Chaos, stochasticity and algebraic topology: Novel methods and their applications to climate

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and Denisse Sciamarella (IFAECI, Buenos Aires)

Based on joint work with Gisela Charó (CIMA-CONICET),
Mickael Chekroun and Christophe Letellier (CORIA-CNRS)

Project-team

DATASHAPE

Understanding the shape of data

DataShape Seminar, LMO 11/5/2023

IFAECI Institut Franco-Argentin d'Études sur le Climat et ses Impacts

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LEFE/MANU:
Noise-Nonlinearity Interaction in Climate Dynamics

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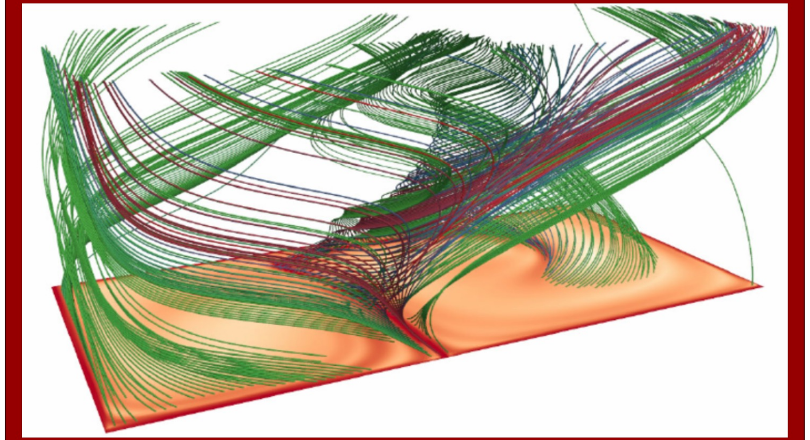

"Mathematical Problems in Climate Dynamics"

Michael Ghil will be visiting CIMA from November 1st to November 13th 2018 and offering a six-lecture course as part of the activities organized by UMI 3351 IFAECI in relation with the project MATH-GEO of the MathAmSud Program, http://mathgeo.cima.fcen.uba.ar/

Organizado y Patrocinado por
El Instituto Franco-Argentino sobre Estudios de Clima y sus Impactos
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Professor Michael Ghil

These lectures were previously offered at the Imperial College, More Info

Michael Ghil has worked on Climate Dynamics, Dynamical and Complex Systems, Extreme Events, Numerical and Statistical Methods, and Mathematical Economics. He is the author or editor of a dozen books and author or co-author of nearly 300 research and review articles. Many can be found on the website of his research group at UCLA. His honors and awards include the L.F. Richardson Medal of the EGU (2004), the E.N. Lorenz Lecture of the AGU (2005), a Plenary Lecture at the 7th International Congress on Industrial and Applied Mathematics (ICIAM 2011), the Alfred Wegener Medal of the EGU (2012), and Membership in the Academia Europaea (1998).

Topology as a data analysis and theoretical tool to understand the fundamental **processes** underlying the **dynamics** of complex systems

https://doi.org/10.5194/egusphere-2023-216
Preprint. Discussion started: 15 February 2023

EGUsphere
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Review Article: Dynamical Systems, Algebraic Topology, and the Climate Sciences

Michael Ghil [1,2] and Denisse Sciamarella [3]

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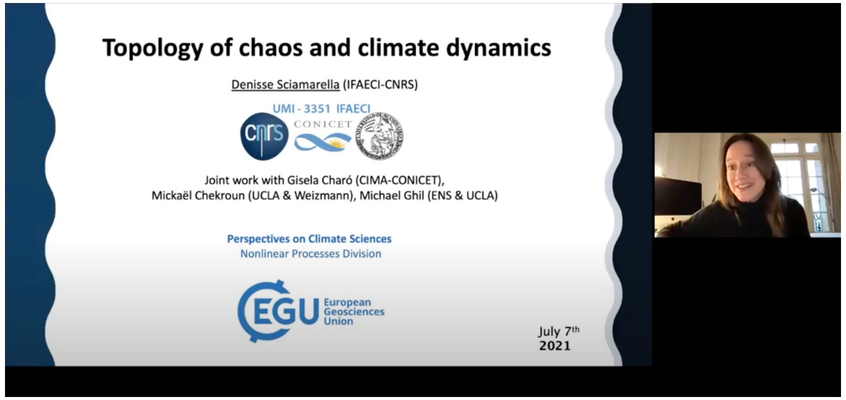
This paper is based on the invited talks given by the two authors in an online series on "**Perspectives on climate sciences: From historical developments to research frontiers**." Its success led to the idea of having a special issue of *Nonlinear Processes in Geophysics.*



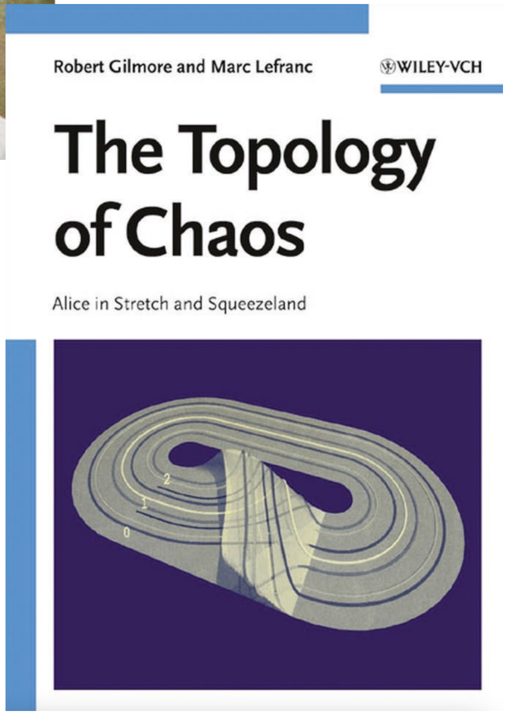
https://youtu.be/W1yndTsvR0g

The first sentence of Leo Tolstoy's novel Anna Karenina is:

"All happy families are alike; each unhappy family is unhappy in its own way".

Following the famous writer, Robert Gilmore and Marc Lefranc, tell us:

"All linear systems are alike; each nonlinear system is nonlinear in its own way".



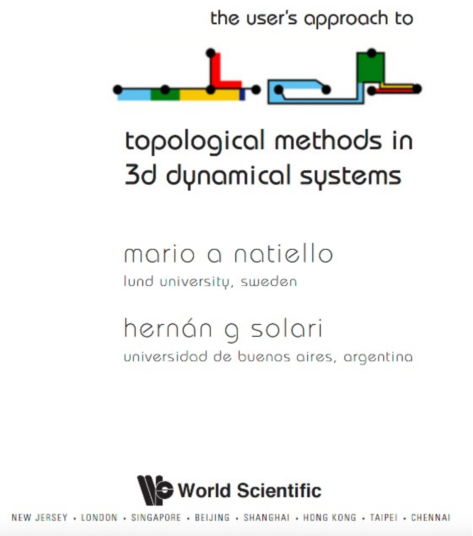
"It was a very happy and shocking discovery that there were structures in nonlinear systems that are always the same if you looked at them the right way."

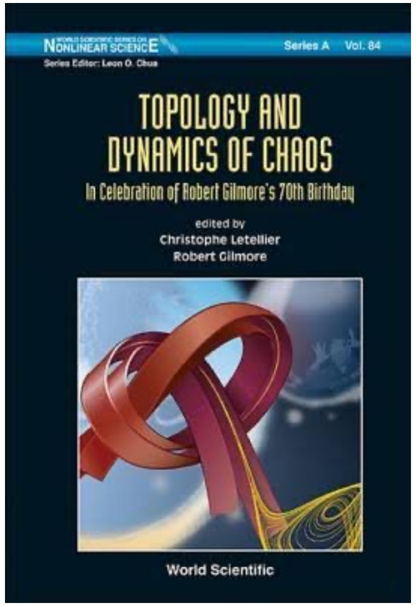
Topological analysis in phase space is the correct way of looking at dynamical systems.

For deterministic systems, the topological structure of a flow is an invariant that provides information on the mechanisms acting in phase space to shape the flow.

A topological analysis involves finding a topological representation of the underlying structure and obtaining an algebraic description allowing for the computation of topological invariants.

Developing methods to accomplish these two steps has involved several false starts as well as partially successful roads, that finally lead to what we now call a templex.





Charó, G. D., Letellier, C., & Sciamarella, D. (2022). Templex: A bridge between homologies and templates for chaotic attractors. Chaos: An Interdisciplinary Journal of Nonlinear Science, 32(8), 083108.

- What is chaos topology and why is it important?

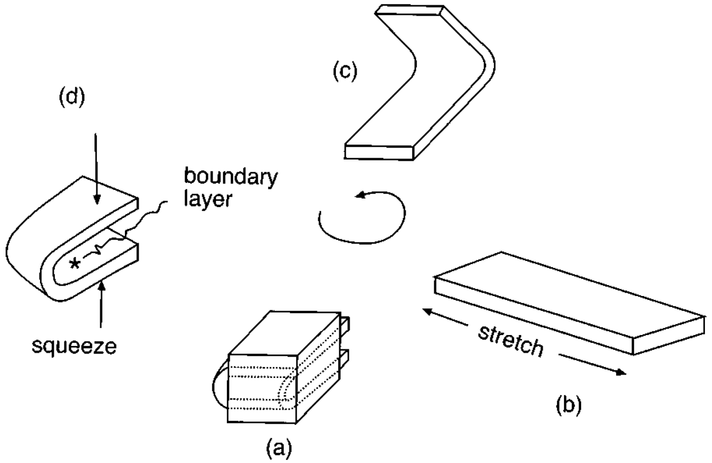
Invariants in phase space can be of different types:

a) Metric: dimensions of various types, e.g., correlation dimension (Grassberger & Procaccia, 1983), multifractal scaling functions (Halsey et al., 1986).

b) Dynamic: Lyapunov exponents (Oseledec, 1968; Wolf et al., 1985), as discussed by Eckmann & Ruelle (1985) and by Abarbanel et al. (1993).

c) Topological: linking numbers, relative rotation rates, Conway polynomials, Branched Manifolds (Birman & Williams, 1983).

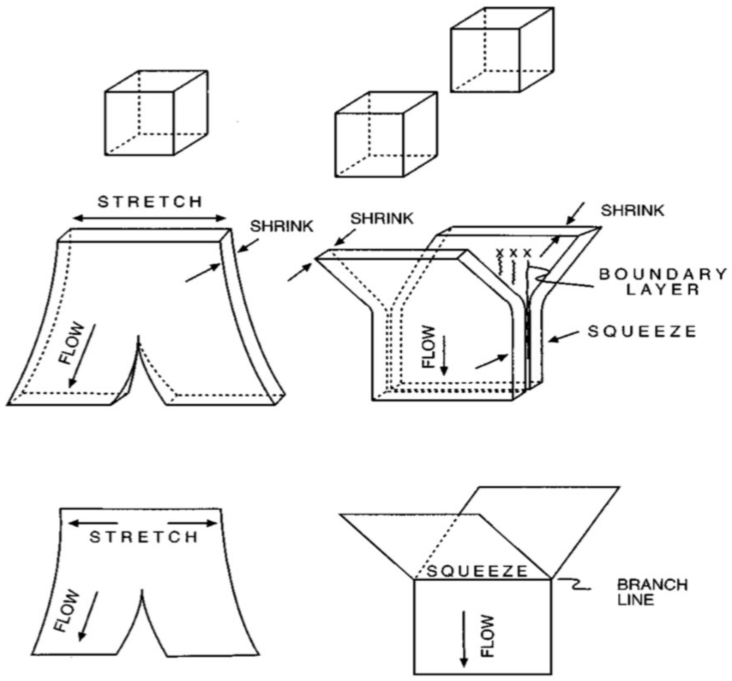
Invariants (a) and (b) do not provide information on how to model the system's dynamics, while (c) actually does!

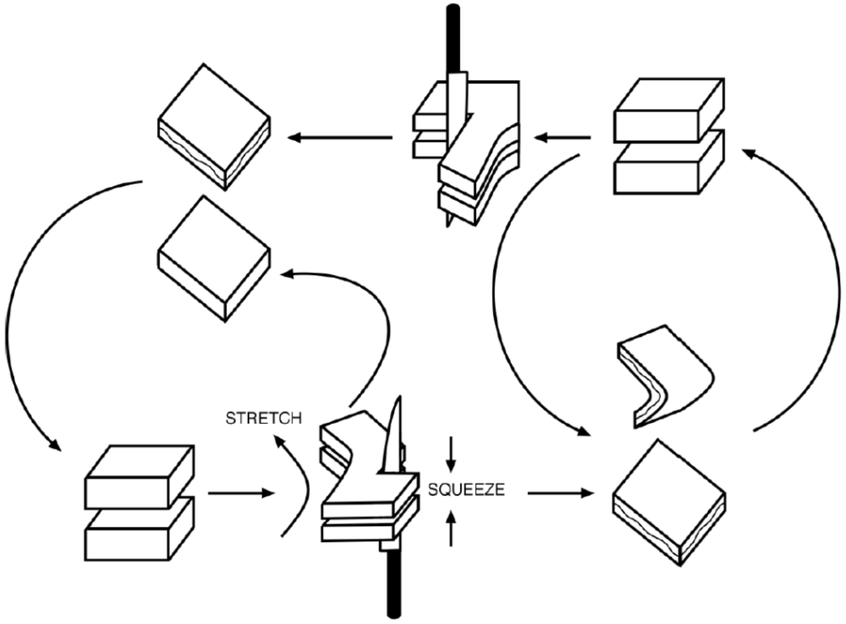


R. Gilmore, *Reviews of Modern Physics*, Vol. 70(4), October 1998

- Why is this so?

Consider a cube of initial conditions under a flow. It can stretched so that it splits into two parts heading to different parts of phase space; it can shrink to form a locally two-dimensional structure. Two cubes can also be squeezed together, so that two inflow regions are joined to the outflow region by a branch line.





Gilmore & Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience, 2002.

The “recipe” to “knead” the Lorenz’63 attractor is a sequence of steps that are topological in nature.

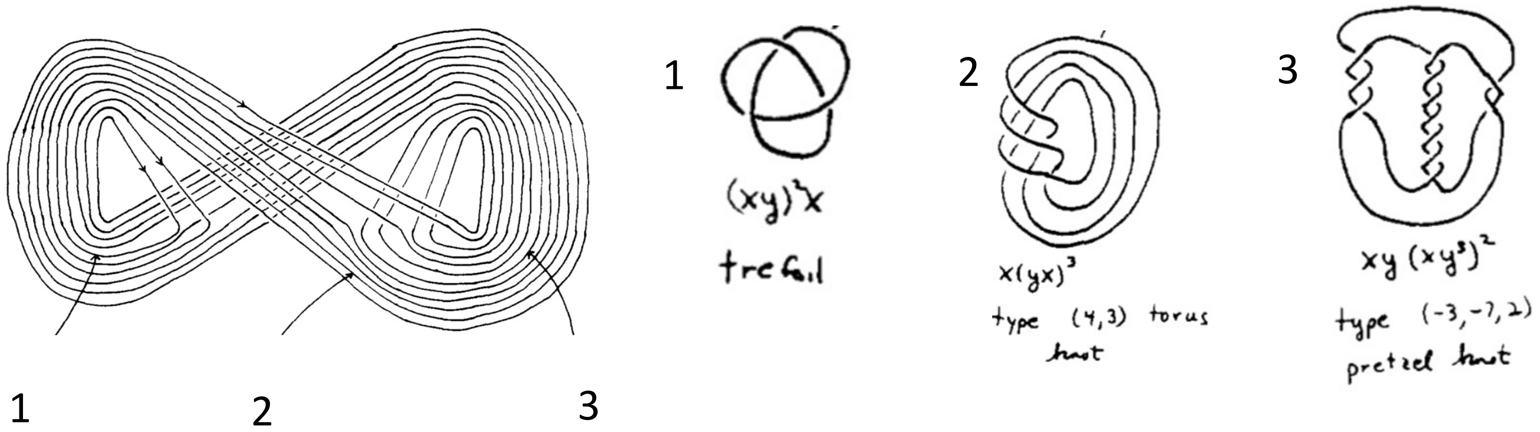
"All linear systems are alike; each nonlinear system is nonlinear in its own way".

- How to unveil the topological structure of a flow in phase space?

1) Approximate trajectories by closed curves.

2) Find a topological representation for the orbit structure.

3) Obtain an algebraic description for the topological structure.



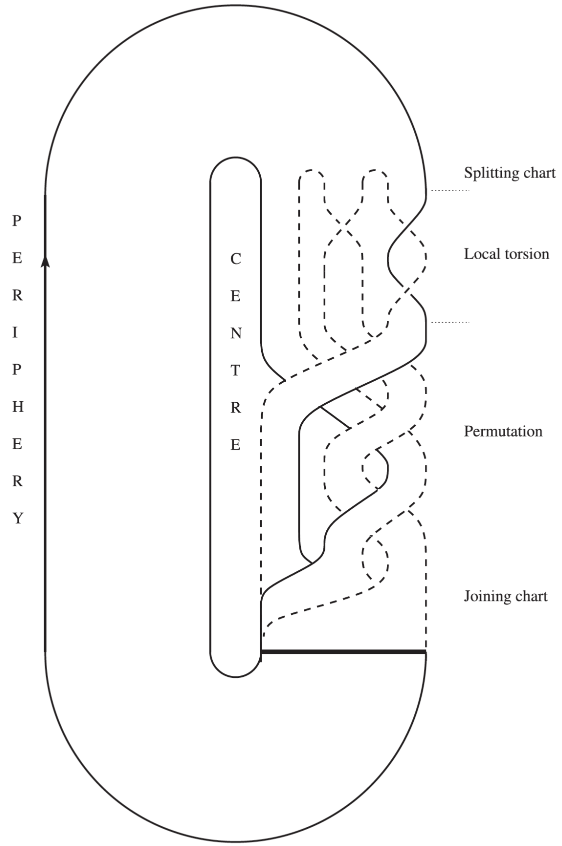
1) Close-return method – time series, though, must be long and noise free.

2) Knot theory – knot ≡ orbit in three dimensions.

3) Knot invariants – *e.g.*, linking numbers, Conway polynomials

Knot information for a chaotic attractor can be condensed in a knot-holder or template.

Computing topological invariants using **knots**…



- How to unveil the topological structure of a flow in phase space?

What's wrong with templates? They are very difficult to construct… they imply the reconstruction of Unstable Periodic Orbits (UPOs) and besides, they are no longer meaningful if a fourth dimension in phase space is involved, since they unknot….

Chapter 7

A braided view of a knotty story

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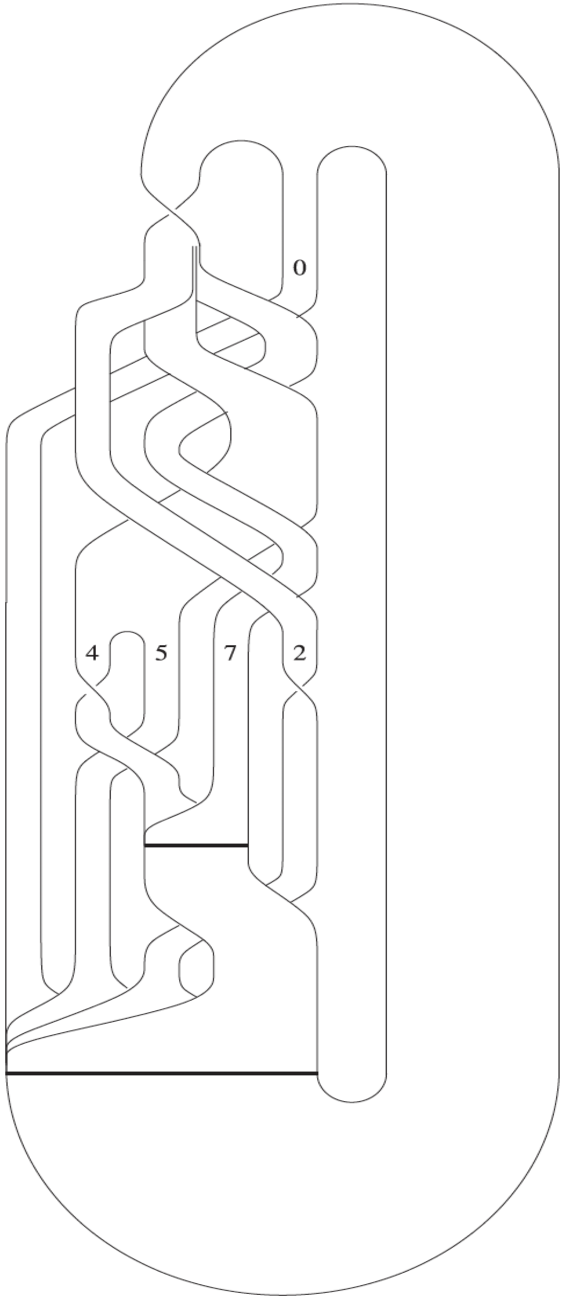
Periodic orbits of 3-d dynamical systems admitting a Poincaré section can be described as braids. This characterisation can be transported to the Poincaré section and Poincaré map, resulting in the braid type. Information from braid types allows to estimate bounds for the topological entropy of the map while revealing detailed orbit information from the original system, such as the orbits that are necessarily present along with the given one(s) and their organisation. We review this characterisation with some examples –from a user-friendly perspective–, focusing on systems whose Poincaré section is homotopic to a disc.



- How to unveil the topological structure of a flow in phase space?

Computing topology using knot theory

3-D trajectory set

Knot invariants

Computing topology using homologies

n-D point-cloud

Homologies

RESTRICTIONS

- Precision and length of time series must be good enough for orbits in phase space to be reconstructed accurately …
- Phase space dimension cannot be higher than three, since knots or braids unknot …

HOMOLOGY GROUPS

- Time series can be shorter and noisy since the method is independent of the reconstruction of trajectories in phase space (no periodic orbit).
- Applicable in *n* dimensions: the method is knotless and braidless.

- How to unveil the topological structure of a flow in phase space?

Computing topological invariants using **homologies**

*n*D point-cloud

Homologies

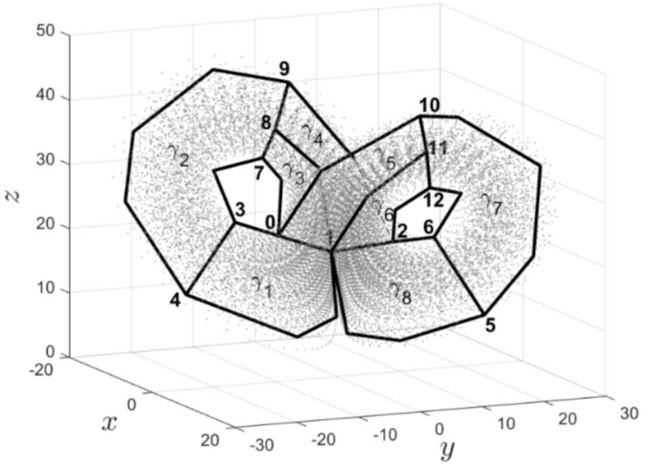
1) Approximate points as lying on a branched manifold.

2) Find a topological representation for the branched manifold.

3) Obtain an algebraic description of the topological structure.

BraMAH COMPLEX construction

BRAMAH



HOMOLOGY GROUP computation

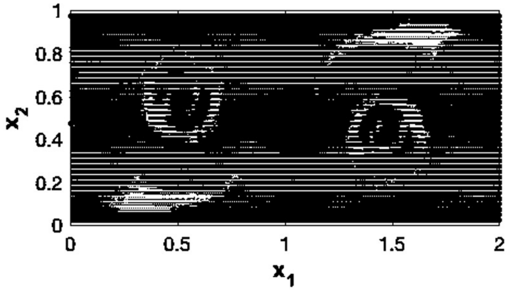
$$H_0 \approx Z^1;$$
$$H_1 \approx Z^2;$$
$$H_2 \approx 0$$

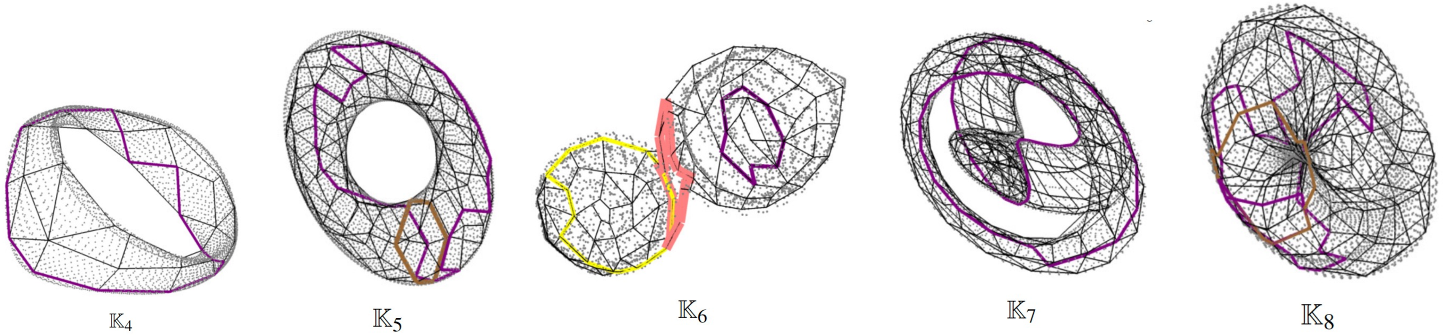
1) Local approximation by *d*-disks using SVD => short and noisy time series *can* be handled.

2) Build a cell complex keeping track of the gluing prescriptions => 3D+ *can* be handled.

3) Compute homologies and orientability properties of the cell complex => the structure can be identified.

- BraMAH used to detect qualitatively different particle dynamics in a fluid flow and therefore Lagrangian Coherent Sets (LCSs)

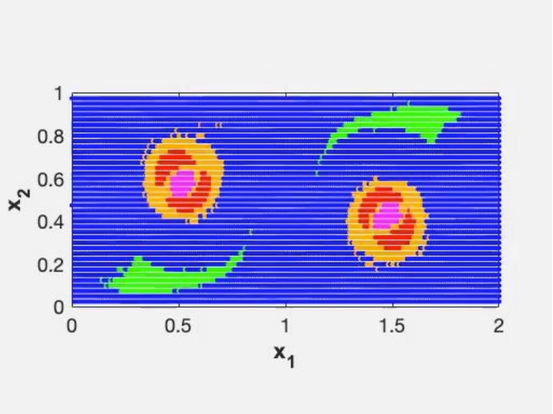




G.D. Charó, G. Artana and D. Sciamarella

Cell complex	β_0	β_1	β_2	o	w	Topological class	Colour
$\mathbb{K}_4$	1	1	0	x	x	I	Green
$\mathbb{K}_5$	1	2	1	x	x	II	Magenta
$\mathbb{K}_6$	1	3	0	✓	x	III	Blue
$\mathbb{K}_7$	1	1	0	✓	✓	IV	Red
$\mathbb{K}_8$	1	2	1	✓	✓	V	Orange

Table 2. Betti numbers β_k ($k = 0, 1, 2$), orientability chain o and weak boundary w for the cell complexes obtained in the BraMAH analysis of the DDG flow example.



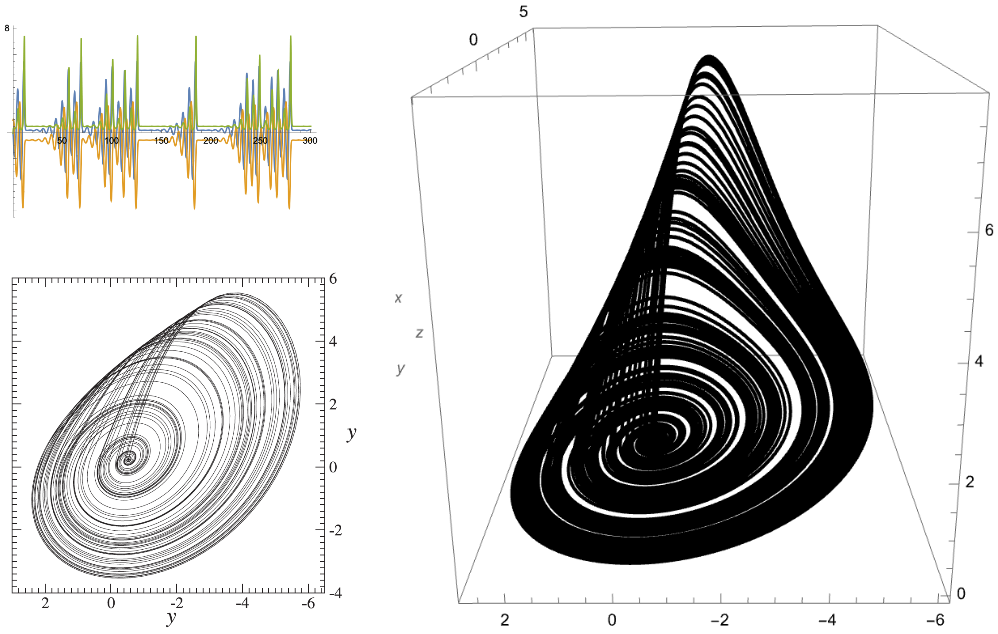
Charó, G., Artana, G., & Sciamarella, D. (2021). Topological colouring of fluid particles unravels finite-time coherent sets. Journal of Fluid Mechanics, 923, A17. doi:10.1017/jfm.2021.561

- Are homologies good enough to unveil the topological structure of a flow?

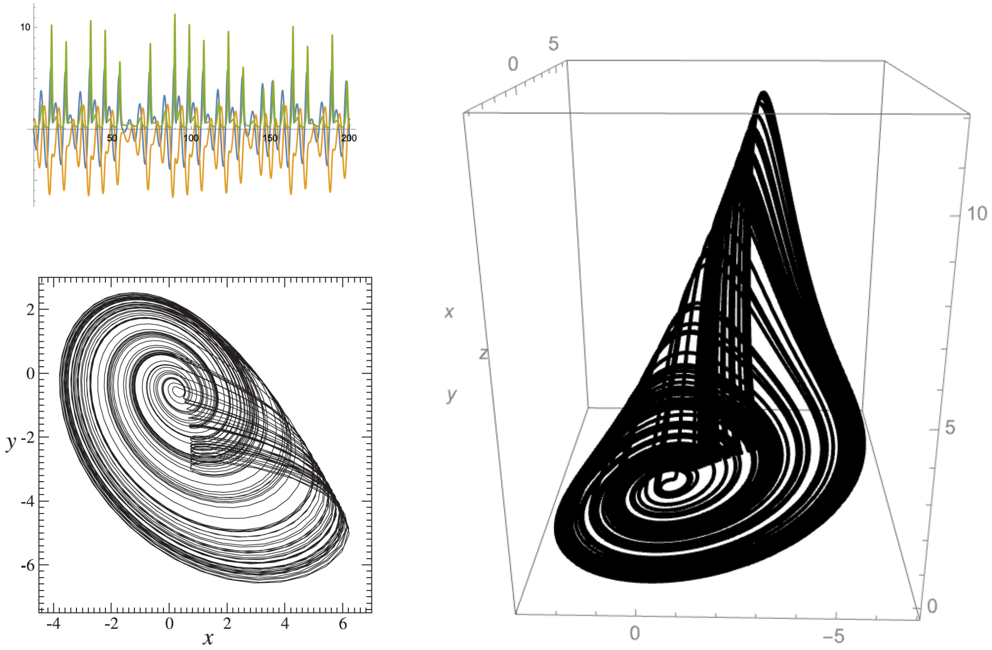
Homologies cannot distinguish between two different attractors produced by the Rössler dynamical system with different parameter values (spiral Rossler attractor with a = 0.343295 on the left and the funnel Rössler attractor with a = 0.492 on the right).

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases}$$

Spiral Rössler attractor



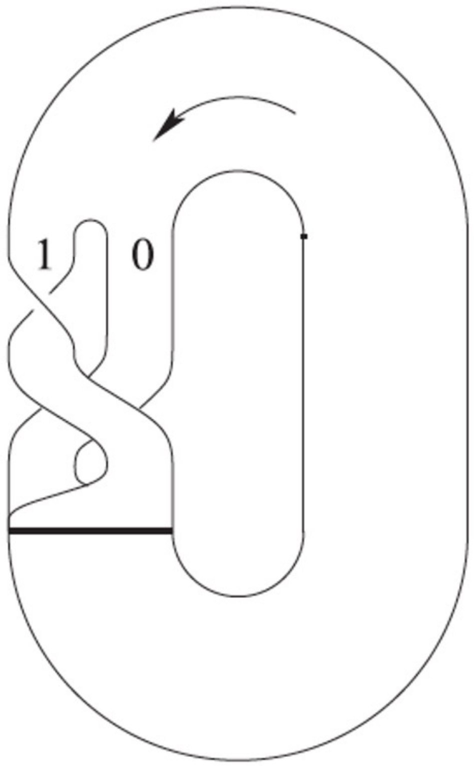
Funnel Rössler attractor



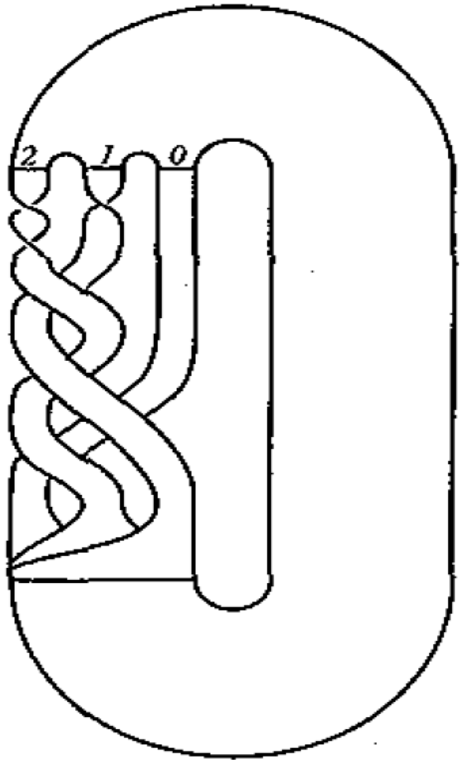
- Are homologies good enough to unveil the topological structure of a flow?

The template or knot-holder does distinguish between both: the spiral Rössler attractor has two strips (0, 1) while the funnel Rössler attractor has three strips (0,1,2).

Spiral Rössler attractor



Funnel Rössler attractor

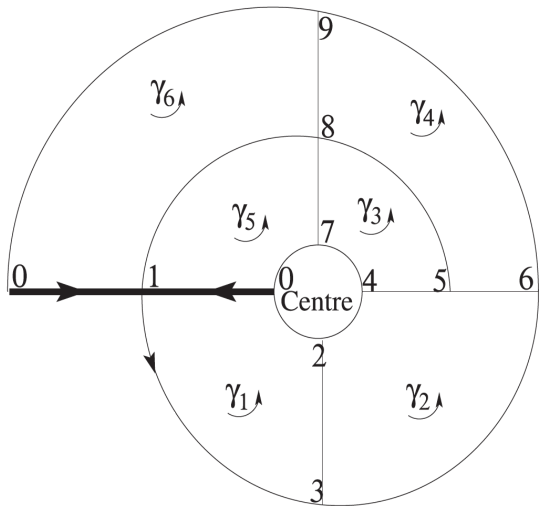


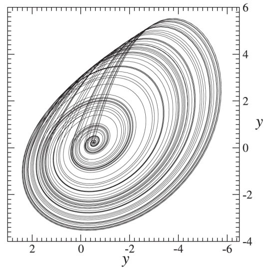
- Uncovering the missing link to unveil the topological structure of a flow

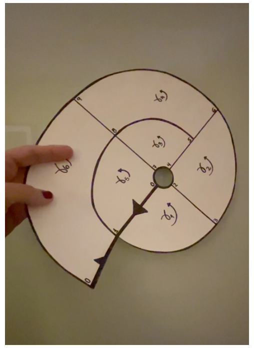
Where are these strips in a BraMAH cell complex? The spiral and funnel Rössler attractors are homologically equivalent: **they have both one hole in the centre** ($H_1 = Z^1$).

But there is more information in a cell complex than the one contained in its homology groups… for instance, the **joining lines**! They can be detected as the 1-cells shared by at least three 2-cells (thick lines).

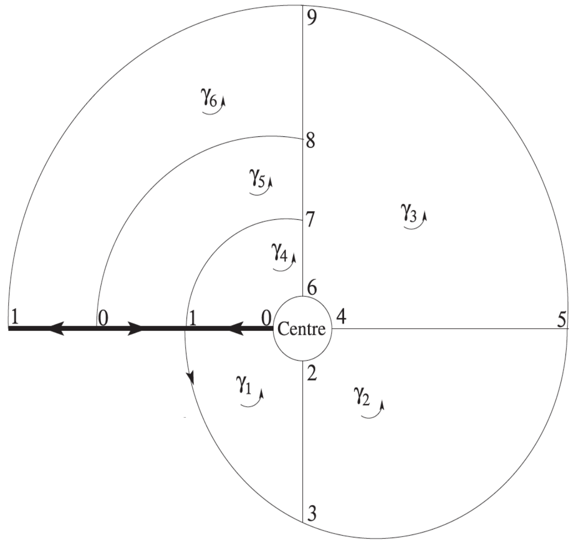
Planar diagrams and kirigamis of cell complexes built from the spiral and funnel Rössler point clouds: notice that the recipe to **scotch** the cell complexes is different.

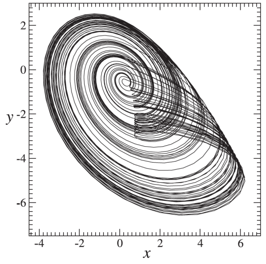






$$\mathcal{H}_1(K(\mathrm{R})) = [[\langle 0,2\rangle - \langle 0,7\rangle + \langle 2,4\rangle + \langle 4,7\rangle]]$$





$$\mathcal{H}_1(K(\mathrm{R}_3)) = [[\langle 0,2\rangle - \langle 0,6\rangle + \langle 2,4\rangle + \langle 4,6\rangle]]$$

- Uncovering the missing link to unveil the topological structure of a flow

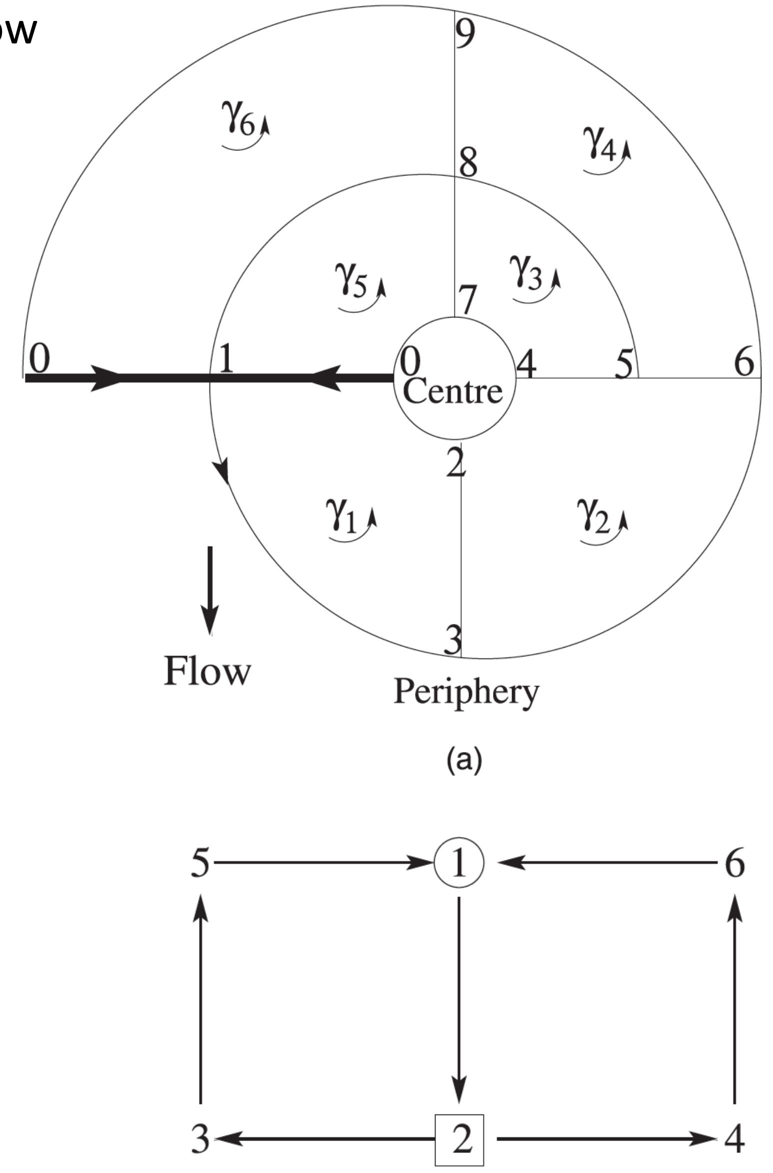
But there is something else that is very important and that is missing in a cell complex representing a branched manifold.

This is the *flow* or, in other words, *time*!

Homologies do not consider the action of the flow on the cell complex…

In order to take the flow on the complex into account, the cell complex will be endowed with a directed graph that prescribes the flow direction between its highest-dimensional cells.

Definition 1. A **templex** $T \equiv (K, G)$ is made of a complex K of dimension $\dim(K) = \kappa$ and a digraph $G = (N, E)$ whose underlying space is a branched κ-manifold associated with a dynamical system, such that (i) the nodes N are the κ-cells of K and (ii) the edges E are the connections between the κ-cells allowed by the flow.



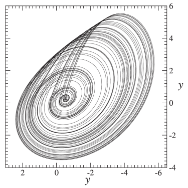
(a)

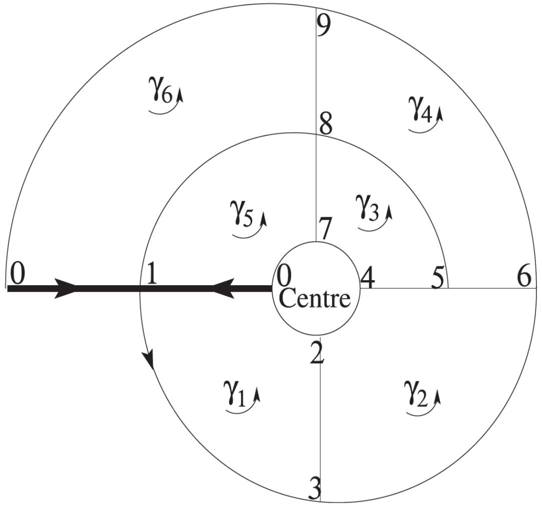
(b)

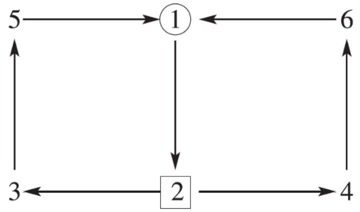
- Template + complex = Templex

Spiral Rössler attractor

T(R)=(K(R),G(R))



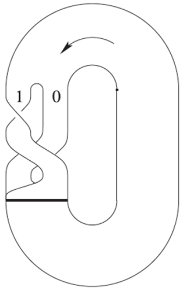




Stripexes in T(R):

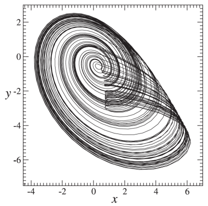
twisted $\underline{1} \to 2 \to 4 \to 6 \to \underline{1}$,

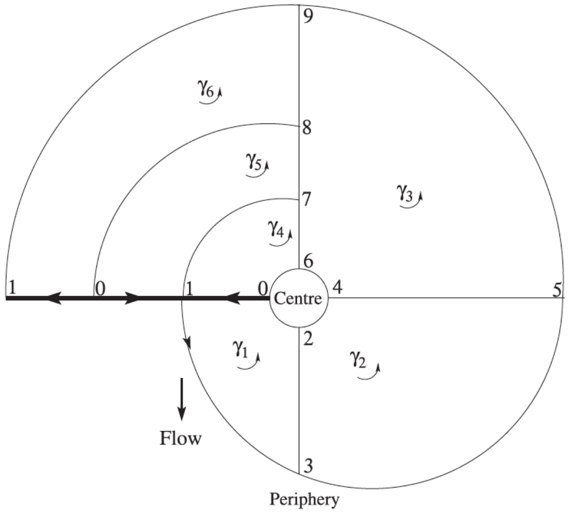
$\underline{1} \to 2 \to 3 \to 5 \to \underline{1}$.

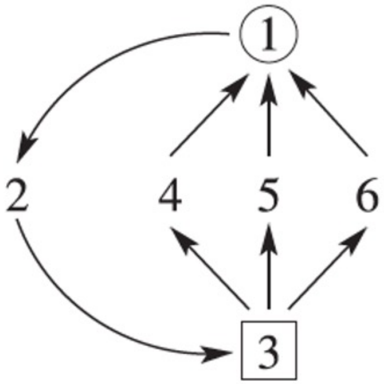


Funnel Rössler attractor

T(R3)=(K(R_3),G(R_3))





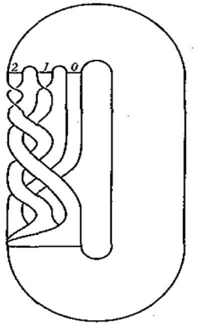


Stripexes in T(R_3):

$\underline{1} \to 2 \to 3 \to 4 \to \underline{1}$,

twisted $\underline{1} \to 2 \to 3 \to 5 \to \underline{1}$,

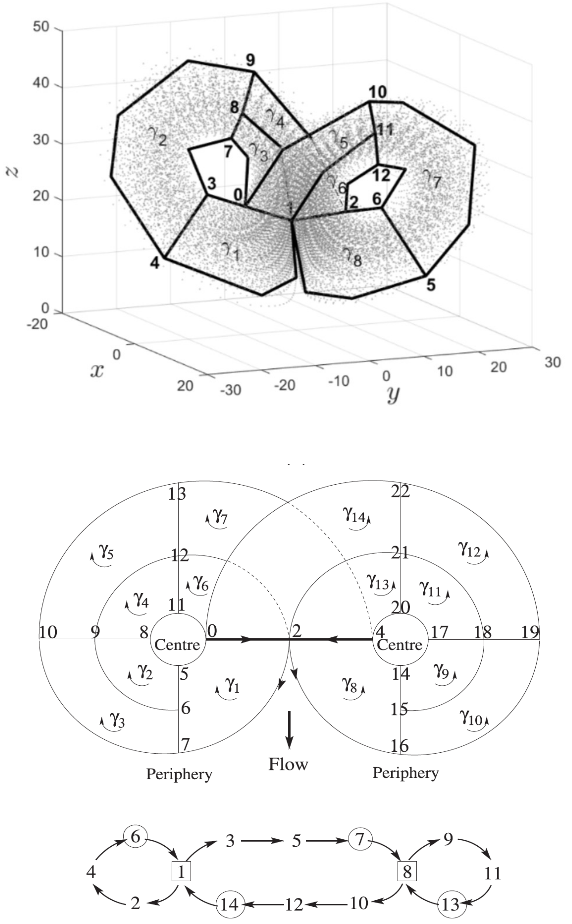
$\underline{1} \to 2 \to 3 \to 6 \to \underline{1}$,



- A templex for the Lorenz 63 attractor

$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = Rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases}$$

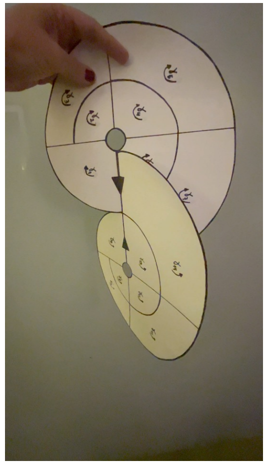
$r = 28, s = 10, \text{and } b = 8/3$

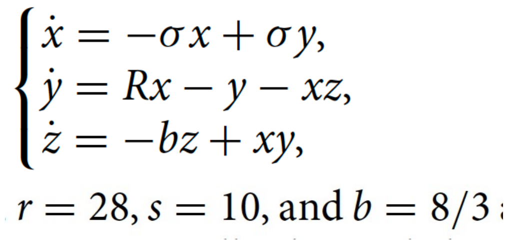


Four stripexes in T(L)=(K(L),G(L)):

$$\left.\begin{array}{l} c_1 \equiv \underline{1} \to 2 \to 4 \to 6 \to \underline{1}, \\ c_2 \equiv \underline{8} \to 9 \to 11 \to 13 \to \underline{8}, \end{array}\right\} \text{strong cycles}$$

$$\left.\begin{array}{l} \boxed{c_{3_1} \equiv \underline{1} \to 3 \to 5 \to 7 \to \underline{8}} \\ \boxed{c_{3_2} \equiv \underline{8} \to 10 \to 12 \to 14 \to \underline{1}.} \end{array}\right\} \text{weak cycles}$$

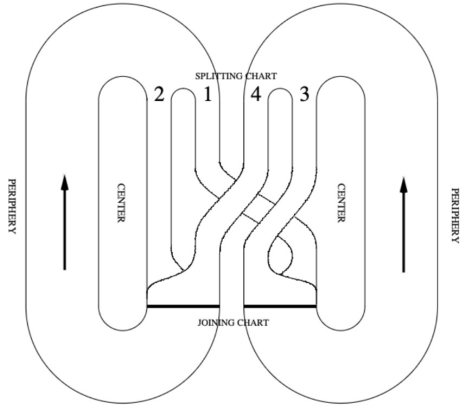




Kirigami L63

The weak cycles that form the **two twisted stripexes** correspond to a single generatex of order 2.

This is consistent with the L63 template that has four stripexes and a Poincaré section divided into two components.



https://git.cima.fcen.uba.ar/sciamarella/templex-properties
Forthcoming version in Python.

- How can we describe the changing topology of a random attractor?

The topological characterization of noise-driven chaos is a challenging issue that is crucial in the understanding of complex systems, where part of the dynamics remains unresolved and is modelled as noise.

While additive noise in a system of equations will blur the topological structure, multiplicative noise may radically change it [Chekroun et al, 2011].

[Charó et al, 2021] extended the concept of a branched manifold to account for the integer-dimensional set in phase space that robustly supports the system's invariant measure at each instant.

Such a branched manifold, however, does not contain any information about the future or the past of the invariant measure. The branched manifold is now itself time-dependent.

LOrenz Random Attractor (LORA)

$$dx = s(y - x)dt + \sigma x dW_t,$$
$$dy = (rx - y - xz)dt + \sigma y dW_t,$$
$$dz = (-bz + xy)dt + \sigma z dW_t.$$
$$r = 28, s = 10, \text{and } b = 8/3$$

- How can we describe the changing topology of a random attractor?

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Open • Submitted: 09 June 2021 • Accepted: 15 September 2021 • Published Online: 12 October 2021

Noise-driven topological changes in chaotic dynamics

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LOrenz Random Attractor (LORA)

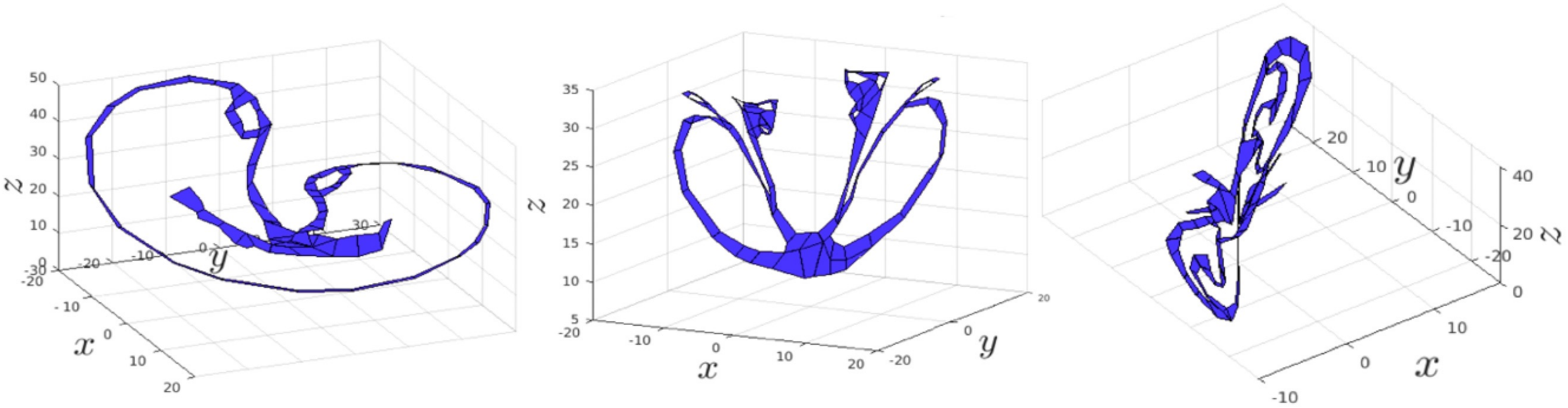
$$\mathrm{d}x = s(y - x)\mathrm{d}t + \sigma x \mathrm{d}W_t,$$

$$\mathrm{d}y = (rx - y - xz)\mathrm{d}t + \sigma y \mathrm{d}W_t,$$

$$\mathrm{d}z = (-bz + xy)\mathrm{d}t + \sigma z \mathrm{d}W_t.$$

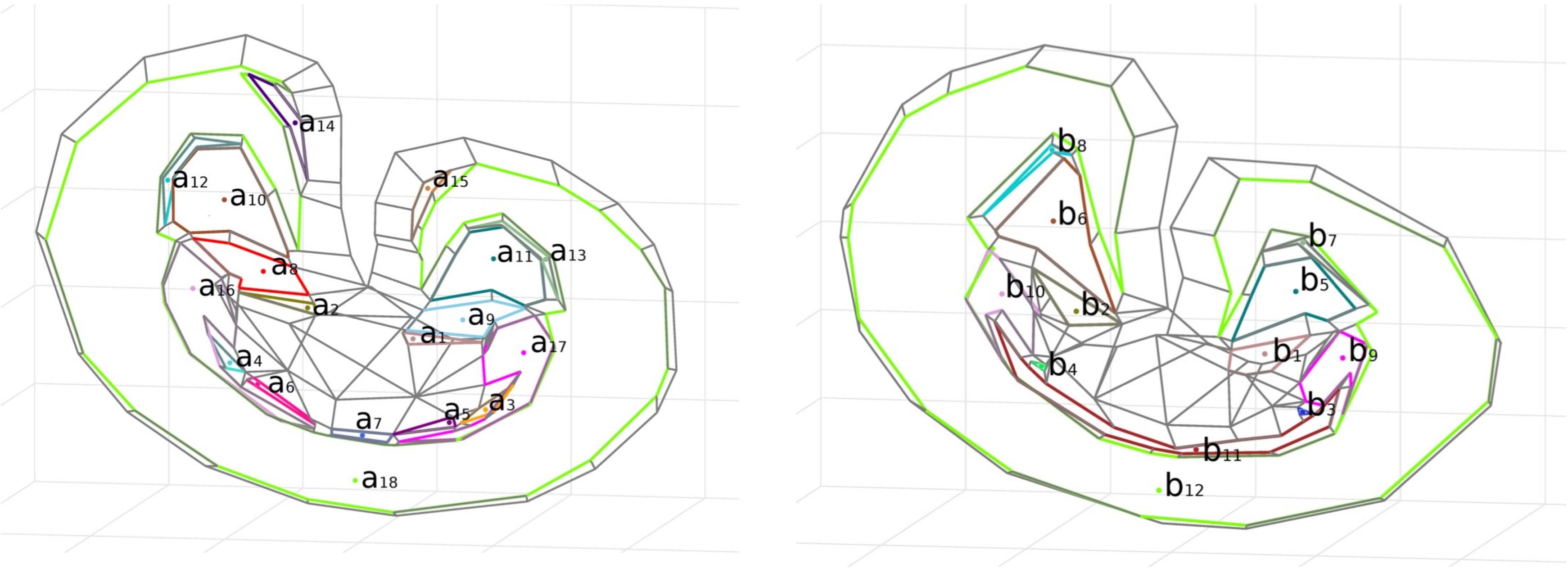
$$r = 28, s = 10, \text{and } b = 8/3$$

One cell complex per snapshot!



- How can we describe the changing topology of a random attractor?

How can we track changes between different cell complexes? Tracking holes!



$a_1 \to b_1$, $a_2 \to b_2$

$a_3, a_4, a_5, a_6, a_7 \to b_{11}$

$a_{10} \to b_6$, $a_{11} \to b_5$, $a_{12} \to b_8$, $a_{13} \to b_7$

$a_{14}, a_{15}, a_{18} \to b_{12}$

$a_{16} \to b_{10}$, $a_{17} \to b_9$

- How can we describe the changing topology of a random attractor?

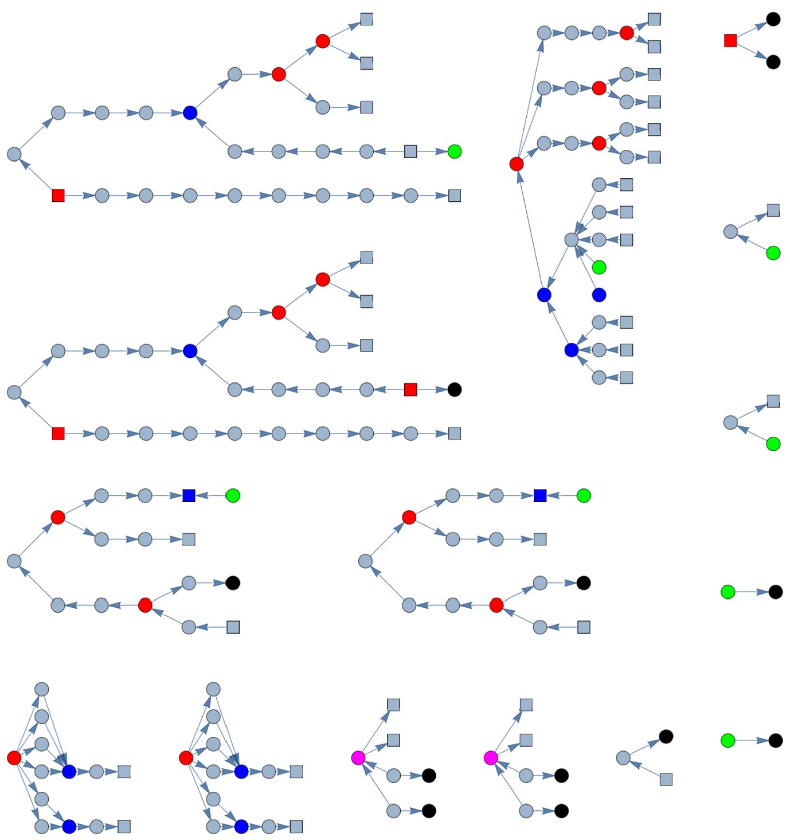
For a random attractor, the digraph does not connect cells, but holes of successive time cell complexes.

***A* random 2-templex R = (K, D)** *is an indexed family* K *of BraMAH 2-complexes and a digraph* D.

The digraph D is presented as a tree plot on the left. It has 15 singly connected components, each of which tells the story of one or several holes.

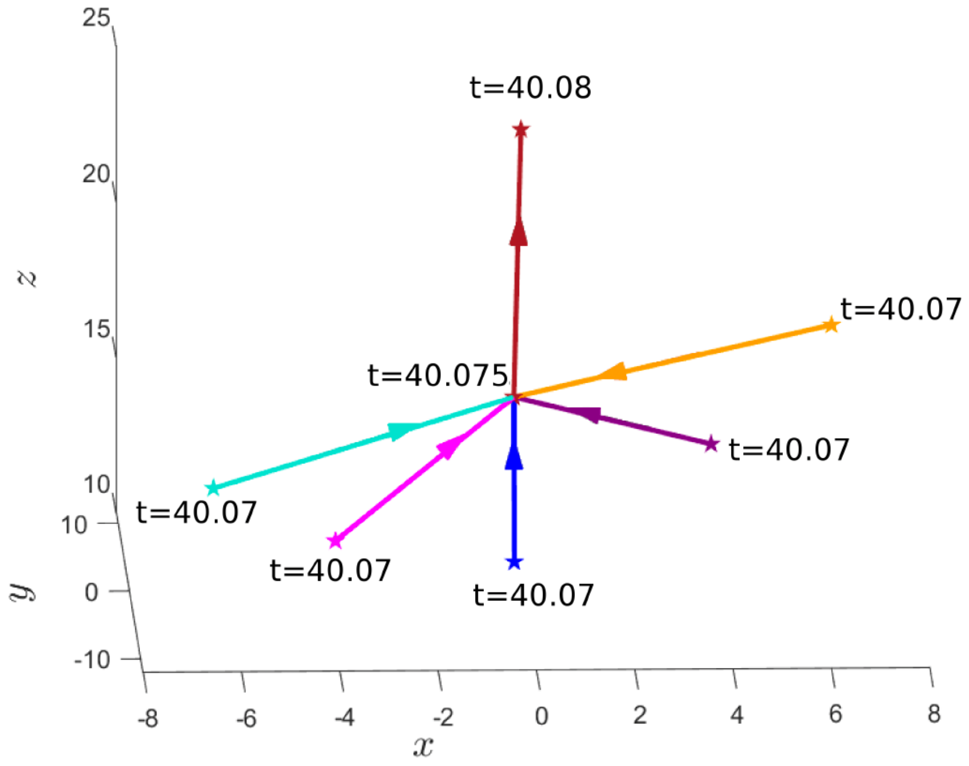
Tipping points can be identified and classified using the digraph.

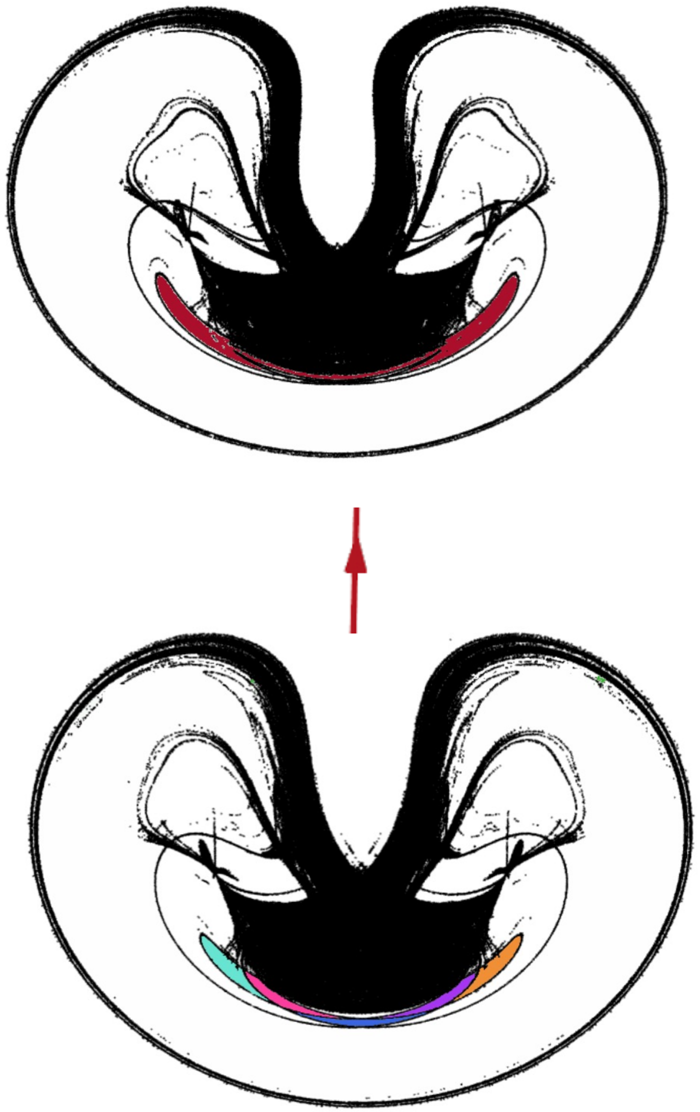
They are highlighted in different colours according to the type of event: creation in green, destruction in black, splitting in red, merging in blue, and merging followed immediately by splitting in magenta.



- How can we describe the changing topology of a random attractor?

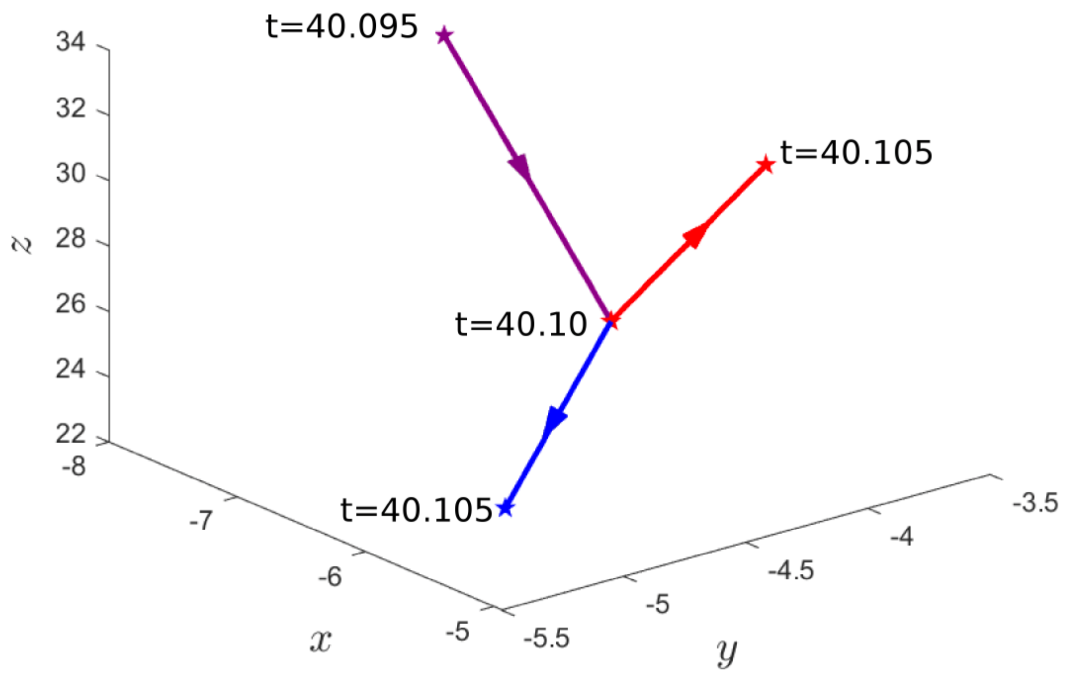
Merging event

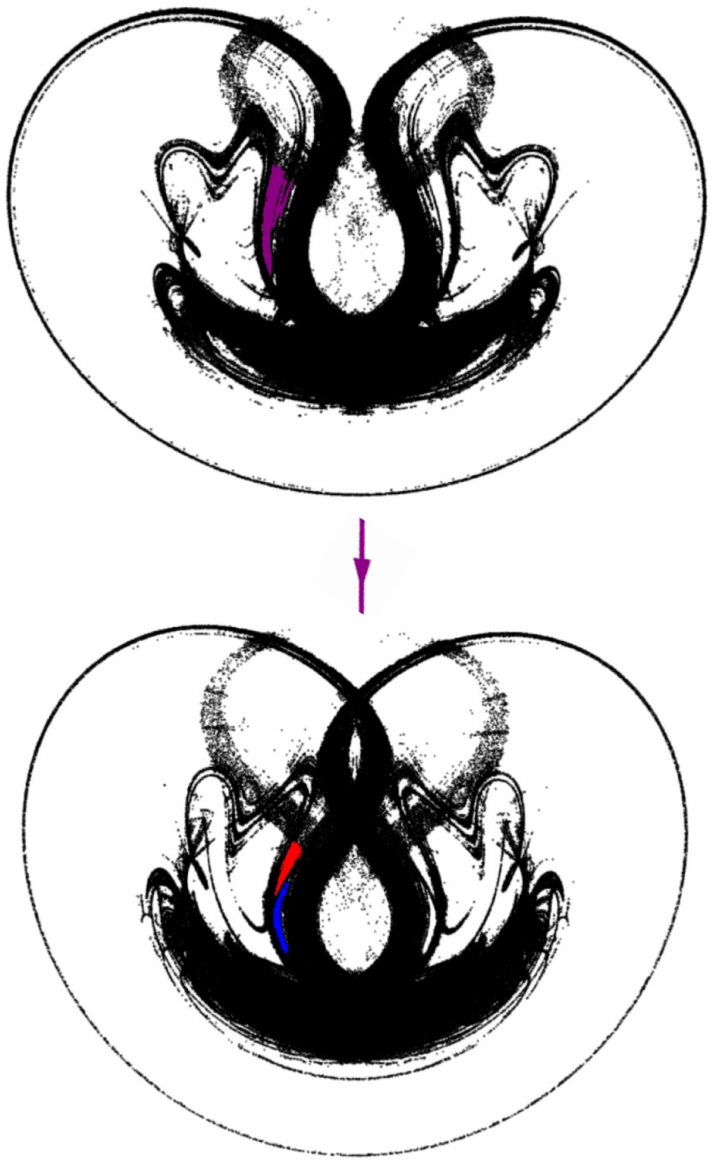




- How can we describe the changing topology of a random attractor?

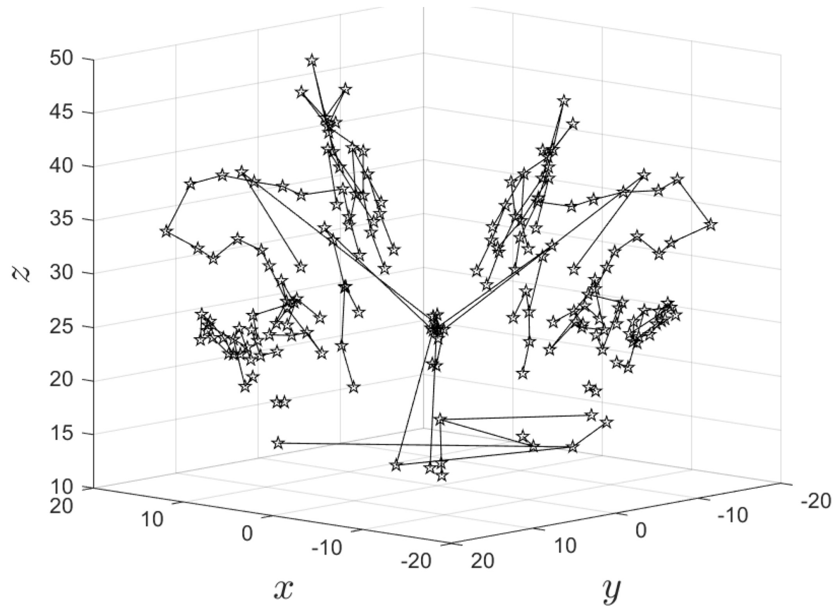
Splitting event





- How can we describe the changing topology of a random attractor?

A better picture of how the holes are evolving in the system's phase space can be gained by using the coordinates of the barycenters of the holes for an embedding of the digraph into this space.



A constellation is the set of immersed nodes and edges forming a connected component in the digraph of a random templex.

Each node is immersed in the phase space using the coordinates of the corresponding hole's barycentre.

- To conclude

In this talk, we introduced novel tools into algebraic topology and used them to provide insights into the behaviour of deterministic and stochastically perturbed chaotic attractors.

- ✓ Attractors in state space can be studied using Branched Manifold Analysis through Homologies: this strategy constructs a cell complex from a cloud of points in state space and uses homology groups to characterize its topology.
- ✓ The approach, however, does not consider the action of the flow on the cell complex. The procedure is here extended to take this fundamental property into account, as done with templates.
- ✓ The goal is achieved endowing the cell complex with a directed graph that prescribes the flow direction between its highest-dimensional cells.
- ✓ The tandem of cell complex and directed graph, baptized templex, is shown to allow for a sophisticated characterization of chaotic attractors and for an accurate classification of them.
- ✓ In a random templex, there is one complex per snapshot of the random attractor and the digraph connects the generators or "holes" of successive cell complexes.
- ✓ Tipping points appear in a random templex as drastic changes of its holes in motion, namely their birth, splitting, merging, or death.

"All linear systems are alike; each nonlinear system is nonlinear according to its templex properties".

https://www.insu.cnrs.fr/fr/cnrsinfo/templex-une-nouvelle-mathematique-pour-comprendre-le-chaos

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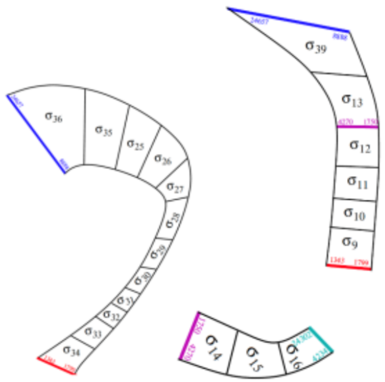

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Templex : une nouvelle mathématique pour comprendre le chaos

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