3 gives

$$\left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \leq C_{d,n,p} \tilde{C}^p 2^{2dj} \int_{2^{-j} \leq |y| \leq 4n 2^{-j}} \int_{\tilde{\lambda} + B(0, 2n 2^{-j})} \frac{|\Delta_y^n f(x)|^p}{\mu(\lambda)^p} dx dy,$$

and by Lemma 2.4, there exists  $C'$  depending on  $(\varepsilon, n, p, \Psi)$  such that

$$\begin{aligned} \left(\frac{|c_\lambda|}{\mu(\lambda)}\right)^p &\leq C_{d,n,p}(C')^p 2^{2dj} \int_{2^{-j} \leq |y| \leq 4n2^{-j}} \int_{\tilde{\lambda}+B(0,2n2^{-j})} |\Delta_y^{\mu^{(+\varepsilon)},n} f(x)|^p dx dy \\ &\leq C_{d,n,p}(C')^p \sum_{k=0}^{j_n} 2^{2dj} \int_{2^{-j+k} \leq |y| \leq 2^{-j+k+1}} \int_{\tilde{\lambda}+B(0,2n2^{-j})} |\Delta_y^{\mu^{(+\varepsilon)},n} f(x)|^p dx dy, \end{aligned}$$

where  $j_n = \lfloor \log_2(4n) \rfloor$ . By (2.4), there exists a constant  $K_{\Psi,n} > 0$  depending on  $(\Psi, n)$  only such that any  $\lambda \in \Lambda_j$  is covered by at most  $K_{\Psi,n}$  sets of the form  $\tilde{\lambda}' + B(0, 2n2^{-j})$  with  $\lambda' \in \Lambda_j$ . It follows that

$$\sum_{\lambda \in \Lambda_j} \left(\frac{|c_\lambda|}{\mu(\lambda)}\right)^p \leq K_{\Psi,n} C_{d,n,p}(C')^p \sum_{j'=0}^{j_n} 2^{2dj} \int_{2^{-j+j'} \leq |y| \leq 2^{-j+j'+1}} \int_{\mathbb{R}^d} |\Delta_y^{\mu^{(+\varepsilon)},n} f(x)|^p dx dy.$$

Recalling the definition (1.4) of  $\omega_n^{\mu^{(+\varepsilon)}}(f, t, \mathbb{R}^d)$ , for every  $j'$  the double integral above is bounded by  $2^{d(-j+j'+1)} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+j'+1}, \mathbb{R}^d)_p$ . Since  $2^{d(j'+1)} \leq 2^{d(j_n+1)} \leq (8n)^d$ , one has

$$\sum_{\lambda \in \Lambda_j} \left(\frac{|c_\lambda|}{\mu(\lambda)}\right)^p \leq C_1^p \sum_{j'=0}^{j_n} 2^{dj} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+j'+1}, \mathbb{R}^d)_p,$$

where  $C_1 = ((8n)^d K_{\Psi,n} C_{d,n,p})^{1/p} C'$  does not depend on  $f$  or  $j$ .

Suppose now that  $q \in [1, +\infty)$  (the case  $q = +\infty$  is obvious). The previous estimates together with the subadditivity of  $t \geq 0 \mapsto t^{1/p}$  and the convexity of  $t \geq 0 \mapsto t^q$  yield

$$\left\| \left(\frac{c_\lambda}{\mu(\lambda)}\right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C_1^q (j_n + 1)^{q-1} \sum_{j'=0}^{j_n} (2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j+j'+1}, \mathbb{R}^d)_p)^q.$$

Summing the last inequality over  $j \in \mathbb{N}$  gives

$$\sum_{j \geq 0} \left\| \left(\frac{c_\lambda}{\mu(\lambda)}\right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C_1^q (j_n + 1)^{q-1} \sum_{j=-j_n-1}^{+\infty} K_j (\omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j}, \mathbb{R}^d)_p)^q,$$

where  $K_j = \begin{cases} \sum_{j'=j+1}^{j+j_n+1} 2^{qdj'/p} & \text{when } j \geq -1 \\ \sum_{j'=0}^{j+j_n+1} 2^{qdj'/p} & \text{when } -j_n-1 \leq j \leq -2 \end{cases}$ . It is easily seen that there is a constant  $C_2 = C_2(n, q, d)$  such that  $C_1^q (j_n + 1)^{q-1} K_j \leq C_2 2^{qdj/p}$ , so

$$\sum_{j \geq 0} \left\| \left(\frac{c_\lambda}{\mu(\lambda)}\right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C_2 \sum_{j=-j_n-1}^{+\infty} (2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j}, \mathbb{R}^d)_p)^q.$$

Observe that there is  $C_3 \geq 1$  depending on  $n$  such that for  $-j_n-1 \leq j \leq 0$ ,  $2^{dj/p} \omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j}, \mathbb{R}^d)_p \leq C_3 \|f\|_{L^p(\mathbb{R}^d)}$ . This follows from the fact that for such a  $j$ :

- $2^{dj/p} \leq 1$ ;
- by periodicity of  $\mu$ ,  $\mu(B[x, x+n2^{-j}]) \geq \mu([0, 1]^d) = 1$ , so  $\frac{|\Delta_y^n f(x)|}{\mu(\lambda)} \leq |\Delta_y^n f(x)|$ , and thus for some constant  $C'''$

$$\omega_n^{\mu^{(+\varepsilon)}}(f, 2^{-j}, \mathbb{R}^d)_p \leq 2^{j\varepsilon} \omega_n(f, 2^{-j}, \mathbb{R}^d)_p \leq (\omega_n(f, 8n, \mathbb{R}^d))_p \leq C_3 \|f\|_{L^p(\mathbb{R}^d)}.$$



Consequently, for some constant  $C$  independent of  $f$ ,

$$\sum_{j \geq 0} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}^q \leq C \left( \|f\|_{L^p(\mathbb{R}^d)}^q + \sum_{j \geq 0} (2^{dj/p} \omega_n^{\mu(+\varepsilon)}(f, 2^{-j}, \mathbb{R}^d)_p)^q \right),$$

which implies that  $\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu, p, q, \Psi} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{B_q^{\mu(+\varepsilon), p}(\mathbb{R}^d)})$ . Hence, (1.16) holds when  $\Psi \in \mathcal{F}_n$ .

### 2.2.2. Proof of inequality (1.17) in Theorem 2

305 Fix  $\varepsilon > 0$  and  $f \in L^p(\mathbb{R}^d)$ .

Define the partial sums  $f_j = \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda$ , for all  $j \geq 0$ .

The following lemma is needed.

**Lemma 2.5.** *Let  $s \in \left(s_2 + \frac{d}{p}, s_2 + \frac{d}{p} + 1\right)$ . There exist a constant  $C > 0$  and a sequence  $(\tilde{\varepsilon}_m)_{m \in \mathbb{N}} \in \ell^q(\mathbb{N})$  bounded by 1, independent of  $f$ , such that for all  $j, J \geq 0$ ,*

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq C 2^{-jd/p} \min(1, 2^{(j-J)(s-s_2)} \tilde{\varepsilon}_{J-j}) \left( \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \right)^{1/p}, \quad (2.6)$$

with the convention that  $\tilde{\varepsilon}_m = 1$  when  $m < 0$ .

*Proof.* Inspired by the proof of [2, Theorem 3.4.3], two cases are separated:

**Case 1:**  $J < j$ . In order to prove (2.6), let us begin by writing that

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p = \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \int_{\lambda'} \frac{\left| \sum_{\lambda \in \Lambda_j} c_\lambda \Delta_h^n \psi_\lambda(x) \right|^p}{\mu(B(x, x + nh))^p} dx. \quad (2.7)$$

310 Consider  $\lambda \in \Lambda_j$ ,  $x \in \mathbb{R}^d$ , and  $h \in \mathbb{R}^d$  such that  $2^{-J-1} \leq |h| \leq 2^{-J}$ . Then:

(i) If  $x \notin \bigcup_{k=0}^n \text{supp}(\psi_\lambda) - kh$ , then  $\Delta_h^n \psi_\lambda(x) = 0$ ;

(ii) Let  $\lambda' = \lambda_J(x)$  the unique cube of generation  $J$  that contains  $x$ .

There exists an integer  $N = N(n, \Psi)$  such that if  $x \in \bigcup_{k=0}^n \text{supp}(\psi_\lambda) - kh$ , then necessarily  $\lambda \subset N\lambda'$ .

(iii) By the almost doubling property of  $\mu$ , there exists a constant  $C = C(\mu, n, \Psi, \varepsilon)$  such that for every  $x \in \bigcup_{k=0}^n \text{supp}(\psi_\lambda) - kh$ ,

$$\mu^{(+\varepsilon)}(\lambda) = 2^{-j\varepsilon} \mu(\lambda) \leq 2^{-j\varepsilon} \mu(N\lambda') \leq C \mu(B(x, x + nh)). \quad (2.8)$$

From the equality  $\Delta_h^n \psi_\lambda = \sum_{k=0}^n (-1)^k \binom{n}{k} \psi_\lambda(\cdot + (n-k)h)$ , the three items (i)-(iii) and (2.7), one obtains that

$$\begin{aligned} \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p &\leq C^p \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \frac{2^{j\varepsilon p}}{\mu(N\lambda')^p} \int_{\lambda'} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \Delta_h^n \psi_\lambda(x) \right|^p dx \\ &\leq C^p \sum_{\lambda' \in \mathcal{D}_J} \frac{2^{j\varepsilon p}}{\mu(N\lambda')^p} T_{j, J, \lambda', \lambda, \psi}, \end{aligned}$$

where

$$T_{j, J, \lambda', \lambda, \psi} = \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \int_{\mathbb{R}^d} \left| \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x + (n-k)h) \right|^p dx.$$

The convexity inequality  $(\sum_{k=0}^n |z_k|)^p \leq (n+1)^{p-1} \sum_{k=0}^n |z_k|^p$  and  $\binom{n}{k} \leq 2^n$  give

$$T_{j,J,\lambda',\lambda,\psi} \leq 2^{np}(n+1)^{p-1} \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{k=0}^n \int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x + (n-k)h) \right|^p dx.$$

Observe that the property (ii) above allows to bound each integral in the above sum by the same term  $\int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x) \right|^p dx$ . Moreover, according to [3, Ch. 6, Prop. 7], there exists  $C' > 0$  depending on  $\Psi$  only such that

$$\int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} c_\lambda \psi_\lambda(x) \right|^p dx \leq C'^p 2^{-jd} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} |c_\lambda|^p.$$

Consequently, using the first inequality of (2.8),

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq (CC')^p (n+1)^{p2^{np}} \sum_{\lambda' \in \mathcal{D}_J} 2^{-jd} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda'} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p.$$

Finally, the number of dyadic cubes  $\lambda' \in \mathcal{D}_J$  such that  $N\lambda'$  intersects a given  $\lambda \in \Lambda_j$  is bounded uniformly with respect of  $j$  and  $J$ , so

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p \leq C 2^{-jd/p} \left( \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \right)^{1/p}$$

315 for some constant  $C$  that depends on  $n, p$  and other constants. This yields (2.6).

**Case 2:**  $J \geq j$ . Let us start with a few observations. First, by assumption,  $\psi^{(i)} \in B_q^{s,p}(\mathbb{R}^d)$ , hence

$$\omega_n(\psi^{(i)}, 2^{j-J}, \mathbb{R}^d)_p \leq 2^{(j-J)s} \tilde{\varepsilon}_{J-j}^{(i)},$$

where  $(\tilde{\varepsilon}_m^{(i)})_{m \geq 1} \in \ell^q(\mathbb{N}^*)$  and  $\|\tilde{\varepsilon}^{(i)}\|_{\ell^q(\mathbb{N}^*)} \leq \|\psi^{(i)}\|_{B_q^{s,p}}$ . Consequently, for all  $\lambda \in \Lambda_j$

$$\omega_n(\psi_\lambda, 2^{-J}, \mathbb{R}^d)_p \leq 2^{(j-J)s} 2^{-jd/p} \tilde{\varepsilon}_{J-j}, \quad (2.9)$$

where  $\tilde{\varepsilon}_{J-j} = \sup_i \tilde{\varepsilon}_{J-j}^{(i)}$ .

Next, observe the following facts:

- (i) There exists an integer  $N$  independent of  $j$  and  $J$  such that for all  $x \in \mathbb{R}^d$  and  $h \in \mathbb{R}^d$  such that  $2^{-J-1} \leq |h| \leq 2^{-J}$ ,  $B[x, x + nh] \subset N\lambda_j(x)$ . Also,  $\Delta_h^n \psi_\lambda(x) \neq 0$  only if  $\lambda \subset N\lambda_j(x)$  (recall that  $\lambda = (i, j, k) \subset E$  means  $\lambda_{j,k} \subset E$ ).
- (ii) There exist two dyadic cubes  $\lambda' \in \mathcal{D}_{J+3}$  and  $\lambda'' \in \mathcal{D}_j$  such that  $\lambda' \subset B(x, x + nh)$  and  $\lambda' \subset \lambda'' \subset N\lambda_j(x)$ . By construction, for all  $\Lambda_j \ni \lambda \subset N\lambda_j(x)$ , one has

$$\mu(B[x, x + nh])^{-1} \leq \mu(\lambda')^{-1} = \frac{\mu(\lambda'')}{\mu(\lambda')} \frac{\mu(\lambda)}{\mu(\lambda'')} \mu(\lambda)^{-1}.$$

Hence, using property (P<sub>2</sub>) to get  $\frac{\mu(\lambda)}{\mu(\lambda'')} = O(2^{N^d \phi(j)})$  and  $\frac{\mu(\lambda'')}{\mu(\lambda')} = O(2^{\phi(j)} 2^{(J-j)s_2})$ , as well as the fact that  $2^{\phi(j)(N^d+1)} \leq |\lambda|^{-\varepsilon}$  since  $\phi \in \Phi$ , there exists a constant  $C$  depending on  $(\mu, n, \varepsilon)$  only such that

$$\mu(B[x, x + nh])^{-1} \leq C 2^{(J-j)s_2} (\mu^{(+\varepsilon)}(\lambda))^{-1}.$$

The two previous observations yield

$$\begin{aligned}\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p &= \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \sum_{\lambda' \in \mathcal{D}_J} \int_{\lambda'} \frac{\left| \sum_{\lambda \in \Lambda_j} c_\lambda \Delta_h^n \psi_\lambda(x) \right|^p}{\mu(B(x, x+nh))^p} dx \\ &\leq C^p 2^{(J-j)s_2 p} \sup_{2^{-J-1} \leq |h| \leq 2^{-J}} \int_{\mathbb{R}^d} \left( \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} |\Delta_h^n \psi_\lambda(x)| \right)^p dx.\end{aligned}$$

Since  $\#\{\lambda \in \Lambda_j : \lambda \subset N\lambda_j(x)\} \leq (2Nd)^d$ , for each  $x \in \mathbb{R}^d$  one has

$$\left( \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} |\Delta_h^n \psi_\lambda(x)| \right)^p \leq (2Nd)^{d(p-1)} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p |\Delta_h^n \psi_\lambda(x)|^p.$$

Also, each  $\lambda \in \mathcal{D}_j$  intersects at most  $(2N)^d$  cubes  $N\lambda'$  with  $\lambda' \in \mathcal{D}_j$ , so

$$\int_{\mathbb{R}^d} \sum_{\lambda \in \Lambda_j, \lambda \subset N\lambda_j(x)} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p |\Delta_h^n \psi_\lambda(x)|^p dx \leq (2N)^{dp} \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \int_{\mathbb{R}^d} |\Delta_h^n \psi_\lambda(x)|^p dx.$$

Finally, taking the supremum over  $h \in [2^{-J-1}, 2^{-J}]$  in the last inequalities gives

$$\omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p^p \leq C^p (2N)^{dp} 2^{(J-j)s_2 p} \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)^p \omega_n(\psi_\lambda, 2^{-J}, \mathbb{R}^d)_p^p,$$

hence the conclusion by (2.9).

By changing the constant  $C$  in (2.9) into  $C\|(\tilde{\varepsilon}_j)_{j \geq 0}\|_{\ell^\infty}$ , one gets  $\tilde{\varepsilon}_j \leq 1$ .  $\square$

We are now in position to prove (1.17). Fix  $\varepsilon \in (0, 1)$ . Setting  $\tilde{f} = f - \sum_{j=0}^\infty f_j$ , the triangle inequality yields

$$\omega_n^\mu(f, 2^{-J}, \mathbb{R}^d)_p \leq \omega_n^\mu(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p + \sum_{j=0}^\infty \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p, \quad (2.10)$$

and our goal is to control the  $\ell_q$  norms of the sequences  $(u_J := 2^{Jd/p} \omega_n^\mu(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p)_{J \in \mathbb{N}}$  and  $(v_J := 2^{Jd/p} \sum_{j=0}^\infty \omega_n^\mu(f_j, 2^{-J}, \mathbb{R}^d)_p)_{J \in \mathbb{N}}$ .

• The terms  $(u_J)_{J \geq 1}$  correspond to low frequencies, and can be controlled as follows. Using property (P), one has  $\mu(B(x, 2^{-J})) \geq 2^{-J(s_2 + \varepsilon)}$  for every  $x \in \mathbb{R}^d$ , and so

$$u_J \leq 2^{J(s_2 + \varepsilon + d/p)} \omega_n(\tilde{f}, 2^{-J}, \mathbb{R}^d)_p. \quad (2.11)$$

325 Observe that, since  $\tilde{f}$  is obtained by removing from  $f$  the high frequency terms,  $\tilde{f} \in B_q^{s', p}(\mathbb{R}^d)$  for all  $s' \in (d/p, r)$  and  $q \in [1, +\infty]$ , as can be checked using (1.13). In addition,  $|\tilde{f}|_{(\mathcal{L}^d)^{\frac{s'+\varepsilon+d/p}{d} - \frac{1}{p}, p, q, \Psi}} = |\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d} - \frac{1}{p}, p, q, \Psi}} = 0$  since the wavelet coefficients  $c_\lambda(\tilde{f})$  of  $\tilde{f}$  vanish for all  $\lambda \in \Lambda_j$ ,  $j \geq 1$ .

Recalling the decomposition (1.12), one notes that the wavelet coefficients  $(\beta(k))_{k \in \mathbb{Z}^d}$  in the wavelet expansions of  $f$  and  $\tilde{f}$  are identical. Hence, using the equivalence of norms recalled after (1.13), there is a constant  $\tilde{C}$  depending on  $(d, \varepsilon, \mu, p, q, \Psi)$  (that may change from line to line) such

that

$$\begin{aligned}
\|(u_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} &\leq \|\tilde{f}\|_{B_q^{s_2+\varepsilon+d/p,p}(\mathbb{R}^d)} \leq \tilde{C}(\|\tilde{f}\|_{L^p(\mathbb{R}^d)} + |\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_2+\varepsilon+d/p}{d}-\frac{1}{p},p,q,\Psi}}) \\
&= \tilde{C}(\|\tilde{f}\|_{L^p(\mathbb{R}^d)} + |\tilde{f}|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p},p,q,\Psi}}) \\
&\leq \tilde{C}(\|\beta(k)\|_{\ell_p(\mathbb{Z}^d)} + |f|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p},p,q,\Psi}}) \\
&\leq \tilde{C}(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{(\mathcal{L}^d)^{\frac{s_1+\varepsilon+d/p}{d}-\frac{1}{p},p,q,\Psi}}) \\
&\leq \tilde{C}(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon),p,q,\Psi}}),
\end{aligned}$$

where the last inequality is a consequence of property (P<sub>1</sub>) (which implies that  $\mu(\lambda) \leq C2^{-js_1} = C\mathcal{L}^d(\lambda)^{\frac{s_1+d/p}{d}-1/p}$  for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ).

330 • Next the  $\ell^q$  norm of  $(v_J)_{J \geq 1}$  is controlled. Set  $A_j = \left\| \left( \frac{|c_\lambda|}{\mu^{(+\varepsilon)}(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell^p(\Lambda_j)}$ .

By Lemma 2.5, when  $j \leq J$  one has  $\omega_n^\mu(f_j, 2^{-j}, \mathbb{R}^d)_p \leq C2^{-jd/p}2^{(j-J)(s-s_2)p}A_j$ , while when  $j > J$ , one has  $\omega_n^\mu(f_j, 2^{-j}, \mathbb{R}^d)_p \leq C2^{-jd/p}A_j$ . Consequently,

$$v_J \leq C2^{Jd/p} \sum_{j=0}^J 2^{-jd/p+(j-J)(s-s_2)} A_j + C2^{Jd/p} \sum_{j=J+1}^{\infty} 2^{-jd/p} A_j,$$

which implies that  $\|(v_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})} \leq C(\|(\alpha_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})} + \|(\beta_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})})$ , where

$$\alpha_J := 2^{Jd/p} \sum_{j=0}^J 2^{-jd/p+(j-J)(s-s_2)} A_j \quad \text{and} \quad \beta_J := 2^{Jd/p} \sum_{j=J+1}^{\infty} 2^{-jd/p} A_j.$$

Recall now the two following Hardy's inequalities (see, e.g. (3.5.27) and (3.5.36) in [2]): let  $q \in [1, +\infty]$  as well as  $0 < \gamma < \delta$ . There exists a constant  $K > 0$  such that :

- if  $(a_j)_{j \in \mathbb{N}}$  is a non negative sequence and for  $J \in \mathbb{N}$  one sets  $b_J = 2^{-\delta J} \sum_{j=0}^J 2^{j\delta} a_j$ , then  $\|(2^{\gamma J} b_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})} \leq K\|(2^{\gamma J} a_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})}$ .
- 335 • if  $(a_j)_{j \in \mathbb{N}}$  is a non negative sequence and for  $J \in \mathbb{N}$  one defines  $b_J = \sum_{j \geq J} a_j$ , then  $\|(2^{\gamma J} b_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K\|(2^{\gamma J} a_J)_{J \geq 0}\|_{\ell^q(\mathbb{N})}$ .

Let  $\delta = s - s_2$  and  $\gamma = d/p$ . The first Hardy's inequality with  $a_j = 2^{-jd/p} A_j$  yields

$$\|(\alpha_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K\|(A_j)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})},$$

while the second one with  $a_j = 2^{-jd/p} A_j$  and  $\gamma = d/p$  gives

$$\|(\beta_J)_{J \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} \leq K\|(A_j)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})}.$$

Since  $\|(A_j)_{j \in \mathbb{N}}\|_{\ell^q(\mathbb{N})} = |f|_{\mu^{(+\varepsilon),p,q,\Psi}}$ , one concludes that

$$\|(v_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})} \leq 2CK(\|f\|_{L^p(\mathbb{R}^d)} + |f|_{\mu^{(+\varepsilon),p,q,\Psi}}),$$

which, together with the control of  $\|(u_J)_{J \geq 1}\|_{\ell^q(\mathbb{N})}$ , implies (1.17).

340 Although we do not elaborate on this in this paper, it is certainly worth investigating the relationship between the Besov spaces in multifractal environment and the following analog of Sobolev space in multifractal environment.

**Definition 2.6.** Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ ,  $s > 0$ ,  $p \geq 1$ . A function  $f$  belongs to  $W_p^{\mu,s}(\mathbb{R}^d)$  if and only if  $\|f\|_{W_p^{\mu,s}(\mathbb{R}^d)} < +\infty$ , where  $\|f\|_{W_p^{\mu,s}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + |f|_{W_p^{\mu,s}(\mathbb{R}^d)}$  and

$$|f|_{W_p^{\mu,s}(\mathbb{R}^d)} := \iint_{([0,1]^d)^2} \frac{|f(x) - f(y)|^p}{\mu(B[x,y])^{sp} |x - y|^{2d}} dx dy < +\infty.$$

### 3. Main properties of the capacities $\mu \in \mathcal{E}_d$

In this section, we gather the main geometric, statistical and approximation properties proved to be associated with the capacities  $\mu \in \mathcal{E}_d$  in [1]. This completes Theorem 3.

#### 3.1. Geometric and statistical properties

Let us introduce the notations for  $\varepsilon > 0$ ,  $\alpha \in \mathbb{R}$ , and  $I = [a, b]$  an interval:

$$\alpha \pm \varepsilon = [\alpha - \varepsilon, \alpha + \varepsilon] \quad \text{and} \quad I \pm \varepsilon = [a - \varepsilon, b + \varepsilon]. \quad (3.1)$$

**Definition 3.1.** Let  $\mu \in \mathcal{C}([0, 1]^d)$  with  $\text{supp}(\mu) \neq \emptyset$ . For  $I \subset \mathbb{R}$  and  $j \in \mathbb{N}^*$  define

$$\mathcal{D}_\mu(j, I) = \left\{ \lambda \subset [0, 1]^d, \lambda \in \mathcal{D}_j : \frac{\log_2 \mu(\lambda)}{-j} \in I \right\}.$$

345 Define the lower and upper large deviations spectra of  $\mu$  respectively as

$$\begin{aligned} \underline{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \liminf_{j \rightarrow \infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j} \\ \text{and } \bar{\sigma}_\mu^{\text{LD}} : \alpha \in \mathbb{R} &\mapsto \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log_2 \#\mathcal{D}_\mu(j, \alpha \pm \varepsilon)}{j}. \end{aligned}$$

Also, define

$$\underline{E}_\mu^{\leq}(\alpha) = \{x \in \text{supp}(\mu) : \underline{h}_\mu(x) \leq \alpha\} \quad \text{and} \quad \bar{E}_\mu^{\geq}(\alpha) = \{x \in \text{supp}(\mu) : \bar{h}_\mu(x) \geq \alpha\}.$$

**Proposition 3.2.** Let  $\mu \in \mathcal{E}_d$ . Then :

1. the concave function  $\tau_\mu^*$  is continuous over its domain

$$\text{dom}(\tau_\mu^*) = [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] = \{\alpha \in \mathbb{R} : \tau_\mu^*(\alpha) \geq 0\} \subset (0, +\infty).$$

2. For every  $\alpha \in \text{dom}(\tau_\mu^*)$ , there exists a Borel probability measure  $\mu_\alpha$  defined on  $[0, 1]^d$  and supported on a set  $\tilde{\Sigma}_\alpha \subset [0, 1]^d$  such that for every  $x \in \tilde{\Sigma}_\alpha$ ,  $h_\mu(x) = \alpha$  and  $h_{\mu_\alpha}(x) = \tau_\mu^*(\alpha)$ .
3. For every  $\alpha \in \mathbb{R}$ , one has

$$\sigma_\mu(\alpha) = \tau_\mu^*(\alpha) = \dim E_\mu(\alpha) = \dim \underline{E}_\mu(\alpha) = \dim \bar{E}_\mu(\alpha) = \underline{\sigma}_\mu^{\text{LD}}(\alpha) = \bar{\sigma}_\mu^{\text{LD}}(\alpha).$$

In particular the SMF holds for  $\mu$ .

- 350 4. For every  $\alpha \leq \tau'_\mu(0^-)$ ,  $\dim \underline{E}_\mu^{\leq}(\alpha) = \tau_\mu^*(\alpha)$ .
5. For every  $\alpha \geq \tau'_\mu(0^+)$ ,  $\dim \bar{E}_\mu^{\geq}(\alpha) = \tau_\mu^*(\alpha)$ .
6. For every  $\eta > 0$  and every interval  $I \subset \text{dom}(\tau_\mu^*)$ , there exists  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for every  $\varepsilon \in (0, \varepsilon_0)$  and  $j \geq J_0$ , for  $\tilde{I} \in \{I, I \pm \varepsilon\}$ ,

$$\left| \frac{\log_2 \#\mathcal{D}_\mu(j, \tilde{I})}{j} - \sup_{\alpha \in I} \tau_\mu^*(\alpha) \right| \leq \eta.$$

7. There exists a positive decreasing sequence  $(\varepsilon_j)_{j \geq 0}$  tending to 0 when  $j \rightarrow +\infty$ , such that for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ ,

$$\tau'_\mu(+\infty) - \varepsilon_j \leq \frac{\log_2 \mu(\lambda)}{-j} \leq \tau'_\mu(-\infty) + \varepsilon_j.$$

**Proposition 3.3.** *Let  $\mu \in \mathcal{E}_d$ .*

1. *For every  $\varepsilon > 0$ , there exists  $j_\varepsilon \in \mathbb{N}$  such that for all  $j' \geq j \geq j_\varepsilon$ , for all  $\lambda, \tilde{\lambda} \in \mathcal{D}_j$  such that  $\partial\lambda \cap \partial\tilde{\lambda} \neq \emptyset$ , and all  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ , one has*

$$\mu(\lambda') \leq \mu(\tilde{\lambda})2^{j\varepsilon}2^{-(j'-j)(\alpha_{\min}-\varepsilon)}. \quad (3.2)$$

2. *For all integers  $j, j' \geq 0$  and  $\lambda \in \mathcal{D}_j$ , one has*

$$\mu(\lambda \cdot [0, 2^{-j'}]^d) = \mu(\lambda)2^{-\phi_\lambda}2^{-j'\alpha_{\min} + \tilde{\phi}_\lambda(j')}, \quad (3.3)$$

where:

- $\lambda \cdot [0, 2^{-j'}]^d$  is the concatenation of  $\lambda$  and  $[0, 2^{-j'}]^d$ , meaning that  $\lambda \cdot [0, 2^{-j'}]^d$  is the image of  $[0, 2^{-j'}]^d$  by the canonical similarity which maps  $[0, 1]^d$  onto  $\lambda$ ,
- $\phi_\lambda \in \mathbb{R}$  and  $\tilde{\phi}_\lambda \in \Phi$  are uniform  $o(j)$  in the sense that

$$\lim_{j \rightarrow +\infty} \sup \left\{ \frac{|\phi_\lambda|}{j} : \lambda \in \mathcal{D}_j \right\} = \lim_{j' \rightarrow +\infty} \sup \left\{ \frac{|\tilde{\phi}_\lambda(j')|}{j'} : \lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j \right\} = 0. \quad (3.4)$$

These inequalities are key to prove the optimal upper bound for the singularity spectrum of typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

### 3.2. Some approximation properties

**Definition 3.4.** *A dyadic vector  $2^{-j}k$ ,  $j \in \mathbb{N}$ ,  $k \in \mathbb{Z}^d$ , is irreducible when  $k \in \mathbb{Z}^d \setminus (2\mathbb{Z})^d$ .*

*The irreducible representation of a dyadic vector  $2^{-j}k$  with  $j \in \mathbb{N}$  and  $k \in \mathbb{Z}^d$  is the unique irreducible dyadic vector  $2^{-\bar{j}}\bar{k}$  such that  $2^{-j}k = 2^{-\bar{j}}\bar{k}$ .*

*If  $\lambda = 2^{-j}(k + [0, 1]^d) \in \mathcal{D}_j$ , then its associated irreducible cube is  $\bar{\lambda} := 2^{-\bar{j}}(\bar{k} + [0, 1]^d) \in \mathcal{D}_{\bar{j}}$ , where  $2^{-\bar{j}}\bar{k}$  is the irreducible representation of  $2^{-j}k$ .*

The following definition invokes an increasing mapping  $\gamma : \mathbb{N} \rightarrow \mathbb{N}$  which is defined in the construction of  $\mathcal{E}_d$  in [1]. The precise definition of  $\gamma(\mathbb{N})$  is not needed here. The only interesting point to mention here is that the integer  $(j)_\delta$  defined below is such that

$$(j)_\delta \sim_{j \rightarrow +\infty} j/\delta. \quad (3.5)$$

**Definition 3.5.** *Let  $\mu \in \mathcal{E}_d$ .*

*For  $\delta > 1$  and  $j \geq 1$ , let  $(j)_\delta$  be the largest integer in  $\gamma(\mathbb{N}) \cap [0, j/\delta]$ .*

*For any positive sequence  $\eta = (\eta_j)_{j \geq 1}$ , define the set*

$$X_j(\delta, \eta) = \left\{ 2^{-(j)_\delta}k \in [0, 1]^d : \begin{cases} k \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d, \\ \mu(2^{-(j)_\delta}(k + [0, 1]^d)) \geq 2^{-(j)_\delta(\alpha_{\min} + \eta_{(j)_\delta})}, \\ \mu(2^{-(j)_\delta}k + 2^{-j}[0, 1]^d) \geq 2^{-j(\alpha_{\min} + \eta_j)} \end{cases} \right\}.$$

*Then, for any increasing sequence of integers  $(j_n)_{n \geq 1}$ , set*

$$S(\delta, \eta, (j_n)_{n \geq 1}) = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{2^{-(j_n)_\delta}k \in X_{j_n}(\delta, \eta)} (2^{-(j_n)_\delta}k + 2^{-j_n}[0, 1]^d).$$

Recall that the lower Hausdorff dimension of a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  is the infimum of the Hausdorff dimension of the Borel sets of positive  $\nu$ -measure (see [12] for instance).

**Proposition 3.6.** Let  $\mu \in \mathcal{E}_d$ . Suppose that  $\sigma_\mu(\alpha_{\min}) > 0$ .

370 There is a positive sequence  $\eta = (\eta_j)_{j \geq 1}$  converging to 0 when  $j \rightarrow +\infty$  such that for any  $\delta > 1$ , for any increasing sequence of integers  $(j_n)_{n \geq 1}$ , there exists a Borel probability measure  $\nu$  on  $\mathbb{R}^d$  of lower Hausdorff dimension larger than or equal to  $\sigma_\mu(\alpha_{\min})/\delta$ , and such that  $\nu(S(\delta, \eta, (j_n)_{n \geq 1})) = 1$ . In particular,  $\dim S(\delta, \eta, (j_n)_{n \geq 1}) \geq \sigma_\mu(\alpha_{\min})/\delta$ .

**Proposition 3.7.** Let  $\mu \in \mathcal{E}_d$ . For every  $x \in [0, 1]^d$ , call  $\overline{\lambda_j(x)} \in \mathcal{D}_{\overline{j(x)}}$  the irreducible representation of  $\lambda_j(x)$ . For every  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$  such that  $\tau_\mu^*(\alpha) > 0$ , for  $\mu_\alpha$ -almost every  $x$ , one has  $\lim_{n \rightarrow +\infty} \frac{\overline{j_n(x)}}{j_n} = 1$ , where  $\mu_\alpha$  is as in Proposition 3.2(2).

#### 4. Main features of the typical singularity spectrum in $B_q^{\mu,p}(\mathbb{R}^d)$

Given  $\mu \in \mathcal{E}_d$ , Theorem 4(2) claims that the singularity spectrum of typical functions in  $B_q^{\mu,p}(\mathbb{R}^d)$  equals the Legendre transform  $\zeta_{\mu,p}^*$  of  $\zeta_{\mu,p}$ , which is explicitly given by (1.22) in terms of  $\tau_\mu$ . In this section, we find an explicit formula for  $\zeta_{\mu,p}^*$  in terms of  $\tau_\mu^*$  ( $= \sigma_\mu$ ) (Proposition 4.2), and we discuss the possible shapes and features of  $\zeta_{\mu,p}^*$  and  $\zeta_{\mu,p}$  (Sections 4.2 and 4.3).

We will need the basic properties listed in the following remark.

**Remark 4.1.** The Legendre pair  $\{\tau_\mu, \tau_\mu^*\}$  has the following properties:

- $\tau_\mu^*$  is increasing over  $[\tau_\mu'(+\infty), \tau_\mu'(0^+))$ .

385 Also,

- if  $t_\infty := (\tau_\mu^*)'(\tau_\mu'(+\infty)^+) < +\infty$ , then  $t_\infty = \inf\{t : \tau_\mu'(t) = \tau_\mu'(+\infty)\}$ ,
- for all  $t \geq t_\infty$  one has  $\tau_\mu(t) = \tau_\mu'(+\infty)t - \tau_\mu^*(\tau_\mu'(+\infty))$ ,
- if  $\tau_\mu$  is linear over the interval  $[p, +\infty)$ , then  $t_\infty \leq p$ .

Similarly,

- 390
- if  $t_{-\infty} := (\tau_\mu^*)'(\tau_\mu'(-\infty)^-) > -\infty$ , then  $t_{-\infty} = \sup\{t : \tau_\mu'(t) = \tau_\mu'(-\infty)\}$ ,
  - for all  $t \leq t_{-\infty}$  one has  $\tau_\mu(t) = \tau_\mu'(-\infty)t - \tau_\mu^*(\tau_\mu'(-\infty))$ ,
  - if  $\tau_\mu$  is linear over the interval  $(-\infty, p)$ , then  $t_{-\infty} \geq p$ .

##### 4.1. Preliminaries and statements

To express  $\zeta_{\mu,p}^*$  in terms of  $\tau_\mu^*$ , the following continuous and concave mapping  $\theta_p$  is introduced:

$$\theta_p : \alpha \in [\tau_\mu'(+\infty), \tau_\mu'(-\infty)] \mapsto \alpha + \frac{\tau_\mu^*(\alpha)}{p}, \quad (4.1)$$

see Figure 2. Notice that  $\theta_\infty$  is just the identity map.

395 The concave sub-differential of a continuous concave function  $g$  whose domain is a non trivial interval is well defined as the opposite  $-\partial(-g)$  of the sub-differential  $\partial(-g)$  of the convex function  $-g$ , and is denoted by  $\widehat{\partial}g$ .

Let us briefly describe the variations of  $\theta_p$ , see Figure 2 for an illustration.

If  $\tau_\mu'(+\infty) = \tau_\mu'(-\infty)$ , then  $\theta_p$  is constant and we set  $\alpha_p = \tau_\mu'(-\infty)$ .

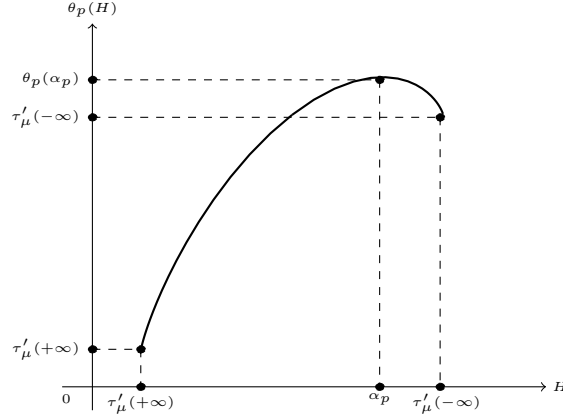


Figure 2: The mapping  $\theta_p$  when  $\sigma_\mu(\tau'_\mu(+\infty)) = \sigma_\mu(\tau'_\mu(-\infty)) = 0$ .

If  $[\tau'_\mu(+\infty), \tau'_\mu(-\infty)]$  is non trivial, using the concavity of  $\tau_\mu^*$ , it is easily seen that the mapping  $\theta_p$  is concave and reaches its maximum at  $\alpha_p$ , where

$$\alpha_p = \begin{cases} \min \{ \alpha \in [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] : -p \in \widehat{\partial}(\tau_\mu^*)(\alpha) \} & \text{if } -p \in \widehat{\partial}(\tau_\mu^*) \\ \tau'_\mu(-\infty) & \text{otherwise} \end{cases}$$

400 (when  $\tau_\mu^*$  is differentiable and strictly concave,  $\alpha_p$  is the unique exponent  $\alpha$  at which  $(\tau_\mu^*)'(\alpha) = -p$  whenever it exists). Moreover,  $\theta_p$  is increasing over  $[\tau'_\mu(+\infty), \alpha_p]$  and if  $\alpha_p < \tau'_\mu(-\infty)$ , then  $\theta_p$  is constant over  $[\alpha_p, \tau'_\mu(-\infty)]$  and decreasing over  $[\tau'_\mu(-\infty), \alpha'_p]$ , where  $\alpha'_p = \max\{\alpha \in [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] : -p \in \widehat{\partial}(\tau_\mu^*)(\alpha)\}$ . Also, one has  $\alpha_p \geq \tau'_\mu(0^+)$  since  $\tau_\mu^*$  is increasing over the interval  $[\tau'_\mu(+\infty), \tau'_\mu(0^+)]$ , and by Legendre duality, if  $-p \in \widehat{\partial}(\tau_\mu^*)$ , then  $\tau_\mu(-p) = (\tau_\mu^*)^*(-p) = -\alpha_p p - \tau_\mu^*(\alpha_p) = -p\theta_p(\alpha_p)$ .

405 Thus, in any case, the range of  $\theta_p$  restricted to the interval  $[\tau'_\mu(+\infty), \alpha_p]$  is the interval  $[\theta_p(\tau'_\mu(+\infty)), \theta_p(\alpha_p)]$ , where

$$\theta_p(\alpha_p) = \begin{cases} \frac{\tau_\mu(-p)}{-p} & \text{if } -p \in \widehat{\partial}(\tau_\mu^*), \\ \tau'_\mu(-\infty) + \frac{\tau_\mu^*(\tau'_\mu(-\infty))}{p} & \text{otherwise.} \end{cases}$$

Note that according to Remark 4.1, if  $-p \notin \widehat{\partial}(\tau_\mu^*)$ , then  $(\tau_\mu^*)'(\tau'_\mu(-\infty)^-) > -\infty$  so that  $\tau_\mu$  is linear near  $-\infty$ . This is also the case for  $\zeta_{\mu,p}$ , with the formula  $\zeta_{\mu,p}(t) = (\tau'_\mu(-\infty) + \frac{\tau_\mu^*(\tau'_\mu(-\infty))}{p})t - \tau_\mu^*(\tau'_\mu(-\infty))$ .

410 Let  $\theta_p^{-1}$  be the inverse branch of  $\theta_p$  over  $[\theta_p(\tau'_\mu(+\infty)), \theta_p(\alpha_p)]$ , see Figure 2. The Legendre transform of  $\zeta_{\mu,p}$  can be written as follows.

**Proposition 4.2.** *Let  $\mu \in \mathcal{E}_d$ . One has*

$$\zeta_{\mu,p}^*(H) = \begin{cases} p(H - \tau'_\mu(+\infty)) & \text{if } H \in [\tau'_\mu(+\infty), \theta_p(\tau'_\mu(+\infty))] \\ \tau_\mu^*(\theta_p^{-1}(H)) & \text{if } H \in [\theta_p(\tau'_\mu(+\infty)), \theta_p(\alpha_p)] \\ -\infty & \text{if } H \notin [\tau'_\mu(+\infty), \theta_p(\alpha_p)]. \end{cases} \quad (4.2)$$

The case  $p = +\infty$  is trivial, since as noticed in Remark 1.13,  $\zeta_{\mu,+\infty} = \tau_\mu$  and  $\theta_\infty$  is the identity map.



#### 4.2. Main features of $\zeta_{\mu,p}$ and $\zeta_{\mu,p}^*$

415 These properties of  $\zeta_{\mu,p}$  and  $\zeta_{\mu,p}^*$  follow from Proposition 4.2, whose proof is given in Section 4.3, or from the definition of  $\zeta_{\mu,p}$ .

- As an immediate consequence of Proposition 4.2,  $\tau'_\mu(+\infty) = \zeta'_{\mu,p}(+\infty)$  and  $\theta_p(\alpha_p) = \zeta'_{\mu,p}(-\infty)$ , although these equalities can be directly checked. Also, by definition of  $\theta_p$ ,  $\zeta'_{\mu,p}(-\infty) \leq \tau'_\mu(-\infty) + \frac{d}{p}$ .

- When  $p = +\infty$ ,  $\zeta_{\mu,+\infty} \equiv \tau_\mu$ .

420 • When  $\tau_\mu^*(\tau'_\mu(+\infty)) = 0$  (i.e. when  $\theta_p(\tau'_\mu(+\infty)) = \tau'_\mu(+\infty)$ ), the function  $\zeta_{\mu,p}^*$  reduces to the map  $H \mapsto \tau_\mu^*(\theta_p^{-1}(H))$  on the interval  $[\theta_p(\tau'_\mu(+\infty)), \theta_p(\alpha_p)]$ , see Figure 1.

- When  $\tau_\mu^*(\tau'_\mu(+\infty)) > 0$  and  $p \in [1, +\infty)$ , (equivalently, when  $\theta_p(\tau'_\mu(+\infty)) > \tau'_\mu(+\infty)$ ),  $\zeta_{\mu,p}^*$  is linear over  $[\tau'_\mu(+\infty), \theta_p(\tau'_\mu(+\infty))]$ . This occurs when  $\zeta_{\mu,p}$  is not differentiable at  $p$ , and in this case  $\zeta'_{\mu,p}(p^+) = \tau'_\mu(+\infty)$  and  $\zeta'_{\mu,p}(p^-) = \theta_p(\tau'_\mu(+\infty))$ .

425 Note that this affine part in the singularity spectrum  $\zeta_{\mu,p}^*$  of typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  follows from the heterogeneous ubiquity property stated in Proposition 3.6.

- When  $[\theta_p(\tau'_\mu(+\infty)), \theta_p(\alpha_p)]$  is non trivial,  $\zeta_{\mu,p}^*$  is concave on this interval.

Moreover, using the notations of Remark 4.1,  $\zeta_{\mu,p}^*$  is differentiable at  $\theta_p(\tau'_\mu(+\infty))$  if and only if  $t_\infty = (\tau_\mu^*)'(\tau'_\mu(+\infty)^+) = +\infty$ . Otherwise, one has  $(\zeta_{\mu,p}^*)'(\theta_p(\tau'_\mu(+\infty))^+) = \frac{t_\infty}{t_\infty+p}p < p =$   

430  $(\zeta_{\mu,p}^*)'(\theta_p(\tau'_\mu(+\infty))^-)$ . This implies that  $\zeta_{\mu,p}$  is affine over the interval  $[\frac{t_\infty}{t_\infty+p}p, p]$ , with slope  $\theta_p(\tau'_\mu(+\infty))$ .

See Figures 1 and 3 for some examples of the shape of the spectrum of typical functions  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

- When  $-p \notin \widehat{\partial}(\tau_\mu^*)$ , one has  $t_{-\infty} > -\infty$ , so both  $\tau_\mu$  and  $\zeta_{\mu,p}$  are affine near  $-\infty$ .

#### 435 4.3. Proof of Proposition 4.2

The case  $p = +\infty$  is trivial. Assume  $p \in [1, +\infty)$ .

Let  $\chi$  be the mapping defined by the right hand side of (4.2). We are going to prove that  $\chi^* = \zeta_{\mu,p}$  (which is defined by (1.22)). Next, the continuity and concavity of  $\chi$  is shown. This and the Legendre duality imply that  $\zeta_{\mu,p}^* = \chi$ .

Denote  $[\tau'_\mu(+\infty), \tau'_\mu(-\infty)]$  by  $[\alpha_{\min}, \alpha_{\max}]$ . It is convenient to write  $\chi^* = \min(\zeta_1, \zeta_2)$  where, for  $t \in \mathbb{R}$ ,

$$\begin{aligned}\zeta_1(t) &= \inf\{tH - p(H - \alpha_{\min}) : H \in [\alpha_{\min}, \theta_p(\alpha_{\min})]\} \\ \zeta_2(t) &= \inf\{tH - \tau_\mu^*(\theta_p^{-1}(H)) : H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]\}.\end{aligned}$$

When  $t \neq p$ , set

$$t_p = \frac{pt}{p-t}.$$

Then, whenever it exists, let  $\tilde{\alpha}_{t_p}$  be the minimum of those real numbers  $\alpha$  such that

$$t_p \in [(\tau_\mu^*)'(\alpha^+), (\tau_\mu^*)'(\alpha^-)].$$

440 Otherwise, set  $\tilde{\alpha}_{t_p} = \alpha_{\min}$ .

4.3.1. Proof of the equality  $\chi^* = \zeta_{\mu,p}$

Recall that  $\zeta_{\mu,p}$  is given by formula (1.22), i.e.  $\zeta_{\mu,p}(t) = \frac{p-t}{p}\tau_\mu\left(\frac{p}{p-t}t\right)$  when  $t < p$ , and  $\zeta_{\mu,p}(t) = t\alpha_{\min}$  when  $t \geq p$ .

**Case  $t > p$ .** In this case,  $t_p < -p$  (as shows a simple verification). Moreover, the mapping  $H \mapsto tH - p(H - \alpha_{\min})$  is increasing, hence  $\zeta_1(t) = t\alpha_{\min}$ .

Setting  $\alpha = \theta_p^{-1}(H)$  for  $H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ , one has

$$\zeta_2(t) = \inf_{\alpha \in [\alpha_{\min}, \alpha_p]} \tilde{\chi}_2(\alpha) \quad \text{where} \quad \tilde{\chi}_2(\alpha) = t\theta_p(\alpha) - \tau_\mu^*(\alpha). \quad (4.3)$$

Suppose that  $\alpha_{\min} < \alpha_p$ . Differentiating (formally)  $\tilde{\chi}_2$  gives

$$\tilde{\chi}'_2(\alpha) = t + \frac{t-p}{p}(\tau_\mu^*)'(\alpha) = \frac{t-p}{p}((\tau_\mu^*)'(\alpha) - t_p). \quad (4.4)$$

Recall that  $\tau_\mu^*$  is concave, non-decreasing over  $[\alpha_{\min}, \tau_\mu'(0^+)]$  and non-increasing over  $[\tau_\mu'(0^+), \alpha_{\max}]$ . Hence, by definition of  $\alpha_p$ ,  $(\tau_\mu^*)'(\alpha^-)$  and  $(\tau_\mu^*)'(\alpha^+)$  are both greater than  $-p$  when  $\alpha \in [\alpha_{\min}, \alpha_p]$ . So formula (4.4) and the fact that  $t-p > 0$  imply that the concave mapping  $\tilde{\chi}_2$  is non-decreasing over  $[\alpha_{\min}, \alpha_p]$ . Thus, the infimum defining  $\zeta_2$  is reached at  $\alpha_{\min}$ , where it equals  $t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min}) \geq t\alpha_{\min}$ .

If  $\alpha_p = \alpha_{\min}$ , then  $\zeta_2(t) = t\theta_p(\alpha_{\min}) - \tau_\mu^*(\alpha_{\min}) = t\alpha_{\min}$ .

In both cases,  $\zeta_2(t) \geq t\alpha_{\min}$ , and so  $\chi^*(t) = \min(\zeta_1(t), \zeta_2(t)) = t\alpha_{\min}$ , and (1.22) holds true.

The case  $t = p$  follows by continuity.

**Case  $t < p$ .** The mapping  $H \mapsto tH - p(H - \alpha_{\min})$  is non increasing, so  $\zeta_1(t) = (t-p)\theta_p(\alpha_{\min}) + p\alpha_{\min} = t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min})$ .

Next we determine  $\zeta_2(t)$ . Since  $t_p > -p$ , using (4.4) and the fact that  $t-p < 0$  now shows that the convex mapping  $\tilde{\chi}_2$  reaches its minimum at  $\tilde{\alpha}_{t_p}$ , which necessarily belongs to  $[\alpha_{\min}, \alpha_p]$ . Consequently,

$$\zeta_2(t) = t\theta_p(\tilde{\alpha}_{t_p}) - \tau_\mu^*(\tilde{\alpha}_{t_p}).$$

Two subcases are distinguished:

- Suppose that  $t_p \leq (\tau_\mu^*)'(\alpha_{\min}^+)$ .

In this case,  $\tilde{\alpha}_{t_p} \geq \alpha_{\min}$ , and one has  $\tau_\mu^*(\tilde{\alpha}_{t_p}) = t_p\tilde{\alpha}_{t_p} - \tau_\mu(t_p)$  (even if  $\tilde{\alpha}_{t_p} = \alpha_{\min}$ , because in this case  $t_p = (\tau_\mu^*)'(\alpha_{\min}^+) = t_\infty$ , hence  $t_\infty < \infty$  and we can use Remark 4.1). After simplification one gets

$$\zeta_2(t) = t \left( \tilde{\alpha}_{t_p} + \frac{\tau_\mu^*(\tilde{\alpha}_{t_p})}{p} \right) - \tau_\mu^*(\tilde{\alpha}_{t_p}) = \frac{p-t}{p}\tau_\mu(t_p).$$

If  $\tilde{\alpha}_{t_p} = \alpha_{\min}$ , then  $\zeta_2(t) = t\theta_p(\alpha_{\min}) - \tau_\mu^*(\alpha_{\min}) = t\alpha_{\min} + (t-p)\tau_\mu^*(\alpha_{\min})/p = \zeta_1(t)$ . And a quick computation shows that  $t\alpha_{\min} + (t-p)\tau_\mu^*(\alpha_{\min})/p = \frac{p-t}{p}\tau_\mu\left(\frac{p}{p-t}t\right)$ .

If  $\tilde{\alpha}_{t_p} > \alpha_{\min}$ , then let us show that  $\zeta_2(t) \geq \zeta_1(t)$ . Indeed, this inequality reads  $t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min}) \geq \frac{p-t}{p}\tau_\mu(t_p) = t\tilde{\alpha}_{t_p} - \frac{p-t}{p}\tau_\mu^*(\tilde{\alpha}_{t_p})$ . The previous inequality is equivalent to  $t(\tilde{\alpha}_{t_p} - \alpha_{\min}) \leq \frac{p-t}{p}(\tau_\mu^*(\tilde{\alpha}_{t_p}) - \tau_\mu^*(\alpha_{\min}))$ , i.e.

$$\frac{\tau_\mu^*(\tilde{\alpha}_{t_p}) - \tau_\mu^*(\alpha_{\min})}{\tilde{\alpha}_{t_p} - \alpha_{\min}} \geq t_p.$$

The concavity of  $\tau_\mu^*$  entails that this last inequality holds true.

Hence, in all cases  $\chi^*(t) = \min(\zeta_1(t), \zeta_2(t)) = \frac{p-t}{p}\tau_\mu\left(\frac{p}{p-t}t\right)$ , so (1.22) holds.

• Suppose that  $t_p > (\tau_\mu^*)'(\alpha_{\min}^+)$ . In this case,  $t_\infty = (\tau_\mu^*)'(\alpha_{\min}^+) < +\infty$ , which implies that  $\tau_\mu(t) = \alpha_{\min}t - \tau_\mu^*(\alpha_{\min})$  for all  $t \geq t_\infty$  (see Remark 4.1).

Also, since  $t_p > (\tau_\mu^*)'(\alpha_{\min}^+) > 0$ ,  $\tilde{\alpha}_{t_p} = \alpha_{\min}$  and the image of  $\partial\tilde{\chi}_2$  is included in  $(0, +\infty)$ . In particular the convex mapping  $\tilde{\chi}_2$  reaches its minimum at  $\alpha_{\min}$ . Consequently,  $\zeta_2(t) = \zeta_1(t) = t\alpha_{\min} + \frac{t-p}{p}\tau_\mu^*(\alpha_{\min})$ . Since  $t_p \geq t_\infty$  and  $\tau_\mu$  is affine on  $[t_\infty, +\infty)$ , it follows that  $\chi^*(t) = \frac{p-t}{p}\tau_\mu(t_p)$ , as stated by (1.22).

Note that the previous case corresponds to  $\frac{t_\infty}{t_\infty+p}p < t < p$ . In regard to the form taken by  $\zeta_{\mu,p}^*$ , it is convenient to rewrite  $\zeta_{\mu,p}(t) = \theta_p(\alpha_{\min})t - \tau_\mu^*(\alpha_{\min})$ .

#### 4.3.2. Concavity of $\chi$

First, observe that  $\chi$  is affine on the interval  $[\alpha_{\min}, \theta_p(\alpha_{\min})]$ .

Let us explain why  $\chi$  is also concave over  $[\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ .

Assume first that  $\tau_\mu^*$  is differentiable over  $(\alpha_{\min}, \theta_p^{-1}(\alpha_p))$ . Then this is also the case for  $\theta_p^{-1}$  over  $(\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ . For  $H \in (\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ , denoting  $\alpha = \theta_p^{-1}(H)$  and  $t = (\tau_\mu^*)'(\alpha)$ , one gets  $\chi'(H) = \frac{t}{1+t/p}$ , which is increasing as a function of  $t$ . Since  $H = \theta_p(\alpha)$  is an increasing function of  $\alpha$  and  $\alpha$  is a decreasing function of  $t$ , it follows that  $\chi'$  is decreasing over  $(\theta_p(\alpha_{\min}), \theta_p(\alpha_p))$ . Hence  $\chi$  is concave over  $[\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ . If  $\tau_\mu^*$  has non differentiability points over  $(\alpha_{\min}, \theta_p^{-1}(\alpha_p))$ , we get the same conclusion by approximating  $\tau_\mu^*$  by the differentiable  $L^q$ -spectra associated with the Bernoulli product generated by the probability vectors used to construct  $\mu$ .

Thus, one knows that  $\chi$  is concave on the two intervals  $[\alpha_{\min}, \theta_p(\alpha_{\min})]$  and on  $[\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ . If  $\theta_p(\alpha_{\min}) = \theta_p(\alpha_p)$ , or if  $\theta_p(\alpha_{\min}) = \alpha_{\min}$ , the conclusion is immediate. Otherwise, to get that  $\chi$  is concave, one must check that  $\chi'(\theta_p(\alpha_{\min}^+)) \leq p = \chi'(\theta_p(\alpha_{\min}^-))$ . With the notations used above, a direct computation then yields  $\chi'(\theta_p(\alpha_{\min}^+)) = p$  if  $(\tau_\mu^*)'(\alpha_{\min}^+) = t_\infty = +\infty$  and  $\chi'(\theta_p(\alpha_{\min}^-)) = \frac{t_\infty}{t_\infty+p}p$  if  $t_\infty < +\infty$ . Hence the conclusion that  $\chi$  is concave.

### 5. Lower bound for the $L^q$ -spectrum, and upper bound for the singularity spectrum in $B_q^{\mu,p}(\mathbb{R}^d)$ , when $\mu \in \mathcal{E}_d$

This section uses the wavelet leaders and  $L^q$ -spectrum of a function introduced in Section 1.5. Item (1) of Theorem 4 is proved by establishing a general lower bound for the  $L^q$ -spectrum of all  $f \in B_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  (Theorem 5(1)).

The main result of this section is the following. Recall the definition (1.21) of  $s_\mu$ .

**Theorem 8.** *Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, +\infty]$ . Let  $\Psi \in \mathcal{F}_{s_\mu}$ . For all  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q,\Psi}| < +\infty$ , one has  $\zeta_{f|\mathbb{R}_+} \geq \zeta_{\mu,p|\mathbb{R}_+}$ .*

It is implicit in Theorem 8 that the semi-norm  $|f_{\mu,p,q,\Psi}|$  defined in (1.14) is computed using the wavelet  $\Psi \in \mathcal{F}_{s_\mu}$  fixed by the statement.

Theorem 8 yields the following corollary.

**Corollary 9.** *Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, +\infty]$ . For all  $f \in B_q^{\mu,p}(\mathbb{R}^d)$ , one has:*

1.  $\zeta_{f|\mathbb{R}_+} \geq \zeta_{\mu,p|\mathbb{R}_+}$ , i.e. the claim of Theorem 5(1) holds true.
2. For all  $H \in \mathbb{R}$ ,

$$\sigma_f(H) \leq \begin{cases} \zeta_{\mu,p}^*(H) & \text{if } H \leq \zeta_{\mu,p}'(0^+) \\ d & \text{if } H > \zeta_{\mu,p}'(0^+), \end{cases}$$

i.e. part (1) of Theorem 4 holds true.

*Proof.* Part (1) follows from the definition of  $B_q^{\mu,p}(\mathbb{R}^d)$  and the continuity of  $\zeta_{\mu(-\varepsilon),p|\mathbb{R}_+}$  as a function of  $\varepsilon$ . Part (2) is then a consequence of (1.28).  $\square$

The wavelets  $\Psi \in \mathcal{F}_{s_\mu}$  are fixed for the rest of this section.

To obtain Theorem 8, one needs to estimate, for any  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q,\Psi}| < +\infty$  and any  $N \in \mathbb{N}$ , the upper large deviations spectrum of the wavelet leaders  $(L_\lambda^f)_{\lambda \subset N[0,1]^d}$  associated with  $\Psi$ , defined as follows. Recall the notations  $H \pm \varepsilon$  introduced in (3.1), and  $N\lambda$  in the introduction.

**Definition 5.1.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  and  $N \in \mathbb{N}^*$ , with wavelet coefficients and leaders computed with the wavelet  $\Psi$ . For any compact subinterval  $I$  of  $\mathbb{R}$ , set

$$\mathcal{D}_f^N(j, I) = \left\{ \lambda \in \mathcal{D}_j : \lambda \subset N[0, 1]^d, \frac{\log_2 |L_\lambda^f|}{-j} \in I \right\},$$

505 The upper wavelet leaders large deviation spectrum of  $f$  associated with  $\Psi$  and  $N[0, 1]^d$  is

$$\bar{\sigma}_f^{\text{LD}, N}(H) = \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log_2 \#\mathcal{D}_f^N(j, H \pm \varepsilon)}{j}.$$

**Proposition 5.2.** Let  $\mu \in \mathcal{E}_d$  and  $p, q \in [1, +\infty]$ . For all  $f \in L^p(\mathbb{R}^d)$  such that  $|f_{\mu,p,q,\Psi}| < +\infty$ , and all  $N \in \mathbb{N}$ , one has

$$\bar{\sigma}_f^{\text{LD}, N}(H) \leq \begin{cases} \zeta_{\mu,p}^*(H) & \text{if } H \leq \zeta'_{\mu,p}(0^+) \\ d & \text{if } H > \zeta'_{\mu,p}(0^+) \end{cases}. \quad (5.1)$$

Assuming that Proposition 5.2 is proved, let us explain how Theorem 8 follows.

*Proof of Theorem 8.* Note that by large deviations theory [13],  $\zeta_f^{N,\Psi}$  defined in (1.26) is the Legendre transform of the concave hull of  $\bar{\sigma}_f^{\text{LD}, N}$ . By Proposition 5.2, this concave hull is dominated by the right hand-side of (5.1). It is easily seen that this right-hand side, as a function of  $H$ , is concave, and that its Legendre transform is equal to  $\zeta_{\mu,p|\mathbb{R}_+}$  over  $\mathbb{R}_+$  and equal to  $-\infty$  over  $\mathbb{R}_+^*$ . Consequently,  $\zeta_f^{N,\Psi}|_{\mathbb{R}_+} \geq \zeta_{\mu,p|\mathbb{R}_+}$ , which allows to conclude since  $\zeta_f^\Psi|_{\mathbb{R}_+} = \lim_{N \rightarrow +\infty} \zeta_f^{N,\Psi}|_{\mathbb{R}_+}$  does not depend on  $\Psi$ .  $\square$

The rest of this section is devoted to the proof of Proposition 5.2. It requires large deviations estimates on the distribution of the wavelet coefficients of  $f$  under the constraint  $|f_{\mu,p,+\infty}| < +\infty$ , which holds automatically if  $|f_{\mu,p,q,\Psi}| < +\infty$ .

### 5.1. Large deviations estimates for wavelet coefficients

**Definition 5.3.** Let  $\mu \in \mathcal{C}(\mathbb{R}^d)$ ,  $I_H$  and  $I_\alpha$  be two compact subintervals of  $\mathbb{R}$ , and  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  with wavelet coefficients  $(c_\lambda)_{\lambda \in \Lambda}$ . Then, define

$$\Lambda_{f,\mu}(j, I_H, I_\alpha) = \left\{ \lambda = (i, j, k) \in \Lambda : \lambda_{j,k} \subset 3[0, 1]^d, \left\{ \begin{array}{l} \frac{\log_2 |c_\lambda|}{-j} \in I_H \\ \frac{\log_2 \mu(\lambda_{j,k})}{-j} \in I_\alpha \end{array} \right\} \right\}. \quad (5.2)$$

In other words,  $\Lambda_{f,\mu}(j, I_H, I_\alpha)$  contains those cubes  $\lambda$  of generation  $j$  such that  $\mu(\lambda) \sim |\lambda|^\alpha$  with  $\alpha \in I_\alpha$  and  $|c_\lambda| \sim 2^{-jh}$  with  $h \in I_H$ . The cube  $3[0, 1]^d$  is considered, rather than  $[0, 1]^d$  because the computation of wavelet leaders on  $[0, 1]^d$  requires some knowledge of  $\mu$  and  $f$  in this neighborhood of  $[0, 1]^d$ .

The cardinality of  $\Lambda_{f,\mu}(j, I_H, I_\alpha)$  is estimated to get a control of the wavelets leaders large deviations spectrum under the assumptions of Proposition 5.2.

In the next lemma, the convention  $\infty \times x = +\infty$  for  $x \geq 0$  is adopted.

**Lemma 5.4.** Let  $\mu \in \mathcal{E}_d$  and  $p \in [1, +\infty]$ . Let  $\alpha_{\min} = \tau'_\mu(+\infty)$  and  $\alpha_{\max} = \tau'_\mu(-\infty)$ . Let  $f \in L^p(\mathbb{R}^d)$  be such that  $|f|_{\mu,p,+\infty,\Psi} < +\infty$  and let  $I_H, I_\alpha$  be two compact subintervals of  $\mathbb{R}$ .

1. If  $\max I_H < \min I_\alpha$ , then  $\Lambda_{f,\mu}(j, I_H, I_\alpha) = \emptyset$  for  $j$  large enough.
2. If  $I_\alpha \subset [\alpha_{\min}, \alpha_{\max}]$  and  $\min I_\alpha \leq \min I_H$ , then for every  $\eta > 0$ , there exists  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for every  $\varepsilon \in [0, \varepsilon_0]$  and  $j \geq J_0$ :

$$\frac{\log_2 \#\Lambda_{f,\mu}(j, I_H \pm \varepsilon, I_\alpha \pm \varepsilon)}{j} \leq \max_{\beta \in I_\alpha \cap [0, \max I_H]} \min(p(\max I_H - \beta), \tau_\mu^*(\beta)) + \eta. \quad (5.3)$$

*Proof.* We treat the case  $p < +\infty$  and leave the simpler case  $p = +\infty$  to the reader.

(1) Recall that by definition  $\sup_{j \in \mathbb{N}} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell_p(\Lambda_j)} < +\infty$ . There is  $C_f \geq 1$  such that

$$\sup_{j \in \mathbb{N}} \left\| \left( \frac{c_\lambda}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_{\ell_p(\Lambda_j)} \leq C_f. \quad (5.4)$$

It follows that item (1) holds true, for otherwise (5.4) would be contradicted.

(2) Fix  $\eta, \varepsilon > 0$  and set  $\tilde{H} = \max(I_H)$ . Since  $I_\alpha$  is compact and  $\tau_\mu^*$  is continuous over its compact domain, there are finitely many numbers  $\alpha_0 < \dots < \alpha_m$  such that  $I_\alpha = \bigcup_{\ell=0}^{m-1} [\alpha_\ell, \alpha_{\ell+1}]$  and for every  $\ell$ ,  $\alpha_{\ell+1} - \alpha_\ell \leq \eta/p$  and  $|\tau_\mu^*(\beta) - \tau_\mu^*(\beta')| \leq \eta$  for all  $\beta, \beta' \in [\alpha_\ell, \alpha_{\ell+1}]$ .

Let  $j \in \mathbb{N}$ . Consider the subset  $\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)$  of  $\Lambda_{f,\mu}(j, I_H \pm \varepsilon, I_\alpha \pm \varepsilon)$ . With each cube  $\lambda \in \Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)$  is associated a wavelet coefficient  $c_\lambda$  whose absolute value is at least equal to  $2^{-j(\tilde{H} + \varepsilon)}$ . Thus, for each  $\ell \in \{0, \dots, m-1\}$ ,

$$C_f^p \geq \sum_{\lambda \in \Lambda_j} \left( \frac{|c_\lambda|}{\mu(\lambda)} \right)^p \geq \sum_{\lambda \in \Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon)} \left( \frac{2^{-j(\tilde{H} + \varepsilon)}}{2^{-j(\alpha_\ell - \varepsilon)}} \right)^p. \quad (5.5)$$

**Remark 5.5.** Recall that for  $\lambda = (i, j, k) \in \Lambda_j$ , we make a slight abuse of notation by identifying  $\lambda$  with  $\lambda_{j,k} \in \mathcal{D}_j$  and writing  $\mu(\lambda)$  for  $\mu(\lambda_{j,k})$  and  $\lambda \subset E$  for  $\lambda_{j,k} \subset E$ .

It follows from (5.5) that

$$\#\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \leq C_f^p 2^{jp(\tilde{H} - \alpha_\ell + 2\varepsilon)}.$$

On the other hand, observe that for each  $j \geq 0$ , one has

$$\Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \subset \left\{ \lambda = (i, j, k) \in \Lambda : \lambda \subset 3[0, 1]^d, \frac{\log_2 \mu(\lambda)}{-j} \in I \right\},$$

where  $I = [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]$ . Applying Proposition 3.2(6) to each interval  $[\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]$ , one finds  $\varepsilon_0 > 0$  and  $J_0 \in \mathbb{N}$  such that for all  $\varepsilon \in (0, \varepsilon_0]$ ,  $0 \leq \ell \leq m-1$  and  $j \geq J_0$ ,

$$\#\mathcal{D}_\mu(j, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon \cap [0, \tilde{H} + \varepsilon]) \leq \#\mathcal{D}_\mu(j, ([\alpha_\ell, \alpha_{\ell+1}] \cap [0, \tilde{H}]) \pm 2\varepsilon) \leq 2^{j(\gamma_\ell + \eta)},$$

where  $\gamma_\ell = \max\{\tau_\mu^*(\beta) : \beta \in [\alpha_\ell, \alpha_{\ell+1}] \cap [0, \tilde{H}]\}$ . Then, taking into account the fact that  $\mu$  is  $\mathbb{Z}^d$ -invariant, as well as the fact that with each dyadic cube  $\lambda_{j,k}$  are associated  $2^d - 1$  wavelet coefficients, one obtains

$$\#\Lambda_{f,\mu}(j, I_H \pm \varepsilon, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \leq 3^d (2^d - 1) 2^{j(\gamma_\ell + \eta)}.$$

Combining the previous estimates, one gets for  $\varepsilon \in (0, \varepsilon_0]$  and  $j \geq J_0$

$$\begin{aligned} \#\Lambda_{f,\mu}(j, I_H, I_\alpha \pm \varepsilon) &\leq \sum_{\ell=0}^{m-1} \#\Lambda_{f,\mu}(j, I_H, [\alpha_\ell, \alpha_{\ell+1}] \pm \varepsilon) \\ &\leq \sum_{\ell=0}^{m-1} \min(C_f^p 2^{jp(\tilde{H} - \alpha_\ell + 2\varepsilon)}, 3^d (2^d - 1) \cdot 2^{j(\gamma_\ell + \eta)}) \\ &\leq 3^d (2^d - 1) C_f^p m \max\{2^{j \min(p(\tilde{H} - \alpha_\ell + 2\varepsilon), \gamma_\ell + \eta)} : \ell = 0, 1, \dots, m-1\}. \end{aligned}$$

Also, the constraints imposed to the exponents  $\alpha_\ell$  and the continuity of  $\tau_\mu^*$  imply that

$$\begin{aligned} & \max \{ \min(p(\tilde{H} - \alpha_\ell + 2\varepsilon), \gamma_\ell + \eta) : \ell = 0, 1, \dots, m-1 \} \\ & \leq \max \{ \min(p(\tilde{H} - \beta), \tau_\mu^*(\beta)) : \beta \in I_\alpha \cap [0, \tilde{H}] \} + 2p\varepsilon + 3\eta. \end{aligned}$$

Taking  $\varepsilon_0 \leq \eta/p$  and  $J_0$  so large that  $2^{J_0\eta} \geq 3^d(2^d - 1)C_f^p m$ , we finally get the desired upper bound (5.3) (with  $6\eta$  instead of  $\eta$ ).  $\square$

We are now ready to get an upper bound for the wavelet leaders upper large deviations spectrum of  $f$ .

### 5.2. Proof of Proposition 5.2

Note that since  $\mu$  is  $\mathbb{Z}^d$ -invariant, and by definition of  $|\cdot|_{\mu,p,q,\Psi}$ , any general upper bound for  $\bar{\sigma}_{f|_{[0,1]^d}}^{\text{LD},1}$  holds for  $\bar{\sigma}_f^{\text{LD},N}$ . Thus, without loss of generality we prove that  $\bar{\sigma}_f^{\text{LD},1}$  is upper bounded by the right hand side of (5.1).

This proof is rather involved because all the possible interactions between the values  $\mu(\lambda)$  and the corresponding wavelet coefficients  $c_\lambda$  must be taken care of.

Note that the inequality  $\bar{\sigma}_f^{\text{LD},1} \leq d$  obviously holds. So it is enough to deal with the case  $H \leq \zeta'_{\mu,p}(0^+)$ .

Fix  $H \leq \zeta'_{\mu,p}(0^+)$ . For  $\varepsilon > 0$  small enough,  $\#\mathcal{D}_f^1(j, H \pm \varepsilon)$  is going to be estimated from above (recall Definition 5.1). We are going to prove that there exist  $C, c > 0$  such that for any  $\eta > 0$ , if  $\varepsilon_0 \in (0, \eta]$  is chosen small enough, then for  $j$  large enough, for all  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\#\mathcal{D}_f^1(j, H \pm \varepsilon) \leq Cj2^{j(\zeta_{\mu,p}^*(H) + c\eta)}. \quad (5.6)$$

It is immediate to check that (5.6) implies (5.1), hence Proposition 5.2.

Since  $|f|_{\mu,p,+\infty} < +\infty$ , there exists  $C > 0$  such that  $|c_\lambda| \leq C\mu(\lambda)$  for every  $\lambda \in \bigcup_{j \geq 0} \Lambda_j$  (recall Remark 5.5). Without loss of generality, suppose that the above constant is equal to 1 and so

$$|c_\lambda| \leq \mu(\lambda) \text{ for every } \lambda \in \bigcup_{j \geq 0} \Lambda_j. \quad (5.7)$$

Recall the definition (1.24) of wavelet leaders:  $L_\lambda^f = \sup\{|c_{\lambda'}| : \lambda' = (i, j, k) \in \Lambda, \lambda' \subset 3\lambda\}$ . The following observations are key.

**Lemma 5.6.** *A dyadic cube  $\lambda$  belongs to  $\mathcal{D}_f^1(j, H \pm \varepsilon)$  if and only if:*

- $\lambda \subset [0, 1]^d$ ;
- There exists a dyadic cube  $\lambda' \subset 3\lambda$  of generation  $j' \geq j$  as well as  $i \in \{1, \dots, 2^d - 1\}$  and  $k' \in \mathbb{Z}^d$  such that  $\lambda' = \lambda_{j', k'}$ , and  $|c_{(i, j', k')}| = 2^{-j'H'}$  with  $H' \in \frac{j}{j'}[H - \varepsilon, H + \varepsilon]$ ;
- when  $j$  is large enough,  $j' \leq 2j(H + \varepsilon)/\alpha_{\min}$ .

*Proof.* The first item is trivial, and the second one follows from the definition (1.24) of the wavelet leaders and the fact that  $\frac{\log_2 |L_\lambda^f|}{-j} \in H \pm \varepsilon$  if and only if there exists some  $\lambda' \subset 3\lambda$  of generation  $j' \geq j$  and  $i \in \{1, \dots, 2^d - 1\}$  such that  $\frac{\log_2 |c_{(i, j', k')}|}{-j} \in H \pm \varepsilon$ .

For the third item, Lemma 5.4(2) implies that  $|c_{(i, j', k')}| \leq 2^{-j'\alpha_{\min}/2}$  when  $j$  (and so  $j'$ ) is large. Hence  $H' \geq \alpha_{\min}/2$  and the fact that  $j' \leq j(H + \varepsilon)/H'$  implies the claim.  $\square$

The second item of Lemma 5.6 is used repeatedly in the forthcoming pages.

560 Three cases are separated.

**Case 1 :**  $H < \alpha_{\min}$ .

Note that  $\zeta_{\mu,p}^*(H) = -\infty$ . Suppose that  $\varepsilon > 0$  is so small that  $\alpha_{\min} - \varepsilon > H + \varepsilon$ . Due to Proposition 3.2(6) and the observation made just above, for  $j$  large enough

$$\#\mathcal{D}_f^1(j, H \pm \varepsilon) \leq \sum_{j \leq j' \leq 2j(H+\varepsilon)/\alpha_{\min}} \#\Lambda_{f,\mu}(j', [0, H + \varepsilon], I_\alpha),$$

with  $I_\alpha = [\alpha_{\min} - \varepsilon, \alpha_{\max} + \varepsilon]$ . However,  $H + \varepsilon < \alpha_{\min} - \varepsilon$ , so by Lemma 5.4,  $\mathcal{D}_f^1(j, H \pm \varepsilon) = \emptyset$ . This implies (5.1), i.e.  $\bar{\sigma}_{f_{[0,1]^d}}^{\text{LD},1}(H) = -\infty$ .

To deal with the other cases, we discretize the interval  $[\alpha_{\min}, H]$ .

565 Fix  $\eta > 0$ ,  $\varepsilon_0 \in (0, \min(1/2, \alpha_{\min}/2, \eta))$ , and split the interval  $[\alpha_{\min}, H]$  into finitely many contiguous closed intervals  $I_1, \dots, I_m$  ( $m = m(\varepsilon_0)$ ) such that

- $|I_\ell| \leq \varepsilon_0$  for every  $\ell \in \{1, \dots, m\}$ ,
- Writing  $I_\ell = [h_\ell, h_{\ell+1}]$ , one has  $1 \leq h_{\ell+1}/h_\ell \leq 1 + \varepsilon_0$  for every  $1 \leq \ell \leq m$ .

In particular,  $H/h_\ell \geq 1$  for every  $\ell$ .

570 By Lemma 5.6, if  $j \geq J_0$  and  $\lambda \in \mathcal{D}_f^1(j, H \pm \varepsilon)$ , there exist  $j' \geq j$  and  $\lambda' = (i, j', k') \in \Lambda_{j'}$  such that  $\lambda' \subset 3\lambda$  and  $|c_{\lambda'}| = 2^{-j'H'}$  with  $H' \in \frac{j}{j'}[H \pm \varepsilon]$ . By (5.7),  $|c_{\lambda'}| \leq \mu(\lambda')$ , so there exist  $1 \leq \ell' \leq \ell \leq m$  such that  $\lambda' \in \Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)$  (recall (5.2)).

In addition,  $H' \in I_\ell \pm \varepsilon \subset I_\ell \pm \varepsilon_0$ ,  $j' \in \frac{j}{H'}[H \pm \varepsilon] \subset \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]$ , and  $h_{\ell+1} \leq H$ . Consequently,

$$\mathcal{D}_f^1(j, H \pm \varepsilon) \subset \bigcup_{1 \leq \ell' \leq \ell \leq m} \mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon), \quad (5.8)$$

where (recall Remark 5.5)

$$\mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon) = \bigcup_{j' \in j \cdot \left[ \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]} \left\{ \lambda \in \mathcal{D}_j \cap [0, 1]^d : \left\{ \begin{array}{l} \exists \lambda' \in \Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon) \\ \text{such that } \lambda' \subset 3\lambda \end{array} \right. \right\}.$$

Next, the cardinality of  $\mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon)$  (and thus of  $\mathcal{D}_f^1(j, H \pm \varepsilon)$ ) is going to be bounded from above using different estimates.

To do so, Lemma 5.4(2) is applied to each pair  $\{I_\ell, I_{\ell'}\}$ : there exist  $\varepsilon \in (0, \varepsilon_0)$  and  $J_0 \in \mathbb{N}$  such that for all  $j' \geq J_0$ , for all  $1 \leq \ell' \leq \ell \leq m$ ,

$$\frac{\log_2 \#\Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)}{j'} \leq d(\ell, \ell') + \eta \quad (5.9)$$

where

$$d(\ell, \ell') = \max \{ \min(p(h_{\ell+1} - \beta), \tau_\mu^*(\beta)) : \beta \in I_{\ell'} \}. \quad (5.10)$$

575 **Case 2:**  $\alpha_{\min} \leq H < \theta_p(\alpha_{\min}) = \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{p}$ . This case occurs only when  $\tau_\mu^*(\alpha_{\min}) > 0$ .

Let  $j \geq J_0$ . For every  $1 \leq \ell' \leq \ell \leq m$ , one has  $p(h_{\ell+1} - h_{\ell'}) \leq p(H - \alpha_{\min}) \leq \tau_\mu^*(\alpha_{\min}) \leq \tau_\mu^*(\beta)$ , for every  $\beta \in I_{\ell'}$ . So, from (5.10) one deduces that  $d(\ell, \ell') \leq p(h_{\ell+1} - \alpha_{\min})$ . Thus, if  $j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_\ell-\varepsilon_0} \right]$ , then  $j'd(\ell, \ell') \leq jp(H+\varepsilon_0) \frac{h_{\ell+1}-\alpha_{\min}}{h_\ell-\varepsilon_0}$ . Then observing that  $\sup_{\ell \in \{1, \dots, m\}} \frac{h_{\ell+1}-\alpha_{\min}}{h_\ell} = \frac{H-\alpha_{\min}}{H} + O(\varepsilon_0)$ , one has

$$j'(d(\ell, \ell') + \eta) \leq j(p(H - \alpha_{\min}) + O(\varepsilon_0) + \eta) = j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta).$$

Consequently, since (5.8) implies

$$\#\mathcal{D}_f^1(j, H \pm \varepsilon) \leq \sum_{1 \leq \ell' \leq \ell \leq m} \sum_{j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} \right]} \#\Lambda_{f,\mu}(j', I_{\ell} \pm \varepsilon, I_{\ell'} \pm \varepsilon),$$

the inequality (5.9) combined with the previous remarks yields

$$\#\mathcal{D}_f^1(j, H \pm \varepsilon) \leq m^2 j \frac{H + \varepsilon_0}{\alpha_{\min} - \varepsilon_0} 2^{j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta)} = C 2^{j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta)},$$

so (5.6) holds true.

**Case 3:**  $\theta_p(\alpha_{\min}) \leq H \leq \zeta_{\mu,p}'(0^+) = \theta_p(\tau_{\mu}'(0^+))$ .

This case is divided into four subcases in order to estimate  $\#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$ .

The term  $d(\ell, \ell')$  can easily be expressed in terms of the mappings  $\theta_p$  defined in (4.1) and  $\tau_{\mu}^*$ .  
 580 The mapping  $\theta_p$  is an increasing map over  $[\alpha_{\min}, \alpha_p]$  and  $\alpha_p \geq \tau_{\mu}'(0^+)$ , so using that  $h_{\ell} \leq H$ , one deduces that

$$d(\ell, \ell') = \begin{cases} \tau_{\mu}^*(h_{\ell'+1}) & \text{if } h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1}), \\ p(h_{\ell+1} - h_{\ell'}) & \text{if } h_{\ell'} \geq \theta_p^{-1}(h_{\ell+1}), \\ \tau_{\mu}^*(\theta_p^{-1}(h_{\ell+1})) = \zeta_{\mu,p}^*(h_{\ell+1}) & \text{otherwise.} \end{cases} \quad (5.11)$$

Moreover, the maximum of the three possible values is always  $\zeta_{\mu,p}^*(h_{\ell+1})$ .

**Subcase (3a):**  $\frac{H}{h_{\ell+1}} d(\ell, \ell') \leq \zeta_{\mu,p}^*(H)$ . Using the definition of  $\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$  and inequality (5.9), for  $j \geq J_0$

$$\begin{aligned} \#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) &\leq \sum_{j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} \right]} \#\Lambda_{f,\mu}(j', I_{\ell} \pm \varepsilon, I_{\ell'} \pm \varepsilon) \\ &\leq \sum_{j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} \right]} 2^{j'(d(\ell,\ell')+\eta)} \leq j \frac{H + \varepsilon_0}{h_{\ell} - \varepsilon_0} 2^{j \frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} (d(\ell,\ell')+\eta)}. \end{aligned}$$

By our assumption,  $\frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} d(\ell, \ell') \leq \left( \frac{H}{h_{\ell+1}} + O(\varepsilon_0) \right) d(\ell, \ell') \leq \zeta_{\mu,p}^*(H) + O(\varepsilon_0)$ , this  $O(\varepsilon_0)$  being uniform with respect to  $\ell$ . So

$$\#\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) \leq C 2^{j(\zeta_{\mu,p}^*(H) + O(\varepsilon_0) + \eta)}.$$

**Subcase (3b):**  $\frac{H}{h_{\ell+1}} d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1})$ .

Recall the definition (4.1) of  $\theta_p$ . A technical lemma is needed.

**Lemma 5.7.** *For every  $j$  large enough,*

$$\mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon) \subset \mathcal{D}_{\mu} \left( j, \left[ \alpha_{\min}, \alpha_{\min} + \frac{H}{h_{\ell+1}} (h_{\ell'+1} - \alpha_{\min}) \right] \pm O(\varepsilon_0) \right),$$

585 where  $O(\varepsilon_0)$  is independent of  $(\ell, \ell')$ .

*Proof.* Take  $\lambda \in \mathcal{D}_f^{\ell,\ell'}(j, H \pm \varepsilon)$  and applying Lemma 5.6, consider  $j' \in \left[ j \frac{H-\varepsilon_0}{h_{\ell+1}+\varepsilon_0}, j \frac{H+\varepsilon_0}{h_{\ell}-\varepsilon_0} \right]$  such that there exists  $\lambda' = (i, j', k') \in \Lambda_{f,\mu}(j', I_{\ell} \pm \varepsilon, I_{\ell'} \pm \varepsilon)$  for which  $\lambda' \subset 3\lambda$ .

Denote by  $\widehat{\lambda}$  the unique dyadic cube of  $\mathcal{D}_j$  containing  $\lambda'$ . Then, note that:



- The two cubes  $\lambda$  and  $\widehat{\lambda}$  are either equal or neighbors. Hence, by property (P<sub>2</sub>) of  $\mu$ ,  $\mu(\lambda) \geq 2^{-j\varepsilon_0} \mu(\widehat{\lambda})$  when  $j$  is large enough.
- $\mu(\widehat{\lambda}) = \mu(\lambda') \frac{\mu(\widehat{\lambda})}{\mu(\lambda')}$ , and by construction of  $\mu$  (see (3.2)),  $\frac{\mu(\widehat{\lambda})}{\mu(\lambda')} \geq 2^{-j\varepsilon_0} 2^{(j'-j)(\alpha_{\min}-\varepsilon_0)}$ .
- Since  $\lambda' \in \Lambda_{f,\mu}(j', I_\ell \pm \varepsilon, I_{\ell'} \pm \varepsilon)$ ,  $\mu(\lambda') \geq 2^{-j'(h_{\ell'+1} + \varepsilon_0)}$ .

Consequently,

$$\begin{aligned} \frac{\log \mu(\lambda)}{-j \log(2)} &\leq \varepsilon_0 + \frac{\log \mu(\widehat{\lambda})}{-j \log(2)} \leq 2\varepsilon_0 + \frac{j'}{j}(h_{\ell'+1} + \varepsilon_0) + (1 - \frac{j'}{j})(\alpha_{\min} - \varepsilon_0) \\ &\leq \alpha_{\min} + \frac{j'}{j}(h_{\ell'+1} - (\alpha_{\min} - 4\varepsilon_0)) \leq \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) + O(\varepsilon_0), \end{aligned}$$

where  $O(\varepsilon_0)$  is independent of  $(\ell, \ell')$ . This yields the result.  $\square$

Let us now bound  $\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min})$  from above. Thanks to (5.11),  $h_{\ell'+1} \leq \theta_p^{-1}(h_{\ell+1})$  implies that  $d(\ell, \ell') = \tau_\mu^*(h_{\ell'+1})$ . Using that  $\theta_p^{-1}(h_{\ell+1}) \leq \theta_p^{-1}(H) \leq \tau_\mu'(0^+)$  and that  $\tau_\mu^*$  is non decreasing over  $[\alpha_{\min}, \tau_\mu'(0^+)]$ , one has

$$\frac{H}{h_{\ell+1}} \tau_\mu^*(\theta_p^{-1}(h_{\ell+1})) \geq \frac{H}{h_{\ell+1}} \tau_\mu^*(h_{\ell'+1}) = \frac{H}{h_{\ell+1}} d(\ell, \ell') > \zeta_{\mu,p}^*(H) = \tau_\mu^*(\theta_p^{-1}(H)),$$

from which one deduces that

$$\frac{\tau_\mu^*(\theta_p^{-1}(h_{\ell+1}))}{h_{\ell+1}} > \frac{\tau_\mu^*(\theta_p^{-1}(H))}{H}. \quad (5.12)$$

Observe that the definition (4.1) of  $\theta_p$  implies that

$$\theta_p^{-1}(\beta) + p^{-1} \tau_\mu^*(\theta_p^{-1}(\beta)) = \beta \quad (5.13)$$

for all  $\beta \in [\alpha_{\min}, \zeta_{\mu,p}'(0^+)]$ . Applying (5.13) to both sides of (5.12) yields

$$\frac{\theta_p^{-1}(h_{\ell+1})}{h_{\ell+1}} < \frac{\theta_p^{-1}(H)}{H}, \quad (5.14)$$

and since  $\frac{H}{h_{\ell+1}} > 1$ , the following series of inequalities holds:

$$\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq \frac{H}{h_{\ell+1}} h_{\ell'+1} \leq \frac{H}{h_{\ell+1}} \theta_p^{-1}(h_{\ell+1}) \leq \theta_p^{-1}(H). \quad (5.15)$$

Consequently, Lemma 5.7 yields

$$\mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon) \subset \mathcal{D}_\mu(j, [\alpha_{\min}, \theta_p^{-1}(H)] \pm O(\varepsilon_0)). \quad (5.16)$$

The function  $\tau_\mu^*$  is continuous and non-decreasing over  $[\alpha_{\min}, \theta_p^{-1}(H)]$ . Hence, choosing initially  $\varepsilon_0$  small enough yields for  $j$  large enough that

$$\#\mathcal{D}_f^{\ell, \ell'}(j, H \pm \varepsilon) \leq 2^{j(\tau_\mu^*(\theta_p^{-1}(H)) + \eta)} = 2^{j(\zeta_{\mu,p}^*(H) + \eta)}. \quad (5.17)$$

**Subcase (3c):**  $\frac{H}{h_{\ell+1}} d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'} < \theta_p^{-1}(h_{\ell+1}) < h_{\ell'+1}$ .

Here one has  $h_{\ell'+1} \leq (1 + \varepsilon_0)h_{\ell'} \leq (1 + \varepsilon_0)\theta_p^{-1}(h_{\ell+1})$ , so

$$\begin{aligned} \alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) &\leq (1 + \varepsilon_0) \frac{H}{h_{\ell+1}} \theta_p^{-1}(h_{\ell+1}) + \alpha_{\min} \left(1 - \frac{H}{h_{\ell'+1}}\right) \\ &\leq (1 + \varepsilon_0) \frac{H}{h_{\ell+1}} \theta_p^{-1}(h_{\ell+1}). \end{aligned} \quad (5.18)$$

Also, (5.11) gives  $d(\ell, \ell') = \tau_\mu^*(\theta_p^{-1}(h_{\ell+1}))$ , so  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  is equivalent to (5.12), and it implies (5.14). Finally, arguing as in the subcase (3b) and using (5.18), one sees that

$$\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq H - \frac{\zeta_{\mu,p}^*(H)}{p} + O(\varepsilon_0) = \theta_p^{-1}(H) + O(\varepsilon_0). \quad (5.19)$$

Applying Lemma 5.7, one deduces that (5.17) holds once again.

**Subcase (3d):**  $\frac{H}{h_{\ell+1}}d(\ell, \ell') > \zeta_{\mu,p}^*(H)$  and  $h_{\ell'} \geq \theta_p^{-1}(h_{\ell+1})$ .

By (5.11),  $d(\ell, \ell') = p(h_{\ell+1} - h_{\ell'})$ . Consequently,  $h_{\ell'} = h_{\ell'} - \frac{d(\ell, \ell')}{p} < h_{\ell+1} - \frac{h_{\ell+1}}{H}\zeta_{\mu,p}^*(H)/p$ , and

$$\alpha_{\min} + \frac{H}{h_{\ell+1}}(h_{\ell'+1} - \alpha_{\min}) \leq \frac{H}{h_{\ell+1}}h_{\ell'+1} < \frac{H}{h_{\ell+1}}\left(h_{\ell+1} - \frac{h_{\ell+1}}{H}\frac{\zeta_{\mu,p}^*(H)}{p}\right) + H\frac{h_{\ell+1} - h_{\ell}}{h_{\ell+1}}.$$

Thus, (5.19) and then (5.17) hold in this subcase as well.

Collecting the estimates obtained along the cases considered above, (5.6) is proved, and so is  
600 Proposition 5.2.

## 6. Typical singularity spectrum in $B_q^{\mu,p}(\mathbb{R}^d)$

In this section, the singularity spectrum of typical functions in  $B_q^{\mu,p}(\mathbb{R}^d)$  when  $\mu \in \mathcal{E}_d$  is computed, proving item (2) of Theorem 4.

The strategy is similar to the one used to derive the generic multifractal behavior in classical  
605 Besov spaces. First, a *saturation* function is built, whose multifractal structure is the one claimed to be generic in  $B_q^{\mu,p}(\mathbb{R}^d)$ . Then, this particular function is used to perturb a countable family of dense sets in  $B_q^{\mu,p}(\mathbb{R}^d)$ , in order to obtain a countable family of dense open sets on the intersection of which the desired multifractal behavior holds. However, the construction of the saturation function and the multifractal analysis of typical functions are much more delicate in  $B_q^{\mu,p}(\mathbb{R}^d)$   
610 than in  $B_q^{s,p}(\mathbb{R}^d)$ .

The environment  $\mu \in \mathcal{E}_d$  is fixed for the rest of this section, as well as  $(p, q) \in [1, +\infty]^2$  and  $\Psi \in \mathcal{F}_{s_\mu}$ .

### 6.1. A saturation function

In this section, a saturation function  $g_{\mu,p,q} \in B_q^{\mu,p}(\mathbb{R}^d)$  is built via its wavelet coefficients, which  
615 are *as large as possible* in  $B_q^{\mu,p}(\mathbb{R}^d)$ , and its wavelet leaders are estimated.

The definition of  $g_{\mu,p,q}$  demands some preparation.

When  $\alpha_{\min} = \alpha_{\max}$ , we set  $(M_N := N^2)_{N \in \mathbb{N}^*}$  and  $I_i^N = \{\alpha_{\min}\}$  for all  $1 \leq i \leq M_N$ .

When  $\alpha_{\min} < \alpha_{\max}$ , for every  $N \in \mathbb{N}^*$ , it is possible to find an integer  $M_N$  such that the interval  $[\alpha_{\min}, \alpha_{\max}] = [\tau_\mu'(+\infty), \tau_\mu'(-\infty)]$  can be split into  $M_N$  non-trivial contiguous closed intervals  $I_1^N, I_2^N, \dots, I_{M_N}^N$  satisfying for every  $i \in \{1, \dots, M_N\}$ ,

$$|I_i^N| \leq 1/N \quad \text{and} \quad \max\{|\tau_\mu^*(\alpha) - \tau_\mu^*(\alpha')| : \alpha, \alpha' \in I_i^N\} \leq 1/N. \quad (6.1)$$

Without loss of generality, we assume that the sequence  $(M_N)_{N \geq 1}$  is increasing.

In any case, item (6) of Proposition 3.2 yields a decreasing sequence  $(\eta_N)_{N \in \mathbb{N}^*}$  converging to 0 as  $N \rightarrow \infty$ , and for all  $N \in \mathbb{N}^*$ ,  $M_N$  integers  $J_{N,1}, J_{N,2}, \dots, J_{N,M_N}$ , such that for every  $i \in \{1, \dots, M_N\}$ , for every  $j \geq J_{N,i}$ ,

$$\left| \frac{\log_2 \#\mathcal{D}_\mu(j, I_i^N \pm 1/N)}{j} - \max_{\alpha \in I_i^N} \tau_\mu^*(\alpha) \right| \leq \eta_N. \quad (6.2)$$

Without loss of generality, we assume that  $\eta_N \geq 1/N$ .

Then, define inductively the non-decreasing sequences of integers  $(J_N)_{N \in \mathbb{N}^*}$  and  $(N_j)_{j \geq 1}$  such that:

$$\begin{cases} \forall N \geq 1, J_N \geq \max\{J_{N,i} : i \in \{1, \dots, M_N\}\} \\ \forall N \geq 2, M_N \leq 2^{J_N \eta_{N-1}}, \\ \forall N \geq 3, J_{N-1} \eta_{N-2} < J_N \eta_{N-1}, \\ \text{for every } J_N \leq j < J_{N+1}, \text{ we set } N_j = N. \end{cases} \quad (6.3)$$

Moreover, Proposition 3.2(7) makes it possible to impose that for every  $j \geq J_N$  and  $\lambda \in \mathcal{D}_j$ ,

$$2^{-j(\alpha_{\max}+1/N)} \leq \mu(\lambda) \leq 2^{-j(\alpha_{\min}-1/N)}.$$

620 Finally, let us introduce some coefficients depending on the elements  $\lambda \in \Lambda_j$ :

- If  $L \in \mathbb{Z}^d$ ,  $j \geq J_2$  and  $\lambda \in \Lambda_j^L = \{\lambda = (i, j, k) \in \Lambda_j : \lambda_{j,k} \subset L + [0, 1]^d\}$ , set

$$w_\lambda = \begin{cases} \frac{2^{-\frac{3j\eta_{N_j}-1}{p}}}{j^{\frac{1}{p}+\frac{2}{q}}(1+\|L\|)^{\frac{d+1}{p}}} & \text{if } p < +\infty \\ j^{-\frac{2}{q}} & \text{if } p = +\infty, \end{cases} \quad (6.4)$$

with the convention  $\frac{2}{\infty} = 0$ .

- If  $j \geq J_2$  and  $\lambda = (i, j, k) \in \Lambda_j$ , set  $\alpha_{j,k} = \frac{\log_2 \mu(\lambda_{j,k})}{-j}$  and

$$\alpha_\lambda = \begin{cases} \alpha_{j,k} & \text{if } \alpha_{j,k} \in [\alpha_{\min}, \alpha_{\max}], \\ \alpha_{\min} & \text{if } \alpha_{j,k} < \alpha_{\min}, \\ \alpha_{\max} & \text{if } \alpha_{j,k} > \alpha_{\max}. \end{cases}$$

**Remark 6.1.** Note that  $\tilde{\varepsilon}_\lambda = \frac{\log_2 \mu(\lambda)}{-j} - \alpha_\lambda$  tends to 0 uniformly in  $\lambda \in \mathcal{D}_j$  as  $j \rightarrow +\infty$ . In other words, there exists  $\tilde{\phi} \in \Phi$  (recall Definition 1.3) such that  $|\frac{\log_2 \mu(\lambda)}{-j} - \alpha_\lambda| \leq \frac{\tilde{\phi}(j)}{j}$ .

Recall the Definition 3.4 of the irreducible dyadic cube  $\bar{\lambda} := \lambda_{\bar{j}, \bar{k}}$ .

625 **Definition 6.2.** The saturation function  $g_{\mu,p,q} : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined by its wavelet coefficients in the wavelet basis associated with  $\Psi$ , denoted by  $(c_\lambda^{\mu,p,q})_{\lambda \in \Lambda}$ , as follows:

- $c_\lambda^{\mu,p,q} = 0$  if  $\lambda \in \bigcup_{j < J_2} \Lambda_j$ .
- If  $j \geq J_2$  and  $\lambda = (i, j, k) \in \Lambda_j$ , set

$$c_\lambda^{\mu,p,q} = \begin{cases} w_\lambda \cdot \mu(\lambda_{j,k}) & \text{if } p = +\infty, \\ w_\lambda \cdot \mu(\lambda_{j,k}) 2^{-\frac{j}{p}} \frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p} & \text{if } p < +\infty. \end{cases} \quad (6.5)$$

**Remark 6.3.** 1. Note that  $c_\lambda^{\mu,p,q}$  does not depend on  $i$  if  $\lambda = (i, j, k)$ . Consequently,  $c_\lambda^{\mu,p,q}$  is defined without ambiguity by the same formula for  $\lambda \in \mathcal{D}_j$ .

630 2. The choice of  $\bar{j}$  and  $\bar{\lambda}$  in the exponent  $2^{-\bar{j}\frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}}$  in (6.5) implies that at a given generation  $j$ , the wavelet coefficients of  $g_{\mu,p,q}$  display several order of magnitudes, which are influenced by the values of  $\mu$  along the  $j$  first generations of dyadic cubes. One can also guess from this choice that approximation by dyadic vectors plays an important role in our analysis, since the local behavior of  $g_{\mu,p,q}$  around a point  $x$  depends on how close  $x$  is to the dyadic vectors.

635 **Lemma 6.4.** The function  $g_{\mu,p,q}$  belongs to  $B_q^{\mu,p}(\mathbb{R}^d)$  and  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

*Proof.* Suppose that  $p < +\infty$ .

For  $j \in \mathbb{N}$  and  $L \in \mathbb{Z}^d$ , set  $\mathcal{D}_j^L = \{\lambda \in \mathcal{D}_j : \lambda \subset L + [0, 1]^d\}$  and  $\Lambda_j^L = \{(i, j, k) \in \Lambda_j : \lambda_{j,k} \in \mathcal{D}_j^L\}$ . Recall that for  $\lambda = (i, j, k)$ ,  $\mu(\lambda)$  stands for  $\mu(\lambda_{j,k})$ .

Let us define, for  $j \geq J_2$  and  $L \in \mathbb{Z}^d$ ,  $A_{j,L} = \sum_{\lambda \in \Lambda_j^L} \left( \frac{|c_\lambda^{\mu,p,q}|}{\mu(\lambda)} \right)^p$ . To prove that  $g_{\mu,p,q} \in$

640  $B_q^{\mu,p}(\mathbb{R}^d) \subset \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , it is enough to show that  $A_j := \left( \sum_{L \in \mathbb{Z}^d} A_{j,L} \right)^{1/p} \in \ell^q(\mathbb{N})$ .

For  $j \in [J_N, J_{N+1})$ , by (6.5) and (6.4), one has

$$A_{j,L} = \sum_{\lambda \in \Lambda_j^L} \left( \frac{2^{-3j\eta_{N_j-1}/p} \mu(\lambda) 2^{-\bar{j}\frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}}}{j^{\frac{1}{p} + \frac{2}{q}} (1 + \|L\|)^{(d+1)/p} \mu(\lambda)} \right)^p = \frac{(2^d - 1) 2^{-3j\eta_{N_j-1}}}{j^{1 + \frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \sum_{\lambda \in \mathcal{D}_j^0} 2^{-\bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}})}, \quad (6.6)$$

where the factor  $2^d - 1$  comes from the fact that  $c_\lambda^{\mu,p,q}$ ,  $\lambda = (i, j, k)$ , is independent of  $i \in \{1, \dots, 2^d - 1\}$ . The periodicity of  $\mu$ , i.e.  $\mu|_{[0,1]^d} = \mu|_{L+[0,1]^d}$  is also used.

Recalling the notations in Proposition (3.3), if  $\lambda \in \mathcal{D}_j$  and  $\bar{\lambda}$  is the cube associated with its irreducible representation, then one can write  $\lambda = \bar{\lambda} \cdot [0, 2^{-(j-\bar{j})}]^d$ .

Then, after regrouping in (6.6) the terms according to the generation of their irreducible representation, one has

$$\begin{aligned} A_{j,L} &= (2^d - 1) \frac{2^{-3j\eta_{N_j-1}}}{j^{1 + \frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( 1 + \sum_{J=1}^j \sum_{\lambda \in \mathcal{D}_J^0 \setminus (\mathcal{D}_{J-1}^0 \cdot [0, 2^{-1}]^d)} 2^{-J\tau_\mu^*(\alpha_\lambda)} \right) \\ &\leq 2^d \frac{2^{-3j\eta_{N_j-1}}}{j^{1 + \frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( 1 + \sum_{J=1}^j \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} \right) \\ &= 2^d \frac{2^{-3j\eta_{N_j-1}}}{j^{1 + \frac{2p}{q}} (1 + \|L\|)^{(d+1)}} \left( \sum_{J=0}^{J_1-1} + \left( \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} \right) + \sum_{J=J_{N_j}}^j \right) \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)}. \end{aligned} \quad (6.7)$$

For each  $J_N \leq J < J_{N+1}$ , using (6.1) and then (6.2), we obtain

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_J^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} &\leq \sum_{i=1}^{M_{N,J}} \sum_{\lambda \in \mathcal{D}_\mu(j, I_i^{N_j} \pm 1/N)} 2^{-J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} - 1/N_j)} \\ &\leq \sum_{i=1}^{M_{N,J}} 2^{J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} + \eta_{N_j})} 2^{-J(\max\{\tau_\mu^*(\alpha) : \alpha \in I_i^{N_j}\} - 1/N_j)} \\ &= M_{N,J} 2^{J(\eta_{N_j} + 1/N_j)} \leq M_{N,J} 2^{2J\eta_{N_j}}. \end{aligned}$$

Consequently, by (6.3),

$$\begin{aligned}
& \left( \left( \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} \right) + \sum_{J=J_{N_j}}^j \right) \sum_{\lambda \in \mathcal{D}_j^0} 2^{-J\tau_\mu^*(\alpha_\lambda)} \\
& \leq \sum_{N=1}^{N_j-1} \sum_{J=J_N}^{J_{N+1}-1} M_N 2^{2J\eta_N} + \sum_{J=J_{N_j}}^j M_{N_j} 2^{2J\eta_{N_j}} \\
& \leq \sum_{N=1}^{N_j-1} (J_{N+1} - J_N) M_N 2^{2J_{N+1}\eta_N} + (j - J_{N_j} + 1) M_{N_j} 2^{2j\eta_{N_j}} \\
& \leq j M_{N_j} 2^{2j\eta_{N_j-1}},
\end{aligned}$$

645 since all terms  $M_N 2^{2J_{N+1}\eta_N}$ , for  $N \leq N_{j-1}$ , are less than  $M_{N_j} 2^{2j\eta_{N_j-1}}$ .

Setting  $C_\mu = \sum_{J=0}^{J_1-1} \sum_{\lambda \in \mathcal{D}_j^0} 2^{-J\tau_\mu^*(\alpha_\lambda)}$ , by (6.3) and  $M_{N_j} 2^{2J_{N_j}\eta_{N_j-1}} \leq 1$  one has

$$A_{j,L} \leq 2^d \frac{M_{N_j} 2^{-J_n \eta_{N_j-1}}}{j^{\frac{2p}{q}} (1 + \|L\|)^{(d+1)}} (C_\mu + 1) \leq \frac{2^d (C_\mu + 1)}{j^{\frac{2p}{q}} (1 + \|L\|)^{(d+1)}}.$$

Finally,

$$\left( \sum_{L \in \mathbb{Z}^d} A_{j,L} \right)^{1/p} = \left\| \left( \frac{c_\lambda^{\mu,p,q}}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p = O(j^{-2/q}),$$

hence  $\left( \left\| \left( \frac{c_\lambda^{\mu,p,q}}{\mu(\lambda)} \right)_{\lambda \in \Lambda_j} \right\|_p \right)_{j \in \mathbb{N}}$  belongs to  $\ell^q(\mathbb{N})$ . This implies that  $g_{\mu,p,q} \in B_q^{\mu,p}(\mathbb{R}^d)$ .

When  $p = +\infty$ , the estimate is much simpler and left to the reader.  $\square$

650 Next lemma shows that the wavelet leader (recall (1.24))  $L_\lambda^{g_{\mu,p,q}}$  of  $g_{\mu,p,q}$  at  $\lambda \in \mathcal{D}_j$  is essentially comparable to the wavelet coefficients  $c_{\lambda'}^{\mu,p,q}$  indexed by the cubes  $\lambda'$  of generation  $j$  which are neighbors of  $\lambda$ . This property is key to estimate the  $L^q$ -spectrum of  $g^{\mu,p,q}$  relative to  $\Psi$ .

**Lemma 6.5.** *Fix  $L \in \mathbb{Z}^d$ . For every  $\varepsilon > 0$ , there exists  $J^\varepsilon \in \mathbb{N}$  such that if  $j \geq J^\varepsilon$ , for every  $\lambda \in \mathcal{D}_j^L$ ,*

$$\tilde{c}_\lambda^{\mu,p,q} \leq L_\lambda^{g_{\mu,p,q}} \leq 2^{j\varepsilon} \tilde{c}_\lambda^{\mu,p,q},$$

where  $\tilde{c}_\lambda^{\mu,p,q} = \max\{c_\lambda^{\mu,p,q} : \lambda \in \mathcal{D}_j, \tilde{\lambda} \subset 3\lambda\}$ .

*Proof.* It is enough to prove the result for  $L = 0$ . Let  $\varepsilon, \varepsilon' \in (0, 1)$ . Let  $j \geq 1$  and  $\lambda \in \mathcal{D}_j^0$ . Let us begin with some remarks:

- in (6.5), the term  $w_\lambda$  depends only on  $j$ , and is decreasing with  $j$ .
- 655 • if  $\lambda' \subset \lambda$ ,  $\mu(\lambda') \leq \mu(\lambda)$  since  $\mu \in \mathcal{C}(\mathbb{R}^d)$ .
- by Remark 6.3(1)  $c_\lambda^{\mu,p,q}$  does not depend on the index  $i$  of  $\lambda = (i, j, k)$ .

Next, observe that if  $\lambda' \subset \lambda$ , the irreducible cubes  $\bar{\lambda}' \in \mathcal{D}_{\bar{j}'}$  and  $\bar{\lambda} \in \mathcal{D}_{\bar{j}}$  respectively associated with  $\lambda'$  and  $\lambda$ , are such that  $\bar{j}' \leq \bar{j}$ .

Then one controls the wavelet coefficients as follows:

- 660 (i) By the property (P<sub>1</sub>) of  $\mu$ , there exists  $M \in \mathbb{N}^*$  such that for every  $\lambda' \in \mathcal{D}_{Mj}$  one has  $\mu(\lambda') \leq 2^{-j(d/p+2\alpha_{\max}+1)}$ . So  $\mu(\lambda) 2^{-\bar{j}' \frac{\tau_\mu^*(\alpha_{\bar{\lambda}'})}{p}} \geq 2^{-j(\alpha_{\max}+1)-jd/p} \geq \mu(\lambda')$ , which implies that for  $j' \geq Mj$ ,  $c_{\lambda'}^{\mu,p,q} \leq c_\lambda^{\mu,p,q}$ . Hence, the only wavelet coefficients  $c_{\lambda'}$  to consider to compute  $L_\lambda^{g_{\mu,p,q}}$  for  $\lambda \in \mathcal{D}_j$  are those of generations  $j'$  such that  $j \leq j' \leq Mj$ .

(ii) if  $j' \leq Mj$  and  $\bar{j} \leq jp\varepsilon/(2d)$ , then  $2^{-\bar{j}\frac{\tau_\mu^*(\alpha_{\bar{\lambda}})}{p}} \geq 2^{-jp\varepsilon/(2d)\cdot d/p} \geq 2^{-j\varepsilon}$ , so  $c_{\lambda'}^{\mu,p,q} \geq w_\lambda\mu(\lambda)2^{-j\varepsilon}$  and by the remarks of the beginning of the proof,

$$c_{\lambda'}^{\mu,p,q} \leq w_{\lambda'}\mu(\lambda') \leq w_\lambda\mu(\lambda) \leq c_\lambda^{\mu,p,q}2^{j\varepsilon}.$$

665 (iii) It is possible to choose  $\varepsilon'$  small enough so that if  $\bar{j}' - \bar{j} \leq \varepsilon'\bar{j}'$ , then since  $\mu$  is almost doubling,  $|\alpha_{\bar{\lambda}'} - \alpha_{\bar{\lambda}}|$  is so small that  $|\bar{j}'\tau_\mu^*(\alpha_{\bar{\lambda}'}) - \bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}})| \leq \bar{j}p\varepsilon$ .

(iv) If  $j' \leq Mj$ ,  $\bar{j} > jp\varepsilon/(2d)$  and  $\bar{j}' - \bar{j} \leq \varepsilon'\bar{j}'$ , then by (iii) one has (for  $j$  is large enough)

$$c_{\lambda'}^{\mu,p,q} \leq c_\lambda^{\mu,p,q}2^{\bar{j}\varepsilon} \leq c_\lambda^{\mu,p,q}2^{j\varepsilon}.$$

(v) If  $j' \leq Mj$ ,  $\bar{j} > jp\varepsilon/(2d)$  and  $\bar{j}' - \bar{j} > \varepsilon'\bar{j}'$ , then

$$\bar{j}'\alpha_{\bar{\lambda}'} = \bar{j}\alpha_{\bar{\lambda}} + (\bar{j}' - \bar{j})\alpha \tag{6.8}$$

for some  $\alpha \in [\alpha_{\min} - \varepsilon, \alpha_{\max} + \varepsilon]$ . The concavity of  $\tau_\mu^*$  then implies that for some  $\varepsilon''$  independent of  $j$  and  $j'$ ,

$$\bar{j}'\tau_\mu^*(\alpha_{\bar{\lambda}'}) \geq \bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}}) + (\bar{j}' - \bar{j})(\tau_\mu^*(\alpha^*) - \varepsilon''), \quad \text{where } \alpha^* = \begin{cases} \alpha & \text{when } \alpha \in [\alpha_{\min}, \alpha_{\max}], \\ \alpha_{\max} & \text{when } \alpha \geq \alpha_{\max}, \\ \alpha_{\min} & \text{when } \alpha \leq \alpha_{\min}. \end{cases}$$

In particular,  $\bar{j}'\tau_\mu^*(\alpha_{\bar{\lambda}'}) \geq \bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}}) - (\bar{j}' - \bar{j})\varepsilon''$ , hence

$$2^{-\bar{j}'\tau_\mu^*(\alpha_{\bar{\lambda}'})/p} \leq 2^{-\bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}})/p}2^{(\bar{j}' - \bar{j})\varepsilon''/p} \leq 2^{-\bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}})/p}2^{\bar{j}'\varepsilon''/p} \leq 2^{-\bar{j}\tau_\mu^*(\alpha_{\bar{\lambda}})/p}2^{Mj\varepsilon''/p}.$$

One checks that  $\varepsilon''$  can be chosen as small as necessary when  $j$  tends to infinity, in particular so that one has for large  $j$  that  $M\varepsilon''/p \leq \varepsilon$ . Finally, with this choice of  $\varepsilon''$ ,  $c_{\lambda'}^{\mu,p,q} \leq c_\lambda^{\mu,p,q}2^{j\varepsilon}$ .

670 Putting together all the previous information yields that when  $j$  is large enough, for all  $\lambda \in \mathcal{D}_j^0$  and all  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \lambda$ , one has  $c_{\lambda'}^{\mu,p,q} \leq c_\lambda^{\mu,p,q}2^{j\varepsilon}$ .

The same property holds true for all  $\tilde{\lambda} \in \mathcal{D}_j$  such that  $\tilde{\lambda} \subset 3[0, 1]^d$  and  $\lambda' \in \mathcal{D}_{j'}$  such that  $\lambda' \subset \tilde{\lambda}$ . This yields the desired property.  $\square$

## 6.2. The singularity spectrum of the saturation function $g_{\mu,p,q}$ and some of its perturbations

675 We now determine the singularity spectrum of  $g_{\mu,p,q}$ , and more generally of any function whose wavelet coefficients are “comparable” to those of  $g_{\mu,p,q}$  over infinitely many generations.

**Proposition 6.6.** *Let  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  such that for any  $L \in \mathbb{Z}^d$ , there exists an increasing sequence of integers  $(j_n)_{n \in \mathbb{N}}$ , and a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0 such that for all  $n \geq 1$  and  $\lambda = (i, j_n, k) \in \Lambda_{j_n}$  such that  $\lambda_{j_n,k} \subset L + 3[0, 1]^d$  the inequality  $2^{-j_n\varepsilon_n}c_\lambda^{\mu,p,q} \leq |c_\lambda^f|$  holds. Then  $\sigma_f = \sigma_{g_{\mu,p,q}} = \zeta_{\mu,p}^*$ .*

680 Only the case  $p < +\infty$  is treated, the case  $p = +\infty$  is simpler and deduced from arguments similar to those developed below. Fix  $(j_n)_{n \in \mathbb{N}}$  and  $(\varepsilon_n)_{n \in \mathbb{N}}$  as in the statement.

It is enough to prove that  $\dim E_f(H) \cap (L + [0, 1]^d) = \zeta_{\mu,p}^*(H)$  for all  $H \in \mathbb{R}$  and  $L \in \mathbb{Z}^d$ . Without loss of generality we work with  $L = 0$  and show that  $\dim E_f(H) \cap [0, 1]^d = \zeta_{\mu,p}^*(H)$  for all  $H \in \mathbb{R}$ .

Note that the characterization (1.25) and the assumptions on  $(j_n)_{n \in \mathbb{N}}$  imply that for all  $x \in [0, 1]^d$ , for the  $\lambda_{j_n} = (i, j_n, k)$  such that  $x \in \lambda_{j_n}$ , one has

$$\liminf_{n \rightarrow +\infty} \frac{\log c_{\lambda_{j_n}}^{\mu,p,q}}{\log 2^{-j_n}} \geq \liminf_{n \rightarrow +\infty} \frac{\log |c_{\lambda_{j_n}}^f|}{\log 2^{-j_n}} \geq \liminf_{j \rightarrow +\infty} \frac{\log L_{j_n}^f(x)}{\log 2^{-j_n}} \geq h_f(x). \tag{6.9}$$

685 Recall that the value of  $c_{\lambda_{j_n}}^{\mu,p,q}$  does not depend on the index  $i$  of  $\lambda_{j_n} = (i, j_n, k)$ .

6.2.1. The upper bound  $\sigma_f \leq \zeta_{\mu,p}^*$ .

Theorem 4(1) gives  $\sigma_f(H) \leq \zeta_{\mu,p}^*(H)$  for all  $H \leq \zeta'_{\mu,p}(0^+)$ . Note also that  $\zeta_{\mu,p}^*(H) = d$  for all  $H \in [\zeta'_{\mu,p}(0^+), \zeta'_{\mu,p}(0^-)]$ . Hence it remains us to treat the case  $H > \zeta'_{\mu,p}(0^-)$ , which corresponds to the decreasing part of the spectrum of  $f$ .

690 Fix  $H > \zeta'_{\mu,p}(0^-)$  and  $x \in [0, 1]^d$  such that  $h_f(x) \geq H$ .

By (6.9), denoting  $\lambda_{j_n}$  any  $\lambda = (i, j_n, k) \in \Lambda_{j_n}$  such that  $x \in \lambda_{j_n}$ , one has

$$\liminf_{n \rightarrow +\infty} \frac{\log c_{\lambda_{j_n}}^{\mu,p,q}}{\log 2^{-j_n}} \geq H. \quad (6.10)$$

Recall that  $\overline{\lambda_{j_n}} \in \mathcal{D}_{\overline{j_n}}$  is the irreducible representation of  $\lambda_{j_n}$ . Using the concatenation of cubes introduced after Definition 3.4, one writes  $\lambda_{j_n} = \overline{\lambda_{j_n}} \cdot [0, 2^{-(j_n - \overline{j_n})}]^d$ , and

$$\frac{\log c_{\lambda_{j_n}}^{\mu,p,q}}{\log 2^{-j_n}} = \frac{\log_2 w_{\lambda_{j_n}}}{j_n} + \frac{\log_2 \mu(\lambda_{j_n})}{-j_n} + \frac{\overline{j_n}}{j_n} \frac{\tau_{\mu}^*(\alpha_{\overline{\lambda_{j_n}}})}{p}. \quad (6.11)$$

Recall (3.3) and the fact that for  $j, j' \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j$ , one has  $\mu(\lambda \cdot [0, 2^{-j'}]^d) = \mu(\lambda) 2^{-\phi_{\lambda}} 2^{-j' \alpha_{\min} + \tilde{\phi}_{\lambda}(j')}$ , where by (3.4)  $|\phi_{\lambda}|$  and  $|\tilde{\phi}_{\lambda}(j')|$  are uniformly bounded by a  $o(j)$  and a  $o(j')$  respectively. So,

$$\frac{\log_2 \mu(\lambda_{j_n})}{-j_n} = \frac{\overline{j_n}}{j_n} \frac{\log_2 \mu(\lambda_{\overline{j_n}})}{-\overline{j_n}} + \frac{\phi_{\overline{\lambda_{j_n}}}}{j_n} + \frac{j_n - \overline{j_n}}{j_n} \alpha_{\min} + \frac{\tilde{\phi}_{\overline{\lambda_{j_n}}}(j_n - \overline{j_n})}{j_n}$$

which combined with (6.11) yields

$$\frac{\log c_{\lambda_{j_n}}^{\mu,p,q}}{\log 2^{-j_n}} = \frac{\overline{j_n}}{j_n} \theta_p(\alpha_{\overline{\lambda_{j_n}}}) + \left(1 - \frac{\overline{j_n}}{j_n}\right) \alpha_{\min} + r_n(x), \quad (6.12)$$

where

$$r_n(x) = \frac{\log_2 w_{\lambda_{j_n}}}{j_n} + \frac{\overline{j_n}}{j_n} \left( \frac{\log_2 \mu(\lambda_{\overline{j_n}})}{\overline{j_n}} - \alpha_{\overline{\lambda_{j_n}}} \right) + \frac{\phi_{\overline{\lambda_{j_n}}}}{j_n} + \frac{\tilde{\phi}_{\overline{\lambda_{j_n}}}(j_n - \overline{j_n})}{j_n}.$$

The dependence of  $r_n(x)$  on  $x$  is explicit, to remember it. But it does not play any role in the bounds above, which are uniform in  $j_n$  and  $j_n - \overline{j_n}$ .

**Lemma 6.7.** *One has  $\lim_{n \rightarrow +\infty} r_n(x) = 0$ .*

*Proof.* The first term in  $r_n(x)$  tends to zero when  $n \rightarrow +\infty$ , by definition (6.4) of  $w_{\lambda}$ .

For the other terms in  $r_n(x)$ , let us define

$$C = \max \left( \sup_{j \geq 1} \left\{ \frac{\tilde{\phi}(j)}{j} \right\}, \sup_{j \geq 1} \left\{ \frac{|\phi_{\lambda}|}{j} : \lambda \in \mathcal{D}_j \right\}, \sup_{j' \geq 1} \left\{ \frac{|\tilde{\phi}_{\lambda}(j')|}{j'} : \lambda \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j \right\} \right).$$

695 By (3.3) and Remark 6.1, one has  $C < +\infty$ .

Now fix  $\eta \in (0, 1)$  and let us treat the second term. Remark 6.1 again gives that  $\frac{\overline{j_n}}{j_n} \left( \frac{\log_2 \mu(\lambda_{\overline{j_n}})}{\overline{j_n}} - \alpha_{\overline{\lambda_{j_n}}} \right) \leq \frac{\tilde{\phi}(\overline{j_n})}{j_n}$ . When  $j_n$  is large, one sees that:

- if  $\frac{\overline{j_n}}{j_n} > \eta$ , then  $\overline{j_n}$  is large and  $\left| \frac{\tilde{\phi}(\overline{j_n})}{j_n} \right| \leq \left| \frac{\tilde{\phi}(\overline{j_n})}{\overline{j_n}} \right| \leq \eta$ ,
- if  $\frac{\overline{j_n}}{j_n} \leq \eta$ , then  $\frac{\overline{j_n}}{j_n} \left| \frac{\tilde{\phi}(\overline{j_n})}{j_n} \right| \leq C\eta$ .

700 In any case, for  $n$  large enough  $\frac{j_n}{j_n} |\tilde{\phi}(\bar{j}_n)| \leq (C+1)\eta$ .

The same argument applies to the third term  $\frac{j_n}{j_n} \phi_{\lambda_{j_n}}$ .

Finally, for the fourth term, one has:

- if  $\frac{j_n - \bar{j}_n}{j_n} > \eta$ , then  $j_n - \bar{j}_n$  is also large and  $\frac{\tilde{\phi}_{\lambda_{j_n}}(j_n - \bar{j}_n)}{j_n} \leq \frac{\tilde{\phi}_{\lambda_{j_n}}(j_n - \bar{j}_n)}{j_n - \bar{j}_n} \leq \eta$ ,
- if  $\frac{j_n - \bar{j}_n}{j_n} \leq \eta$ , then  $\frac{|\tilde{\phi}_{\lambda_{j_n}}(j_n - \bar{j}_n)|}{j_n} = \frac{|\tilde{\phi}_{\lambda_{j_n}}(j_n - \bar{j}_n)|}{j_n - \bar{j}_n} \frac{j_n - \bar{j}_n}{j_n} \leq C\eta$ .

705 This concludes the proof of the Lemma. □

Note now that  $\theta_p(\alpha) \geq \alpha_{\min}$  for all  $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ . Since  $\alpha_{\min} \leq \zeta'_{\mu,p}(0^-) < H$ , (6.10) and (6.12) together imply that necessarily, for every  $\varepsilon > 0$ ,  $\theta_p(\alpha_{\lambda_{j_n}}) \geq H - \varepsilon$  for infinitely many integers  $n$ . Hence, on one hand  $H \leq \theta_p(\alpha_p)$  and in particular  $E_f(H) = \emptyset$  if  $H > \theta_p(\alpha_p)$ , and on the other hand

$$\bar{h}_\mu(x) \geq \limsup_{j \rightarrow +\infty} \frac{\log_2 \mu(\lambda_j(x))}{-j} \geq \limsup_{n \rightarrow +\infty} \alpha_{\lambda_{j_n}} \geq \theta_p^{-1}(H),$$

where the same notations as above are used, i.e.  $\lambda_{j_n}$  is here the unique cube of generation  $j_n$  that contains  $x$ . This implies that  $x \in \bar{E}_\mu^{\geq}(\theta_p^{-1}(H))$ .

As a conclusion,  $H \leq \theta_p(\alpha_p)$  and  $E_f(H) \subset \bar{E}_\mu^{\geq}(\theta_p^{-1}(H))$ . Since  $\theta_p^{-1}(H) \geq \tau'_\mu(0^-)$  lies in the decreasing part of the singularity spectrum of  $\mu$ , Proposition 3.2(5) yields that  $\dim E_f(H) \leq$   
 710  $\dim \bar{E}_\mu^{\geq}(\theta_p^{-1}(H)) = \tau_\mu^*(\theta_p^{-1}(H))$ . This is the desired upper bound.

6.2.2. The lower bound  $\sigma_f \geq \zeta_{\mu,p}^*$  over the range  $[\alpha_{\min}, \theta_p(\alpha_p)] = [\zeta'_{\mu,p}(+\infty), \zeta'_{\mu,p}(-\infty)]$ .

Two cases must be separated.

**Case 1:**  $H \in [\theta_p(\alpha_{\min}), \theta_p(\alpha_p)]$ .

Let  $\alpha \in [\alpha_{\min}, \alpha_p]$  such that  $H = \theta_p(\alpha) (= \alpha + \tau_\mu^*(\alpha)/p)$ . Our goal is to show that  $\sigma_f(H) =$   
 715  $\dim E_f(H) \geq \zeta_{\mu,p}^*(H) = \tau_\mu^*(\alpha)$ . To achieve this, we prove that  $\mu_\alpha(E_f(H)) > 0$ , where  $\mu_\alpha$  is the measure described in Proposition 3.2(2). Since  $\mu_\alpha$  is exact dimensional with exponent  $\tau_\mu^*(\alpha)$ , this yields the claim.

For any  $H' \geq 0$  set

$$E_{\bar{f}}^{\leq}(H') := \{y \in [0, 1]^d : h_f(y) \leq H'\}.$$

Let us start with one technical lemma.

**Lemma 6.8.** For every  $\eta > 0$ ,  $\mu_\alpha(E_\mu(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)) = 0$ .

*Proof.* Fix  $\eta > 0$ ,  $J_0 \in \mathbb{N}$ , and set

$$E_{\mu,\eta,J_0}(\alpha) = \left\{ x \in [0, 1]^d : \begin{cases} \forall J \geq J_0, \forall \lambda \in \mathcal{D}_J \text{ such that } \lambda \subset 3\lambda_J(x), \\ 2^{-J(\alpha + \frac{\eta}{3})} \leq \mu(\lambda) \leq 2^{-J(\alpha - \frac{\eta}{3})} \end{cases} \right\}$$

and for  $j \geq J \geq J_0$

$$\mathcal{D}_{\eta,J,j}(\alpha) = \left\{ \lambda \in \mathcal{D}_J : \begin{cases} \lambda \cap E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta) \neq \emptyset \text{ and} \\ \exists \lambda' = (i, j, k) \in \Lambda_j, \lambda' \subset 3\lambda, |c_{\lambda'}^f| \geq 2^{-J(H - \frac{\eta}{2})} \end{cases} \right\}. \quad (6.13)$$

720 Recall the following fact stated along the proof of Lemma 6.5: there exists a constant  $M$  such that the only wavelet coefficients  $c_{\lambda'}$  to consider to compute  $L_\lambda^{g_{\mu,p,q}}$  for  $\lambda \in \mathcal{D}_j$  are the  $j'$  such that  $j \leq j' \leq Mj$ .



**Lemma 6.9.** *There exists  $C > 0$  such that for  $J_0 \leq J \leq j \leq MJ$ ,*

$$\#\mathcal{D}_{\eta,J,j}(\alpha) \leq C 2^{-(j-J)p \frac{\alpha_{\min}}{2}} 2^{J(\tau_{\mu}^*(\alpha) - p \frac{\eta}{8})},$$

and when  $j > MJ$ ,  $\mathcal{D}_{\eta,J,j}(\alpha)$  is empty.

*Proof.* The case  $j > MJ$  follows from the remark just before the Lemma.

725 Let  $x \in E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)$ . By (1.25), there are infinitely many integers  $J \geq J_0$  for which  $L_J^f(x) \geq 2^{-J(H-\eta/2)}$ . For such a generation  $J$ , the definition of the wavelet leader as a supremum implies that there exist  $MJ \geq j \geq J$  and  $\lambda = (i, j, k) \in \Lambda_j$  with  $\lambda \subset 3\lambda_J(x)$  such that  $|c_{\lambda}^f| \geq 2^{-J(H-\eta/2)}$ . This means that  $\lambda_J(x) \in \mathcal{D}_{\eta,J,j}(\alpha)$ .

Recalling (3.2), assume that  $J_0$  is so large that  $\mu(\lambda) \leq \mu(\lambda_J(x)) 2^{J\eta/8} 2^{-(j-J)\alpha_{\min}/2}$ .

Then, the definition of  $E_{\mu,\eta,J_0}(\alpha)$  and the fact that  $\alpha + \tau_{\mu}^*(\alpha)/p = H$  give

$$\frac{|c_{\lambda}^f|}{\mu(\lambda)} \geq 2^{-J\eta/8} 2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J(H-\frac{\eta}{2})} 2^{J(\alpha-\eta/8)} \geq 2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J\left(\frac{\tau_{\mu}^*(\alpha)}{p} - \frac{\eta}{4}\right)}. \quad (6.14)$$

Since  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ ,  $f \in B_q^{\mu(-\frac{\eta}{8M}),p}(\mathbb{R}^d)$ , and so  $\sum_{\lambda \in \Lambda_j} \left(2^{-j\frac{\eta}{8M}} \frac{|c_{\lambda}^f|}{\mu(\lambda)}\right)^p = C < \infty$ . Thus,

$$C \geq \sum_{\lambda \in \Lambda_j} \left(2^{-j\frac{\eta}{8M}} \frac{|c_{\lambda}^f|}{\mu(\lambda)}\right)^p \mathbf{1}_{\frac{|c_{\lambda}^f|}{\mu(\lambda)} \geq 2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J\left(\frac{\tau_{\mu}^*(\alpha)}{p} - \frac{\eta}{4}\right)}}.$$

The number of cubes  $\lambda \in \Lambda_j$  such that the above indicator function is 1 is by (6.13) larger than the cardinality of  $\mathcal{D}_{\eta,J,j}(\alpha)$ . It follows that

$$C \geq \#\mathcal{D}_{\eta,J,j}(\alpha) 2^{-jp\frac{\eta}{8M}} \left(2^{(j-J)\frac{\alpha_{\min}}{2}} 2^{-J\left(\frac{\tau_{\mu}^*(\alpha)}{p} - \frac{\eta}{4}\right)}\right)^p.$$

730 Noting that  $j \leq MJ$  implies  $2^{jp\frac{\eta}{8M}} \leq 2^{Jp\frac{\eta}{8}}$ , the last inequality yields the result.  $\square$

In particular,  $\mathcal{D}_{\eta,J,j} = \emptyset$  for  $j \geq J(p\frac{\alpha_{\min}}{2} + \tau_{\mu}^*(\alpha_{\min}))$ .

Note that

$$E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta) \subset \bigcap_{J \geq J_0} \bigcup_{j \geq J} \bigcup_{\lambda \in \mathcal{D}_{\eta,J,j}(\alpha)} \lambda.$$

For any  $\delta > 0$ , denote by  $\mathcal{H}_{\delta}^s$  the pre- $s$ -Hausdorff measure on  $\mathbb{R}^d$  associated with coverings by sets of diameter less than or equal to  $\delta$ . Using  $\bigcup_{j \geq J} \bigcup_{\lambda \in \mathcal{D}_{\eta,J,j}(\alpha)} \lambda$  as covering of  $E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)$ , one deduces that for every  $J \geq J_0$ ,

$$\begin{aligned} \mathcal{H}_{\sqrt{d}, 2^{-J}}^s(E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)) &\leq \sum_{J \leq j \leq J(p\frac{\alpha_{\min}}{2} + \tau_{\mu}^*(\alpha_{\min}))} (\#\mathcal{D}_{\eta,J,j}(\alpha)) (\sqrt{d} \cdot 2^{-j})^s \\ &\leq (\sqrt{d})^s C \left( \sum_{m \geq 0} 2^{-mp\frac{\alpha_{\min}}{2}} \right) 2^{J(\tau_{\mu}^*(\alpha) - p\frac{\eta}{8} - s)}, \end{aligned}$$

which tends to zero as soon as  $s > \tau_{\mu}^*(\alpha) - p\frac{\eta}{8}$ . It follows that

$$\dim(E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)) \leq \tau_{\mu}^*(\alpha) - p\frac{\eta}{8},$$

and thus  $\mu_{\alpha}(E_{\mu,\eta,J_0}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)) = 0$ , because  $\mu_{\alpha}$  may give a positive mass to a set  $E$  only if  $\dim E \geq \tau_{\mu}^*(\alpha)$ .

To conclude, observe that the almost doubling property of  $\mu$  yields

$$E_{\mu}(\alpha) = \bigcap_{m \geq 1} \bigcup_{J_0 \in \mathbb{N}} E_{\mu, \frac{1}{m}, J_0}(\alpha).$$

This, combined with the previous estimate on  $\mu_{\alpha}$  gives  $\mu_{\alpha}(E_{\mu}(\alpha) \cap E_{\bar{f}}^{\leq}(H - \eta)) = 0$ .  $\square$

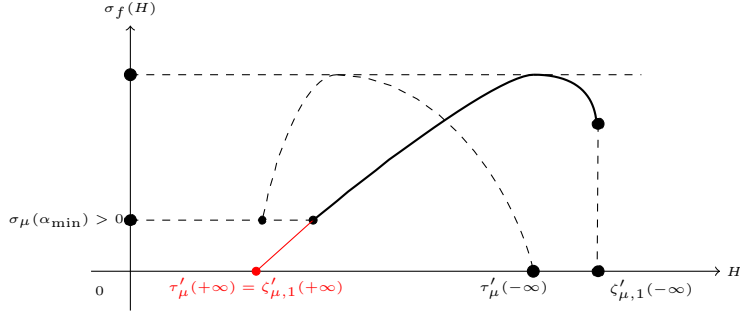


Figure 3: Case where  $\sigma_\mu(\alpha_{\min}) > 0$  and  $p = 1$ : the dashed graph represents the spectrum of  $\mu$ , the plain graph represents the multifractal spectrum  $\sigma_f$  of typical functions  $f \in \mathbf{B}_q^{\mu,1}(\mathbb{R}^d)$ . An affine segment (in red) with slope  $p = 1$  appears in the spectrum  $\sigma_f$ .

735 We are now equipped to prove the lower bound  $\dim E_f(H) \geq \tau_\mu^*(\alpha)$ .

First, (6.12) states that  $\frac{\log c_{\lambda_{j_n}(x)}^{\mu,p,q}}{\log 2^{-j_n}} = \frac{\overline{j_n(x)}}{j_n(x)} \theta_p(\alpha_{\lambda_{j_n}(x)}) + \left(1 - \frac{\overline{j_n(x)}}{j_n(x)}\right) \alpha_{\min} + r_n(x)$ .

By Proposition 3.2(2), for  $\mu_\alpha$ -almost every  $x$ ,  $\lim_{j \rightarrow +\infty} \alpha_{\lambda_j(x)} = \alpha$ . By Proposition 3.7, for  $\mu_\alpha$ -almost every  $x$ ,  $\lim_{n \rightarrow +\infty} \frac{\overline{j_n(x)}}{j_n(x)} = 1$ .

740 One deduces that  $h_f(x) \leq \theta_p(\alpha) = H$  for  $\mu_\alpha$ -almost every  $x$ , i.e.  $\mu_\alpha(E_f^{\leq}(H)) = 1$  (the equality  $h_f(x) = H$  does not hold in general, since (6.12) is true only for a subsequence of integers  $(j_n)_{n \geq 1}$ ).

Combining all the above results, one concludes that

$$\begin{aligned} \mu_\alpha(E_f(H)) &= \mu_\alpha(E_\mu(\alpha) \cap E_f(H)) \\ &\geq \mu_\alpha(E_\mu(\alpha) \cap E_f^{\leq}(H)) - \sum_{m \geq 1} \mu_\alpha(E_\mu(\alpha) \cap E_f^{\leq}(H - 1/m)) = 1. \end{aligned}$$

This proves that necessarily  $\dim E_f(H) \geq \tau_\mu^*(\alpha)$ , as expected.

**Case 2:**  $H \in [\alpha_{\min}, \theta_p(\alpha_{\min}))$ : this corresponds to the affine part of the spectrum, which occurs only when  $\sigma_\mu(\alpha_{\min}) = \tau_\mu^*(\alpha_{\min}) > 0$ , see Figure 3.

745 If  $H \in [\alpha_{\min}, \theta_p(\alpha_{\min}))$ , write  $H = \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{\delta p}$ , where  $\delta > 1$ . By Proposition 3.6 applied to the sequence  $(j_n)_{n \in \mathbb{N}}$  given by Proposition 6.6, the set  $S(\delta, (\eta_j)_{j \in \mathbb{N}^*}, (j_n)_{n \in \mathbb{N}})$  supports a Borel probability measure  $\nu$  of lower Hausdorff dimension at least equal to  $\tau_\mu^*(\alpha_{\min})/\delta = p(H - \alpha_{\min}) = \zeta_{\mu,p}^*(H)$ . Note that  $(\eta_j)_{j \in \mathbb{N}^*}$  depends only on  $\mu$ .

For  $x \in S(\delta, (\eta_j)_{j \in \mathbb{N}^*}, (j_n)_{n \in \mathbb{N}})$ , one checks that

$$h_f(x) \leq \liminf_{n \rightarrow +\infty} \frac{\log c_{j_n}^{\mu,p,q}(x)}{\log 2^{-j_n}} \leq \alpha_{\min} + \frac{\tau_\mu^*(\alpha_{\min})}{\delta p} = H.$$

750 In addition,  $\{y \in [0, 1]^d : h_f(y) < H\} = \bigcup_{m \geq 1} E_f^{\leq}(H - 1/m)$ , and each set  $E_f^{\leq}(H - 1/m)$  has a  $\nu$ -measure equal to 0, since due to Proposition 3.2(2) applied to the capacity provided by the leaders of  $f$ ,  $\dim E_f^{\leq}(H - 1/m) \leq (\zeta_f^\Psi)^*(H - 1/m) < \zeta_{\mu,p}^*(H)$ . Consequently,  $\nu(E_f(H)) = 1$  and  $\dim E_f(H) \geq \zeta_{\mu,p}^*(H)$ .

755 Finally, if  $H = \alpha_{\min}$ , the set  $F = \bigcap_{p \in \mathbb{N}} S(p, (\eta_j)_{j \geq 1}, (j_n)_{n \in \mathbb{N}})$  is easily seen to be non empty (by taking  $\delta = p$  at step  $p$  of the construction in the proof of proposition 3.6) and to be included in  $E_f^{\leq}(\alpha_{\min})$ , by using the previous estimates. However we know that  $E_f^{\leq}(h) = \emptyset$  for all  $h < \alpha_{\min}$  by Theorem 4. Consequently,  $E_f^{\leq}(\alpha_{\min}) = E_f^{\leq}(\alpha_{\min}) \neq \emptyset$ , so  $\sigma_f(\alpha_{\min}) = \dim E_f(\alpha_{\min}) \geq 0$ .

### 6.3. Typical multifractal behavior in $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$

We finally prove item (2) of Theorem 4, hence obtaining the multifractal behavior of typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

Recall the definition (1.18) of the basis  $\{\mathcal{N}_m\}_{m \in \mathbb{N}}$  of neighborhoods of the origin in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ .

760 For every integer  $m > m_0 = \lfloor \max(1, s_1^{-1}) \rfloor + 1$ , set

$$V_m = \left\{ f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d) : \forall j \geq J_2, \forall \lambda \in \Lambda_j, \frac{|c_\lambda^f|}{c_\lambda^{\mu,p,q}} \in m^{-1}\{1, \dots, m^2\} \right\}.$$

Then let

$$\mathcal{G} = \limsup_{m \rightarrow \infty} (V_m + \mathcal{N}_{2^{\lfloor m \log(m) \rfloor}}). \quad (6.15)$$

Each  $\bigcup_{\ell \geq m} V_\ell$ ,  $m \geq m_0$ , is dense in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , so  $\mathcal{G}$  contains a dense  $G_\delta$  set.

When  $f \in \mathcal{G}$ , there exists an increasing sequence  $(j_n)_{n \geq 0}$  such that  $f \in V_{j_n} + \mathcal{N}_{2^{\lfloor j_n \log(j_n) \rfloor}}$  for all  $n \geq 0$ .

765 Fix  $L \in \mathbb{Z}^d$ . Looking at the particular generation  $j_n$ , for all  $\lambda \in \Lambda_{j_n}$  such that  $\lambda \subset L + 3[0, 1]^d$ , by definition of  $V_{j_n}$  and  $\tilde{\mathcal{N}}_{2^{\lfloor j_n \log(j_n) \rfloor}}$ , the lower bound  $|c_\lambda^f| \geq j_n^{-1} c_\lambda^{\mu,p,q} - 2^{-\lfloor j_n \log(j_n) \rfloor} \mu(\lambda) 2^{j_n 2^{-j_n \log(j_n)}}$  holds. By construction of the coefficients  $c_\lambda^{\mu,p,q}$ , this implies that for  $n$  large enough one has  $|c_\lambda^f| \geq j_n^{-1} c_\lambda^{\mu,p,q} / 2$ , hence there exists a positive sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0 such that  $|c_\lambda^f| \geq 2^{-j_n \varepsilon_n} |c_\lambda^{\mu,p,q}|$  for all  $\lambda \in \Lambda_{j_n}$  such that  $\lambda \subset L + 3[0, 1]^d$ . Consequently, Proposition 6.6 yields  $\sigma_f = \sigma_{g_{\mu,p,q}} = \zeta_{\mu,p}^*$ .

**Remark 6.10.** In fact, the definitions of  $V_{j_n}$ ,  $\tilde{\mathcal{N}}_{2^{\lfloor j_n \log(j_n) \rfloor}}$ , and  $c_\lambda^{\mu,p,q}$ , imply that if  $(j_n)_{n \geq 1}$  is an increasing sequence of integers and  $f \in \bigcap_{n \geq 1} V_{j_n} + \mathcal{N}_{2^{\lfloor j_n \log(j_n) \rfloor}}$ , then for all  $N, K \in \mathbb{N}^*$ , for all  $n \geq 1$  large enough and  $\lambda \in \bigcup_{j=j_n}^{K j_n} \Lambda_j$  such that  $\lambda \subset N[0, 1]^d$ , one has

$$\frac{1}{2j_n} c_\lambda^{\mu,p,q} \leq |c_\lambda^f| \leq 2j_n c_\lambda^{\mu,p,q}.$$

770 These bounds will be useful to estimate the  $L^q$ -spectrum of  $f$ .

## 7. Validity of the WMF and the WWMF in $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$

775 Recall that the multifractal formalisms for functions were defined in Section 1.5. In this last section, we first discuss the validity of the WMF for the saturation function  $g_{\mu,p,q}$ . This helps in establishing part (3) of Theorem 5 in Section 7.3, while Section 7.2 provides the proof of part (2) of Theorem 5.

### 7.1. WMF and WWMF for the saturation function $g_{\mu,p,q}$

Recall that the wavelet  $\Psi$  is fixed, and that  $g_{\mu,p,q}$  is built via its wavelet coefficients in the wavelet basis generated by  $\Psi$ . Also, recall (1.26) for the definition of  $\zeta_{g_{\mu,p,q},j}^{N,\Psi}$ , and the various notations concerning  $L^q$ -spectra for functions.

780 **Proposition 7.1.** *The WMF holds for  $g_{\mu,p,q}$  on the interval  $[\zeta'_{\mu,p}(+\infty), \zeta'_{\mu,p}(0^+)]$ , and the WWMF holds for  $g_{\mu,p,q}$  on the interval  $[\zeta'_{\mu,p}(+\infty), \zeta'_{\mu,p}(-\infty)]$ .*

Moreover, for all  $N \in \mathbb{N}^*$ , one has  $\lim_{j \rightarrow +\infty} \zeta_{g_{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$ .

785 The second part of the statement shows that the convergence of the sequence  $(\zeta_{g_{\mu,p,q},j}^{N,\Psi})_{j \geq 1}$  is stronger than what is required for the WWMF to hold (only the convergence over a subsequence is needed).

*Proof.* Suppose that it is established that for all  $N \in \mathbb{N}^*$ , one has  $\lim_{j \rightarrow +\infty} \zeta_{g_{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$ . In particular  $\zeta_{g_{\mu,p,q}}^{N,\Psi} = \zeta_{\mu,p}$  for all  $N \in \mathbb{N}^*$ , so  $\zeta_{g_{\mu,p,q}}^{\Psi} = \zeta_{\mu,p}$ . Since it was shown in the previous section that  $\sigma_{g_{\mu,p,q}} = \zeta_{\mu,p}^*$ , one concludes that  $g_{\mu,p,q}$  satisfies the WMF.

Now, fix  $N \in \mathbb{N}^*$ . Let us prove that  $\lim_{j \rightarrow +\infty} \zeta_{g_{\mu,p,q},j}^{N,\Psi} = \zeta_{\mu,p}$ .

790 The  $\mathbb{Z}^d$ -invariance of  $\mu$  and the definition of  $g_{\mu,p,q}$  show that it is enough to work on  $[0, 1]^d$  and to prove that  $\lim_{j \rightarrow +\infty} j^{-1} \log \sum_{\lambda \in \mathcal{D}_j^0} (L_{\lambda}^{g_{\mu,p,q}})^t = \zeta_{\mu,p}(t)$ .

Fix  $t \in \mathbb{R}$ . Recall Remark 6.3(1) and Lemma 6.5. The reader can check that due to these two facts,

$$\lim_{j \rightarrow \infty} j^{-1} \log \frac{\sum_{\lambda \in \mathcal{D}_j^0} (L_{\lambda}^{g_{\mu,p,q}})^t}{\sum_{\lambda \in \mathcal{D}_j^0} (c_{\lambda}^{g_{\mu,p,q}})^t} = 0.$$

Moreover, by definition of the coefficients  $c_{\lambda}^{g_{\mu,p,q}}$ , and since  $\log(w_{\lambda}) = o(\log(\mu(\lambda)))$  uniformly in  $\lambda \in \Lambda_j$  as  $j \rightarrow +\infty$ ,

$$\lim_{j \rightarrow \infty} j^{-1} \log \frac{\sum_{\lambda \in \mathcal{D}_j^0} (c_{\lambda}^{g_{\mu,p,q}})^t}{B(j,t)} = 0, \text{ where } B(j,t) = \sum_{\lambda \in \mathcal{D}_j^0} \left( \mu(\lambda) 2^{-\frac{j}{p} J \tau_{\mu}^*(\alpha_{\lambda})} \right)^t.$$

Thus, one must prove that

$$\lim_{j \rightarrow +\infty} j^{-1} \log_2 B(j,t) = \zeta_{\mu,p}(t). \quad (7.1)$$

When  $p = +\infty$ , this was established in Section 3 of [1], but in the general case where  $\mu$  is a positive power of such a measure the result holds as well by a direct calculation.

Assume now that  $p < +\infty$ . Fix  $t \in \mathbb{R}^*$ , the case  $t = 0$  being obvious.

Fix  $\varepsilon > 0$ . Using the same decomposition as that used in the proof of Lemma 6.4,

$$B(j,t) = \sum_{J=0}^j \sum_{\lambda \in \mathcal{D}_j^0 \setminus (\mathcal{D}_{j-1}^0 \cdot [0, 2^{-1}]^d)} \mu(\lambda \cdot [0, 2^{-(j-J)}]^d) 2^{-\frac{t}{p} J \tau_{\mu}^*(\alpha_{\lambda})}.$$

Then, from (3.3) we deduce that there exists a positive sequence  $(C_j)_{j \geq 1}$  depending on  $t$  and  $\mu$  such that  $\lim_{j \rightarrow +\infty} \frac{\log(C_j)}{j} = 0$  and for all  $j \geq 1$ ,

$$2^{(j-J)(\alpha_{\min} + \varepsilon)} C_j^{-1} \leq \frac{\mu(\lambda)}{\mu(\lambda \cdot [0, 2^{-(j-J)}]^d)} \leq C_j 2^{(j-J)(\alpha_{\min} - \varepsilon)}.$$

795 Observe that when  $\lambda$  and  $\lambda'$  are neighbors in  $\Lambda_J$ , the two numbers  $\mu(\lambda)^t 2^{-\frac{t}{p} J \tau_{\mu}^*(\alpha_{\lambda})}$  and  $\mu(\lambda')^t 2^{-\frac{t}{p} J \tau_{\mu}^*(\alpha_{\lambda'})}$  differ by a factor at most  $2^{J\varepsilon}$ . This follows from the almost doubling property (P<sub>2</sub>) of  $\mu$  and the continuity of  $\tau_{\mu}^*$ .

These considerations prove that there exists another positive sequence  $(\tilde{C}_j)_{j \geq 1}$  depending on  $t$  and  $\mu$  such that  $\lim_{j \rightarrow +\infty} \frac{\log(\tilde{C}_j)}{j} = 0$  and

$$\tilde{C}_j^{-1} \tilde{B}(j,t, \alpha_{\min}, s(t)\varepsilon) \leq B(j,t) \leq \tilde{C}_j \tilde{B}(j,t, \alpha_{\min}, -s(t)\varepsilon), \quad (7.2)$$

where  $s(t)$  is the sign of  $t$  and

$$\tilde{B}(j,t, \beta, \gamma) = \sum_{J=0}^j 2^{-(j-J)t(\beta+\gamma)} 2^{-J\gamma} \sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda)^t 2^{-\frac{t}{p} J \tau_{\mu}^*(\alpha_{\lambda})}, \quad (7.3)$$

The quantity  $\sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)}$  is now controlled. Using Proposition 3.2(6), the interval  $[\alpha_{\min}, \alpha_{\max}]$  can be split into  $M$  contiguous intervals  $I_i = [\alpha_i, \alpha_{i+1}]$ ,  $i = 1, \dots, M$ , of length less than  $\varepsilon$  such that for every  $i \in \{1, \dots, M\}$ ,

$$\left| \sup_{\alpha \in I_i} \tau_\mu^*(\alpha) - \frac{\log_2 \#\mathcal{D}_\mu(j, I_i)}{j} \right| \leq \varepsilon \quad \text{and} \quad \sup_{\alpha, \alpha' \in I_i} |\tau_\mu^*(\alpha) - \tau_\mu^*(\alpha')| \leq \varepsilon.$$

Define the mapping  $\chi_3 : \alpha \in [\alpha_{\min}, \alpha_{\max}] \mapsto t\theta_p(\alpha) - \tau_\mu^*(\alpha)$  (in (4.3) its restriction  $\tilde{\chi}_2$  to the interval  $[\alpha_{\min}, \alpha_p]$  was considered). Without loss of generality, suppose that there exists  $1 \leq i_0 \leq M$  such that  $t\theta_p(\alpha_{i_0}) - \tau_\mu^*(\alpha_{i_0}) = \min\{\chi_3(\alpha) : \alpha \in [\alpha_{\min}, \alpha_{\max}]\} := \zeta_3(t)$ .

Also, by Remark 6.1, there exists  $C \geq 1$  such that for all  $j \in \mathbb{N}$  and  $\lambda \in \mathcal{D}_j^0$ , one has  $C^{-1}2^{-j(\alpha_\lambda + \varepsilon)} \leq \mu(\lambda) \leq C2^{-j(\alpha_\lambda - \varepsilon)}$ .

It follows from the previous information that

$$\sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)} \begin{cases} \leq C^{|t|} \sum_{i=1}^M 2^{J(\tau_\mu^*(\alpha_i) + \varepsilon)} 2^{-Jt(\alpha_i - 2s(t)\varepsilon)} 2^{-\frac{t}{p} J(\tau_\mu^*(\alpha_i) - s(t)\varepsilon)} \\ \geq C^{-|t|} \sum_{i=1}^M 2^{J(\tau_\mu^*(\alpha_i) - \varepsilon)} 2^{-Jt(\alpha_i + 2s(t)\varepsilon)} 2^{-\frac{t}{p} J(\tau_\mu^*(\alpha_i) + s(t)\varepsilon)}, \end{cases}$$

which implies that

$$2^{Js(t)\varepsilon} \sum_{\lambda \in \mathcal{D}_j^0} \mu(\lambda) t 2^{-\frac{t}{p} J \tau_\mu^*(\alpha_\lambda)} = m_J(t, \varepsilon) \sum_{i=1}^M 2^{-J\chi_3(\alpha_i)} \quad (7.4)$$

where  $|\log(m_J(t, \varepsilon))| \leq |t| \log(C) + (2 + 2|t| + \frac{|t|}{p})J\varepsilon$ .

Then, incorporating (7.4) in (7.3) and using that the infimum of  $\chi_3(\alpha_i)$  is reached at  $i_0$ , i.e.  $\chi_3(\alpha_{i_0}) = \zeta_3(t)$ , one gets

$$\tilde{B}(j, t, \beta, \pm\varepsilon) = \sum_{J=0}^j 2^{-(j-J)t(\beta + \gamma)} \tilde{m}_J(t, \varepsilon) 2^{-J\zeta_3(t)}, \quad (7.5)$$

where  $|\log(\tilde{m}_J(t, \varepsilon))| \leq \log(M) + |t| \log(C) + (2 + 2|t| + \frac{|t|}{p})J\varepsilon$ . Incorporating (7.5) in (7.2) then implies

$$B_j = \widehat{m}_j(t, \varepsilon) 2^{-jt\alpha_{\min}} \sum_{J=0}^j \overline{m}_J(t, \varepsilon) \tilde{m}_J(t, \varepsilon) 2^{-J(\zeta_3(t) - t\alpha_{\min})}, \quad (7.6)$$

where  $\max(|\log(\overline{m}_j(t, \varepsilon))|, |\log(\widehat{m}_j(t, \varepsilon))|) \leq j|t|\varepsilon + \log(\tilde{C}_j)$ .

It follows from (7.6) and the fact that  $\varepsilon$  is arbitrary, that:

- $\zeta_3(t) - t\alpha_{\min} \geq 0$  implies  $\lim_{j \rightarrow +\infty} -j^{-1} \log_2 B(j, t) = t\alpha_{\min}$ ,
- $\zeta_3(t) - t\alpha_{\min} \leq 0$  implies  $\lim_{j \rightarrow +\infty} -j^{-1} \log_2 B(j, t) = \zeta_3(t)$ .

hence, to prove (7.1) and Proposition 7.1, the value of  $\zeta_3(t)$  and the sign of  $\zeta_3(t) - t\alpha_{\min}$  must be investigated. According to the previous observations, this will give the desired conclusion.

The two cases  $\alpha_{\min} = \alpha_{\max}$  and  $\alpha_{\min} < \alpha_{\max}$  are split.

Suppose first that  $\alpha_{\min} = \alpha_{\max}$ . Then,  $\tau_\mu(t) = \alpha_{\min}t - d$  for all  $t \in \mathbb{R}$ , and

$$\zeta_{\mu, p}(t) = \begin{cases} (\alpha_{\min} + \frac{d}{p})t - d & \text{when } t < p, \\ \alpha_{\min}t & \text{when } t \geq p. \end{cases}$$

A straightforward computation gives  $\zeta_3(t) = t\alpha_{\min} + \left(\frac{t}{p} - 1\right)d$ . Thus when  $t < p$ ,  $\zeta_3(t) = \zeta_{\mu,p}(t)$ . Moreover,  $\zeta_3(t) - t\alpha_{\min} = \left(\frac{t}{p} - 1\right)\tau_{\mu}^*(\alpha_{\min})$  is non negative if and only if  $t \geq p$ , i.e.  $\zeta_{\mu,p}(t) = \alpha_{\min}t$ ; and when  $p > t$  one has  $\zeta_{\mu,p}(t) = \zeta_3(t)$ , hence the result.

Assume next that  $[\alpha_{\min}, \alpha_{\max}]$  is non trivial.

815 When  $t \geq p$ , the mapping  $\chi_3$  rewrites  $\chi_3(\alpha) = t\alpha + \left(\frac{t}{p} - 1\right)\tau_{\mu}^*(\alpha)$  so it is concave, and it reaches its minimum  $\zeta_3(t)$  either at  $\alpha_{\min}$  or at  $\alpha_{\max}$ . In either case,  $\zeta_3(t) - t\alpha_{\min} \geq 0$ . Moreover, in this range  $\zeta_{\mu,p}(t) = t\alpha_{\min}$ , so (7.1) holds true.

When  $t < p$ , recall the notations introduced and the fact established in the proof of Proposition 4.2.

820 If  $t_p = \frac{pt}{p-t} \leq t_{\infty} = (\tau_{\mu}^*)'(\alpha_{\min}^+)$ , the convex function  $\chi_3$  reaches its minimum  $\frac{p-t}{p}\tau_{\mu}(\frac{p}{p-t}t) = \zeta_{\mu,p}(t)$  at  $\tilde{\alpha}_t$ , i.e.  $\zeta_3(t) = \zeta_{\mu,p}(t)$ .

If  $t_p > t_{\infty}$ , then  $\chi_3$  is increasing and reaches at  $\alpha_{\min}$  its minimum equal to  $t\alpha_{\min} + \left(\frac{t}{p} - 1\right)\tau_{\mu}^*(\alpha_{\min}) = \zeta_{\mu,p}(t)$  (here  $\zeta_3(t) = \zeta_{\mu,p}(t)$  as well). In both cases,  $\zeta_3(t) - t\alpha_{\min} \leq \zeta_3(t) - \chi_3(\alpha_{\min}) \leq 0$  and (7.1) holds true.  $\square$

### 825 7.2. Proof of Theorem 5(2)

As recalled in the introduction, it is known [10] that for any smooth function  $f$ , one has  $\sigma_f \leq \zeta_f^*$ . Since it was shown in Section 6.3 that  $\sigma_f = (\zeta_{\mu,p})^*$  for typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , for such functions one necessarily has  $\zeta_f \leq \zeta_{\mu,p}$  by inverse Legendre transform. Simultaneously, Theorem 8 states that  $\zeta_f|_{\mathbb{R}_+} = \zeta_f^{\Psi}|_{\mathbb{R}_+} \geq \zeta_{\mu,p}|_{\mathbb{R}_+}$ , which yields the desired result.

### 830 7.3. Proof of Theorem 5(3)

It is enough to get part (i). Then part (ii) follows from the fact that the class of residual sets is stable by countable intersection.

Let  $f \in \mathcal{G}$ , where  $\mathcal{G}$  is the  $G_{\delta}$  set defined by (6.15), and consider a sequence  $(j_n)_{n \geq 1}$  such that  $f \in V_{j_n} + \mathcal{N}_{2^{\lceil m \log(m) \rceil}}$  for all  $n \geq 1$ . Fix  $N \in \mathbb{N}^*$ . We prove that  $\zeta_{f,j_n}^{\Psi,N}$  converges pointwise to  $\zeta_{\mu,p}$  as  $n \rightarrow$   
835  $+\infty$ , which is enough to show that the WWMF holds relatively to  $\Psi$  over  $[\zeta'_{\mu,p}(+\infty), \zeta'_{\mu,p}(-\infty)]$ , since it was established that  $\sigma_f = \zeta_{\mu,p}^*$ .

Since a function  $f \in \mathcal{G}$  necessarily belongs to  $\mathcal{C}^{\alpha_{\min} - \varepsilon}(\mathbb{R}^d)$  (for every  $\varepsilon > 0$ ), one has  $|c_{\lambda}^f| \leq 2^{-j(\alpha_{\min} - \varepsilon)}$  for every large  $j$  and  $\lambda \in \Lambda_j$  such that  $\lambda \subset (N+1)[0, 1]^d$ .

840 Fix  $\varepsilon = \alpha_{\min}/2$ . By construction, when  $j$  is large and  $\lambda \in \Lambda_j$ ,  $c_{\lambda}^{\mu,p,q} \geq 2^{-2j\alpha_{\max}}$ . Hence, from the previous fact and Remark 6.10 applied with  $K = \lfloor 4\alpha_{\max}/\alpha_{\min} \rfloor + 1$ , one sees that when  $n$  becomes large, for all  $j \geq j_n$  and  $\lambda \in \Lambda_j$  such that  $\lambda \subset (N+1)[0, 1]^d$ :

- either  $j \in \{j_n, \dots, Kj_n\}$  and the wavelet coefficient  $c_{\lambda}^f$  of  $f$  satisfies  $\frac{1}{2j_n}c_{\lambda}^{\mu,p,q} \leq |c_{\lambda}^f| \leq 2j_n c_{\lambda}^{\mu,p,q}$ , item or  $j > Kj_n$  and  $|c_{\lambda}^f| \leq c_{\lambda}^{\mu,p,q}$ . This implies that for all  $\lambda \in \mathcal{D}_{j_n}$  such that  $\lambda \subset N[0, 1]^d$ , the wavelets leader  $L_{\lambda}^f$  of  $f$  satisfies

$$\frac{1}{2j_n} L_{\lambda}^{g_{\mu,p,q}} \leq L_{\lambda}^f \leq 2j_n L_{\lambda}^{g_{\mu,p,q}}.$$

Consequently,  $\lim_{n \rightarrow +\infty} j_n^{-1} \log_2 \left( \frac{\zeta_{f,j_n}^{\Psi,N}}{\zeta_{g,j_n}^{\Psi,N}} \right) = 0$ , and by Proposition 7.1,  $\zeta_{f,j_n}^{\Psi,N}$  indeed converges to  $\zeta_{\mu,p}$  as  $n \rightarrow \infty$ .

Finally, when  $q < +\infty$ , to establish that for a typical  $f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  one has  $\zeta_f^{\Psi}|_{\mathbb{R}_-} = -\infty$ , consider for all  $m \in \mathbb{N}^*$  the set

$$\widetilde{V}_m = \left\{ f \in \mathbf{B}_q^{\mu,p}(\mathbb{R}^d) : \forall m \leq j \leq m \log(m), \forall \lambda \in \Lambda_j, c_{\lambda}^f = 0 \right\}.$$

The set  $\limsup_{m \rightarrow \infty} \widetilde{V}_m$  is dense in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  and

$$\widetilde{\mathcal{G}} = \mathcal{G} \cap \limsup_{m \rightarrow \infty} (\widetilde{V}_m + \mathcal{N}_{2^{\lceil m \log(m) \rceil}}).$$

is a dense  $G_\delta$ -set. When  $f \in \widetilde{\mathcal{G}}$ , there exists an increasing sequence of integers  $(m_n)_{n \in \mathbb{N}}$  such that  $f \in \widetilde{V}_{m_n} + \mathcal{N}_{2^{\lceil m_n \log(m_n) \rceil}}$  for all  $n \in \mathbb{N}$ . It is easily checked that for any  $A > 0$  and  $N \in \mathbb{N}$ , for  $n$  large enough, if  $\lambda \in \mathcal{D}_{m_n}$  and  $\lambda \subset N[0, 1]^d$ , one has  $L_\lambda^f \leq 2^{-Am_n}$ . This implies that for  $t < 0$ ,

$$\sum_{\lambda \in \mathcal{D}_{m_n}, \lambda \subset N[0,1]^d} \mathbf{1}_{L_\lambda^f > 0} (L_\lambda^f)^t \geq \#\{\lambda \in \mathcal{D}_{m_n}, \lambda \subset N[0,1]^d : L_\lambda^f > 0\} \cdot 2^{-Atm_n},$$

hence  $\zeta_f^{\Psi,N}(t) \leq At$ . Thus,  $A$  being arbitrary and  $t < 0$ , the desired conclusion holds.

## 845 8. Proof of Theorem 7

Part (1) follows from the fact that for  $\sigma \in \mathcal{S}_s$  to be the typical singularity spectrum in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  with  $p < +\infty$ , by Theorem 4 and Proposition 4.2 it is necessary that  $\sigma(H_{\min}) = 0$ , and by Theorem 4 the function  $\sigma^*$  is linear over  $[p, +\infty)$  so  $\sigma'(H_{\min}^+) \leq p$  by Remark 4.1.

To prove part (2), the cases  $p \notin \widehat{\partial}\sigma((H_{\min}, H_{\max}])$  and  $p \in \widehat{\partial}\sigma((H_{\min}, H_{\max}])$  are separated.

**Case  $p \notin \widehat{\partial}\sigma((H_{\min}, H_{\max}])$ :** Define the mapping

$$A : H \in [H_{\min}, H_{\max}] \mapsto H - \frac{\sigma(H)}{p}.$$

It is a continuous increasing bijection onto its image, that we denote by  $I = [\alpha_{\min}, \alpha_{\max}]$ . For  $\alpha \in I$ , denote  $A^{-1}(\alpha)$  by  $H(\alpha)$ . It is easily checked that the mapping

$$\widetilde{\sigma} : \alpha \in I \mapsto p(H(\alpha) - \alpha)$$

850 belongs to  $\mathcal{S}_d$  as well, and that if  $\mu \in \mathcal{E}_d$  is chosen such that  $\sigma_\mu = \widetilde{\sigma}$ , the study achieved in Section 4 implies that  $\sigma$  is the singularity spectrum of the typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , for all  $q \in [1, +\infty)$  (the function  $A$  is then the inverse of the function  $\theta_p$  defined in (4.1)).

855 Suppose, moreover, that  $\sigma'(H_{\max}^-) = -\infty$  and  $\sigma(H_{\max}) > 0$ . This is equivalent to suppose that  $\widetilde{\sigma}'(\alpha_{\max}^-) = -p$  and  $\widetilde{\sigma}(\alpha_{\max}) > 0$ . Again, the study achieved in Section 4 shows that for any element  $\widehat{\sigma}$  of  $\mathcal{S}_d$  whose domain takes the form  $[\alpha_{\min}, \alpha'_{\max}]$  with  $\alpha'_{\max} > \alpha_{\max}$  and  $\widehat{\sigma}|_{[\alpha_{\min}, \alpha_{\max}]} = \widetilde{\sigma}$ , for any  $\nu \in \mathcal{E}_d$  such that  $\sigma_\nu = \widehat{\sigma}$ ,  $\sigma$  is still the singularity spectrum of the typical functions in  $\mathbf{B}_q^{\nu,p}(\mathbb{R}^d)$ , for all  $q \in [1, +\infty)$ . Note that there are infinitely many ways to consider such an extension. On the contrary, if  $\sigma'(H_{\max}^-) > -\infty$  or  $\sigma(H_{\max}) = 0$ ,  $\mu$  is the unique element of  $\mathcal{E}_d$  such that typical elements of  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$  do have a singularity spectrum equal to  $\sigma$ .

860 **Case  $p \in \widehat{\partial}\sigma((H_{\min}, H_{\max}])$ :** this means that there is a non-trivial maximal subinterval  $[H_{\min}, \widetilde{H}_{\min}]$  of  $[H_{\min}, H_{\max}]$  such that for all  $H \in [H_{\min}, \widetilde{H}_{\min}]$  one has  $\sigma(H) = p(H - H_{\min})$ .

If  $\widetilde{H}_{\min} = H_{\max}$ , one chooses  $\widetilde{\sigma} = d \cdot \mathbf{1}_{H_{\max}}$ , so that  $\mu = (\mathcal{L}^d)^{\frac{H_{\max}}{d}}$  is such that  $\sigma_\mu = \widetilde{\sigma}$  and  $\sigma$  is the singularity spectrum of the typical functions in  $\mathbf{B}_q^{\mu,p}(\mathbb{R}^d)$ , for all  $q \in [1, +\infty)$ .

865 If  $\widetilde{H}_{\min} < H_{\max}$ , the same approach as in the case  $p \notin \widehat{\partial}\sigma((H_{\min}, H_{\max}])$  works, except that  $\sigma$  is replaced by its restriction to  $[\widetilde{H}_{\min}, H_{\max}]$ , and with the difference that now one necessarily has  $\sigma(\widetilde{H}_{\min}) > 0$ .

The claim about the validity of the WMF and WWMF follows from Theorem 5.

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