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Epidemic change-point detection in general causal time series

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Abstract: We consider an epidemic change-point detection in a large class of causal time series models, including among other processes, AR(∞), ARCH(∞), TARCH(∞), ARMA-GARCH. A test statistic based on the Gaussian quasi-maximum likelihood estimator of the parameter is proposed. It is shown that, under the null hypothesis of no change, the test statistic converges to a distribution obtained from a difference of two Brownian bridge and diverges to infinity under the epidemic alternative. Numerical results for simulation and real data example are provided.

Keywords: causal processes, epidemic change-point, semi-parametric statistic, quasi-maximum likelihood estimator.

1 Introduction

We consider a general class of affine causal time series models in a semiparametric setting. Let $(\xi_t)_{t \in \mathbb{Z}}$ be a sequence of centered independent and identically distributed (iid) random variables satisfying $\mathbb{E}\xi_0^2 = 1$ and Θ a compact subset of \mathbb{R}^d ($d \in \mathbb{N}$). For $\mathcal{T} \subset \mathbb{Z}$ and any $\theta \in \Theta$, define

Class $\mathcal{AC}_{\mathcal{T}}(M_{\theta}, f_{\theta})$: A process $\{X_t, t \in \mathcal{T}\}$ belongs to $\mathcal{AC}_{\mathcal{T}}(M_{\theta}, f_{\theta})$ if it satisfies:

$$X_t = M_{\theta}(X_{t-1}, X_{t-2}, \dots) \cdot \xi_t + f_{\theta}(X_{t-1}, X_{t-2}, \dots) \quad \forall t \in \mathcal{T}, \quad (1.1)$$

where $M_{\theta}, f_{\theta} : \mathbb{R}^{\infty} \rightarrow \mathbb{R}$ are two measurable functions.

Numerous classical time series such as AR(∞), ARCH(∞), TARCH(∞) or ARMA-GARCH models belong to this class. The existence of a stationary and ergodic solution as well as the inference for the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$ have been addressed by Bardet and Wintenberger (2009). This class of models is now well studied, see for instance, Bardet *et al.* (2012), Kengne (2012) for change-point detection on this class; Bardet *et al.* (2017) for inference based on the Laplacian quasi-likelihood; Bardet *et al.* (2020), Kengne (2020) for works relating to model selection problems.

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We focus here on the epidemic change-point detection in the class $\mathcal{AC}_{\mathcal{T}}(M_{\theta}, f_{\theta})$. Assume that a trajectory (X_1, \dots, X_n) of the process $\{X_t, t \in \mathbb{Z}\}$ is observed and consider the following test hypotheses:

H_0 : (X_1, \dots, X_n) is a trajectory of the process $\{X_t, t \in \mathbb{Z}\} \in \mathcal{AC}_{\mathbb{Z}}(M_{\theta_0^*}, f_{\theta_0^*})$ with $\theta_0^* \in \Theta$.

H_1 : there exists $(\theta_1^*, \theta_2^*, t_1^*, t_2^*) \in \Theta^2 \times \{2, 3, \dots, n-1\}^2$ (with $\theta_1^* \neq \theta_2^*$ and $t_1^* < t_2^*$) such that (X_1, \dots, X_n) belongs to $\mathcal{AC}_{\{1, \dots, t_1^*\}}(M_{\theta_1^*}, f_{\theta_1^*}) \cap \mathcal{AC}_{\{t_1^*+1, \dots, t_2^*\}}(M_{\theta_2^*}, f_{\theta_2^*}) \cap \mathcal{AC}_{\{t_2^*+1, \dots, n\}}(M_{\theta_1^*}, f_{\theta_1^*})$.

H_1 is the epidemic alternative; that is, there are two change-points (i.e. three regimes) such that the structure of the data reverts back to its original state (the first regime) after the second change-point (i.e., the structure of the first and the third regime is the same, and different from the second regime). This hypothesis refers to the so-called epidemic period, which runs from t_1^* to t_2^* . Such models can be used to perform "crisis phenomenon" or some seasonal variation with a dependence structure. For example, it is well known that, the GARCH model can be used for many stocks index return. During a financial crisis, one could see high variations in the volatility, while the post-crisis behaviour tend to close to the pre-crisis volatility. Some macroeconomic indicators, such as energy consumption/demand, which exhibit an autoregressive component (see, for instance, Contreras *et al.* (2003) or Burman and Shumway (2006)) are known to follow such phenomena during an economic depression period. See also Subsection 4.2 for an application to the concentrations of carbon monoxide; which could be higher during a period of heat stress.

Several works in the literature are devoted to the epidemic change-point detection in time series. We refer among others, to Levin and Kline (1985), Yao (1993), Csörgö and Horváth (1997), Račkauskas and Suquet (2004), Guan (2007), Jarušková and Piterbarg (2011), Aston and Kirch (2012a, 2012b), Bucchia (2014), Graiche *et al.* (2016). As pointed out by Diop and Kengne (2021), most of these procedures are developed for the epidemic change-point detection in the mean of random variables. The latter authors addressed this issue for a general class of integer valued time series.

In this new contribution, we propose a test based on the Gaussian quasi-likelihood for the epidemic change-point detection in the class of affine causal models $\mathcal{AC}_{\mathcal{T}}(M_{\theta}, f_{\theta})$. Under the null hypothesis of no change, the proposed statistic converges to a distribution obtained from a difference between two Brownian bridges; this statistic diverges to infinity under the epidemic alternative. These findings lead to a test which has correct size asymptotically and is consistent.

The rest of the paper is outlined as follows. Section 2 provides some assumptions and the definition of the Gaussian quasi-likelihood. Section 3 focuses on the construction of the test statistic and the asymptotic studies under the null and the epidemic alternative. Some numerical results for simulation and real data example are displayed in Section 4. Section 5 is devoted to the proofs of the main results.

2 Assumptions and QMLE

Throughout the sequel, we use the following notations:

- $\|x\| := \sqrt{\sum_{i=1}^p |x_i|^2}$, for any $x \in \mathbb{R}^p$;
- $\|x\| := \sqrt{\sum_{i=1}^p \sum_{j=1}^q |x_{i,j}|^2}$, for any matrix $x = (x_{i,j}) \in M_{p,q}(\mathbb{R})$; where $M_{p,q}(\mathbb{R})$ denotes the set of matrices of dimension $p \times q$ with coefficients in \mathbb{R} ;

- $\|f\|_{\Theta} := \sup_{\theta \in \Theta} (\|f(\theta)\|)$ for any function $f : \Theta \rightarrow M_{p,q}(\mathbb{R})$;
- $\|Y\|_r := \mathbb{E}(\|Y\|^r)^{1/r}$, where Y is a random vector with finite r -order moments;
- $T_{\ell,\ell'} = \{\ell, \ell+1, \dots, \ell'\}$ for any $(\ell, \ell') \in \mathbb{N}^2$ such as $\ell \leq \ell'$.

In the sequel, 0 denote the null vector of any space. For $\Psi_{\theta} = f_{\theta}$, M_{θ} , M_{θ}^2 and any compact set $\mathcal{K} \subseteq \Theta$, define

Assumption $\mathbf{A}_i(\Psi_{\theta}, \mathcal{K})$ ($i = 0, 1, 2$): Assume that $\|\partial^i \Psi_{\theta}(0)/\partial \theta^i\|_{\mathcal{K}} < \infty$ and there exists a sequence of non-negative real number $(\alpha_i^{(k)}(\Psi_{\theta}, \mathcal{K}))_{i \geq 1}$ such that $\sum_{k=1}^{\infty} \alpha_k^{(i)}(\Psi_{\theta}, \mathcal{K}) < \infty$ satisfying

$$\left\| \frac{\partial^i \Psi_{\theta}(x)}{\partial \theta^i} - \frac{\partial^i \Psi_{\theta}(y)}{\partial \theta^i} \right\|_{\mathcal{K}} \leq \sum_{k=1}^{\infty} \alpha_k^{(i)}(\Psi_{\theta}, \mathcal{K}) |x_k - y_k| \quad \text{for all } x, y \in \mathbb{R}^{\infty},$$

where x, y, x_k, y_k are respectively replaced by x^2, y^2, x_k^2, y_k^2 if $\Psi_{\theta} = h_{\theta} := M_{\theta}^2$.

For any $r \geq 1$, define

$$\Theta(r) = \left\{ \theta \in \mathbb{R}^d / \mathbf{A}_0(f_{\theta}, \{\theta\}) \text{ and } \mathbf{A}_0(M_{\theta}, \{\theta\}) \text{ hold with } \sum_{k=1}^{\infty} \left\{ \alpha_k^{(0)}(f_{\theta}, \{\theta\}) + \|\xi_0\|_r \alpha_k^{(0)}(M_{\theta}, \{\theta\}) \right\} < 1 \right\}$$

$$\bigcup \left\{ \theta \in \mathbb{R}^d / f_{\theta} = 0 \text{ and } \mathbf{A}_0(h_{\theta}, \{\theta\}) \text{ holds with } \|\xi_0\|_r^2 \sum_{k=1}^{\infty} \alpha_k^{(0)}(h_{\theta}, \{\theta\}) < 1 \right\}.$$

These Lipschitz-type conditions are notably useful when studying the existence of solutions of the class $\mathcal{AC}_{\mathcal{T}}(M_{\theta}, f_{\theta})$. If $\theta \in \Theta(r)$, then there exists a stationary and ergodic solution $X = (X_t)_{t \in \mathbb{Z}} \in \mathcal{AC}_{\mathbb{Z}}(M_{\theta}, f_{\theta})$ satisfying $\|X_0\|_r < \infty$ (see Bardet and Wintenberger (2009)).

Consider a trajectory (X_1, \dots, X_n) of a process $X = (X_t)_{t \in \mathbb{Z}}$. If $(X_1, \dots, X_n) \in \mathcal{AC}_{\{1, \dots, n\}}(M_{\theta}, f_{\theta})$, then for any segment $T \subset \{1, \dots, n\}$, the conditional Gaussian quasi-(log)likelihood computed on T is given by,

$$L(T, \theta) := -\frac{1}{2} \sum_{t \in T} q_t(\theta) \quad \text{with} \quad q_t(\theta) = \frac{(X_t - f_{\theta}^t)^2}{h_{\theta}^t} + \log(h_{\theta}^t) \quad (2.1)$$

where $f_{\theta}^t = f_{\theta}(X_{t-1}, X_{t-2}, \dots)$, $M_{\theta}^t = M_{\theta}(X_{t-1}, X_{t-2}, \dots)$ and $h_{\theta}^t = (M_{\theta}^t)^2$. In the sequel, we deal with an approximated quasi-(log)likelihood contrast given for any segment $T \subset \{1, \dots, n\}$ by,

$$\widehat{L}(T, \theta) := -\frac{1}{2} \sum_{t \in T} \widehat{q}_t(\theta) \quad \text{where} \quad \widehat{q}_t(\theta) := \frac{(X_t - \widehat{f}_{\theta}^t)^2}{\widehat{h}_{\theta}^t} + \log(\widehat{h}_{\theta}^t)$$

with $\widehat{f}_{\theta}^t = f_{\theta}(X_{t-1}, \dots, X_1, 0, 0, \dots)$, $\widehat{M}_{\theta}^t = M_{\theta}(X_{t-1}, \dots, X_1, 0, 0, \dots)$ and $\widehat{h}_{\theta}^t = (\widehat{M}_{\theta}^t)^2$; and define,

$$\widehat{\theta}_n(T) := \operatorname{argmax}_{\theta \in \Theta} (\widehat{L}(T, \theta)) \quad (2.2)$$

The following assumptions are needed to study the asymptotic behavior of the estimator defined in (2.2).

Assumption $\mathbf{D}(\Theta)$: $\exists \underline{h} > 0$ such that $\inf_{\theta \in \Theta} (h_{\theta}(x)) \geq \underline{h}$ for all $x \in \mathbb{R}^{\mathbb{N}}$.

Assumption $\mathbf{Id}(\Theta)$: For a process $(X_t)_{t \in \mathbb{Z}} \in \mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$ and for all $\theta \in \Theta$,

$$\left(f_{\theta^*}(X_0, X_{-1}, \dots) = f_{\theta}(X_0, X_{-1}, \dots) \text{ and } h_{\theta^*}(X_0, X_{-1}, \dots) = h_{\theta}(X_0, X_{-1}, \dots) \text{ a.s.} \right) \Rightarrow \theta^* = \theta.$$

Assumption Var(Θ): For a process $(X_t)_{t \in \mathbb{Z}} \in \mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$, one of the families $(\frac{\partial f_{\theta^*}}{\partial \theta^i}(X_0, X_{-1}, \dots))_{1 \leq i \leq d}$ or $(\frac{\partial h_{\theta^*}}{\partial \theta^i}(X_0, X_{-1}, \dots))_{1 \leq i \leq d}$ is *a.e.* linearly independent.

Under H_0 and the above assumptions, Bardet and Wintenberger (2009) established the consistency and the asymptotic normality of the estimator $\widehat{\theta}_n(T_{1,n})$ for the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$.

3 Test statistic and asymptotic results

Under H_0 , recall that (see Bardet and Wintenberger (2009)), for the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$, it holds that

$$\sqrt{n}(\widehat{\theta}(T_{1,n}) - \theta_0^*) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, F^{-1} G F^{-1}), \text{ with } G := \mathbb{E} \left[\frac{\partial q_0(\theta_0^*)}{\partial \theta} \frac{\partial q_0(\theta_0^*)'}{\partial \theta} \right] \text{ and } F := \mathbb{E} \left[\frac{\partial^2 q_0(\theta_0^*)}{\partial \theta \partial \theta'} \right], \quad (3.1)$$

where $'$ denotes the transpose. For any segment $T \subset \{1, \dots, n\}$, consider the following matrices,

$$\widehat{G}(T) := \frac{1}{\text{Card}(T)} \sum_{t \in T} \left(\frac{\partial \widehat{q}_t(\widehat{\theta}(T))}{\partial \theta} \right) \left(\frac{\partial \widehat{q}_t(\widehat{\theta}(T))}{\partial \theta} \right)' \text{ and } \widehat{F}(T) := \frac{1}{\text{Card}(T)} \sum_{t \in T} \frac{\partial^2 \widehat{q}_t(\widehat{\theta}(T))}{\partial \theta \partial \theta'}. \quad (3.2)$$

Under H_0 , $\widehat{G}(T_{1,n})$ and $\widehat{F}(T_{1,n})$ are consistent estimators of G and F , respectively.

In the sequel, we follow the idea of Diop and Kengne (2021). Let $(u_n)_{n \geq 1}$, $(v_n)_{n \geq 1}$ be two integer valued sequences such that: $(u_n, v_n) = o(n)$ and $u_n, v_n \xrightarrow[n \rightarrow \infty]{} +\infty$. For all $n \geq 1$, define the matrix

$$\widehat{\Sigma}(u_n) = \frac{1}{3} \left[\widehat{F}(T_{1,u_n}) \widehat{G}(T_{1,u_n})^{-1} \widehat{F}(T_{1,u_n}) + \widehat{F}(T_{u_n+1, n-u_n}) \widehat{G}(T_{u_n+1, n-u_n})^{-1} \widehat{F}(T_{u_n+1, n-u_n}) \right. \\ \left. + \widehat{F}(T_{n-u_n+1, n}) \widehat{G}(T_{n-u_n+1, n})^{-1} \widehat{F}(T_{n-u_n+1, n}) \right] \quad (3.3)$$

where $\widehat{G}(T_{1,u_n})^{-1}$, $\widehat{G}(T_{u_n+1, n-u_n})^{-1}$, $\widehat{G}(T_{n-u_n+1, n})^{-1}$ are replaced by 0 if these matrices are not invertible. Also, define the set

$$\mathcal{T}_n = \left\{ (k_1, k_2) \in ([v_n, n - v_n] \cap \mathbb{N})^2 \text{ with } k_2 - k_1 \geq v_n \right\}.$$

For all $(k_1, k_2) \in \mathcal{T}_n$, set

$$C_{n, k_1, k_2} = \frac{(k_2 - k_1)}{n^{3/2}} \left[(n - (k_2 - k_1)) \widehat{\theta}(T_{k_1+1, k_2}) - k_1 \widehat{\theta}(T_{1, k_1}) - (n - k_2) \widehat{\theta}(T_{k_2+1, n}) \right], \quad (3.4)$$

and consider the test statistic

$$\widehat{Q}_n = \max_{(k_1, k_2) \in \mathcal{T}_n} \widehat{Q}_{n, k_1, k_2} \text{ with } \widehat{Q}_{n, k_1, k_2} = C'_{n, k_1, k_2} \widehat{\Sigma}(u_n) C_{n, k_1, k_2}. \quad (3.5)$$

As pointed out by Diop and Kengne (2021), this test statistic coincides with those proposed by Rackauskas and Suquet (2004) (statistic $UI(n, \rho)$), Jarusková and Piterbarg (2011) (statistic T_1^2), Bucchia (2014) (statistic $T_n(\alpha, \beta)$) or Aston and Kirch (2012) (statistic $T_n^{B^2}$) for the particular case of epidemic change-point detection in the mean. In this sense, the test considered here can be seen as a generalization of these procedures.

The following theorem provides the asymptotic behavior of the statistic \widehat{Q}_n under the null hypothesis. In the condition (3.6) in this theorem, we make the convention that if $\mathbf{A}_i(M_\theta, \Theta)$ holds, then $\alpha_k^{(i)}(h_\theta, \Theta) = 0$ for all $k \in \mathbb{N}$ and if $\mathbf{A}_i(h_\theta, \Theta)$ holds, then $\alpha_k^{(i)}(M_\theta, \Theta) = 0$ for all $k \in \mathbb{N}$.

Theorem 3.1 Under H_0 with $\theta_0^* \in \overset{\circ}{\Theta} \cap \Theta(4)$, assume that $\mathbf{D}(\Theta)$, $\mathbf{Id}(\Theta)$, $\mathbf{Var}(\Theta)$ (for the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta_0^*}, f_{\theta_0^*})$), $\mathbf{A}_i(f_{\theta}, \Theta)$, $\mathbf{A}_i(M_{\theta}, \Theta)$ (or $\mathbf{A}_i(h_{\theta}, \Theta)$) hold with

$$\alpha_k^{(i)}(f_{\theta}, \Theta) + \alpha_k^{(i)}(M_{\theta}, \Theta) + \alpha_k^{(i)}(h_{\theta}, \Theta) = O(k^{-\gamma}) \text{ for } i = 0, 1, 2 \text{ and some } \gamma > 3/2. \quad (3.6)$$

Then,

$$\widehat{Q}_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sup_{0 \leq \tau_1 < \tau_2 \leq 1} \|W_d(\tau_1) - W_d(\tau_2)\|^2, \quad (3.7)$$

where W_d is a d -dimensional Brownian bridge.

For any $\alpha \in (0, 1)$, denote $c_{d,\alpha}$ the $(1 - \alpha)$ -quantile of the distribution of $\sup_{0 \leq \tau_1 < \tau_2 \leq 1} \|W_d(\tau_1) - W_d(\tau_2)\|^2$. Therefore, at a nominal level $\alpha \in (0, 1)$, the critical region of the test is $(\widehat{Q}_n > c_{d,\alpha})$; which leads to a procedure with correct size asymptotically. Table 1 of Diop and Kengne (2021) provides the values of $c_{d,\alpha}$ for $\alpha = 0.01, 0.05, 0.10$ and $d = 1, \dots, 5$.

For asymptotic under the epidemic alternative, the following additional condition is needed.

Assumption B: There exists $(\tau_1^*, \tau_2^*) \in (0, 1)^2$ with $\tau_1^* < \tau_2^*$, such that $(t_1^*, t_2^*) = ([n\tau_1^*], [n\tau_2^*])$ (where $[\cdot]$ is the integer part).

We have the following result.

Theorem 3.2 Under H_1 with $\theta_1^*, \theta_2^* \in \overset{\circ}{\Theta} \cap \Theta(4)$, assume that $\mathbf{D}(\Theta)$, $\mathbf{Id}(\Theta)$, $\mathbf{Var}(\Theta)$ (for the classes $\mathcal{AC}_{\mathbb{Z}}(M_{\theta_1^*}, f_{\theta_1^*})$ and $\mathcal{AC}_{\mathbb{Z}}(M_{\theta_2^*}, f_{\theta_2^*})$), $\mathbf{A}_i(f_{\theta}, \Theta)$, $\mathbf{A}_i(M_{\theta}, \Theta)$ (or $\mathbf{A}_i(h_{\theta}, \Theta)$) and (3.6) hold. Then,

$$\widehat{Q}_n \xrightarrow[n \rightarrow \infty]{\mathcal{P}} + \infty. \quad (3.8)$$

This theorem shows that the proposed procedure is consistent. An estimator of the change-points $\underline{t}^* = (t_1^*, t_2^*)$ under the epidemic alternative is given by

$$\widehat{\underline{t}}_n = \underset{(k_1, k_2) \in \mathcal{T}_n}{\operatorname{argmax}} C'_{n, k_1, k_2} \widehat{\Sigma}(u_n) C_{n, k_1, k_2}.$$

Remark 3.3 The nonlinear framework used in the present paper imposes a stronger condition on the Lipschitz-type coefficients when the model is linear, but the results of Theorem 3.1 and 3.2 apply to a wider class of models. For example, consider an ARMA(p, q) model with

$$P_{\theta_0^*}(L)X_t = Q_{\theta_0^*}(L)\xi_t, \text{ for all } t \in \mathbb{Z}, \quad (3.9)$$

where $P_{\theta}(z) = 1 - \alpha_1 z - \dots - \alpha_p z^p$, $Q_{\theta_0^*}(z) = 1 + \beta_1 z + \dots + \beta_q z^q$, $\theta = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q)$ and θ_0^* the true parameter. We know that (see Brockwell and Davis (1991)) if the polynomials $P_{\theta_0^*}$ and $Q_{\theta_0^*}$ are coprime with the roots outside the unit circle, then there exists a stationary solution of (3.9) which is causal and invertible. For such example, a weaker condition with Θ a compact subset of $\{\theta = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \in \mathbb{R}^{p+q}, P_{\theta}$ and Q_{θ} are coprime with the roots outside the unit circle\} is enough. That is, in the case of the model (3.9), the condition " $\theta_0^* \in \overset{\circ}{\Theta} \cap \Theta(4)$ " can be replaced by " $\|\xi_0\|_4 < \infty, \Theta = \{\theta = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \in \mathbb{R}^{p+q}, P_{\theta}$ and Q_{θ} are coprime with the roots outside the unit circle\}, $\theta_0^* \in \overset{\circ}{\Theta}$ " and the result of Theorem 3.1 holds. See other details in Subsection 4.1 for the case of an ARMA(1,1). This remark is also applied to Theorem 3.2.

4 Some numerical results

This section presents some results of a simulation study and a real data example. For a sample size n , the statistic \widehat{Q}_n is computed with $u_n = [(\log(n))^{5/2}]$ and $v_n = [(\log(n))^2]$ (see also Remark 1 in Kengne (2012)). The empirical levels and powers are obtained after 200 replications at the nominal level $\alpha = 0.05$.

4.1 Simulation study

We consider the following models:

(i) ARMA(1,1) processes:

$$X_t = \alpha_0^* + \alpha_1^* X_{t-1} + \xi_t + \beta_1^* \xi_{t-1} \quad \text{for all } t \in \mathbb{Z}. \quad (4.1)$$

The parameter of the model is $\theta^* = (\alpha_0^*, \alpha_1^*, \beta_1^*) \in \Theta$, where Θ is a compact subset of \mathbb{R}^3 such as: for all $\theta = (\alpha_0, \alpha_1, \beta_1) \in \Theta$, $|\alpha_1| < 1$ and $|\beta_1| < 1$. Since we can write for all $t \in \mathbb{Z}$,

$$X_t = \frac{\alpha_0^*}{1 + \beta_1^*} + (\alpha_1^* + \beta_1^*)(X_{t-1} + \sum_{k \geq 2} (-\beta_1^*)^{k-1} X_{t-k}) + \xi_t,$$

the model (4.1) belongs to the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$ with $f_{\theta}(x_1, \dots) = \frac{\alpha_0}{1 + \beta_1} + (\alpha_1 + \beta_1)(x_1 + \sum_{k \geq 2} (-\beta_1)^{k-1} x_k)$ and $M_{\theta} \equiv 1$ for all $\theta = (\alpha_0, \alpha_1, \beta_1) \in \Theta$. In accordance with Remark 3.3, the conditions $|\alpha_1^*| < 1$ and $|\beta_1^*| < 1$, with are weaker than the Lipschitz-type assumptions $A_0(\Psi_{\theta}, \Theta)$ (which would require that the parameter space Θ be a subset of $\{(\alpha_0, \alpha_1, \beta_1) \in \mathbb{R}^3 \text{ with } 0 < |\alpha_1 + \beta_1| / (1 - |\beta_1|) < 1\}$) suffices to get a stationary solution of (4.1) which is causal and invertible. Moreover, if ξ_0 is a non-degenerate random variable, then the assumptions $\text{Id}(\Theta)$ and $\text{Var}(\Theta)$ hold; and for any $r \geq 1$ such that $\mathbb{E}\xi_0^r < \infty$, $\Theta(r) = \Theta$. In the sequel, we deal with an ARMA(1,1) with a non zero mean ($\theta^* = (\alpha_0^*, \alpha_1^*, \beta_1^*)$), an ARMA(1,1) with mean zero ($\theta^* = (\alpha_1^*, \beta_1^*)$) and an AR(1) with a non zero mean ($\theta^* = (\alpha_0^*, \alpha_1^*)$).

We consider the change-point test with an epidemic alternative where the parameter of the model is θ_0^* under H_0 , and θ_1^*, θ_2^* under H_1 . Firstly, two trajectories with $n = 500$ of an ARMA(1,1) (with mean zero) are generated: a trajectory under H_0 with $\theta_0^* = (-0.4, -0.25)$ and a trajectory under H_1 with breaks at $(t_1^*, t_2^*) = (150, 350)$, $\theta_1^* = (-0.4, -0.25)$, $\theta_2^* = (-0.4, 0.1)$. Figure 1 displays the statistic $\widehat{Q}_{n, k_1, k_2}$. One can see that, for the scenario without change, the values of this statistic are below the horizontal triangle which represents the limit of the critical region (see Figure 1(a)). Under the epidemic alternative, $\max_{(k_1, k_2) \in \mathcal{T}_n} \widehat{Q}_{n, k_1, k_2}$ is greater than the critical value of the test and is reached around the points where the changes occur (see the dotted lines in Figure 1(b)).

(ii) GARCH(1,1) processes:

$$X_t = \sigma_t \xi_t \quad \text{with} \quad \sigma_t^2 = \alpha_0^* + \alpha_1^* X_{t-1}^2 + \beta_1^* \sigma_{t-1}^2, \quad (4.2)$$

the parameter $\theta^* = (\alpha_0^*, \alpha_1^*, \beta_1^*) \in \Theta$, a compact subset of $]0, \infty[\times]0, \infty[^2$ such as: for all $\theta = (\alpha_0, \alpha_1, \beta_1) \in \Theta$, $\alpha_1 + \beta_1 < 1$. For all $t \in \mathbb{Z}$, we get

$$X_t = \xi_t \sqrt{\alpha_0^* / (1 - \beta_1^*) + \alpha_1^* \sum_{k \geq 1} (\beta_1^*)^{k-1} X_{t-k}^2}.$$

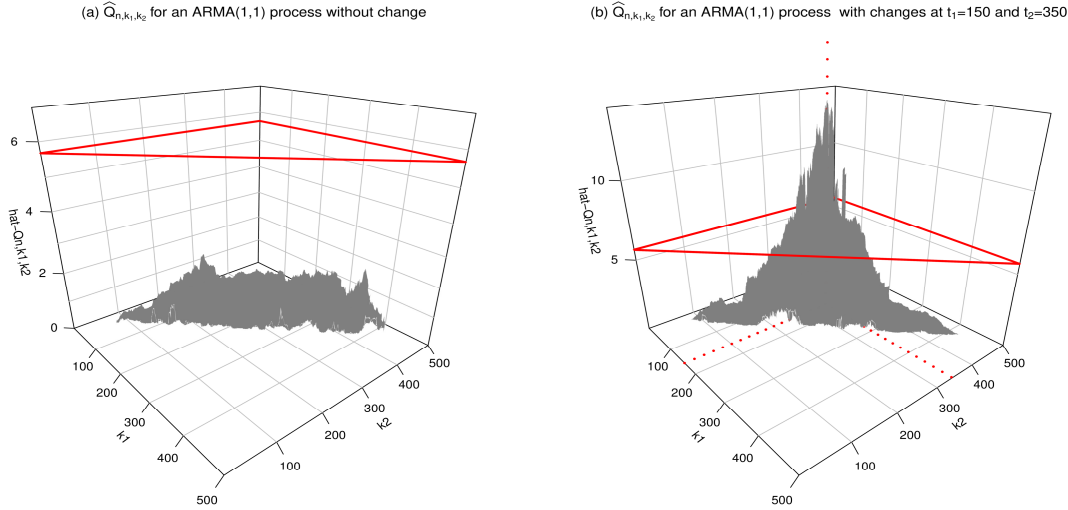


Figure 1: The statistics \widehat{Q}_{n,k_1,k_2} for trajectories of an ARMA(1,1) with mean zero: (a) a scenario without change, where the true parameter $\theta_0^* = (-0.4, -0.25)$ is constant; (b) a scenario under the epidemic alternative where the parameter $\theta_1^* = (-0.4, -0.25)$ changes to $\theta_2^* = (-0.4, 0.1)$ at $t_1 = 150$ and reverts back to θ_1^* at $t_2 = 350$.

Therefore, the model (4.2) belongs to the class $\mathcal{AC}_{\mathbb{Z}}(M_{\theta^*}, f_{\theta^*})$ with $M_{\theta}(x_1, \dots) = \sqrt{\frac{\alpha_0}{1-\beta_1} + \alpha_1 \sum_{k \geq 1} \beta_1^{k-1} x_k^2}$ and $f_{\theta} \equiv 0$ for all $\theta = (\alpha_0, \alpha_1, \beta_1) \in \Theta$. The Lipschitz-type conditions $A_i(\Psi_{\theta}, \Theta)$ ($i = 0, 1, 2$) hold automatically and $D(\Theta)$ is satisfied with $\underline{h} = \inf_{\theta=(\alpha_0, \alpha_1, \beta_1) \in \Theta} \sqrt{\frac{\alpha_0}{1-\beta_1}} > 0$. In addition, if ξ_0 is a non-degenerate random variable, then the assumptions $\text{Id}(\Theta)$ and $\text{Var}(\Theta)$ hold; and for any $r \geq 1$ such that $\mathbb{E}\xi_0^r < \infty$, $\Theta(r) = \{\theta = (\alpha_0, \alpha_1, \beta_1) \in \Theta; \|\xi_0\|_r^2(\alpha_1 + \beta_1) < 1\}$. In the sequel, we consider a GARCH(1,1) ($\theta^* = (\alpha_0^*, \alpha_1^*, \beta_1^*)$) and an ARCH(1) ($\theta^* = (\alpha_0^*, \alpha_1^*)$).

For both the ARMA and GARCH model, we carry out the change-point test with an epidemic alternative where the parameter of the model is θ_0^* under H_0 , and θ_1^*, θ_2^* under H_1 with change-points at $(t_1^*, t_2^*) = (0.3n, 0.7n)$ for sample size $n = 500, 1000$. The empirical levels and powers are displayed in Table 1. The AR(1) example is related to the real data application, see Subsection 4.2. The results in this table show that, the empirical level approaching the nominal one when n increases and the empirical power increases with n and is overall close to one when $n = 1000$. These findings are consistent with the asymptotic results of Theorems 3.1 and 3.2.

4.2 Real data example

We consider the daily concentrations of carbon monoxide in the Vitória metropolitan area. These daily levels are obtained from the State Environment and Water Resources Institute, where the data were collected at eight monitoring stations. There are 455 available observations that represent the average concentrations from September 11, 2009 through December 09, 2010 (see Figure 2(a)). The data are a part of a large dataset (available at <https://rss.onlinelibrary.wiley.com/pb-assets/hub-assets/rss/Datasets/RSSC%2067.2/C1239deSouza-1531120585220.zip>) which were analyzed by Souza *et al.* (2018) to quantify the association between respiratory disease and air pollution concentrations.

To test the presence of an epidemic change in this series, we apply our detection procedure with the

Table 1: Empirical sizes and powers for the epidemic change-point detection in the models (4.1) and (4.2).

		$n = 500$	$n = 1000$	
AR(1)	Empirical levels:	$\theta_0^* = (813, 0.3)$	0.025	0.035
	Empirical powers:	$\theta_1^* = (813, 0.3);$	$\theta_2^* = (933, 0.24);$	1.000
ARMA(1,1) with zero mean	Empirical levels:	$\theta_0^* = (-0.4, -0.25)$	0.045	0.050
	Empirical powers:	$\theta_1^* = (-0.4, -0.25);$	$\theta_2^* = (-0.4, 0.1);$	0.760
ARMA(1,1) with non zero mean	Empirical levels:	$\theta_0^* = (1, 0.15, 0.2)$	0.065	0.060
	Empirical powers:	$\theta_1^* = (1, 0.15, 0.2);$	$\theta_2^* = (1, 0.5, 0.2);$	0.925
ARCH(1)	Empirical levels:	$\theta_0^* = (0.6, 0.4)$	0.035	0.045
	Empirical powers:	$\theta_1^* = (0.6, 0.4);$	$\theta_2^* = (0.2, 0.4);$	0.910
GARCH(1,1)	Empirical levels:	$\theta_0^* = (0.15, 0.3, 0.25)$	0.080	0.060
	Empirical powers:	$\theta_1^* = (0.15, 0.3, 0.25);$	$\theta_2^* = (0.15, 0.3, 0.55);$	0.730

ARMA(p, q) model. We have applied the test with several values of p and q ; and the results after change-point detection show a preference (in the sense of AIC and BIC) for an AR(1). Figure 2(b) displays the values of \widehat{Q}_{n,k_1,k_2} for all $(k_1, k_2) \in \mathcal{T}_n$. The critical value on nominal level $\alpha = 5\%$ is $c_{d,\alpha} = 5.69$ and the resulting test statistic is $\widehat{Q}_n = 6.61$; which implies that the null hypothesis H_0 is rejected (i.e., an epidemic change-point is detected). The vector of the break-points estimated is $\widehat{\underline{t}}_n = (143, 330)$; i.e, the point where the peak in the graph is reached (see Figure 2(b)). The locations of the changes correspond to the dates January 31 and August 06, 2010. This corresponds to the period where the winds are weaker and the austral winter; these meteorological factors are noticeable to increase the concentration of the carbon monoxide. The estimated model on each regime is given by:

$$X_t = \begin{cases} 813.39 + 0.309X_{t-1} + \xi_t & \text{for } t \leq 143, \\ \begin{matrix} (19.72) & (0.08) \end{matrix} \\ 933.27 + 0.240X_{t-1} + \xi_t & \text{for } 144 \leq t \leq 330, \\ \begin{matrix} (22.43) & (0.07) \end{matrix} \\ 822.83 + 0.293X_{t-1} + \xi_t & \text{for } t \geq 331, \\ \begin{matrix} (22.82) & (0.09) \end{matrix} \end{cases} \quad (4.3)$$

where in parentheses are the standard errors of the estimators. From (4.3), one remark that, the parameter of the first regime is close to that of the third regime; which strengthens the hypothesis of the existence of an epidemic change-point.

5 Proofs of the main results

To simplify the expressions, in this section, we will use the conditional Gaussian quasi-log-likelihood up to multiplication by 1/2, given by $L(T, \theta) := -\sum_{t \in T} q_t(\theta)$ and $\widehat{L}(T, \theta) := -\sum_{t \in T} \widehat{q}_t(\theta)$.

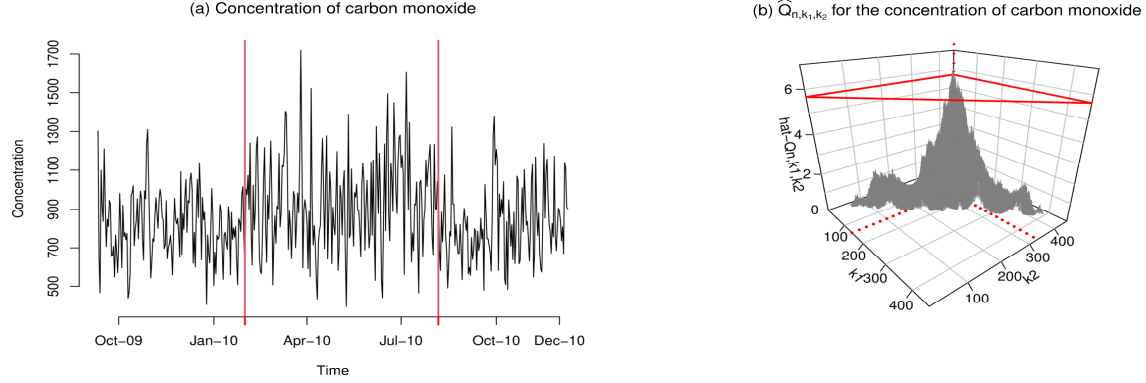


Figure 2: Plot of \widehat{Q}_{n,k_1,k_2} for the epidemic change-point detection applied to the daily averages concentrations for carbon monoxide in the Vitória metropolitan area, between September 11, 2009 and December 09, 2010 with an AR(1) process. The vertical lines in (a) are the estimated breakpoints. The horizontal triangle in (b) represents the limit of the critical region of the test, whereas the dotted lines show the point where the maximum of \widehat{Q}_{n,k_1,k_2} is reached.

5.1 Proof of Theorem 3.1

Let $\Sigma := F^{-1}GF^{-1}$, where F and G are the matrices defined in (3.1). Define the statistic

$$Q_n = \max_{(k_1,k_2) \in \mathcal{T}_n} Q_{n,k_1,k_2} \quad \text{with} \quad Q_{n,k_1,k_2} = C'_{n,k_1,k_2} \Sigma C_{n,k_1,k_2}.$$

Consider the following lemma; we can go along similar lines as in the proof of Lemma 6.3 in [12] to show the part (i). The part (ii) is established in [6].

Lemma 5.1 *Suppose that the assumptions of Theorem 3.1 hold. Then,*

- (i) $\max_{(k_1,k_2) \in \mathcal{T}_n} |\widehat{Q}_{n,k_1,k_2} - Q_{n,k_1,k_2}| = o_P(1)$;
- (ii) $(\frac{\partial}{\partial \theta} q_t(\theta_0^*), \mathcal{F}_t)_{t \in \mathbb{Z}}$ is a stationary ergodic, square integrable martingale difference sequence with covariance matrix G .

Let two integers $k, k' \in [1, n]$, $\bar{\theta} \in \Theta$ and $i \in \{1, 2, \dots, d\}$. Applying the mean value theorem to $\theta \mapsto \frac{\partial}{\partial \theta_i} L(T_{k,k'}, \theta)$, there exists $\theta_{n,i}$ between $\bar{\theta}$ and θ_0^* such that

$$\frac{\partial}{\partial \theta_i} L(T_{k,k'}, \bar{\theta}) = \frac{\partial}{\partial \theta_i} L(T_{k,k'}, \theta_0^*) + \frac{\partial^2}{\partial \theta \partial \theta_i} L(T_{k,k'}, \theta_{n,i})(\bar{\theta} - \theta_0^*);$$

i.e.,

$$(k' - k + 1)F_n(T_{k,k'}, \bar{\theta})(\bar{\theta} - \theta_0^*) = \frac{\partial}{\partial \theta} L(T_{k,k'}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{k,k'}, \bar{\theta}) \quad (5.1)$$

with

$$F_n(T_{k,k'}, \bar{\theta}) = -\frac{1}{(k' - k + 1)} \frac{\partial^2}{\partial \theta \partial \theta_i} L(T_{k,k'}, \theta_{n,i})_{1 \leq i \leq d}. \quad (5.2)$$

The following lemma will be useful in the sequel.

Lemma 5.2 *Suppose that the assumptions of Theorem 3.1 hold. If $(j_n)_{n \geq 1}$ and $(k_n)_{n \geq 1}$ are two integer valued sequences such that $j_n < k_n$, $k_n \xrightarrow[n \rightarrow \infty]{} \infty$ and $k_n - j_n \xrightarrow[n \rightarrow \infty]{} \infty$, then*

- (i) $\frac{1}{\sqrt{k_n - j_n}} \left\| \frac{\partial}{\partial \theta} \widehat{L}(T_{j_n, k_n}, \theta) - \frac{\partial}{\partial \theta} L(T_{j_n, k_n}, \theta) \right\|_{\Theta} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0$;
- (ii) $F_n(T_{j_n, k_n}, \widehat{\theta}(T_{j_n, k_n})) \xrightarrow[n \rightarrow \infty]{a.s.} F$.

Proof.

(i) See the proof of Theorem 2 in [6].

(ii) Applying (5.2) with $\bar{\theta} = \widehat{\theta}(T_{j_n, k_n})$, we obtain

$$F_n(T_{j_n, k_n}, \widehat{\theta}(T_{j_n, k_n})) = \left(\frac{1}{k_n - j_n + 1} \frac{\partial^2}{\partial \theta \partial \theta_i} L(T_{j_n, k_n}, \theta_{n,i}) \right)_{1 \leq i \leq d} = \frac{1}{k_n - j_n + 1} \left(\sum_{t \in T_{j_n, k_n}} \frac{\partial^2 q_t(\theta_{n,i})}{\partial \theta \partial \theta_i} \right)_{1 \leq i \leq d},$$

where $\theta_{n,i}$ belongs between $\widehat{\theta}(T_{j_n, k_n})$ and θ_0^* . Since $\widehat{\theta}(T_{j_n, k_n}) \xrightarrow[n \rightarrow \infty]{a.s.} \theta_0^*$, $\theta_{n,i} \xrightarrow[n \rightarrow \infty]{a.s.} \theta_0^*$ for any $i = 1, \dots, d$ (from the consistency of the QMLE) and that $F = \mathbb{E} \left[\frac{\partial^2 q_0(\theta_0^*)}{\partial \theta \partial \theta'} \right]$ exists (see [6]), by the uniform strong law of large numbers, for any $i = 1, \dots, d$, we get

$$\begin{aligned} & \left\| \frac{1}{k_n - j_n + 1} \sum_{t \in T_{j_n, k_n}} \frac{\partial^2 q_t(\theta_{n,i})}{\partial \theta \partial \theta_i} - \mathbb{E} \left[\frac{\partial^2 q_0(\theta_0^*)}{\partial \theta \partial \theta_i} \right] \right\| \\ & \leq \left\| \frac{1}{k_n - j_n + 1} \sum_{t \in T_{j_n, k_n}} \frac{\partial^2 q_t(\theta_{n,i})}{\partial \theta \partial \theta_i} - \mathbb{E} \left[\frac{\partial^2 q_0(\theta_{n,i})}{\partial \theta \partial \theta_i} \right] \right\| + \left\| \mathbb{E} \left[\frac{\partial^2 q_0(\theta_{n,i})}{\partial \theta \partial \theta_i} \right] - \mathbb{E} \left[\frac{\partial^2 q_0(\theta_0^*)}{\partial \theta \partial \theta_i} \right] \right\| \\ & \leq \left\| \frac{1}{k_n - j_n + 1} \sum_{t \in T_{j_n, k_n}} \frac{\partial^2 q_t(\theta)}{\partial \theta \partial \theta_i} - \mathbb{E} \left[\frac{\partial^2 q_0(\theta)}{\partial \theta \partial \theta_i} \right] \right\|_{\Theta} + o(1) = o(1) + o(1) = o(1). \end{aligned}$$

This establishes the lemma. ■

Now, let us show that

$$Q_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sup_{0 \leq \tau_1 < \tau_2 \leq 1} \|W_d(\tau_1) - W_d(\tau_2)\|^2. \quad (5.3)$$

Remark that, Q_{n, k_1, k_2} can be rewritten as

$$Q_{n, k_1, k_2} = \|G^{-1/2} F \cdot C_{n, k_1, k_2}\|^2$$

where C_{n, k_1, k_2} is defined in (3.4). Let $(k_1, k_2) \in \mathcal{T}_n$. Applying (5.1) with $\bar{\theta} = \widehat{\theta}(T_{k_1+1, k_2})$ and $T_{k, k'} = T_{k_1+1, k_2}$, we obtain

$$F_n(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0^*) = \frac{1}{k_2 - k_1} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right). \quad (5.4)$$

With $\bar{\theta} = \widehat{\theta}(T_{k_2+1, n})$ and $T_{k, k'} = T_{k_2+1, n}$, (5.1) implies

$$F_n(T_{k_2+1, n}, \widehat{\theta}(T_{k_2+1, n})) \cdot (\widehat{\theta}(T_{k_2+1, n}) - \theta_0^*) = \frac{1}{n - k_2} \left(\frac{\partial}{\partial \theta} L(T_{k_2+1, n}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{k_2+1, n}, \widehat{\theta}(T_{k_2+1, n})) \right). \quad (5.5)$$

Moreover, as $n \rightarrow +\infty$, from the asymptotic normality of the QMLE (see [6]) and Lemma 5.2(ii), we have

$$\left\{ \begin{array}{l} \|\sqrt{k_1} (\widehat{\theta}(T_{1, k_1}) - \theta_0^*)\| = O_P(1), \quad \|\sqrt{k_2 - k_1} (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0^*)\| = O_P(1); \\ \|\sqrt{n - k_2} (\widehat{\theta}(T_{k_2+1, n}) - \theta_0^*)\| = O_P(1); \\ \|F_n(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) - F\| = o(1) \quad \text{and} \quad \|F_n(T_{k_2+1, n}, \widehat{\theta}(T_{k_2+1, n})) - F\| = o(1). \end{array} \right. \quad (5.6)$$

Then, for n large enough, it holds from (5.4) that

$$\begin{aligned}
\sqrt{k_2 - k_1} F \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0^*) &= \frac{1}{\sqrt{k_2 - k_1}} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) \\
&\quad - \sqrt{k_2 - k_1} \left((F_n(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) - F) (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0) \right) \\
&= \frac{1}{\sqrt{k_2 - k_1}} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) + o_P(1) \\
&= \frac{1}{\sqrt{k_2 - k_1}} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} \widehat{L}(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) + o_P(1) \\
&\quad + \frac{1}{\sqrt{k_2 - k_1}} \left(\frac{\partial}{\partial \theta} \widehat{L}(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) - \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) \\
&= \frac{1}{\sqrt{k_2 - k_1}} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} \widehat{L}(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) + o_P(1),
\end{aligned}$$

where the last equality is obtained from Lemma 5.2(i). This is equivalent to

$$F \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0^*) = \frac{1}{k_2 - k_1} \left(\frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} \widehat{L}(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) \right) + o_P\left(\frac{1}{\sqrt{k_2 - k_1}}\right). \quad (5.7)$$

For n large enough, $\widehat{\theta}(T_{k_1+1, k_2})$ is an interior point of Θ and we have $\frac{\partial}{\partial \theta} \widehat{L}(T_{k_1+1, k_2}, \widehat{\theta}(T_{k_1+1, k_2})) = 0$. Thus, from (5.7), we obtain

$$F \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \theta_0^*) = \frac{1}{k_2 - k_1} \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) + o_P\left(\frac{1}{\sqrt{k_2 - k_1}}\right). \quad (5.8)$$

Similarly, using (5.5), we also obtain

$$F \cdot (\widehat{\theta}(T_{k_2+1, n}) - \theta_0^*) = \frac{1}{n - k_2} \frac{\partial}{\partial \theta} L(T_{k_2+1, n}, \theta_0^*) + o_P\left(\frac{1}{\sqrt{n - k_2}}\right). \quad (5.9)$$

The subtraction of (5.8) and (5.9) gives

$$\begin{aligned}
F \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \widehat{\theta}(T_{k_2+1, n})) &= \frac{1}{k_2 - k_1} \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - \frac{1}{n - k_2} \frac{\partial}{\partial \theta} L(T_{k_2+1, n}, \theta_0^*) \\
&\quad + o_P\left(\frac{1}{\sqrt{k_2 - k_1}} + \frac{1}{\sqrt{n - k_2}}\right);
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{(k_2 - k_1)(n - k_2)}{n^{3/2}} F \cdot (\widehat{\theta}(T_{k_1+1, k_2}) - \widehat{\theta}(T_{k_2+1, n})) &= \\
&= \frac{1}{n^{3/2}} \left[(n - k_2) \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) - (k_2 - k_1) \frac{\partial}{\partial \theta} L(T_{k_2+1, n}, \theta_0^*) \right] + o_P(1). \quad (5.10)
\end{aligned}$$

By going along similar lines, we can also show that

$$\begin{aligned}
\frac{k_1(k_2 - k_1)}{n^{3/2}} F \cdot (\widehat{\theta}(T_{1, k_1}) - \widehat{\theta}(T_{k_1+1, k_2})) &= \\
&= \frac{1}{n^{3/2}} \left[(k_2 - k_1) \frac{\partial}{\partial \theta} L(T_{1, k_1}, \theta_0^*) - k_1 \frac{\partial}{\partial \theta} L(T_{k_1+1, k_2}, \theta_0^*) \right] + o_P(1). \quad (5.11)
\end{aligned}$$

Combining (5.10) and (5.11), we get

$$\begin{aligned}
F \cdot C_{n,k_1,k_2} &= \frac{1}{n^{3/2}} \left[(n - (k_2 - k_1)) \frac{\partial}{\partial \theta} L(T_{k_1+1,k_2}, \theta_0^*) - (k_2 - k_1) \left(\frac{\partial}{\partial \theta} L(T_{k_2+1,n}, \theta_0^*) + \frac{\partial}{\partial \theta} L(T_{1,k_1}, \theta_0^*) \right) \right] + o_P(1). \\
&= \frac{1}{\sqrt{n}} \left[\frac{\partial}{\partial \theta} L(T_{k_1+1,k_2}, \theta_0^*) - \frac{(k_2 - k_1)}{n} L(T_{1,n}) \right] + o_P(1) \\
&= \frac{1}{\sqrt{n}} \left[\frac{\partial}{\partial \theta} L(T_{1,k_2}, \theta_0^*) - \frac{\partial}{\partial \theta} L(T_{1,k_1}, \theta_0^*) - \frac{(k_2 - k_1)}{n} L(T_{1,n}) \right] + o_P(1) \\
&= \frac{1}{\sqrt{n}} \left[\left(\frac{\partial}{\partial \theta} L(T_{1,k_2}, \theta_0^*) - \frac{k_2}{n} L(T_{1,n}) \right) - \left(\frac{\partial}{\partial \theta} L(T_{1,k_1}, \theta_0^*) - \frac{k_1}{n} L(T_{1,n}) \right) \right] + o_P(1);
\end{aligned}$$

i.e.,

$$G^{-1/2} F \cdot C_{n,k_1,k_2} = \frac{G^{-1/2}}{\sqrt{n}} \left[\left(\frac{\partial}{\partial \theta} L(T_{1,k_2}, \theta_0^*) - \frac{k_2}{n} L(T_{1,n}) \right) - \left(\frac{\partial}{\partial \theta} L(T_{1,k_1}, \theta_0^*) - \frac{k_1}{n} L(T_{1,n}) \right) \right] + o_P(1). \quad (5.12)$$

According to Lemma 5.1(ii), applying the central limit theorem for the sequence $(\frac{\partial}{\partial \theta} q_t(\theta_0^*), \mathcal{F}_t)_{t \in \mathbb{Z}}$, we obtain

$$\begin{aligned}
\frac{1}{\sqrt{n}} \left(\frac{\partial}{\partial \theta} L(T_{1,[n\tau_1]}, \theta_0^*) - \frac{[n\tau_1]}{n} \frac{\partial}{\partial \theta} L(T_{1,n}, \theta_0^*) \right) &= \frac{1}{\sqrt{n}} \left(\sum_{t=1}^{[n\tau_1]} \frac{\partial}{\partial \theta} q_t(\theta_0^*) - \frac{[n\tau_1]}{n} \sum_{t=1}^n \frac{\partial}{\partial \theta} q_t(\theta_0^*) \right) \\
&\xrightarrow[n \rightarrow \infty]{\mathcal{D}} B_G(\tau_1) - \tau_1 B_G(1),
\end{aligned}$$

where B_G is a Gaussian process with covariance matrix $\min(s, t)G$. Hence,

$$\frac{G^{-1/2}}{\sqrt{n}} \left(\frac{\partial}{\partial \theta} L(T_{1,[n\tau_1]}, \theta_0^*) - \frac{[n\tau_1]}{n} \frac{\partial}{\partial \theta} L(T_{1,n}, \theta_0^*) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} B_d(\tau_1) - \tau_1 B_d(1) = W_d(\tau_1)$$

in $D([0, 1])$, where B_d is a d -dimensional standard motion, and W_d is a d -dimensional Brownian bridge.

Similarly, we get

$$\frac{G^{-1/2}}{\sqrt{n}} \left(\frac{\partial}{\partial \theta} L(T_{1,[n\tau_2]}, \theta_0^*) - \frac{[n\tau_2]}{n} \frac{\partial}{\partial \theta} L(T_{1,n}, \theta_0^*) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} B_d(\tau_2) - \tau_2 B_d(1) = W_d(\tau_2).$$

Thus, as $n \rightarrow \infty$, it comes from (5.12) that

$$Q_{n,[n\tau_1],[n\tau_2]} = \left\| G^{-1/2} F \cdot C_{n,[n\tau_1],[n\tau_2]} \right\|^2 \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \|W_d(\tau_1) - W_d(\tau_2)\|^2 \text{ in } D([0, 1]).$$

Hence, for n large enough, we have

$$Q_n = \max_{\substack{v_n \leq k_1 < k_2 \leq n - v_n \\ k_1 < k_2 - v_n}} Q_{n,k_1,k_2} = \sup_{\substack{v_n \leq \tau_1 < \tau_2 \leq 1 - v_n \\ \tau_1 < \tau_2 - v_n}} Q_{n,[n\tau_1],[n\tau_2]} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sup_{0 \leq \tau_1 < \tau_2 \leq 1} \|W_d(\tau_1) - W_d(\tau_2)\|^2;$$

which establishes (5.3). Using Lemma 5.1(i), we can conclude the proof of the theorem. \blacksquare

5.2 Proof of Theorem 3.2

Under the epidemic alternative, (Y_1, \dots, Y_n) is a trajectory of $Y = \{Y_t, t \in \mathbb{Z}\}$ which belongs to

$\mathcal{AC}_{\{\dots, -1, 0, 1, \dots, t_1^*\}}(M_{\theta_1^*}, f_{\theta_1^*}) \cap \mathcal{AC}_{\{t_1^*+1, \dots, t_2^*\}}(M_{\theta_2^*}, f_{\theta_2^*}) \cap \mathcal{AC}_{\{t_2^*+1, \dots\}}(M_{\theta_1^*}, f_{\theta_1^*})$, $(t_1^*, t_2^*) = ([\tau_1^* n], [\tau_2^* n])$ (with $0 < \tau_1^* < \tau_2^* < 1$) and $\theta_1^* \neq \theta_2^*$.

For any $n \in \mathbb{N}$, we have

$$\widehat{Q}_n = \max_{(k_1, k_2) \in \mathcal{T}_n} \widehat{Q}_{n, k_1, k_2} \geq \widehat{Q}_{n, t_1^*, t_2^*} = C'_{n, t_1^*, t_2^*} \widehat{\Sigma}(u_n) C_{n, t_1^*, t_2^*},$$

where C_{n, t_1^*, t_2^*} and $\widehat{\Sigma}(u_n)$ are defined in (3.3) and (3.4). We can also write,

$$C_{n, t_1^*, t_2^*} = -\frac{(t_2^* - t_1^*)(n - (t_2^* - t_1^*))}{n^{3/2}} \left(\widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) + \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \right).$$

Moreover, by definition, the three matrices in the formula of $\widehat{\Sigma}_n(u_n)$ are positive semi-definite. Then, according to the assumption **B** and for n large enough, we can find a constant $C > 0$ such that, it holds *a.s.*

$$\begin{aligned} \widehat{Q}_n &\geq \widehat{Q}_{n, t_1^*, t_2^*} \\ &\geq \frac{(t_2^* - t_1^*)^2 (n - (t_2^* - t_1^*))^2}{n^3} \left(\widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) + \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \right)' \\ &\quad \times \left[\widehat{F}(T_{1, u_n}) \widehat{G}(T_{1, u_n})^{-1} \widehat{F}(T_{1, u_n}) + \widehat{F}(T_{n-u_n+1, n}) \widehat{G}(T_{n-u_n+1, n})^{-1} \widehat{F}(T_{n-u_n+1, n}) \right] \\ &\quad \times \left(\widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) + \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \right) \\ &\geq C \times n \left(\widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) + \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \right)' \\ &\quad \times \left[\widehat{F}(T_{1, u_n}) \widehat{G}(T_{1, u_n})^{-1} \widehat{F}(T_{1, u_n}) + \widehat{F}(T_{n-u_n+1, n}) \widehat{G}(T_{n-u_n+1, n})^{-1} \widehat{F}(T_{n-u_n+1, n}) \right] \\ &\quad \times \left(\widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) + \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \right). \end{aligned} \quad (5.13)$$

From the asymptotic properties of the QMLE, and the study of the stationary regime approximation developed by Bardet *et al.* (2012) (see Proposition 6.1 and Corollary 6.1 of these authors), we deduce

$$\left\{ \begin{array}{l} \bullet \widehat{\theta}(T_{1, u_n}) \xrightarrow[n \rightarrow \infty]{a.s.} \theta_1^*, \quad \widehat{\theta}(T_{n-u_n+1, n}) \xrightarrow[n \rightarrow \infty]{a.s.} \theta_1^*, \quad \widehat{\theta}(T_{1, t_1^*}) - \widehat{\theta}(T_{t_1^*+1, t_2^*}) \xrightarrow[n \rightarrow \infty]{a.s.} \theta_1^* - \theta_2^* \neq 0; \\ \bullet \frac{n - t_2^*}{n - (t_2^* - t_1^*)} (\widehat{\theta}(T_{t_2^*+1, n}) - \widehat{\theta}(T_{1, t_1^*})) \xrightarrow[n \rightarrow \infty]{a.s.} 0; \\ \bullet \widehat{F}(T_{1, u_n}) \widehat{G}(T_{1, u_n})^{-1} \widehat{F}(T_{1, u_n}) + \widehat{F}(T_{n-u_n+1, n}) \widehat{G}(T_{n-u_n+1, n})^{-1} \widehat{F}(T_{n-u_n+1, n}) \xrightarrow[n \rightarrow \infty]{a.s.} 2\Sigma^{(1)}, \end{array} \right.$$

where $\Sigma^{(1)}$ denotes the covariance matrix of the stationary model of the first and third regimes, defined as in (3.1) and computed at θ_1^* . This matrix, which is defined on the stationary regime is positive definite (see [6]). This implies $\widehat{Q}_{n, t_1^*, t_2^*} \xrightarrow[n \rightarrow \infty]{a.s.} +\infty$. Thus, the theorem is obtained from (5.13). \blacksquare

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