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Products of the iteration of the divisor function

JOHN M. CAMPBELL

Abstract

Let $d(n)$ denote the number of divisors of n . Erdős, in 1968, proved the remarkable result whereby there exists a positive constant $d_2 < \infty$ such that

$$\lim_{x \rightarrow \infty} \frac{\sum_{n \leq x} d(d(n))}{x \log \log x} = d_2,$$

and this was improved by Rieger in 1972. If we compare the above result to Ramanujan's formula for $\prod_{n=1}^x d(n)$ and its relation to the Dirichlet formula for $\sum_{n=1}^x d(n)$, this raises questions as to the behaviour of products of the form $\prod_{n=1}^x d(d(n))$. By the AM-GM inequality, Erdős' result gives us that

$$\frac{(\prod_{n=1}^x d(d(n)))^{\frac{1}{x}}}{\log \log x}$$

is bounded above by a positive, finite constant, but this does not give that the limit of the above expression exists as a positive constant, and known bounds for expressions such as $d(n)$ and $\log d(d(n))$ cannot be used in any direct way. Moreover, as we discuss, it seems that Erdős' and Rieger's summation techniques cannot be altered so as to be applicable to expressions such as $\sum_{n \leq x} \log d(d(n))$ or $\prod_{n \leq x} d(d(n))$. We prove that: For all $\epsilon > 0$, the bounds

$$\log 2 - \epsilon \leq \frac{\sum_{n \leq x} \log d(d(n))}{x \log \log \log x} \leq 1 + \epsilon$$

hold for all sufficiently large x , yielding bounds for the product obtained by replacing $d(n)$ with $d(d(n))$ in Ramanujan's product.

Keywords: divisor function, highly composite number, divisor, prime omega function

AMS Classification: 11N37, 11A25

1 Introduction

The arithmetic function $d(n)$ providing the number of divisors of a given natural number n is of fundamental importance in number theory. A classic result given by Dirichlet in 1849 [6] provides the following formula for determining the average order for the divisor function:

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}), \quad (1)$$

letting $\gamma = \lim_{n \rightarrow \infty} (H_n - \log n)$ denote the Euler–Mascheroni constant, and where $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$. In 1968, Erdős [7] proved that

$$\lim_{x \rightarrow \infty} \frac{1}{x \log \log x} \sum_{n \leq x} d(d(n)) = d_2 \quad (2)$$

for a constant $0 < d_2 < \infty$ (see also [4, 14]), and Kátai [13] later proved the same result. Rieger [18] later proved that

$$\sum_{n \leq x} d(d(n)) = d_2 x \log \log x + d_3 x + O\left(\frac{x}{\log x}\right), \quad (3)$$

and the error term in (3) was improved by Heppner [12]. If we compare (1) to (2), and then compare both (1) and (2) to Ramanujan’s formula, as in (6) below, for products of the form

$$\prod_{n \leq x} d(n), \quad (4)$$

then this naturally leads us to consider products of the form

$$\prod_{n \leq x} d(d(n)). \quad (5)$$

As indicated above, a main reason as to the research interest in products as in (5) is due to Ramanujan’s product formula

$$d(1)d(2) \cdots d(x) = 2^{x(\log \log x + C) + \Phi(x)}, \quad (6)$$

where

$$C = \gamma + \sum_{\nu=2}^{\infty} \left(\log_2 \left(1 + \frac{1}{\nu} \right) - \frac{1}{\nu} \right) (2^{-\nu} + 3^{-\nu} + 5^{-\nu} + \dots), \quad (7)$$

and where $\Phi(x)$ is defined as in [17], with $\Phi(n) \sim \frac{n}{\log n}(\gamma - 1)$. In this article, we introduce and prove an analogue of (2) for the expression obtained from the finite sum in (2) by replacing the partial sum operator $\sum_{n \leq x} \cdot$ with the corresponding product operator $\prod_{n \leq x} \cdot$.

The iteration $d(d(n))$ was considered by Ramanujan in 1915 in relation to highly composite numbers [16]. A problem concerning the study of the sequence $(d(d(n)) : n \in \mathbb{N})$ is due to the fact that this sequence is not multiplicative, in contrast to $(d(n) : n \in \mathbb{N})$ being multiplicative. So, summation techniques such as the Dirichlet hyperbola method cannot be applied, at least in any direct way, using expressions such as $d(d(n))$ or $\log(d(d(n)))$. It appears that products as in (5) have not previously been considered, noting that the integer sequence given by expressions of the form in (5) for $x \in \mathbb{N}$ is not currently in the On-Line Encyclopedia of Integer Sequences. The research interest in the problem of devising a product analogue of Erdős' asymptotic result in (2) [7] is further motivated by past work as in [2, 12, 18] concerning sums of the form

$$\sum_{n \leq x} d(d(n)). \quad (8)$$

As expressed in [2], the study of (8) is in the hope of obtaining a deeper understanding as to the multiplicative structure of the natural numbers, and this again leads us to investigate the behaviour of the product indicated in (5) corresponding to (8).

A main difficulty associated with the problem of “converting” Erdős' finite sum in (2) [7] into a corresponding product and in such a way so as to obtain an asymptotic evaluation as in (2) has to do with the irregular behaviour of the divisor function. For example, one might hope to approximate $d(d(n))$ with an integrable function f in such a way so that an asymptotic relation such as $\int_1^x f(x) dx \sim d_2 x \log \log x$, for $0 < d_2 < \infty$ as in (2), could somehow be manipulated, say, by an Abel-type summation method or by an analogue of the Fundamental Theorem of Calculus, in order to obtain an asymptotic formula for

$$\sum_{n \leq x} \log d(d(n)), \quad (9)$$

but the irregular behaviour of $d(n)$ and of $d(d(n))$ would interfere with this approach. For example, observe how Ramanujan's product formula in (6) cannot be obtained via such an approach from the Dirichlet formula in (1).

Similar problems arise would arise if one were to attempt to use the Euler-Maclaurin summation method.

Ramanujan's product formula in (6) relies on the identity

$$\sum_{n=1}^{\infty} \frac{\log_2(d(n))}{n^s} = \zeta(s) \sum_p \sum_{\nu=1}^{\infty} \frac{\log_2(1 + \frac{1}{\nu})}{p^{\nu s}}$$

for $\Re(s) > 1$ (cf. [5, §6]). To the best of our knowledge, there are no known results of a simliar nature for expressions involving $\log_2(d(d(n)))$, recalling that $(d(d(n)) : n \in \mathbb{N}_0)$ is not multiplicative. DeKoninck and Luca [5, p. 88] provided a way of generalizing a weaker version of Ramanujan's formula for $d(1)d(2) \cdots d(x)$, but this only applies to certain multiplicative functions, again with $(d(d(n)) : n \in \mathbb{N}_0)$ being non-multiplicative. Moreover, as we briefly discuss in Section 4, it seems that Erdős' [7] and Rieger's [18] summation techniques that were applied to prove (2) and (3) cannot be "translated" or altered so as to be applicable to (5) or to the logarithm of (5).

2 Background and preliminaries

By the AM-GM inequality together with the Erdős result in (2), we can conclude that

$$0 < \frac{(\prod_{n=1}^x d(d(n)))^{\frac{1}{x}}}{\log \log x} < c \tag{10}$$

for some positive constant c and for all $x \geq 3$. However, the behaviour of quotients of geometric means to arithmetic means even for number-theoretic sequences that are more well behaved than

$$(d(n) : n \in \mathbb{N}_0) \quad \text{and} \quad (d(d(n)) : n \in \mathbb{N}_0) \tag{11}$$

is poorly understood, as we briefly review below. This leads us to conclude that the inequalities in (10) would not be enough to prove that the limit as $x \rightarrow \infty$ of the central expression in (10) would even exist as a strictly *positive* constant. Furthermore, inequalities as in (10) would not be enough to establish the existence of a positive lower bound, and would not be enough to obtain any closed form for any positive lower bound.

For a positive sequence $(a_n : n \in \mathbb{N})$, by letting $\text{AM}(a_1, a_2, \dots, a_n) = \frac{a_1 + a_2 + \dots + a_n}{n}$ denote the arithmetic mean of the first n entries, and letting

$\text{GM}(a_1, a_2, \dots, a_n) = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$ denote the geometric mean of the first n entries, we have that $0 < \frac{\text{GM}(a_1, a_2, \dots, a_n)}{\text{AM}(a_1, a_2, \dots, a_n)} \leq 1$ for all n , but

$$\lim_{n \rightarrow \infty} \frac{\text{GM}(a_1, a_2, \dots, a_n)}{\text{AM}(a_1, a_2, \dots, a_n)} \quad (12)$$

may vanish and may be strictly in the interval $(0, 1)$. Furthermore, based on the relevant literature on limits of ratios as in (12) [1, 10, 11], it seems that not much is known about the behaviour of (12), especially for number-theoretic sequences as in $(p_n : n \in \mathbb{N})$ [1, 10, 11]. Notably, even for the $a_n = p_n$ case, with $p_n \sim n \log n$ much more well behaved compared to (11), it is still not even known whether or not the ratio sequence

$$\frac{\text{AM}(p_1, p_2, \dots, p_n)}{\text{GM}(p_1, p_2, \dots, p_n)}$$

is strictly decreasing after a fixed point [1, 11]. This open problem illustrates the difficulty of proving that

$$\lim_{x \rightarrow \infty} \frac{(\prod_{n \leq x} d(d(n)))^{\frac{1}{x}}}{\log \log x} \quad (13)$$

exists as a positive value and of proving a closed form for (13). Such results would not follow from known upper bounds for expressions such as $d(n)$ and $d(d(n))$, including the upper bounds shown in (17) and (18). Furthermore, Hassani [10] has considered the behaviour of ratios as in (12) for the $a_n = d(n)$ case, which motivates our exploration of the corresponding ratio for the much more difficult $a_n = d(d(n))$ case.

2.1 Estimates for the divisor function

While it is classically known [9, §18] that

$$\limsup_{x \rightarrow \infty} \frac{\log d(n) \log \log n}{\log 2 \log n} = 1,$$

the following more explicit result is to be used in a key way in our work. This remarkable result is proved in De Koninck and Luca's classic text on analytic number theory.

Theorem 1. *The formula*

$$d(n) = (\log n)^{\log 2 + o(1)} \quad (14)$$

holds for almost all n [5, p. 99].

This result is closely related to Hardy and Ramanujan's results concerning the behaviour of the prime omega functions, referring to [5, §7] for details.

A 1906 result due to Wigert [19] is such that: For a given $\varepsilon > 0$, the inequality

$$d(n) < 2^{\frac{\log n}{\log \log n} (1 + \varepsilon)} \quad (15)$$

holds for all sufficiently large natural numbers n . This recalls Ramanujan's result such that

$$d(n) < 2^{\frac{\log n}{\log \log n} + O\left(\frac{\log n}{(\log \log n)^2}\right)} \quad (16)$$

for all n [3, p. 87]. By rewriting the upper bounds in (15) and (16) with $2^{\log n}$ replaced by $n^{\log 2}$, this illustrates how the estimate in (14) is a dramatic improvement, but only for *almost all* n . As suggested by Erdős and Kátai [8], one of the most basic upper bounds for the divisor function is such that $d(n) < n$ for $n \geq 3$. This seems to reflect a result due to Nicolas and Robin [15] which would imply that

$$d(n) < n^{\frac{1.06602}{\log(\log(n))}} \quad (17)$$

for each $n \geq 3$.

The study of the product in (5) and of the equivalent expression in (9) is also motivated by the work of Buttkewitz et al. [4] on the iterated divisor function, in which the inequality

$$\log d(d(n)) \leq \frac{\sqrt{\log n}}{\log \log n} \left(C + O\left(\frac{\log \log \log n}{\log \log n}\right) \right) \quad (18)$$

was introduced, where the constant C in (18) is optimal and given explicitly in terms of a logarithmic sum. However, sums as in (9) are not considered in [4], and bounds obtained from applying $\sum_{n \leq x} \cdot$ to both sides of (18) are not enough to obtain explicit results on the growth of products as in (5), and similarly with respect to Ramanujan's result whereby

$$\frac{2\sqrt{2} \log(2)}{n \sqrt{\log(n) \log(\log(n))}} < d(d(n))$$

for infinitely many n , as introduced in Ramanujan's seminal work on highly composite numbers [16].

3 Main result

Let us write

$$d(n) = (\log n)^{\log 2 + \varepsilon(n)}, \quad (19)$$

where the error function $\varepsilon(n)$ in (19) is the unique function $f(n)$ such that $d(n) = (\log n)^{\log 2 + f(n)}$ holds for all natural numbers n . The notation for ε in (19) is distinct from that for the value $\epsilon > 0$ indicated in the following main result.

Theorem 2. *For all $\epsilon > 0$, the bounds*

$$\log 2 - \epsilon \leq \frac{\sum_{n \leq x} \log d(d(n))}{x \log \log \log x} \leq 1 + \epsilon \quad (20)$$

hold for all sufficiently large x

Proof. Let $(\mathbf{g}(m) : m \in \mathbb{N})$ denote any fixed, strictly increasing sequence of natural numbers such that $\varepsilon(\mathbf{g}(n)) = o(1)$ and

$$\lim_{x \rightarrow \infty} \frac{\# \text{ of integers of the form } \mathbf{g}(m) \text{ such that } \mathbf{g}(m) \leq x}{x} = 1. \quad (21)$$

Since (14) holds for almost all natural numbers, we find that \mathbf{g} -sequences satisfying the specified properties exist. Also observe that since \mathbf{g} is a strictly increasing sequence of natural numbers satisfying (21), we may deduce that $\mathbf{g}(x) \sim x$.

Now, let us consider natural numbers m that can be expressed in such a way so that

$$\begin{aligned} \mathbf{g}(m) &= (\log \mathbf{g}(n))^{\log 2 + \varepsilon(\mathbf{g}(n))} \\ &= d(\mathbf{g}(n)) \end{aligned} \quad (22)$$

for some n . Since $\mathbf{g}(\ell) \sim \ell$ and $\varepsilon(\mathbf{g}(\ell)) = o(1)$, the expression $d(\mathbf{g}(\ell))$, for a free variable ℓ , is asymptotically equivalent to $(\log \ell)^{\log 2}$, and by comparing the image of

$$(\log \mathbf{g}(\ell))^{\log 2 + \varepsilon(\mathbf{g}(\ell))} \sim (\log \ell)^{\log 2} \quad (23)$$

as a function on \mathbb{N} to the image of \mathbf{g} , we see that almost all natural numbers m satisfy the desired property concerning (22).

Applying the divisor function to (22), writing

$$\begin{aligned} d(\mathfrak{g}(m)) &= d\left((\log \mathfrak{g}(n))^{\log 2 + \varepsilon(\mathfrak{g}(n))}\right) \\ &= d(d(\mathfrak{g}(n))), \end{aligned} \quad (24)$$

we then find that (24) then gives us that

$$d(d(\mathfrak{g}(n))) = (\log \mathfrak{g}(m))^{\log 2 + \varepsilon(\mathfrak{g}(m))}. \quad (25)$$

By rewriting $\mathfrak{g}(m)$ in (25) according to (22), we obtain

$$d(d(\mathfrak{g}(n))) = \left(\log \left((\log \mathfrak{g}(n))^{\log 2 + \varepsilon(\mathfrak{g}(n))}\right)\right)^{\log 2 + \varepsilon(\mathfrak{g}(m))}. \quad (26)$$

Starting with any fixed $m \in \mathbb{N}$ such that

$$\mathfrak{g}(m) = (\log \mathfrak{g}(n))^{\log 2 + \varepsilon(\mathfrak{g}(n))} \quad (27)$$

holds for some n , we see that there may be multiple natural numbers n such that (27) holds, again for a fixed natural number m , and this depends on the way the \mathfrak{g} -sequence and the error function ε are defined. So, for $m \in \mathbb{N}$, if we take all natural numbers n such that (27) holds, and then sort the resultant sequence of n -values for all possible values of $m \in \mathbb{N}$, we find that almost all natural numbers are in this resultant sequence, by comparing the equivalence $\mathfrak{g}(m) \sim m$ with the growth of (23) for a free variable ℓ .

So, if we repeat our derivation of (26), but with all possible n -values for a given m -value, this leads us to rewrite m as a function of n , with

$$\begin{aligned} d(d(\mathfrak{g}(n))) &= \\ &(\log 2 + \varepsilon(\mathfrak{g}(n)))^{\log 2 + \varepsilon(\mathfrak{g}(m(n)))} (\log (\log \mathfrak{g}(n)))^{\log 2 + \varepsilon(\mathfrak{g}(m(n)))} \end{aligned}$$

holding for almost all natural numbers n . Since $\varepsilon(\mathfrak{g}(\ell)) = o(1)$ and since $m(n) \sim (\log n)^{\log 2}$, we may obtain that $\varepsilon(\mathfrak{g}(m(n))) = o(1)$. This leads us to define an error function $\delta(n) = \varepsilon(\mathfrak{g}(m(n)))$, with $\delta(n) = o(1)$.

Now, let $(\mathfrak{h}(\ell) : \ell \in \mathbb{N})$ denote any strictly increasing sequence such that

$$\begin{aligned} d(d(\mathfrak{g}(\mathfrak{h}(\ell)))) &= \\ &(\log 2 + \varepsilon(\mathfrak{g}(\mathfrak{h}(\ell))))^{\log 2 + \delta(\mathfrak{h}(\ell))} (\log (\log \mathfrak{g}(\mathfrak{h}(\ell))))^{\log 2 + \delta(\mathfrak{h}(\ell))} \end{aligned}$$

holds for all $\ell \in \mathbb{N}$ and such that

$$\lim_{x \rightarrow \infty} \frac{\#\text{ of integers of the form } \mathfrak{h}(\ell) \text{ such that } \mathfrak{h}(\ell) \leq x}{x} = 1,$$

noting that $\mathfrak{h}(\ell) \sim \ell$.

Now, let \mathfrak{i} denote the composition $\mathfrak{g} \circ \mathfrak{h}$. Let $\Pi^{\mathfrak{i}}(x)$ denote the number of integers of the form $\mathfrak{i}(\ell)$ such that $\mathfrak{i}(\ell) \leq x$, with

$$\lim_{x \rightarrow \infty} \frac{\Pi^{\mathfrak{i}}(x)}{x} = 1 \quad \text{and} \quad \mathfrak{i}(\ell) \sim \ell \quad (28)$$

and

$$\begin{aligned} d(d(\mathfrak{i}(\ell))) &= \\ &(\log 2 + \varepsilon(\mathfrak{i}(\ell)))^{\log 2 + \bar{\delta}(\ell)} (\log(\log \mathfrak{i}(\ell)))^{\log 2 + \bar{\delta}(\ell)}, \end{aligned} \quad (29)$$

where the error function $\bar{\delta}$ is such that $\bar{\delta}(\ell) = o(1)$, and noting that $\varepsilon(\mathfrak{i}(\ell)) = o(1)$ since $\mathfrak{h}(\ell) \sim \ell$.

Now, let $\Pi^{\mathfrak{b}}(x)$ denote the number of natural numbers that are less than or equal to x and that are not of the form $\mathfrak{i}(\ell)$. Let the strictly increasing sequence of natural numbers that are not of the form $\mathfrak{i}(\ell)$ be denoted as $(\mathfrak{b}(\ell) : \ell \in \mathbb{N})$. This leads us to the decomposition

$$\begin{aligned} \sum_{n \leq x} \log d(d(n)) &= \\ &\sum_{n \leq \Pi^{\mathfrak{b}}(x)} \log d(d(\mathfrak{b}(n))) + \sum_{n \leq \Pi^{\mathfrak{i}}(x)} \log d(d(\mathfrak{i}(n))). \end{aligned} \quad (30)$$

By rewriting the final sum in (30) according to (29), and by then applying the asymptotic equivalences in (28), and by applying the usual identity for converting a sum into an integral [5, §1], we find that

$$\begin{aligned} \sum_{n \leq \Pi^{\mathfrak{i}}(x)} \log d(d(\mathfrak{i}(n))) &\sim \\ &x(\log 2)(\log \log 2) + (\log 2) \int_3^x \log \log \log t \, dt, \end{aligned} \quad (31)$$

and integration by parts applied to (31) can be used to obtain that

$$\sum_{n \leq \Pi^{\mathfrak{i}}(x)} \log d(d(\mathfrak{i}(n))) \sim \log 2 (x \log \log \log x). \quad (32)$$

So, by dividing each sum in (30) by $x \log \log \log x$, the quotient obtained from the second sum on the right approaches $\log 2$ as $x \rightarrow \infty$, and the quotient obtained from the first sum on the right is bounded below by 0, giving us the desired lower bound in (20).

By the AM-GM inequality, Erdős' formula in (2) [7] implies that

$$\left(\prod_{n \leq x} d(d(n)) \right)^{\frac{1}{x}} < C \log \log x$$

for a positive constant C and $x \geq 3$, so that

$$\frac{1}{x \log \log \log x} \sum_{n \leq x} \log d(d(n)) < \frac{\log(C \log \log x)}{\log \log \log x}, \quad (33)$$

with the right-hand side of (33) approaching 1 as $x \rightarrow \infty$. This gives us the desired upper bound in (20). \square

So, we obtain the lower bound $\exp((\log 2 - \epsilon)x \log \log \log x)$ for the product $\prod_{n \leq x} d(d(n))$.

3.1 Computational problems

An interesting aspect about Theorem 2 has to do with the extent of the computational problems related to extremely slow convergences related to Theorem 2, as illustrated in Table 1. A key step in our proof required the asymptotic equivalence of the right-hand sides of (31) and (32), and the extremely slow convergence associated with this equivalence is illustrated in Table 2.

4 Conclusion

One might hope to mimic Erdős' summation techniques [7] and/or Rieger's summation techniques [18] in pursuit of the goal of obtaining a more precise understanding of the behaviour of the expressions in (5) and (9). However, this seems to be problematic, as described below.

x	$\frac{1}{x \log \log \log x} \sum_{n \leq x} \log d(d(n))$
5,000,000	1.49829...
9,000,000	1.48833...
16,000,000	1.4793...
31,000,000	1.46968...
64,000,000	1.45998...
145,000,000	1.44995...
353,000,000	1.43999...
943,000,000	1.42999...

Table 1: The behaviour of the central expression in (20).

x	Quotient of the right-hand sides of (31) and (32)
10^{10}	0.66649...
10^{20}	0.72271...
10^{30}	0.743671...
10^{40}	0.755528...
10^{50}	0.763463...
10^{100}	0.78312...
10^{200}	0.797711...
10^{300}	0.804679...

Table 2: Estimates illustrating the asymptotic equivalences of (31) and (32) obtained via NIntegrate.

Erdős [7] and Rieger [18] followed a similar approach toward decomposing sums as in (8). Following [18], we write

$$E(x) := \sum_{n \leq x} d(d(n)) = \sum_{s_j \leq x} \sum_{\substack{q \leq \frac{x}{s_j} \\ (q, s_j) = 1 \\ \mu(q) \neq 0}} d(d(qs_j)), \quad (34)$$

referring to [18] for details. Again following [18], we write

$$d(d(qs_j)) = d(b_j)(v(q) + d_j + 1), \quad (35)$$

again referring to [18] for details, as in with the arithmetic function such that $v(n) := \sum_{p|n} 1$. Rieger's decomposition method [18] then relied on rewriting

the inner sum in

$$\sum_{n \leq x} d(d(n)) = \sum_{s_j \leq x} d(b_j) \sum_{\substack{q \leq \frac{x}{s_j} \\ (q, s_j) = 1 \\ \mu(q) \neq 0}} (v(q) + d_j + 1), \quad (36)$$

according to the Q -function such that

$$Q(x, k) := \sum_{\substack{n \leq x \\ (n, k) = 1 \\ \mu(n) \neq 0}} 1. \quad (37)$$

However, to rewrite the inner sum of (36) in terms of (37), this would require identities such as

$$\sum_{\substack{q \leq y \\ (q, s) = 1 \\ \mu(q) \neq 0}} v(q) = \sum_{\substack{p \leq y \\ p \nmid s}} Q\left(\frac{y}{p}, ps\right), \quad (38)$$

but we encounter problems if we were to try to mimic this approach with (9), due to log not preserving addition, as clarified below.

Mimicking (34), we may write

$$\sum_{n \leq x} \log d(d(n)) = \sum_{s_j \leq x} \sum_{\substack{q \leq \frac{x}{s_j} \\ (q, s_j) = 1 \\ \mu(q) \neq 0}} \log d(d(qs_j)), \quad (39)$$

so that (35) gives us that

$$\sum_{n \leq x} \log d(d(n)) = \sum_j \sum_q (\log(d(b_j)) + \log(v(q) + d_j + 1)),$$

being consistent with the indexing in (39). We thus encounter the problem of rewriting a sum such as

$$\sum_q \log(v(q) + d_j + 1)$$

in terms of (37), since identities as in (38) cannot be applied, again since log does not preserve addition. This leads us to conclude with some conjectures

and problems concerning products of the iterate of the divisor function, as below.

From Ramanujan's formula for products of consecutive entries of the divisor sequence, it seems natural to conjecture that the limit of the central expression in (20) as $x \rightarrow \infty$ exists and is equal to $\log 2$. We leave it as an open problem to prove this. Moreover, Ramanujan's constant in (7) leads us to consider how

$$\lim_{x \rightarrow \infty} \left(\frac{1}{x} \sum_{n \leq x} \log_2 d(d(n)) - \log \log \log x \right) \quad (40)$$

could be expressed similarly, under the assumption that the limit in (40) exists.

Erdős' formula in (2) [7] together with an analogue of (30) can be used to prove that the limit

$$\lim_{x \rightarrow \infty} \frac{\sum_{n \leq \Pi^b(x)} d(d(\mathbf{b}(n)))}{x \log \log x}$$

exists and equals the same constant d_2 in (2). Could this be somehow applied to show that the limit

$$\lim_{x \rightarrow \infty} \frac{\sum_{n \leq \Pi^b(x)} \log d(d(\mathbf{b}(n)))}{x \log \log x} \quad (41)$$

exists and vanishes? If so, this would prove the conjecture given in the preceding paragraph. The AM-GM inequality is not enough to prove that (41) exists/vanishes, and known upper bounds for $d(n)$ and for $\log d(d(n))$ are not enough. How could we obtain a better upper bound for the sum in (33)? Ideally, this could be applied to prove the conjecture given in the preceding paragraph.

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Conflict of interest statement

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