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► To cite this version:

H. Kamalinia, Andrea Barbarulo, Bing Tie. A coupled acoustic/elastic discontinuous Galerkin finite element method: Application to ultrasonic imaging of 3D-printed synthetic materials. *Computers & Structures*, 2023, 291, pp.107208. 10.1016/j.compstruc.2023.107208 . hal-04276338

HAL Id: hal-04276338

<https://hal.science/hal-04276338>

Submitted on 23 Nov 2023

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A coupled acoustic/elastic discontinuous Galerkin finite element method: application to ultrasonic imaging of 3D-printed synthetic materials

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Abstract

We present the derivation of upwind numerical fluxes for the space discontinuous Galerkin (dG) finite element method applied to the numerical modeling of wave propagation in multidimensional coupled acoustic/elastic media. The space dG method is formulated using the first-order velocity-pressure and velocity-stress systems for acoustic and elastic media, respectively. After eigenanalysis of the first-order hyperbolic systems highlighting the eigenmodes of wave propagation, the upwind numerical fluxes on the acoustic/acoustic and acoustic/elastic interface are obtained in terms of exact solutions of relevant Riemann problems. Thanks to the proposed approach, explicit closed-form expressions of the upwind numerical fluxes are obtained on the acoustic/elastic interface for the general case of multidimensional anisotropic heterogeneous solid media coupled with acoustic fluids. The developed numerical fluxes are validated by analytical/numerical comparison considering the example of an acoustic domain with a circular elastic inclusion. Finally, the coupled solver is used to perform a multiparametric study on the microstructure's echogenicity in a 3D-printed synthetic material under ultrasonic imaging.

Keywords: Space discontinuous Galerkin method; Upwind numerical flux; Wave propagation; Acoustic/elastic coupling; Echogenicity.

1. Introduction

The coupled acoustic/elastic wave propagation exists in a broad range of problems, including seismic waves that interact with the outer core of the earth or the ocean, or ultrasonic waves, generated by transducers for medical imaging of patients, which propagate through human tissues and bones, and the blood inside the organs. In these problems, fluid-solid interfaces exist, and their interactions with the wave should be carefully taken into account.

An accurate numerical simulation of the acoustic and elastic wave propagation in a heterogeneous medium can be a difficult task if the characteristic size of the medium is comparable to the wavelength such that the wave interacts with the heterogeneities. This problem becomes even more challenging when dealing with multi-physics coupling. In this case, high-order methods are required to reduce the numerical dispersion and dissipation errors, and more effort is needed to maintain the high accuracy on the coupled interface [1].

Several numerical methods are introduced in the literature for solving both second-order and first-order wave equations in elastic and acoustic domains, including the acoustic/elastic interface. Di Bartolo *et al.* propose a finite difference framework with an optimized memory use based on equivalent staggered-grid schemes [2]. Among high-order finite element (FE) methods, the spectral element method is used in [3–5] to model seismic waves with different strategies of taking into account the acoustic/elastic coupling on the fluid-solid interfaces. The spatial discontinuous Galerkin (dG) is a well-established finite element method

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18 in which strategies to treat flux on element interfaces developed in the finite volume framework can be
19 integrated, which has proven to be particularly advantageous for hyperbolic problems such as the one of
20 elastic or acoustic wave propagation. Moreover, as the unknown fields are discontinuous from one element to
21 another, the mass matrix is split into independent elementary matrices. Hence, the development of massively
22 parallel solvers thus becomes straightforward [6, 7].

23 Unlike the continuous FE method, the space dG FE method is based on the use of discontinuous basis
24 functions between finite elements. However, this discontinuity must be controlled by defining appropriated
25 numerical fluxes on element interfaces. Hence, developing and implementing the appropriate numerical fluxes
26 is essential to the success of the dG method. A rigorous and accurate numerical flux can be obtained from the
27 exact solution of the Riemann problem on element interfaces [1, 6–11]. Otherwise, the Lax-Friedrich flux or
28 the penalty flux is another rigorous alternative for calculating the numerical fluxes [12–18]. In the past two
29 decades, the dG method was extensively used for numerical modeling of wave propagation in elastic media
30 [6–9, 19–21], as well as in more complex media involving multi-physical interfaces. More particularly for
31 the elastic-acoustic coupling, Wilcox *et al.* introduced a unified dG framework for isotropic elastic/acoustic
32 media [1], while Zhan *et al.* considered a dG framework for arbitrary anisotropic elastic/acoustic media
33 [10]. They all used the velocity-strain formulation and developed the upwind numerical fluxes by exactly
34 solving the Riemann interface problem. Zhan *et al.* then extended their approach to a more general case
35 involving poroelastic media [11]. However, unlike the present work, the method developed by Zhan *et al.*
36 uses the velocity-strain formulation, and the numerical fluxes are developed in cartesian coordinates. Here
37 we keep the most general coordinate-free expressions for development of the numerical fluxes. Besides, using
38 the first-order elastic velocity-strain and acoustic velocity-pressure formulations, Ye *et al.* obtained a stable
39 algorithm with a penalty flux defined on element interfaces [13], while Guo *et al.*, using the first-order elastic
40 velocity-stress and acoustic velocity-pressure formulations, presented a weight-adjusted dG method also with
41 a dissipative penalty flux defined on element interfaces [16].

42 This work aims to present a development of the upwind numerical fluxes in the most general case of
43 multidimensional anisotropic elastic/acoustic media with discontinuous material properties using the wave-
44 oriented variational framework previously proposed in [9, 21]. Within the framework of the first-order
45 elastic velocity-stress and acoustic velocity-pressure formulations, the numerical fluxes on the interfaces,
46 including the elastic/acoustic interfaces, are obtained in terms of the exact solution of the associated Riemann
47 problem, unlike the penalty flux method implemented in [13]. Moreover, unlike the approach used in
48 [10, 11], we use a coordinate-free vector and tensor notation and a wave-oriented eigenanalysis of the first-
49 order hyperbolic system. Thanks to the proposed approach, for the general anisotropic elasticity tensor,
50 the Riemann problem is analytically solved, and explicit closed-form expressions of the numerical fluxes in
51 terms of wave propagation modes are obtained. These explicit expressions of fluxes are implemented in our
52 code, and they are computed only once and stored at the beginning of each dynamic calculation.

53 The developed coupled acoustic/elastic solver is then employed to investigate the impact of the mi-
54 crostructure of 3D-printed synthetic tissues on medical ultrasonic imaging. The microstructure of biological
55 tissues plays a crucial role in their visibility under ultrasonic imaging. At the microstructural level, biological
56 tissues are composed of cells, extracellular matrix, and blood vessels. Each has distinct acoustic proper-
57 ties that can affect the echoes returned to the ultrasonic probe and the final reconstructed medical image.
58 Recently, anatomical phantom twins are beginning to be used for surgical training and preparation for oper-
59 ations in some specific medical applications. These synthetic tissues created using different manufacturing
60 techniques, including 3D-printing, are expected to mimic the acoustic properties of biological tissues in the
61 3-7 MHz ultrasonic imaging frequency range. The matrix of these synthetic composite materials is typically
62 composed of a quasi-incompressible material such as elastomers or gels, closely resembling the material
63 properties of biological tissues. The selection of different materials with different mechanical properties for
64 the inclusions in this synthetic composite is intended to enhance the visibility of the material thickness in
65 ultrasonic imaging by scattering the ultrasonic wave and to make the resulting image more similar to those
66 of the biological tissue [22–24].

67 In this work, a two-dimensional model of a 3D printed composite material containing circular inclusions
68 is studied numerically to examine how the size and area fraction of the inclusion phase affect the ultrasonic
69 images. The resulting ultrasonic images, called B-mode images, are obtained using a simplified model of a

70 sequential linear array transducer (see [25] for details).

71 The paper is organized as follows. In Section 2, we first present the first-order acoustic velocity-pressure
72 formulation and then review the first-order elastic velocity-stress wave equations. An intrinsic tensorial
73 notation is used and allows us to develop unified strong and variational frameworks [9]. Second, an analysis
74 of the eigenstructure of the governing hyperbolic systems is given. In Section 3, the numerical flux for
75 the acoustic/acoustic and acoustic/elastic interfaces are developed by exactly solving the relevant Riemann
76 problem, and their explicit closed-form expressions are presented. In Section 4, the coupled acoustic/elastic
77 dG solver using the developed upwind numerical fluxes is validated by numerical/analytical comparison
78 considering the example of an acoustic domain with a circular elastic inclusion. Finally, in Section 5, the
79 developed solver is applied to a 3D-printed microstructure, and virtual B-mode images of the ultrasonic wave
80 propagation in it are reconstructed. A first parametric study on the effects of microstructural parameters
81 on the echogenicity of the printed material is carried out.

82 2. Wave governing equations: strong form, variational framework, and characteristic structure

83 To apply the space dG FE method to the coupled acoustic/elastic wave propagation problem, the first-
84 order hyperbolic governing equations in both elastic and acoustic media are discussed in the following
85 sections, with the introduction of unified strong and variational frameworks for the coupled system.

86 2.1. First-order \mathbf{v} - p acoustic wave governing equations

87 Let us consider an acoustic fluid $\Omega \subset \mathbb{R}^d$ of space dimension d ($d = 1, 2, 3$) and in a time interval $[0, t_f]$.
88 The governing equations of acoustic wave propagation in the form of a first-order velocity-pressure system
89 can be written in the following generic form: $\forall (\mathbf{x}, t) \in \Omega \times]0, t_f[$

$$89 \quad \mathbf{M}(\partial_t \mathbf{U}) + \mathbf{A}^{\partial_x}(\mathbf{U}) = \mathbf{0} \quad \text{or} \quad \begin{cases} \rho_f \partial_t \mathbf{v} - \nabla_{\mathbf{x}} p & = \mathbf{0} \\ \lambda_f^{-1} \partial_t p - \operatorname{div}_{\mathbf{x}} \mathbf{v} & = 0 \end{cases} \quad (1)$$

90 where the velocity and pressure fields are the primary unknowns, and ρ_f and λ_f respectively denote the
91 density and the bulk modulus of the fluid. The tensorial compact form in (1) has been proposed by the
92 authors in [21] within the frameworks of elastic waves. No source term is considered in the equilibrium
93 equation without loss of generality of the purpose of the present work. The generalized unknown $\mathbf{U}(\mathbf{x}, t) =$
94 $(\mathbf{v}(\mathbf{x}, t) \ p(\mathbf{x}, t))^T$ consists of \mathbf{v} the velocity and p the pressure, with $(\cdot)^T$ the adjoint operator. Hence,
95 $\mathbf{U}(\mathbf{x}, t)$ is a field in $\mathbb{R}^d \times \mathbb{R}$ and defined over the open set $\Omega \times]0, t_f[$. The operator \mathbf{M} and space derivative
96 operator \mathbf{A}^{∂_x} are defined as follows: $\forall \mathbf{W} = (\mathbf{w} \ q)^T$

$$96 \quad \mathbf{M} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} \rho_f \mathbf{w} \\ \lambda_f^{-1} q \end{pmatrix}, \quad \mathbf{A}^{\partial_x} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} -\nabla_{\mathbf{x}} q \\ -\operatorname{div}_{\mathbf{x}} \mathbf{w} \end{pmatrix} \quad (2)$$

97 with the usual space gradient and divergence operators $\nabla_{\mathbf{x}}$ and $\operatorname{div}_{\mathbf{x}}$.

98 It is also useful to define the dot product in the vectorial space $\mathbb{R}^d \times \mathbb{R}$: $\forall \mathbf{W}_i = (\mathbf{w}_i \ q_i)^T, (i = 1, 2)$,

$$98 \quad \mathbf{W}_1 \cdot \mathbf{W}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2 + q_1 q_2 \quad (3)$$

99 Herein, all the vectors and tensors are denoted using bold letters.

100 On the boundary ∂D of any subdomain $D \subseteq \Omega$, the flux operator $\mathbf{F}_{\mathbf{n}}$ for $\mathbf{n} = n_i \mathbf{e}_i$, (herein, the Einstein
101 summation convention is systematically used), the outward unit normal vector defined on ∂D , and associated
102 to the first-order system (1), is: $\forall \mathbf{W} = (\mathbf{w} \ q)^T$,

$$102 \quad \mathbf{F}_{\mathbf{n}}(\mathbf{W}) = \mathbf{A}_{\mathbf{n}}(\mathbf{W}) = \mathbf{A}_{\mathbf{n}} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} -q \mathbf{n} \\ -\mathbf{n} \cdot \mathbf{w} \end{pmatrix} \quad (4)$$

103 In (4), the subscript index “ \mathbf{n} ” indicates the dependency of $\mathbf{F}_{\mathbf{n}}$ and of $\mathbf{A}_{\mathbf{n}}$ on \mathbf{n} .

104 Finally, to complete the definition of the acoustic wave propagation framework, the following boundary
 105 conditions with a prescribed pressure p_D and a prescribed normal velocity v_{nN} are considered:

$$p = p_D \text{ on } \partial\Omega_D \times]0, t_f[; \mathbf{v} \cdot \mathbf{n} = v_{nN} \text{ on } \partial\Omega_N \times]0, t_f[\quad (5)$$

106 with $\partial\Omega_N \cup \partial\Omega_D = \partial\Omega$ and $\partial\Omega_N \cap \partial\Omega_D = \emptyset$. On the other hand, for the initial conditions, we have:

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}) \text{ and } \partial_t p(\mathbf{x}, 0) = \lambda_f \operatorname{div}_{\mathbf{x}} \mathbf{v}_0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \quad (6)$$

107 **Remark 2.1.** The first-order velocity-pressure governing equations (1) can also be written in the following
 108 equivalent form:

$$\partial_t \mathbf{U} + \overline{\mathbf{A}}^{\partial_x}(\mathbf{U}) = \mathbf{0} \quad \text{or} \quad \begin{cases} \partial_t \mathbf{v} - \rho_f^{-1} \nabla_{\mathbf{x}} p = \mathbf{0} \\ \partial_t p - \lambda_f \operatorname{div}_{\mathbf{x}} \mathbf{v} = 0 \end{cases} \quad (7)$$

109 with:

$$\overline{\mathbf{A}}^{\partial_x} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} -\rho_f^{-1} \nabla_{\mathbf{x}} q \\ -\lambda_f \operatorname{div}_{\mathbf{x}} \mathbf{w} \end{pmatrix}, \quad \overline{\mathbf{A}}_{\mathbf{n}} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} -\rho_f^{-1} q \mathbf{n} \\ -\lambda_f \mathbf{n} \cdot \mathbf{w} \end{pmatrix} = \mathbf{M}^{-1} \cdot \mathbf{A}_{\mathbf{n}} \quad (8)$$

110 The direct use of the strong form (7) leads to an inconsistent numerical flux on the physical interfaces.
 111 This issue has already been discussed in the elastic case in [9], so in-depth consideration is omitted in this
 112 paper. Hence, hereafter, the strong form (1) and the associated Jacobian operator are considered for the
 113 development of the numerical fluxes.

114 **Remark 2.2.** When applied to the first-order velocity-strain wave equations used in [1], the corresponding
 115 tensorial compact form can be written as follows for the generalized unknown $\mathbf{U}(\mathbf{x}, t) = (\mathbf{v}(\mathbf{x}, t) \boldsymbol{\varepsilon}(\mathbf{x}, t))^T$
 116 with $\boldsymbol{\varepsilon}$ the strain field: $\forall (\mathbf{x}, t) \in \Omega \times]0, t_f[$,

$$\mathbf{M}(\partial_t \mathbf{U}) + \mathbf{A}^{\partial_x}(\mathbf{U}) = \mathbf{0} \quad \text{or} \quad \begin{cases} \rho_f \partial_t \mathbf{v} - \nabla_{\mathbf{x}}(\lambda_f \operatorname{tr}(\boldsymbol{\varepsilon})) = \mathbf{0} \\ \partial_t \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0} \end{cases} \quad (9)$$

118 with the operator \mathbf{M} and the space derivative operator \mathbf{A}^{∂_x} defined as follows: $\forall \mathbf{W} = (\mathbf{w} \boldsymbol{\tau})^T$

$$\mathbf{M} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} \rho_f \mathbf{w} \\ \boldsymbol{\tau} \end{pmatrix}, \quad \mathbf{A}^{\partial_x} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} -\nabla_{\mathbf{x}}(\lambda_f \operatorname{tr}(\boldsymbol{\tau})) \\ -\boldsymbol{\varepsilon}(\mathbf{w}) \end{pmatrix} \quad (10)$$

119 In (9), the infinitesimal strain operator $\boldsymbol{\varepsilon}(\cdot)$ is defined as follows:

$$\boldsymbol{\varepsilon}(\mathbf{w}) = \frac{1}{2} \left(\mathbf{D}_{\mathbf{x}} \mathbf{w} + \mathbf{D}_{\mathbf{x}}^T \mathbf{w} \right) = \frac{\partial \mathbf{w}}{\partial x_i} \otimes_s \mathbf{e}_i \quad (11)$$

120 with “ \otimes_s ” the symmetrized tensor product defined as: $(\mathbf{a} \otimes_s \mathbf{b})_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$. Then, the flux operator
 121 $\mathbf{F}_{\mathbf{n}}$ associated to the first-order system (9) is in fact equal to $\mathbf{A}_{\mathbf{n}}$ the Jacobian operator in the \mathbf{n} direction:
 122 $\forall \mathbf{W} = (\mathbf{w} \boldsymbol{\tau})^T$,

$$\mathbf{F}_{\mathbf{n}}(\mathbf{W}) = \mathbf{A}_{\mathbf{n}}(\mathbf{W}) = \begin{pmatrix} -\lambda_f \operatorname{tr}(\boldsymbol{\tau}) \mathbf{n} \\ -\mathbf{n} \otimes_s \mathbf{w} \end{pmatrix} \quad (12)$$

123 It is worth noticing that the method proposed in this work can be easily applied to this velocity-strain
 124 acoustic formulation.

125 2.2. First-order \mathbf{v} - $\boldsymbol{\sigma}$ elastic wave equations

126 In this section, the governing equations of elastic wave propagation previously introduced in [9, 21] is
 127 presented. We consider the wave propagation in an elastic medium $\Omega \subset \mathbb{R}^d$ of space dimension d ($d = 1, 2, 3$)
 128 and in a time interval $[0, t_f]$. The first-order velocity-stress governing equations is also written in the generic
 129 form: $\forall (\mathbf{x}, t) \in \Omega \times]0, t_f[$

$$\mathbf{M}(\partial_t \mathbf{U}) + \mathbf{A}^{\partial_x}(\mathbf{U}) = \mathbf{0} \quad \text{or} \quad \begin{cases} \rho \partial_t \mathbf{v} - \operatorname{Div}_{\mathbf{x}} \boldsymbol{\sigma} = \mathbf{0} \\ \mathbf{C}^{-1} : \partial_t \boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0} \end{cases} \quad (13)$$

130 with the following definitions of operators:

$$131 \quad \mathbf{M} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} \rho \mathbf{w} \\ \mathbf{C}^{-1} : \boldsymbol{\tau} \end{pmatrix}, \quad \mathbf{A}^{\partial_x} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} -\mathbf{Div}_x \boldsymbol{\tau} \\ -\boldsymbol{\varepsilon}(\mathbf{w}) \end{pmatrix} \quad (14)$$

132 In (14), \mathbf{Div}_x denotes the usual space divergence operator of a tensor.

133 In the elastic case, the generalized unknown is $\mathbf{U}(\mathbf{x}, t) = (\mathbf{v}(\mathbf{x}, t) \boldsymbol{\sigma}(\mathbf{x}, t))^T$ with $\boldsymbol{\sigma}$ the stress unknown. In
134 (13) and (14), ρ is the density of the solid, \mathbf{C} the fourth-order elasticity tensor, and “:” the usual double-dot
135 product between two tensors defined as $(\mathbf{C} : \boldsymbol{\tau})_{ij} = C_{ijkl} \tau_{kl}$.

136 Similar to the acoustic wave framework, it is useful to define the dot product this time in the vectorial
space of $\mathbb{R}^d \times \mathbb{R}^{d \times d_{sym}}$: $\forall \mathbf{W}_i = (\mathbf{w}_i \boldsymbol{\tau}_i)^T, (i = 1, 2)$,

$$\mathbf{W}_1 \cdot \mathbf{W}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2 + \boldsymbol{\tau}_1 : \boldsymbol{\tau}_2 \quad (15)$$

137 The flux operator \mathbf{F}_n on the boundary ∂D of any subdomain $D \subseteq \Omega$ is defined as: $\forall \mathbf{W} = (\mathbf{w} \boldsymbol{\tau})^T$,

$$\mathbf{F}_n(\mathbf{W}) = \mathbf{A}_n(\mathbf{W}) = \begin{pmatrix} -\boldsymbol{\tau} \cdot \mathbf{n} \\ -\mathbf{n} \otimes_s \mathbf{w} \end{pmatrix} \quad (16)$$

138 2.3. dG variational framework

139 Now that all the required operators and parameters are defined, the unified variational framework can be
140 introduced. Within the framework of the discontinuous Galerkin FE method, we look for an approximated
141 solution \mathbf{U}_h , discontinuous across the element interfaces, of the generalized unknown \mathbf{U} . This discontinuous
142 character of \mathbf{U}_h makes it possible to integrate the concept of well-established numerical fluxes within the
143 framework of the finite volume method [6].

144 Let us consider $\mathcal{M}_h = \{\Omega_k\}_k$ a FE mesh of the domain Ω . For the sake of simplicity, from now on, any
145 element Ω_k of the mesh \mathcal{M}_h will be denoted by E and any of the neighboring elements of E by E' . To
146 obtain the dG variational formulation of the coupled acoustic/elastic system (1) and (13) for any element
147 E , the integration by parts formula is used, and the discontinuous flux $\mathbf{F}_n(\mathbf{U}_h)$ on the element boundary
148 ∂E is replaced by a numerical flux $\hat{\mathbf{F}}_n(\mathbf{U}_h, \mathbf{U}'_h)$, which depends on the solution in both E and the adjacent
149 element E' . Then we get: $\forall \mathbf{W}_h(\mathbf{x})$,

$$(\mathbf{W}_h, \mathbf{M}(\partial_t \mathbf{U}_h))_E - (\mathbf{A}^{\partial_x, T}(\mathbf{W}_h), \mathbf{U}_h)_{E+} + \langle \mathbf{W}_h, \hat{\mathbf{F}}_n(\mathbf{U}_h, \mathbf{U}'_h) \rangle_{\partial E} = 0 \quad (17)$$

150 where \mathbf{U}_h and \mathbf{U}'_h are elementary solutions in E and E' , respectively, and $\mathbf{U}_h \neq \mathbf{U}'_h$ on $\partial E \cap \partial E'$. $\mathbf{W}_h(\mathbf{x})$
151 are test functions, with $\mathbf{W}_h(\mathbf{x}) = (\mathbf{w}_h(\mathbf{x}) q_h(\mathbf{x}))^T$ in the acoustic case and $\mathbf{W}_h(\mathbf{x}) = (\mathbf{w}_h(\mathbf{x}) \boldsymbol{\tau}_h(\mathbf{x}))^T$ in
152 the elastic case. In (17), following notations are adopted for volume and surface integrations, respectively:

$$(\mathbf{W}_1, \mathbf{W}_2)_E = \int_E \mathbf{W}_1 \cdot \mathbf{W}_2 \, dV, \quad \langle \mathbf{W}_1, \mathbf{W}_2 \rangle_{\partial E} = \int_{\partial E} \mathbf{W}_1 \cdot \mathbf{W}_2 \, dS \quad (18)$$

153 Applying integration by parts again to (17) leads to the following equivalent form of (17): $\forall \mathbf{W}_h(\mathbf{x})$,

$$(\mathbf{W}_h, \mathbf{M}(\partial_t \mathbf{U}_h))_{E+} + (\mathbf{W}_h, \mathbf{A}^{\partial_x}(\mathbf{U}_h))_{E+} + \langle \mathbf{W}_h, \hat{\mathbf{F}}_n(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}(\mathbf{U}_h) \rangle_{\partial E} = 0 \quad (19)$$

154 It is this variational formulation (19) that is used in the present work to implement the dG solver because
155 it involves the slightly more familiar operator \mathbf{A}^{∂_x} , instead of $\mathbf{A}^{\partial_x, T}$.

156 The study of the numerical fluxes on internal element boundaries $\partial E_{int} = \partial E \setminus (\partial E \cap \partial \Omega)$ is presented
157 in Section 3. For external element boundaries $\partial E_{ext} = \partial E \cap \partial \Omega$, a ghost neighbor element E' with the
158 same mechanical behavior as E is assumed [20]. In the case of an acoustic medium, one should prescribe
159 $p'_h = 2p_D - p_h$ and $\mathbf{v}'_h \cdot \mathbf{n} = \mathbf{v}_h \cdot \mathbf{n}$ on $\partial E \cap \partial \Omega_D$, and $p'_h = p_h$ and $\mathbf{v}'_h \cdot \mathbf{n} = 2v_{nN} - \mathbf{v}_h \cdot \mathbf{n}$ on $\partial E \cap \partial \Omega_N$.
160 The same approach is considered for the elastic external element boundary elaborated in [9].

161 *2.4. Characteristic structure of the wave governing equations*

162 Before giving the definition of Riemann problems on element interfaces in the next section, it is necessary
 163 to analyze the characteristic structure of the wave equations (1) and (13) for both acoustic and elastic cases.
 164 To do this, we analyze the corresponding eigenvalue problem defined with the operator \mathbf{M} and the Jacobian
 165 operator \mathbf{A}_n as follows

$$\mathbf{A}_n(\mathbf{R}_n) = \lambda_n \mathbf{M}(\mathbf{R}_n) \quad (20)$$

166 *2.4.1. Eigenanalysis of the acoustic wave equation*

167 In the acoustic case, we get $m = d + 1$ eigenvalues $\{\lambda_{n,k}\}_{k=1,\dots,m}$ and m associated right eigenvectors
 168 $\{\mathbf{R}_{n,k}\}_{k=1,\dots,m}$ from (20). It is noteworthy that the associated eigenvalue and eigenvectors of the two
 169 equivalent forms of the governing equations (1) and (7) are identical. In other words, one can use (8) to
 170 rewrite the eigenproblem (20) in the following form:

$$\overline{\mathbf{A}}_n(\mathbf{R}_n) = \lambda_n \mathbf{R}_n \quad (21)$$

171 Solving (21) results in two nonzero eigenvalues and the associated right eigenvectors:

$$\lambda_n^\pm = \pm \sqrt{\frac{\lambda_f}{\rho_f}}, \quad \mathbf{R}_n^\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \mathbf{n} \\ -\frac{1}{\sqrt{2}} \lambda_f (\lambda_n^\pm)^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \mathbf{n} \\ -\frac{1}{\sqrt{2}} z_n^\pm \end{pmatrix} \quad (22)$$

172 where z_n is the acoustic impedance defined as $z_n^\pm = \lambda_f (\lambda_n^\pm)^{-1} = \rho_f \lambda_n^\pm$.

173 For the calculation of the left eigenvectors of (21), which are also the right eigenvectors of the tensor $\overline{\mathbf{A}}_n^T$
 174 defined as:

$$\overline{\mathbf{A}}_n^T(\mathbf{W}) = \begin{pmatrix} -\lambda_f q \mathbf{n} \\ -\rho_f^{-1} \mathbf{n} \cdot \mathbf{w} \end{pmatrix} \quad (23)$$

175 It can be shown that:

$$\mathbf{L}_n^\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \mathbf{n} \\ -\frac{1}{\sqrt{2}} (z_n^\pm)^{-1} \end{pmatrix}, \quad \mathbf{M}(\mathbf{R}_n^\pm) = \rho_f \mathbf{L}_n^\pm \quad (24)$$

176 Hence, we get the following decompositions of the flux (or jacobian) tensors:

$$\overline{\mathbf{A}}_n = \lambda_n^\pm \mathbf{R}_n^\pm \otimes \mathbf{L}_n^\pm, \quad \mathbf{A}_n = z_n^\pm \mathbf{L}_n^\pm \otimes \mathbf{L}_n^\pm \quad (25)$$

177 The system (20) is therefore symmetric and has \mathbf{R}_n^\pm as eigenvectors.

178 *2.4.2. Eigenanalysis of the elastic wave equation*

179 The characteristic structure of the first-order hyperbolic velocity-stress equation (13) is studied in [21].
 180 Here, some of the important results are recalled.

181 Among the $m = d + d(d+1)/2$ eigenvalues of \mathbf{A}_n , there are d strictly negative eigenvalues and d strictly
 182 positive eigenvalues, representing the propagation speed of quasi-longitudinal “ qL ” and quasi transverse
 183 “ qT ” wave modes propagating in the \mathbf{n} direction. The right and left eigenmodes corresponding to the
 184 nonzero eigenvalues of $\overline{\mathbf{A}}_n = \mathbf{M}^{-1} \cdot \mathbf{A}_n$ are the following: $\forall k = qL, \{qT_\alpha\}_{\alpha=1,\dots,d-1}$

$$\mathbf{R}_{n,k}^\pm = \begin{pmatrix} \mathbf{w}_{n,k} \\ -\rho (z_{n,k}^\pm)^{-1} \mathbf{C} : (\mathbf{n} \otimes_s \mathbf{w}_{n,k}) \end{pmatrix}, \quad \mathbf{L}_{n,k}^\pm = \begin{pmatrix} \mathbf{w}_{n,k} \\ -(z_{n,k}^\pm)^{-1} \mathbf{n} \otimes_s \mathbf{w}_{n,k} \end{pmatrix} \quad (26)$$

185 where $z_{n,k}^\pm = \rho \lambda_{n,k}^\pm$ denotes the acoustic impedance of k th eigenmode and $\mathbf{w}_{n,k} = \frac{1}{\sqrt{2}} \boldsymbol{\gamma}_{n,k}$ with $\boldsymbol{\gamma}_{n,k}$ unit
 186 eigenvectors of $\boldsymbol{\Gamma}_n$ the Christoffel tensor:

$$\boldsymbol{\Gamma}_n \cdot \boldsymbol{\gamma}_{n,k} = \lambda_{n,k}^2 \boldsymbol{\gamma}_{n,k}, \quad k = qL, \{qT_\alpha\}_{\alpha=1,\dots,d-1} \quad (27)$$

187 and $\boldsymbol{\Gamma}_n$ is defined as:

$$\boldsymbol{\Gamma}_n \cdot \mathbf{w} = (\rho^{-1} \mathbf{C} : (\mathbf{n} \otimes_s \mathbf{w}_{n,k})) \cdot \mathbf{n}, \quad \forall \mathbf{w} \quad (28)$$

188 Finally, as in the acoustic case, the following equations hold:

$$\mathbf{M}(\mathbf{R}_{n,k}^\pm) = \rho \mathbf{L}_{n,k}^\pm, \quad \mathbf{A}_n = z_{n,k}^\pm \mathbf{L}_{n,k}^\pm \otimes \mathbf{L}_{n,k}^\pm \quad (29)$$

189 **3. Upwind numerical fluxes**

190 This section develops upwind numerical fluxes for multidimensional coupled anisotropic elastic-acoustic
 191 wave equations with discontinuous material properties (*i.e.*, media including physical interfaces), more par-
 192 ticularly, those for acoustic-acoustic and acoustic-elastic interfaces. Upwind numerical fluxes for elastic-
 193 elastic interfaces for heterogeneous anisotropic solids have been studied in detail in [9, 21]. The acoustic-
 194 acoustic case is simple and has already been treated in the literature. Still, for the sake of completeness,
 195 it is considered below in order to have the numerical fluxes expressed with the same notations used in the
 196 present work. Our main contribution is the development of numeric fluxes on acoustic/elastic interfaces.

197 *3.1. Numerical fluxes across an acoustic-acoustic interface*

We consider the interface of two adjacent elements E and E' , governed by the previously presented velocity-pressure acoustic wave equations and having respectively $(\rho_f, \lambda_f, \mathbf{U}_h)$ and $(\rho'_f, \lambda'_f, \mathbf{U}'_h)$ as densities, bulk modulus and initial states (Figure 1). The Riemann problem defines the states that result from the discontinuity $(\mathbf{U}_h - \mathbf{U}'_h)$'s propagation. All the following equations are written in the 3D case without loss of generality.

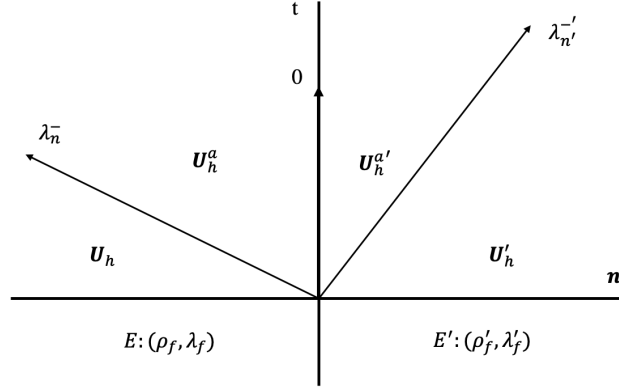


Figure 1. Sketch illustration of the Rankine-Hugoniot jump condition in the Riemann problem at an acoustic-acoustic element interface

$$\mathbf{A}_n(\mathbf{U}_h - \mathbf{U}_h^a) = \lambda_n^- \mathbf{M}(\mathbf{U}_h - \mathbf{U}_h^a) \quad (30a)$$

$$\mathbf{A}_n(\mathbf{U}_h^a) + \mathbf{A}'_{n'}(\mathbf{U}_h^{a'}) = \mathbf{0} \quad (30b)$$

$$\mathbf{A}'_{n'}(\mathbf{U}'_h - \mathbf{U}_h^{a'}) = \lambda_n'^- \mathbf{M}'(\mathbf{U}'_h - \mathbf{U}_h^{a'}) \quad (30c)$$

198 We note that the two outward unit normal vectors \mathbf{n} and \mathbf{n}' of E and E' on their interface verify $\mathbf{n} + \mathbf{n}' = \mathbf{0}$.
 199 According to the definition of the eigenvalues and eigenvectors (20), the discontinuity terms in (30) can be
 200 decomposed as follows within the right eigenvectors basis:

$$\mathbf{U}_h - \mathbf{U}_h^a = \alpha \mathbf{R}_n^-, \quad \mathbf{U}'_h - \mathbf{U}_h^{a'} = \alpha' \mathbf{R}_{n'}^- \quad (31)$$

By applying (31) and the second identity of (24), the Rankine-Hugoniot jump conditions (30) finally become:

$$\mathbf{A}_n(\mathbf{U}_h) - \mathbf{A}_n(\mathbf{U}_h^a) = \alpha z_n^- \mathbf{L}_n^- \quad (32a)$$

$$\mathbf{A}_n(\mathbf{U}_h^a) + \mathbf{A}'_{n'}(\mathbf{U}_h^{a'}) = \mathbf{0} \quad (32b)$$

$$\mathbf{A}'_{n'}(\mathbf{U}'_h) - \mathbf{A}'_{n'}(\mathbf{U}_h^{a'}) = \alpha' z_{n'}^- \mathbf{L}_{n'}^- \quad (32c)$$

201 It is worth noting that, according to the definition of the Jacobian operator (8), the equation (32b) corre-
 202 sponds to the following classical interface conditions for perfect fluids:

$$p_h^a - p_h^{a'} = 0, \quad \mathbf{v}_h^a \cdot \mathbf{n} + \mathbf{v}_h^{a'} \cdot \mathbf{n}' = 0 \quad (33)$$

203 Solving the Riemann problem (32) leads to the determination of the two unknown states $\{\mathbf{U}^a, \mathbf{U}^{a'}\}$,
 204 *i.e.*, the two characteristic coefficients $\{\alpha, \alpha'\}$. By eliminating the two unknown states \mathbf{U}_h^a and $\mathbf{U}_h^{a'}$ in
 205 (32) and using (33), the Riemann problem (32) is solved, we obtain the following formulas for $\{\alpha, \alpha'\}$ (see
 206 Appendix A1):

$$\alpha = \tilde{\mathbf{L}}_{\mathbf{n}}^- \cdot (\mathbf{U}_h - \mathbf{U}_h'), \quad \alpha' = \tilde{\mathbf{L}}_{\mathbf{n}'}'^- \cdot (\mathbf{U}_h' - \mathbf{U}_h) \quad (34)$$

207 In (34), the two ‘‘perturbed’’ left eigenmodes $\{\tilde{\mathbf{L}}_{\mathbf{n},k}^-, \tilde{\mathbf{L}}_{\mathbf{n}',k}^-'\}$ are calculated as follows and couple the material
 208 properties of the adjacent elements E and E' :

$$\tilde{\mathbf{L}}_{\mathbf{n}}^- = \begin{pmatrix} C_z^- \wp_{vect}(\mathbf{L}_{\mathbf{n}}^-) \\ C_z^- \wp_{scalar}(\mathbf{L}_{\mathbf{n}}^-) \end{pmatrix}, \quad \tilde{\mathbf{L}}_{\mathbf{n}'}'^- = \begin{pmatrix} C_z^- \wp_{vect}(\mathbf{L}_{\mathbf{n}'}^-) \\ C_z^- \wp_{scalar}(\mathbf{L}_{\mathbf{n}'}^-) \end{pmatrix} \quad (35)$$

209 with:

$$C_z^- = \frac{\overline{\overline{R}}}{\overline{\overline{z}}_{\mathbf{n}}} = \frac{z_{\mathbf{n}'}^-}{\overline{\overline{z}}_{\mathbf{n}}} > 0, \quad C_z^- = \frac{\overline{\overline{R}}}{\overline{\overline{z}}_{\mathbf{n}'}} = \frac{z_{\mathbf{n}}^-}{\overline{\overline{z}}_{\mathbf{n}'}} > 0 \quad (36)$$

210 $\overline{\overline{z}}_{\mathbf{n}}^R$ and $\overline{\overline{z}}_{\mathbf{n}}^V$ respectively denote the harmonic and arithmetic means, between E and E' , of the acoustic
 211 impedance of the eigenvector. Furthermore, two operators $\wp_{vect}(\cdot)$ and $\wp_{scalar}(\cdot)$ are introduced to treat
 212 the vectorial and scalar part of a generalized field $\mathbf{W} = (\mathbf{w} \ q)^T$ separately, and they are defined as follows:

$$\wp_{vect}(\mathbf{W}) = \mathbf{w}, \quad \wp_{scalar}(\mathbf{W}) = q \quad (37)$$

Finally, the upwind numerical fluxes defined as $\hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}_h') = \mathbf{A}_{\mathbf{n}}(\mathbf{U}_h^a) = \frac{1}{2}(\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h^a) - \mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h^{a'}))$ can
 be calculated by replacing the calculated characteristic coefficients in following equivalent equations:

$$\hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}_h') = \frac{1}{2}(\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h) - \mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h') - \alpha z_{\mathbf{n}}^- \mathbf{L}_{\mathbf{n}}^- + \alpha' z_{\mathbf{n}'}^- \mathbf{L}_{\mathbf{n}'}^-) \quad (38a)$$

$$= \mathbf{A}_{\mathbf{n}}(\mathbf{U}_h) - \alpha z_{\mathbf{n}}^- \mathbf{L}_{\mathbf{n}}^- \quad (38b)$$

213 3.2. Numerical fluxes across an acoustic-elastic interface

We again consider the interface of two adjacent elements E and E' . But now, the element E is governed by
 the acoustic wave equations with $(\rho_f, \lambda_f, \mathbf{U}_h)$ as density, bulk modulus, and initial state, and E' is governed
 by the elastic wave equations with $(\rho', C', \mathbf{U}_h')$ as density, elasticity tensor, and initial state (Figure 2). In
 this case, the Rankine-Hugoniot jump conditions are written as follows:

$$\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h - \mathbf{U}_h^a) = \lambda_{\mathbf{n}}^- \mathbf{M}(\mathbf{U}_h - \mathbf{U}_h^a) \quad (39a)$$

$$\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h^a) + \mathbf{\Pi}_{\mathbf{n}'}'(\mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h^{c'})) = \mathbf{0} \quad (39b)$$

$$\mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h^{b'} - \mathbf{U}_h^{c'}) = \lambda_{\mathbf{n}',qT_2}^- \mathbf{M}'(\mathbf{U}_h^{b'} - \mathbf{U}_h^{c'}) \quad (39c)$$

$$\mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h^{a'} - \mathbf{U}_h^{b'}) = \lambda_{\mathbf{n}',qT_1}^- \mathbf{M}'(\mathbf{U}_h^{a'} - \mathbf{U}_h^{b'}) \quad (39d)$$

$$\mathbf{A}_{\mathbf{n}'}'(\mathbf{U}_h' - \mathbf{U}_h^{a'}) = \lambda_{\mathbf{n},qL}^- \mathbf{M}'(\mathbf{U}_h' - \mathbf{U}_h^{a'}) \quad (39e)$$

214 In (39), the operator $\mathbf{\Pi}_{\mathbf{n}'}'$ from $\mathbb{R}^d \times \mathbb{R}^{d \times d_{sym}}$ to $\mathbb{R}^d \times \mathbb{R}$ is defined as follows: $\forall \mathbf{W} = (\mathbf{w} \ \boldsymbol{\tau})^T \in \mathbb{R}^d \times \mathbb{R}^{d \times d_{sym}}$,

$$\mathbf{\Pi}_{\mathbf{n}'}'(\mathbf{W}) = \begin{pmatrix} \mathbf{w} \\ (\mathbf{n}' \otimes \mathbf{n}') : \boldsymbol{\tau} \end{pmatrix} \quad (40)$$

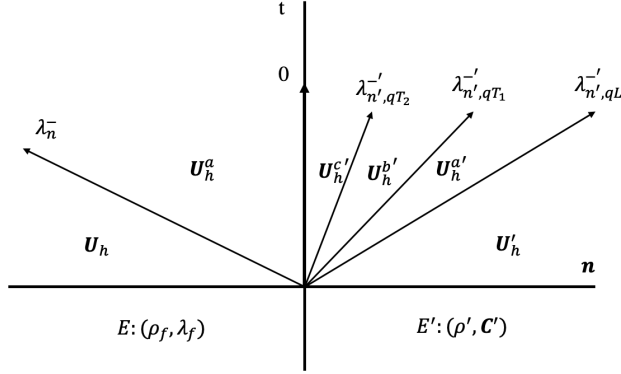


Figure 2. Sketch illustration of the Rankine-Hugoniot jump condition in the Riemann problem at an acoustic-elastic element interface

215 Then, it can be shown that according to the definition of \mathbf{A}_n in both acoustic and elastic media and
 216 the definition of $\mathbf{\Pi}_{n'}$, the equation (39b) corresponds to the following classical perfect fluid-solid interface
 217 conditions:

$$p_h^a \mathbf{n} + \boldsymbol{\sigma}_h^{c'} \cdot \mathbf{n}' = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{v}_h^a + \mathbf{n}' \cdot \mathbf{v}_h^{c'} = 0 \quad (41)$$

Now, as in the acoustic-acoustic case, the discontinuity terms in (39) are decomposed as follows using the right eigenvectors basis:

$$\mathbf{U}_h - \mathbf{U}_h^a = \alpha \mathbf{R}_n^- \quad (42a)$$

$$\mathbf{U}_h' - \mathbf{U}_h^{a'} = \alpha'_{qL} \mathbf{R}_{n',qL}^-, \quad \mathbf{U}_h^a - \mathbf{U}_h^{b'} = \alpha'_{qT_1} \mathbf{R}_{n',qT_1}^-, \quad \mathbf{U}_h^{b'} - \mathbf{U}_h^{c'} = \alpha'_{qT_2} \mathbf{R}_{n',qT_2}^- \quad (42b)$$

By adding the three last equations given by (39c)–(39e), and by applying (42), the second identity of (24), and the first identity of (29), the Rankine–Hugoniot jump conditions (39) finally become:

$$\mathbf{A}_n(\mathbf{U}_h) - \mathbf{A}_n(\mathbf{U}_h^a) = \alpha z_n^- \mathbf{L}_n^- \quad (43a)$$

$$\mathbf{A}_n(\mathbf{U}_h^a) + \mathbf{\Pi}_{n'}(\mathbf{A}'_{n'}(\mathbf{U}_h^{c'})) = \mathbf{0} \quad (43b)$$

$$\mathbf{A}'_{n'}(\mathbf{U}_h') - \mathbf{A}'_{n'}(\mathbf{U}_h^{c'}) = \alpha'_k z_{n',k}^- \mathbf{L}_{n',k}^- \quad (43c)$$

Again by eliminating the two unknown states \mathbf{U}_h^a and $\mathbf{U}_h^{c'}$ in (43), it can be shown that the characteristic coefficients $\{\alpha, \alpha'_k\}$ of the Riemann problem (43) are solutions of the following linear system of equations (see Appendix A2):

$$\begin{bmatrix} 1 & [B^{ae}]_{1 \times 3} \\ [B^{ae'}]_{3 \times 1} & [Id]_{3 \times 3} \end{bmatrix} \cdot \begin{pmatrix} \alpha \\ \{\alpha'_k\} \end{pmatrix} = \begin{bmatrix} \mathbf{L}_n^- \cdot (\mathbf{U}_h - \mathbf{\Pi}_{n'}(\mathbf{U}_h')) \\ \{2\boldsymbol{\wp}_{tens}(\mathbf{L}_{n',k}^-) : \boldsymbol{\wp}_{tens}(\mathbf{U}_h' - \mathbf{\Psi}_n(\mathbf{U}_h))\} \end{bmatrix} \quad (44)$$

218 where the components of the matrices $[B^{ae}]$ and $[B^{ae'}]$ are:

$$B_{1k}^{ae} = -\frac{z_n^- - z_{n',k}^-}{2z_n^-} \mathbf{n} \cdot \boldsymbol{\gamma}'_{n',k}, \quad B_{k1}^{ae'} = \frac{z_n^-}{z_{n',k}^-} \mathbf{n} \cdot \boldsymbol{\gamma}'_{n',k} \quad (45)$$

219 in (44), the operator $\mathbf{\Psi}_n$ from $\mathbb{R}^d \times \mathbb{R}$ to $\mathbb{R}^d \times \mathbb{R}^{d \times d_{sym}}$ is defined as follows: $\forall \mathbf{W} = (\mathbf{w} \ q)^T \in \mathbb{R}^d \times \mathbb{R}$,

$$\mathbf{\Psi}_n(\mathbf{W}) = \begin{pmatrix} \mathbf{w} \\ q \mathbf{n} \otimes \mathbf{n} \end{pmatrix} \quad (46)$$

220 After solving the system of equations (44), one finally gets:

$$\alpha = D_{11}^{ae} \mathbf{L}_n^- \cdot (\mathbf{U}_h - \mathbf{\Pi}_{n'}(\mathbf{U}_h')) - 2H_{1k}^{ae} \boldsymbol{\wp}_{tens}(\mathbf{L}_{n',k}^-) : \boldsymbol{\wp}_{tens}(\mathbf{U}_h' - \mathbf{\Psi}_n(\mathbf{U}_h)) \quad (47)$$

with D_{11}^{ae} and H_{1k}^{ae} defined by:

$$[R^{ae}]^{-1} = \begin{bmatrix} 1 & [B^{ae}]_{1 \times 3} \\ [B^{ae'}]_{3 \times 1} & [Id]_{3 \times 3} \end{bmatrix}^{-1} = \begin{bmatrix} D_{11}^{ae} & -[H^{ae}]_{1 \times 3} \\ -[H^{ae'}]_{3 \times 1} & [D^{ae'}]_{3 \times 3} \end{bmatrix} \quad (48)$$

221 Then, the numerical flux is calculated within the acoustic element E using the equation (38b).

222 When the elastic element E' is isotropic, we get one pure longitudinal “ L ” mode and two pure transverse
223 “ T ” modes, i.e.,:

$$\mathbf{n} \cdot \boldsymbol{\gamma}'_{\mathbf{n}',L} = 0, \quad \mathbf{n} \cdot \boldsymbol{\gamma}'_{\mathbf{n}',T_1} = \mathbf{n} \cdot \boldsymbol{\gamma}'_{\mathbf{n}',T_2} = 0 \quad (49)$$

224 According to (44)-(45), α , the acoustic wave in E , is only coupled to α'_L , the elastic longitudinal wave in
225 E' , which is a well-known classical result. Otherwise, the quasi-transverse elastic waves are always coupled
226 to the acoustic wave through an acoustic-elastic interface.

227 **Remark 3.1** For the case in which E is governed by elastic wave equations and the neighboring E' is
governed by acoustic wave equations, by reversing the role of E and E' in the preceding equations (39), (42)
and (43), after calculation of the characteristic coefficients $\{\{\alpha_k\}, \alpha'\}$, the characteristic coefficients $\{\alpha, \alpha'_k\}$
of the Riemann problem (43) are solutions of the following linear system of equations (see Appendix A2):

$$\begin{bmatrix} [Id]_{3 \times 3} & [B^{ea}]_{3 \times 1} \\ [B^{ea'}]_{1 \times 3} & 1 \end{bmatrix} \cdot \begin{pmatrix} \{\alpha_k\} \\ \alpha' \end{pmatrix} = \begin{bmatrix} \{2\boldsymbol{\varphi}_{tens}(\mathbf{L}_{\mathbf{n},k}^-) : \boldsymbol{\varphi}_{tens}(\mathbf{U}_h - \boldsymbol{\Psi}_{\mathbf{n}'}(\mathbf{U}'_h))\} \\ \mathbf{L}_{\mathbf{n}'}^- \cdot (\boldsymbol{\Pi}_{\mathbf{n}}(\mathbf{U}_h) - \mathbf{U}'_h) \end{bmatrix} \quad (50)$$

228 where the components of the matrices $[B^{ea}]$ and $[B^{ea'}]$ are:

$$B_{k1}^{ea} = \frac{z_{\mathbf{n}'}^-}{z_{\mathbf{n},k}^-} \mathbf{n}' \cdot \boldsymbol{\gamma}_{\mathbf{n},k}, \quad B_{1k}^{ea'} = -\frac{z_{\mathbf{n},k}^- - z_{\mathbf{n}'}^-}{2z_{\mathbf{n}'}^-} \mathbf{n}' \cdot \boldsymbol{\gamma}_{\mathbf{n},k} \quad (51)$$

229 After solving the system of equations (50) for characteristic equations, one finally gets:

$$\alpha_k = 2D_{kl}^{ea} \boldsymbol{\varphi}_{tens}(\mathbf{L}_{\mathbf{n},l}^-) : \boldsymbol{\varphi}_{tens}(\mathbf{U}_h - \boldsymbol{\Psi}_{\mathbf{n}'}(\mathbf{U}'_h)) - H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot (\boldsymbol{\Pi}_{\mathbf{n}}(\mathbf{U}_h) - \mathbf{U}'_h) \quad (52)$$

with D_{kl}^{ea} and H_{k1}^{ea} defined by:

$$[R^{ea}]^{-1} = \begin{bmatrix} [Id]_{3 \times 3} & [B^{ea}]_{3 \times 1} \\ [B^{ea'}]_{1 \times 3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} [D^{ea}]_{3 \times 3} & -[H^{ea}]_{3 \times 1} \\ -[H^{ea'}]_{1 \times 3} & D_{11}^{ea'} \end{bmatrix} \quad (53)$$

230 Then, the numerical flux is calculated within the elastic element E using the following equation [9]:

$$\hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) = \mathbf{A}_{\mathbf{n}}(\mathbf{U}_h) - \alpha_k z_{\mathbf{n},k}^- \mathbf{L}_{\mathbf{n},k}^- \quad (54)$$

231 3.3. Implementation and calculation of numerical fluxes in the variational formulation

232 For an acoustic-elastic interface, we briefly present how to implement and compute the terms associated
233 with the numerical fluxes in the variational formulation (19).

234 Let us consider first the acoustic element, denoted E . In this case, the interface is consequently denoted
235 ∂E^{ae} and the elastic element E' . Taking into account (38b) and (47), The flux term in (19) becomes:

$$\begin{aligned} & \langle \mathbf{W}_h, \hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}(\mathbf{U}_h) \rangle_{\partial E^{ae}} \\ &= - \langle \mathbf{W}_h, \alpha z_{\mathbf{n}}^- \mathbf{L}_{\mathbf{n}}^- \rangle_{\partial E^{ae}} \\ &= - \int_{\partial E^{ae}} (\mathbf{W}_h \cdot \mathbf{L}_{\mathbf{n}}^-) z_{\mathbf{n}}^- \left(D_{11}^{ae} \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{U}_h + 2H_{1k}^{ae} \boldsymbol{\varphi}_{tens}(\mathbf{L}_{\mathbf{n}',k}^-) : \boldsymbol{\varphi}_{tens}(\boldsymbol{\Psi}_{\mathbf{n}}(\mathbf{U}_h)) \right) \\ &+ \int_{\partial E^{ae}} (\mathbf{W}_h \cdot \mathbf{L}_{\mathbf{n}}^-) z_{\mathbf{n}}^- \left(D_{11}^{ae} \mathbf{L}_{\mathbf{n}}^- \cdot \boldsymbol{\Pi}_{\mathbf{n}'}(\mathbf{U}'_h) + 2H_{1k}^{ae} \boldsymbol{\varphi}_{tens}(\mathbf{L}_{\mathbf{n}',k}^-) : \boldsymbol{\varphi}_{tens}(\mathbf{U}'_h) \right) \end{aligned} \quad (55)$$

236 Then, we exchange the role of acoustic and elastic elements. Let us consider the elastic element, now
 237 denoted E . In this case, the same interface is consequently denoted ∂E^{ea} and the acoustic element E' .
 238 Taking into account (54) and (52), The flux term in (19) becomes:

$$\begin{aligned}
 & \langle \mathbf{W}_h, \hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}(\mathbf{U}_h) \rangle_{\partial E^{ea}} \\
 &= - \langle \mathbf{W}_h, \alpha_k z_{\mathbf{n},k}^- \mathbf{L}_{\mathbf{n},k}^- \rangle_{\partial E^{ea}} \\
 &= - \int_{\partial E^{ea}} \left(\mathbf{W}_h \cdot \mathbf{L}_{\mathbf{n},k}^- \right) z_{\mathbf{n},k}^- \left(2D_{kl}^{ea} \wp_{tens}(\mathbf{L}_{\mathbf{n},l}^-) \right) : \wp_{tens}(\mathbf{U}_h) - H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{\Pi}_{\mathbf{n}}(\mathbf{U}_h) \Big) \\
 &+ \int_{\partial E^{ea}} \left(\mathbf{W}_h \cdot \mathbf{L}_{\mathbf{n},k}^- \right) z_{\mathbf{n},k}^- \left(2D_{kl}^{ea} \wp_{tens}(\mathbf{L}_{\mathbf{n},l}^-) \right) : \wp_{tens}(\mathbf{\Psi}_{\mathbf{n}'}(\mathbf{U}'_h)) - H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{\Pi}_{\mathbf{n}}(\mathbf{U}'_h) \Big)
 \end{aligned} \tag{56}$$

239 In both cases, (55) and (56), when a FE discretization is applied, we obtained two flux matrices, one for
 240 E itself and the other coupling E with E' . We note that the two matrices $[R^{ae}]$ and $[R^{ea}]$ being analytically
 241 inverted, all terms in (55) and (56) are explicitly implemented, and the flux matrices are calculated once at
 242 the beginning of each dynamic calculation and stored. In other words, they are not calculated in each time
 243 step, so their computational cost is limited.

244 4. Validation of the upwind numerical fluxes

245 This section aims to validate the proposed upwind numerical fluxes, particularly on the acoustic/elastic
 246 interface, and demonstrate the performance of the coupled dG solver. For this purpose, a numerical/analytical
 247 comparison is carried out by considering an example with a circular acoustic/elastic interface for which an-
 248 alytical solutions are available [26].

249 We consider $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 a square fluid domain $[0, L_x] \times [0, L_y]$ with $L_x = L_y = 600\text{m}$, inside
 250 which there is a circular solid inclusion Ω_2 centered at $(x_0, y_0) = (330, 299)\text{m}$ and with radius $a = 119\text{m}$ (see
 251 Figure 3a). The acoustic wave speed in the fluid domain Ω_1 with a density of $\rho_1 = 1000 \frac{\text{kg}}{\text{m}^3}$ is $c = 1500\text{m/s}$,
 252 while the speeds of the pressure and shear waves in the solid domain Ω_2 with a density of $\rho_2 = 2600 \frac{\text{kg}}{\text{m}^3}$ are
 253 $c_p = 4000\text{m/s}$ and $c_s = 2000\text{m/s}$ respectively.

254 An external loading, a uniformly distributed pressure, is applied to the left side of Ω_1 . Its time dependence
 255 is a sinusoid signal defined by the following function (Figure 3b):

$$f(t) = \sin(\omega_c t) - \frac{1}{2} \sin(2\omega_c t), \quad t \in [0, 0.025\text{s}] \tag{57}$$

256 The frequency content of the signal is centered at 40Hz (Figure 3c), and a cutoff frequency $f_c = 150\text{Hz}$ can
 257 be defined, which gives rise to the shortest involved wavelength $\lambda_{min} = \min\{c, c_p, c_s\}/f_c = 10\text{m}$. In this
 258 work, the non-reflecting boundary condition is handled by using the same approach proposed by Käser *et*
 259 *al.* [6]. The main idea resides in the choice of the numerical fluxes, which keeps only the outgoing wave
 260 modes and forbids incoming modes.

261 Finite element meshes with the 4-node quadrilateral (Q4) elements are used in the present work. It is
 262 noteworthy that the element size h^E is chosen with respect to λ_{min} . Herein, four different element sizes are
 263 considered: $h_1^E = 4\text{m}$, $h_2^E = 2\text{m}$, $h_3^E = 1\text{m}$ and $h_4^E = 0.5\text{m}$, which correspond respectively to 2.5, 5, 10, and
 264 20 elements per shortest wavelength.

265 The total simulation time is chosen to be 0.3s so that the incident wavefront passes entirely through the
 266 circular inclusion. For the choice of time steps, in the present work, the time integration is performed by
 267 using the fourth-order Runge-Kutta iterative method, which is explicit and conditionally stable. Hence, it
 268 is necessary to respect the following stability condition [9]:

$$\Delta t \leq \frac{CFL}{2N_p + 1} \min_E \left\{ \frac{h^E}{c_{max}} \right\} \tag{58}$$

269 where N_p is the order of FE basis function and $N_p = 1$ with the use of the Q4 elements, h^E is the size
 270 of element E , c_{max} is the fastest wave speed in E , and CFL is the Courant-Friedrichs-Levy number. Our
 271 numerical experiences demonstrate that $CFL = 0.6$ guarantees the stability of our space DG solver.

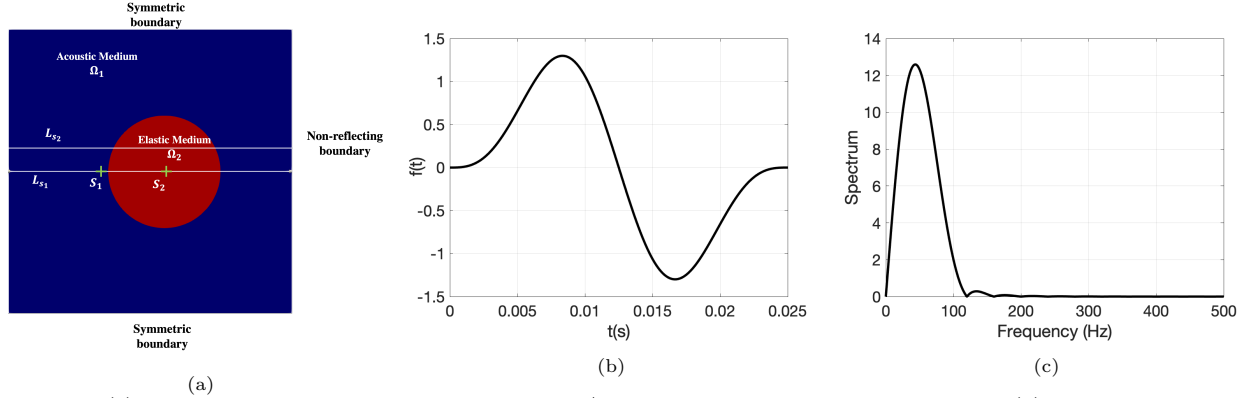


Figure 3. (a) Considered geometry with a circular acoustic/elastic interface and boundary conditions; (b) Initial sinusoid pulse placed over the left edge of the fluid medium, and (c) its frequency content

4.1. Convergence analysis and analytical/numerical comparison

First, a convergence analysis is performed using previously defined four different element sizes. Time signals in velocity, pressure, and stress are output and compared at two sampling points $S_1 = (200, 300)\text{m} \in \Omega_1$ and $S_2 = (330, 300)\text{m} \in \Omega_2$ (Figure 3a). Figure 4 shows that, for an element size less or equal to 2m , the convergence is reached as suggested by the negligible differences between the obtained signals.

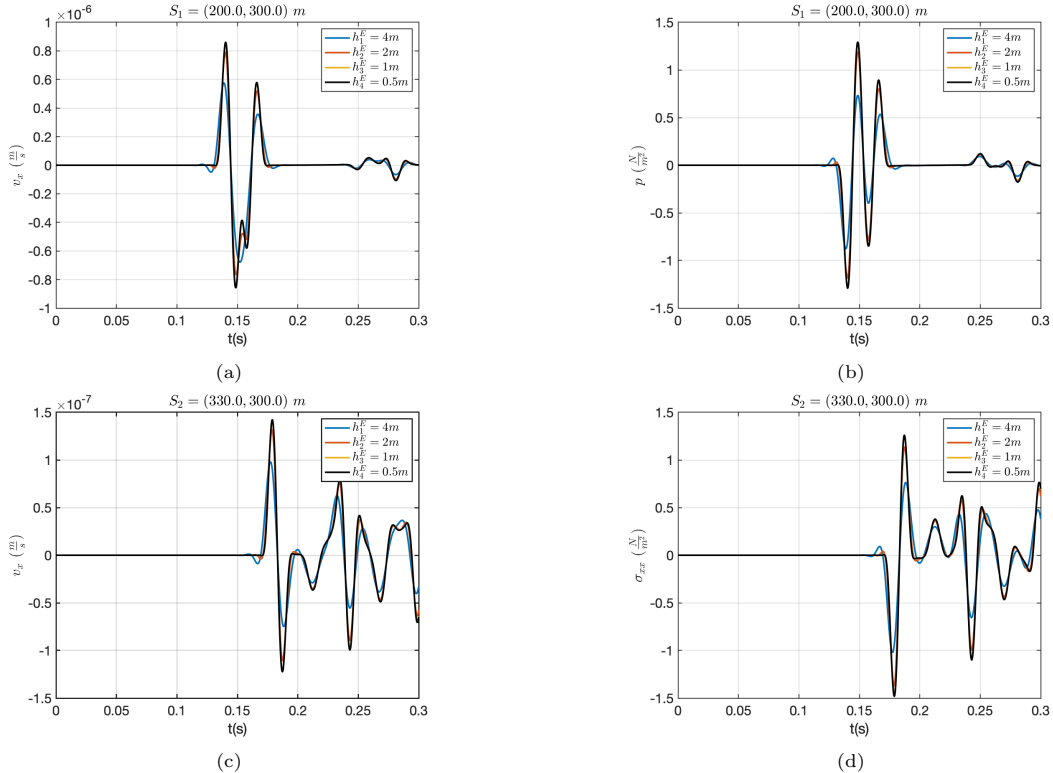


Figure 4. Convergence analysis of the space DG solver using six different element sizes by considering time signals output at $S_1 = (200, 300)\text{m} \in \Omega_1$ and $S_2 = (330, 300)\text{m} \in \Omega_2$. (a) v_x at S_1 ; (b) p at S_1 ; (c) v_x at S_2 ; (d) σ_{xx} at S_2

Then, the wave propagation phenomena are analyzed. For the numerical simulation using $h_4^E = 0.5\text{m}$ and $\Delta t = 2.5\mu\text{s}$, three snapshots of the evolving longitudinal (pressure) and transverse (shear) wavefronts are displayed in Figure 5 by means of the Helmholtz decomposition. By this method, in the case of isotropic

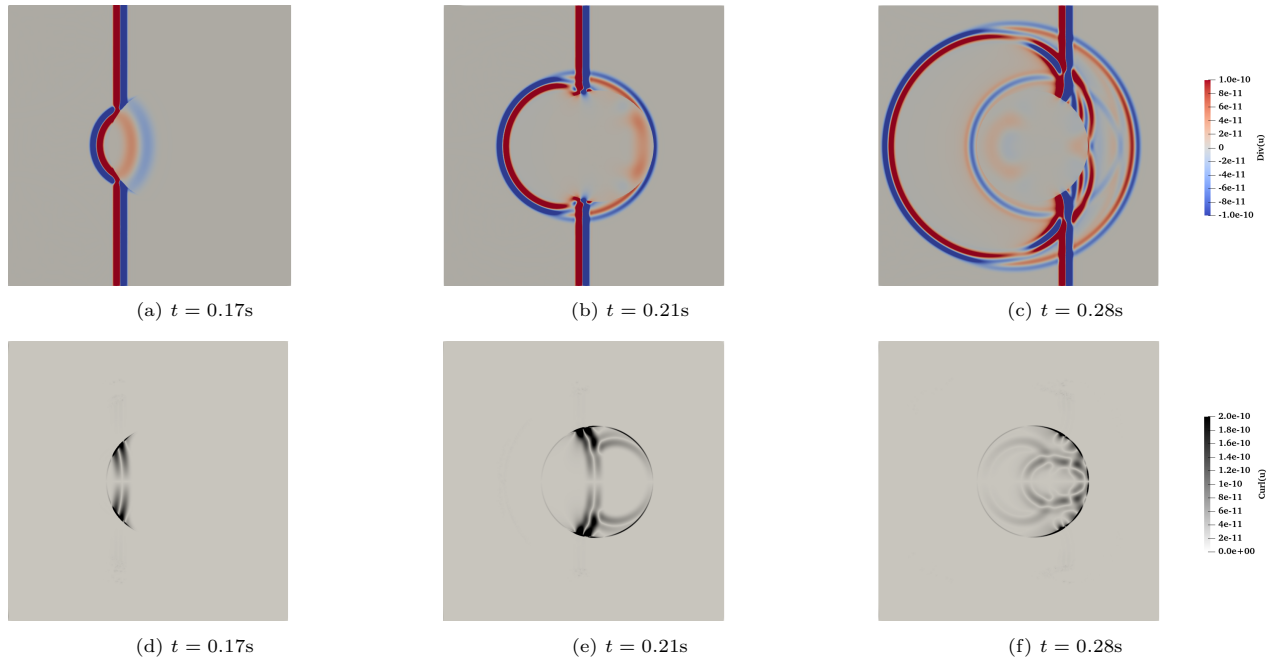


Figure 5. Wave propagation in the fluid-solid domain at different time instances; (a), (b) and (c): Longitudinal (pressure) wavefronts; (d), (e) and (f): Transverse (shear) wavefronts

280 and homogeneous elastic 2D medium, the scalar field $div(\mathbf{u})$ and the vector field $\mathbf{curl}(\mathbf{u})$ represent the
 281 longitudinal and transverse wavefronts, respectively. Expected wave phenomena are observed. The figures
 282 Figure 5(a-c) show the propagation of longitudinal waves in both media, which is faster in the solid inclusion,
 283 and wave reflection and transmission at the fluid-solid interface. Transverse waves are generated at the
 284 fluid-solid interface and only propagate inside the solid inclusion Ω_2 with a slower propagation speed than
 285 the longitudinal waves. Moreover, the minimum amplitude of the transverse waves is obtained along the
 286 horizontal diameter of the inclusion because no longitudinal to transverse wave conversion takes place due
 287 to the normal incidence of the incident wave at the fluid-solid interface (Figure 5(d-f)).

288 Finally, an analytical/numerical comparison is made using the analytical solution provided by Lombard
 289 et al. [26]. Figure 6 presents the comparison along the horizontal line $L_{s_1} = \{(x, y) | y = 300\text{m}\}$ and
 290 $L_{s_2} = \{(x, y) | y = 344\text{m}\}$ (Figure 3a) of the numerical stress and pressure fields obtained using $h_4^E = 0.5\text{m}$
 291 and the corresponding analytical solution. Figure 6a shows the pressure field over the line L_s at $t = 0.1380\text{s}$,
 292 which is before the arrival of the wavefront at the acoustic/elastic interface. It is noteworthy that the points
 293 with $x \in [0, 211 \cup] 449, 600]$ for L_{s_1} lie in the acoustic medium and the points with $x \in]211, 449[$ are in the
 294 elastic medium. Hence, the y-axis in Figure 6(b-f) reports either the pressure p or the σ_{xx} component of
 295 the stress tensor, normalized by the maximum amplitude incident pressure p_{max} . It can be seen there is
 296 a good agreement between the numerical results and the analytical solution. In addition, Figure 7 shows
 297 the comparison on L_s of the numerical results of different element sizes with the analytical solution, which
 298 shows the convergence when the mesh gets finer.

299 Although visual comparison of the spatial signals shows that the proposed method and, in particular,
 300 the implemented numerical fluxes are working properly, these qualitative comparisons do not provide pre-
 301 cise quantification and characterization of the differences between the results. Therefore, a more rigorous
 302 approach is presented in the next section to quantify the mismatch between the numerical and analytical
 303 solutions.

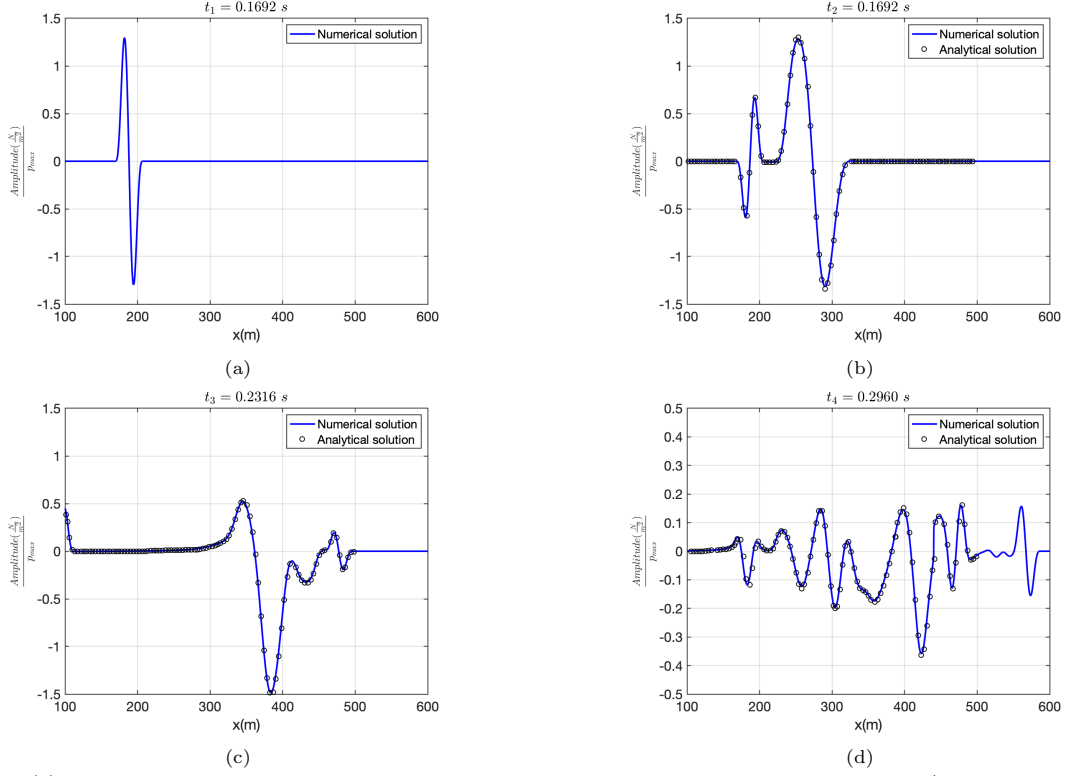


Figure 6. (a) Pressure wavefront in the acoustic medium Ω_1 before its interaction with the acoustic/elastic interface; (b)-(c) Numerical/analytical comparison along the horizontal line $L_{s_1} = \{(x, y) | y = 300\text{m}\}$; (d) Numerical/analytical comparison along the horizontal line $L_{s_2} = \{(x, y) | y = 344\text{m}\}$

;

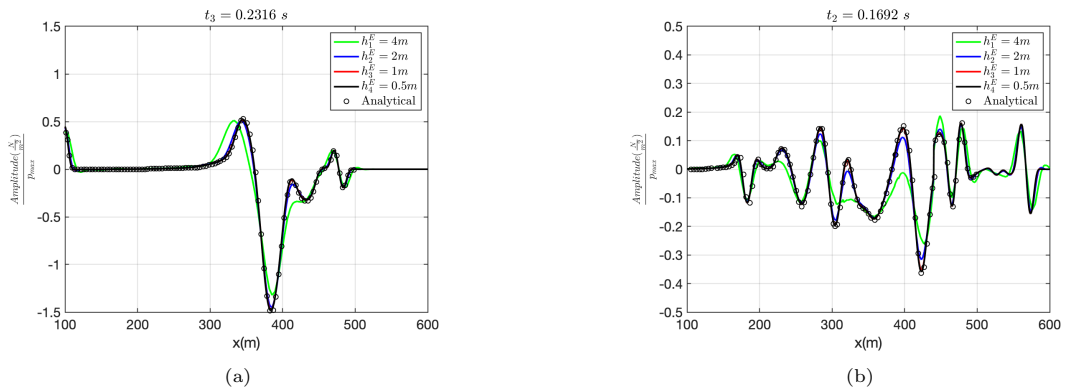


Figure 7. (a) Numerical/analytical comparison along (a) L_{s_1} and (b) L_{s_2} for different element sizes

304 *4.2. Space-Wavenumber misfit and goodness-of-fit criteria*

305 Kristekova et al. proposed a criterion that quantifies and characterizes the misfit between two temporal
 306 signals using their time-frequency representations (TFR) [27, 28]. This criterion is shown to be able to
 307 detect envelope (or amplitude) and phase misfits and has a better performance compared to the standard
 308 Root-Mean-Squared (RMS) misfit criterion that overestimates the misfits up to 300% [27]. In the present
 309 work, we use a similar approach applied to spatial signals instead of temporal signals.

310 For this purpose, the Continuous Wavelet Transform (CWT) is used to obtain the space-wavenumber
 311 representation (SWR) of a 1-D spatial signal, denoted $g(x)$ below [27, 29]:

$$CWT(\xi, \ell)\{g\} = \frac{1}{\sqrt{\ell}} \int_{-\infty}^{+\infty} g(x)\psi^*\left(\frac{x-\xi}{\ell}\right) dx \quad (59)$$

312 where ξ is the translational parameter, ℓ is the scale parameter inversely proportional to wavenumber k , ψ is
 313 the analyzing wavelet (or basic wavelet), and $(\cdot)^*$ is the complex conjugate operator. Using ℓ , the analyzing
 314 wavelet is stretched in space at different scales.

315 Among plenty of signals that can be used as the analyzing wavelet, the Morlet wavelet is selected for
 316 the rest of the calculations in this work. It is an analytical signal whose spectrum has zero amplitudes at
 317 negative frequencies and is written as follows:

$$\psi(\eta) = \pi^{-1/4} e^{i\chi\eta - 0.5\eta^2} \quad (60)$$

318 where χ is dimensionless wavenumber, and η is dimensionless space length. By choosing the appropriate
 319 ℓ and ξ , with a relation $k = \chi/\ell$ between the scale parameter ℓ and the wavenumber k , the SWR of the
 320 spatial signal $g(x)$ is defined as:

$$W(x, k)\{g\} = CWT\left(x, \frac{\chi}{k}\right)\{g\} \quad (61)$$

The SWRs are obtained using the “cwt” function in MATLAB[®] with the parameter $\chi = 6$. Then a local
 SW envelope difference ΔE and a local SW phase difference ΔP with respect to a reference signal $g_{ref}(x)$
 are defined for $g(x)$ as follows:

$$\Delta E(x, k) = |W(x, k)\{g\}| - |W(x, k)\{g_{ref}\}| \quad (62a)$$

$$\Delta P(x, k) = |W(x, k)\{g_{ref}\}| \frac{\text{Arg}[W(x, k)\{g\}] - \text{Arg}[W(x, k)\{g_{ref}\}]}{\pi} \quad (62b)$$

321 Afterward, local envelope and phase misfits, respectively called space-wavenumber envelope misfit (SWEM)
 322 and space-wavenumber phase misfit (SWPM), are defined as follows:

$$SWEM(x, k) = \frac{\Delta E(x, k)}{\max_{(x', k')} (|W(x', k')\{g_{ref}\}|)}, \quad SWPM(x, k) = \frac{\Delta P(x, k)}{\max_{(x', k')} (|W(x', k')\{g_{ref}\}|)} \quad (63)$$

323 Figure 8 presents the local envelope and phase misfits SWEM and SWPM of the spatial signals recorded
 324 on the previously defined line L_{s_1} and L_{s_2} . In each subfigure, the x-axis is the horizontal position of the
 325 points on the sampling line, the y-axis is the wavenumber, and the corresponding numerical signal and the
 326 analytical solution are also shown below the x-axis. It is noteworthy that the investigated wavenumber
 327 range is $k \in [0, 0.65]$ where the maximum wavenumber is obtained as $k_{max} = \frac{2\pi f_c}{c}$ in the acoustic domain.
 328 A general decreasing trend is observed in the local envelope and phase misfits as the element size is reduced.
 329 In addition, we note that the phase misfit is smaller than the envelope misfit for a given element size.

330 On the other hand, Figure 9 shows the envelope and phase misfits on L_s at different time instances for
 331 the finest element size of $h_4^E = 0.5m$. Similar to Figure 8, here again, the phase misfit is smaller compared
 332 to the envelope misfit at a given time. Moreover, both the envelope and phase misfit are bigger in the
 333 elastic domain compared to the acoustic domain. According to [28], a fit is considered as a “good” one
 334 if the envelope and phase misfits are less than 0.41 and 0.35, and an “excellent” one if the envelope and
 335

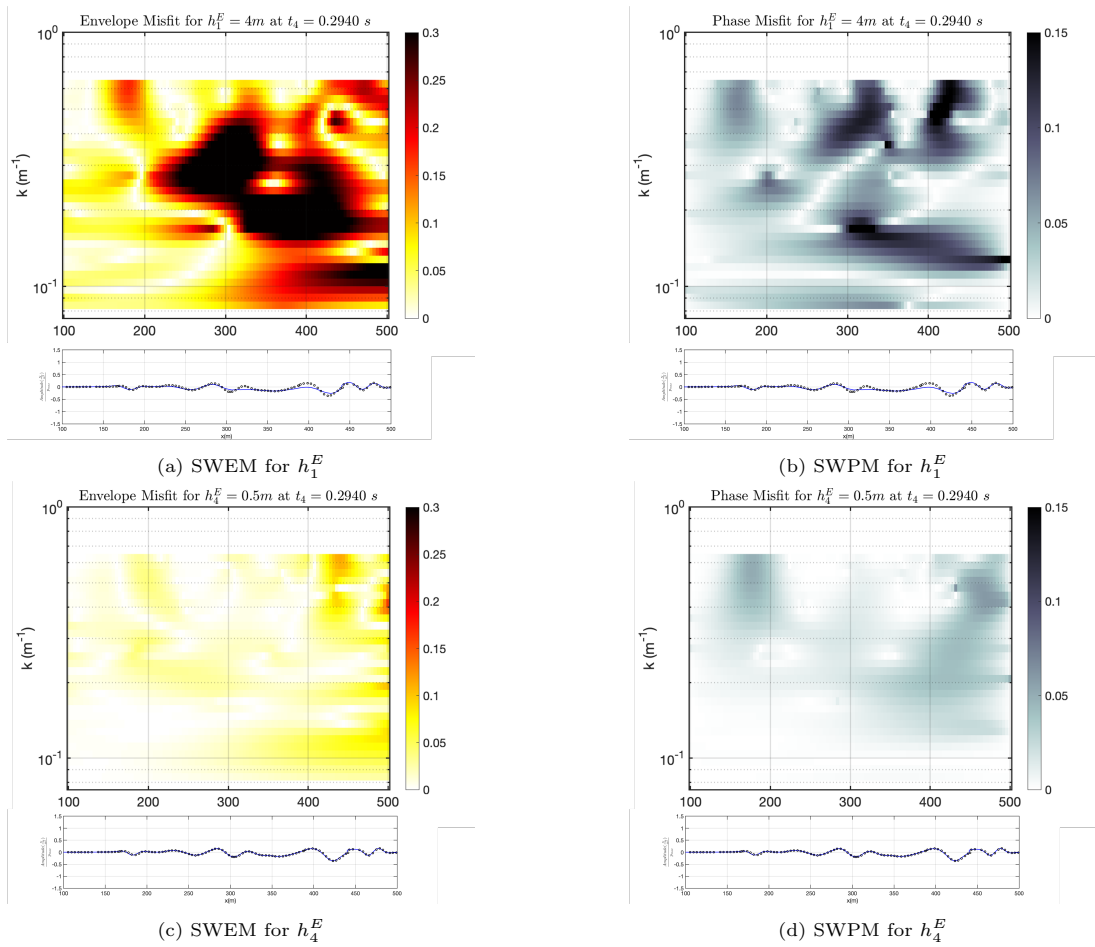


Figure 8. Space-wavenumber misfits between the numerical signal and the analytical solution on the sampling line L_{s_2} with different element sizes at $t_6 = 0.2940\text{ s}$

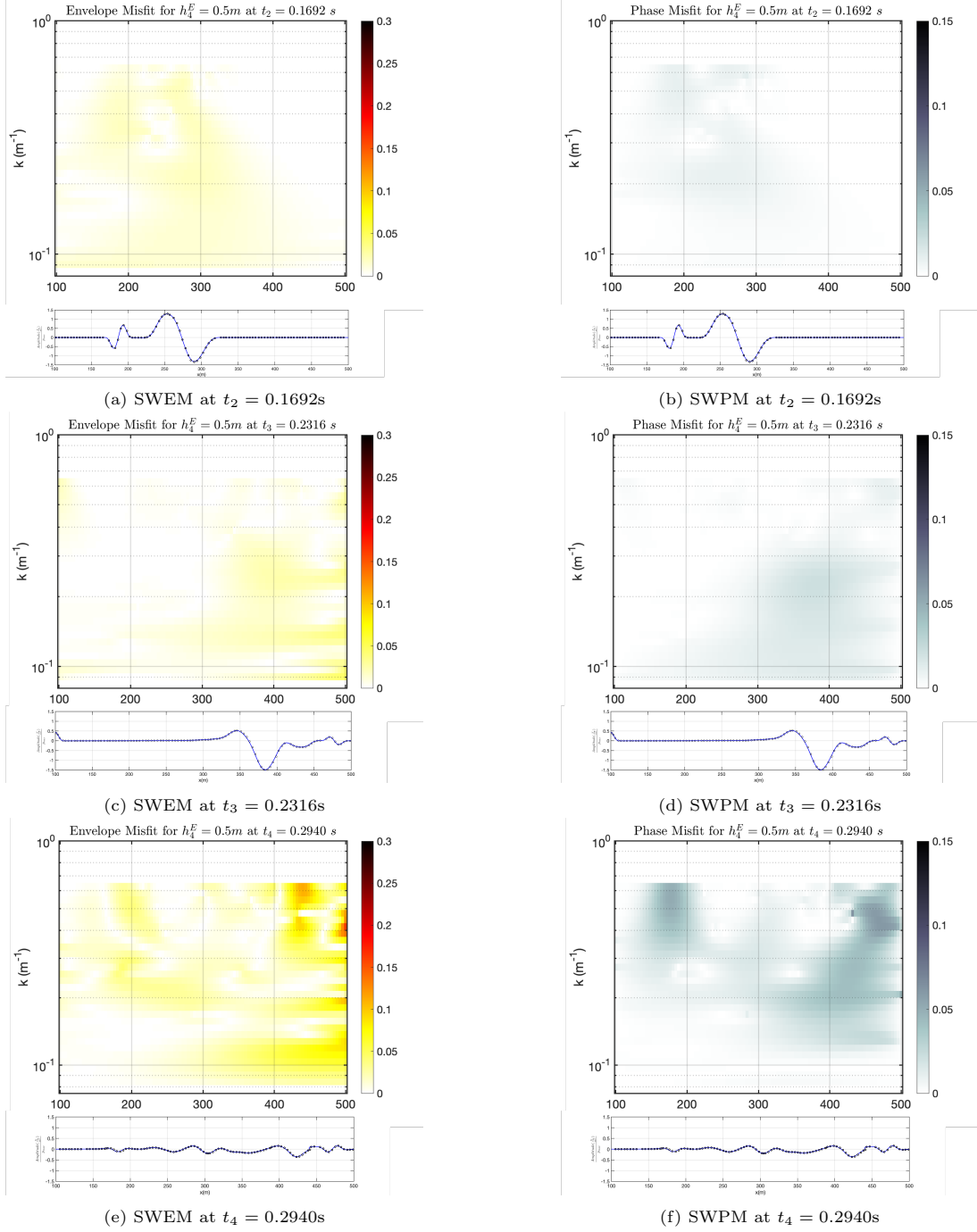


Figure 9. Space-wavenumber misfits between the numerical signal and the analytical solution on the sampling line L_{s_1} and L_{s_2} at different time instances for the element size $h_4^E = 0.5m$

336 phase misfits are less than 0.16 and 0.15, respectively. Based on this Goodness-of-Fit criterion, there is an
 337 excellent match between the numerical result and the analytical solution all over the sampling lines L_{s_1} and
 338 L_{s_2} , for the element sizes of $h_3^E = 1m$ and $h_4^E = 0.5m$.

339 Significant insights could have enriched this study if the analytical solutions over the entire domain was
 340 available to consider the L^2 - or H^1 - error norms. Nonetheless, obtaining the analytical solution over the
 341 entire domain has a high, even prohibitive, numerical cost.

342 5. Ultrasonic imaging of the simplified 3D-printed microstructure

343 The proposed coupled acoustic/elastic solver is used to simulate ultrasonic wave propagation in a 2D
 344 representation of a synthetic 3D-printed material microstructure that mimics the acoustic properties of
 345 biological tissues. The study aims to examine the influence of two key microstructure parameters, the area
 346 fraction and size of inclusion, on the resulting ultrasonic imaging.

347 For that purpose, as a printed microstructure, a rectangular domain made of a matrix with circular
 348 inclusions is considered (Figure 10a). Furthermore, the printed layer is assumed to be submerged in water,
 349 just as in the actual ultrasonic imaging process where tissues are usually surrounded by a fluid (blood or
 350 water). Hence, a water layer is added above and below the solid material layer (Figure 10a). The geometry
 351 size of the modeled domain is $6mm \times 4.5mm$, of which the thickness of each water layer is $1mm$ and
 352 the thickness of the 3D-printed material is $2.5mm$. As depicted in Figure 10a, a non-reflecting boundary
 condition is applied to the sides and bottom edge of the rectangular geometry.

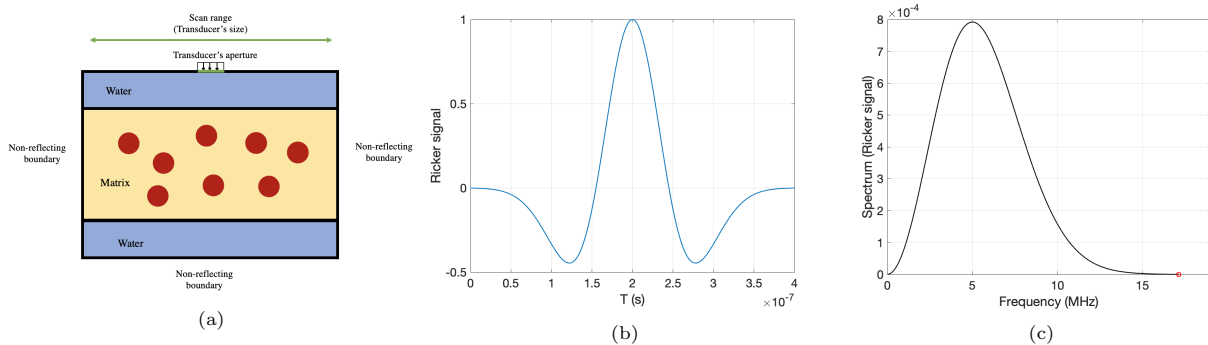


Figure 10. (a) 2D representative geometry of the studied 3D-printed microstructure embedded in water and boundary conditions; (b) Initial Ricker pulse generated by the transducer's piezoelectric element and (c) its frequency content

353 The ultrasonic wave is initiated in the domain with a simple sequential linear array transducer model.
 354 A sequential linear array transducer arranges multiple small piezoelectric elements along a straight line to
 355 produce 2D images. It works by transmitting on each element or group of elements (to increase the aperture
 356 size), receiving the echo information with the same element or group of elements, and reconstructing each
 357 line in the final 2D image displayed. To simplify the process, It is assumed that the aperture size is that of a
 358 single piezoelectric element. The operation of the transducer is performed sequentially; that is, after the first
 359 element has completed the transmit-receive procedure, the second element initiates the pulse and completes
 360 the same procedure, and so on, until the last element. Hence, the total length of the image is equivalent to
 361 the length of the transducer. The final image is composed of n different lines, with n corresponding to the
 362 total number of elements in the transducer.
 363

364 In the examples presented in this paper, the transducer is assumed to have a length of $5.6mm$ and is
 365 composed of $n = 28$ piezoelectric elements, each with a size of $200\mu m$. The vertical component of the
 366 velocity signal recorded on the transducer is used to perform the post-processing and to reconstruct the
 367 final image.

368 The ultrasonic pulse generated by the transducer is chosen to be a Ricker wavelet with the frequency
 369 content centered at $f_{max} = 5MHz$, and a cutoff frequency of $f_c = 17MHz$ (Figure 10b and Figure 10c). The

Table I. Material properties of the quasi-incompressible matrix and the inclusions

| Material | $\rho(kg/m^3)$ | $\kappa(GPa)$ | $\mu(MPa)$ | $c_p(m/s)$ | $c_s(m/s)$ | $\lambda_p(\mu m)$ | $\lambda_s(\mu m)$ |
|-----------|----------------|---------------|------------|------------|------------|--------------------|--------------------|
| water | 1000 | 2.10 | – | 1500 | – | [215 – 500] | – |
| matrix | 1150 | 3.10 | 0.3 | 1620 | 17 | [230 – 540] | [2.4 – 5.6] |
| inclusion | 1200 | 6.78 | 1830 | 2680 | 1235 | [380 – 890] | [175 – 410] |

material properties and the range of ultrasonic wave wavelengths corresponding to the ultrasonic imaging frequency range considered here, 3 – 7MHz, are presented in Table I.

One can observe that in the quasi-compressible matrix, the propagation speed of the shear wave is two orders of magnitude smaller than the one of the compression wave. Therefore, to capture the shear wave in the matrix, an unusually small element size is required, which makes the numerical simulation extremely computationally expensive. In addition, the shear wave generated at the matrix/inclusion interface propagates very slowly in the matrix and needs much more time to reach the transducer. As a result, we argue that its propagation in the matrix has a negligible impact on the velocity signal recorded by the transducer. Therefore, the finite element size in the numerical simulations is chosen based on the next shortest wavelength in the domain, the one determined by the shear wave in the inclusion. By selecting the element size of $h = 10\mu m$, we have respectively 10 and 18 elements per this shortest wavelength at the cutoff frequency 12.5MHz of the ricker signal and the maximum frequency 7MHz of the ultrasonic imaging frequency range considered in this work. The time step $\Delta t = 0.7\mu s$ is calculated using (58) to ensure stability.

5.1. Advantages of modeling the quasi-incompressible material as an equivalent acoustic medium

In this section, the quasi-incompressible matrix in the composite is modeled using an equivalent acoustic material, i.e., both quasi-incompressible elastic material and acoustic material have the same compression wave speed (Table II). The objective is to see whether this modification impacts the velocity signal recorded on the transducer and evaluate the performance of the developed coupled acoustic/elastic solver compared to the pure elastic solver. In this precise case. For this purpose, the layer of water is removed, and the transducer is placed directly in contact with the matrix over the entire top boundary (Figure 11a).

Table II. Material properties of the quasi-incompressible matrix, using acoustic and elastic models

| Material | $\rho(kg/m^3)$ | $\kappa(GPa)$ | $\mu(MPa)$ | $c_p(m/s)$ | $c_s(m/s)$ | $\lambda_p(\mu m)$ | $\lambda_s(\mu m)$ |
|-------------------|----------------|---------------|------------|------------|------------|--------------------|--------------------|
| matrix (acoustic) | 1150 | 3.03 | – | 1620 | – | [230 – 540] | – |
| matrix (elastic) | 1150 | 3.10 | 0.3 | 1620 | 17 | [230 – 540] | [2.4 – 5.6] |

Figure 11b shows the vertical component of velocity recorded on the transducer using two different models, and there is an excellent agreement in the results. However, it was observed that by using the coupled solver, the total calculation time is decreased by 19%. The reason is that for the acoustic element in a 2D problem, we have 3 degrees of freedom per node (2 for velocity and 1 for pressure, while for the elastic problem, we have 5 degrees of freedom per node (2 for velocity and 3 for stress). Moreover, the size of the flux matrices on an acoustic/elastic interface is smaller than those on an elastic/elastic interface. Therefore the coupled acoustic/elastic is faster and less expensive in memory than the elastic solver. This advantage is more pronounced in 3D because, for 3D problems, we have for the acoustic case 4 degrees of freedom per node (3 for velocity and 1 for pressure), whereas for the elastic case 9 degrees of freedom per node (3 for velocity and 6 for stress). Hence, we chose to model the quasi-incompressible matrix by its equivalent acoustic medium for all the results presented below.

5.2. Reconstructed medical image: a multiparametric study

A B-mode image is a type of ultrasound image that displays the relative reflectivity of the target material. They are obtained by a simple post-processing procedure on the vertical component of the velocity signal

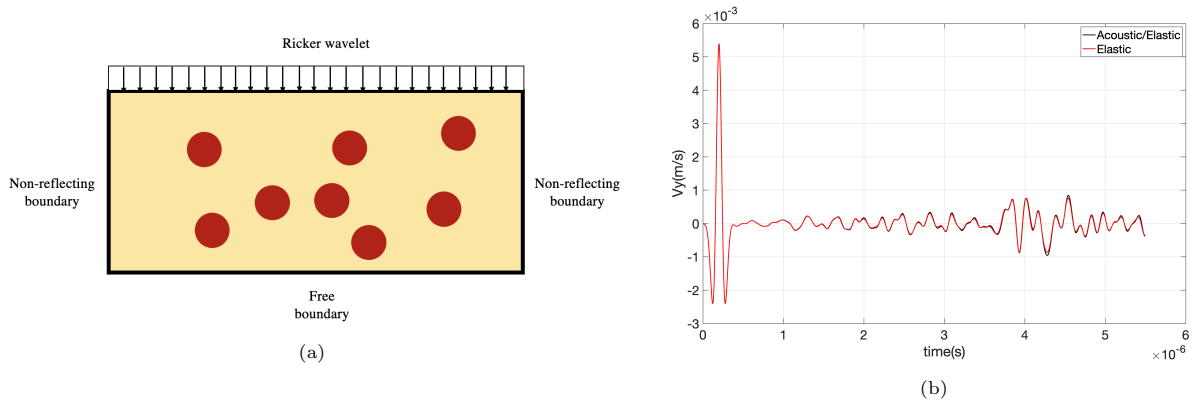


Figure 11. (a) Considered geometry and boundary conditions for comparing the elastic and acoustic model of the quasi-incompressible matrix, (b) Comparison of the vertical component of the velocity signal recorded on the transducer, obtained by coupled acoustic/elastic and pure elastic solver

405 recorded on the transducer. Indeed, the amplitude of the reflected echoes is used to generate an image, in
 406 which the brightness of each pixel representing the reflectivity of the object at that location (converted from
 407 time using an estimated effective propagation velocity of the medium). The image is typically displayed
 408 using a rectangular format, with parallel scan lines perpendicular to the direction of the ultrasound beam.
 409 The gray levels are represented by a 8-bit scale (0-255), with higher values corresponding to brighter pixels
 410 and lower values corresponding to darker pixels.

411 Here we investigate the impact of the area fraction (ϕ) and the diameter (D) of circular inclusions on the
 412 echogenicity of a single layer of 3D-printed composites (Figure 10a). For that purpose, the B-mode images
 413 are obtained for four different area fractions of inclusions: $\phi_1 = 1\%$, $\phi_2 = 3\%$, $\phi_3 = 5\%$ and $\phi_4 = 10\%$ and
 414 three different inclusion diameters $D_1 = 200\mu\text{m}$, $D_2 = 300\mu\text{m}$ and $D_3 = 400\mu\text{m}$.

415 In the reconstruction of B-mode images, a series of post-processing steps are applied to the raw received
 416 signal from each piezoelectric element:

- 417 (1) Obtain the envelope of the received signal;
- 418 (2) Convert the amplitude to gray levels (G), with higher amplitudes resulting in brighter points;
- 419 (3) Convert the time to the distance of the reflector from the transducer by multiplying the time by the
 420 propagation speed of the compression wave in water, $c_{p,water} = 1500\text{m/s}$, and dividing by two. This is
 421 done to account for the wave traveling to the object and back to the transducer.

422 Figure 12 shows these steps applied to the raw signal obtained for a piezoelectric element for the case of
 423 $\phi = 10\%$ and $D = 200\mu\text{m}$. The final image is obtained by assembling all the gray level vectors, where the
 424 vertical axis shows the distance of the reflector from the transducer, and the lateral axis shows the total
 425 scan range, which in this paper is the size of the transducer.

426 As an example, Figure 13 depicts two different geometries and the corresponding B-mode images (all
 427 the numbers are in millimeters). One can observe that the water-matrix interface at the distance of 1mm is
 428 perfectly depicted. A horizontal dashed red line is also added to the B-mode image to show the ideal location
 429 of the matrix-water interface at the distance of 3.5mm. This interface might not be easily distinguishable
 430 in some cases, while it is very important if we want to obtain precisely the thickness of the imaged layer.

431 Figure 14 presents the reconstructed B-mode images for all the mentioned area fractions and sizes of
 432 inclusions. It can be observed that as the area fraction of the inclusion decreases, the matrix-water interface
 433 becomes more visible. However, the interface appears to be closer to the transducer than it actually is.
 434 This is because the post-processing steps assume the medium in which the wave is propagating is water and
 435 therefore calculate distances based on the speed of sound in water. Since the speed of sound in the matrix
 436 is faster than that of water, the wave travels faster in the matrix and appears thus closer to the transducer.

437 We also observe that as the area fraction of the inclusions increases and the size of the inclusion decreases,
 438 the echogenicity of the material to ultrasound waves improves. However, this makes it more challenging to

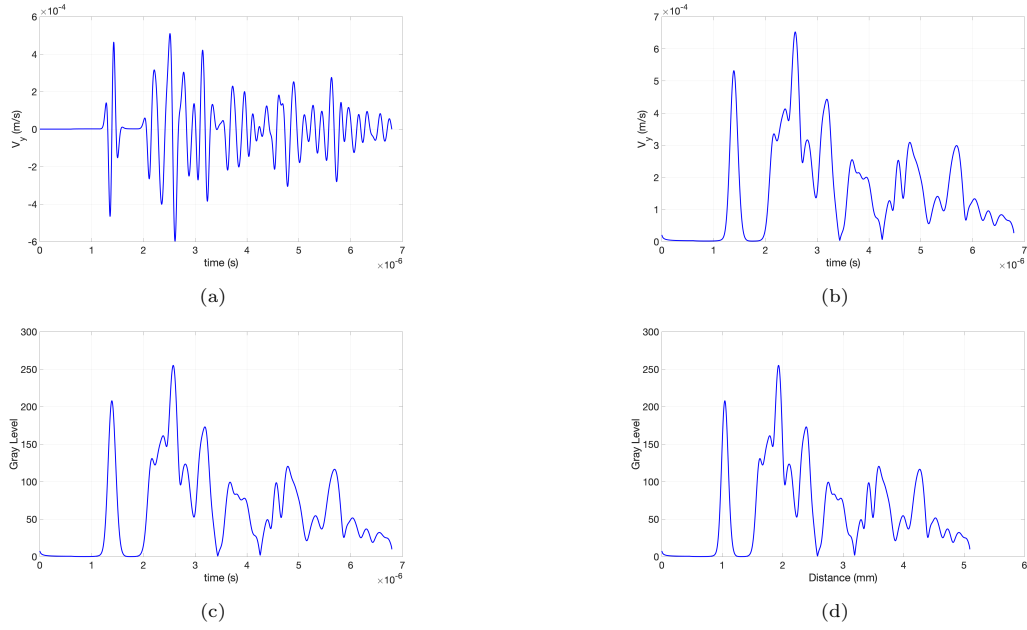


Figure 12. Post-processing steps for reconstruction of the B-mode medical image applied to a signal obtained for a single piezoelectric element for the case of $\phi = 10\%$ and $D = 200\mu\text{m}$ (a) raw received signal, (b) extracted envelope, (c) converted to gray levels ranging from 0 to 255, and (d) time converted to distance of the reflector from the transducer.

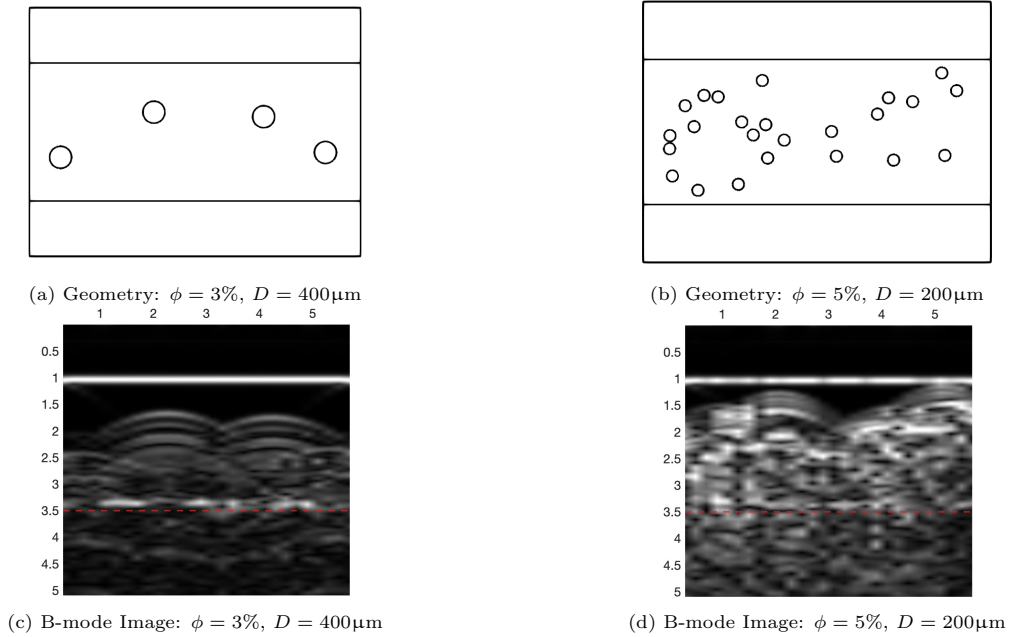


Figure 13. Two different configurations with various area fractions and diameter sizes of the inclusions, and corresponding B-mode images (all the sizes are in millimeters)

439 distinguish the matrix-water interface located at a distance of 3.5mm from the transducer. Therefore, it is
 440 challenging to see the actual thickness of the material, which is an important objective.

441 To have a more precise quantitative measure of echogenicity, the average gray level over the region

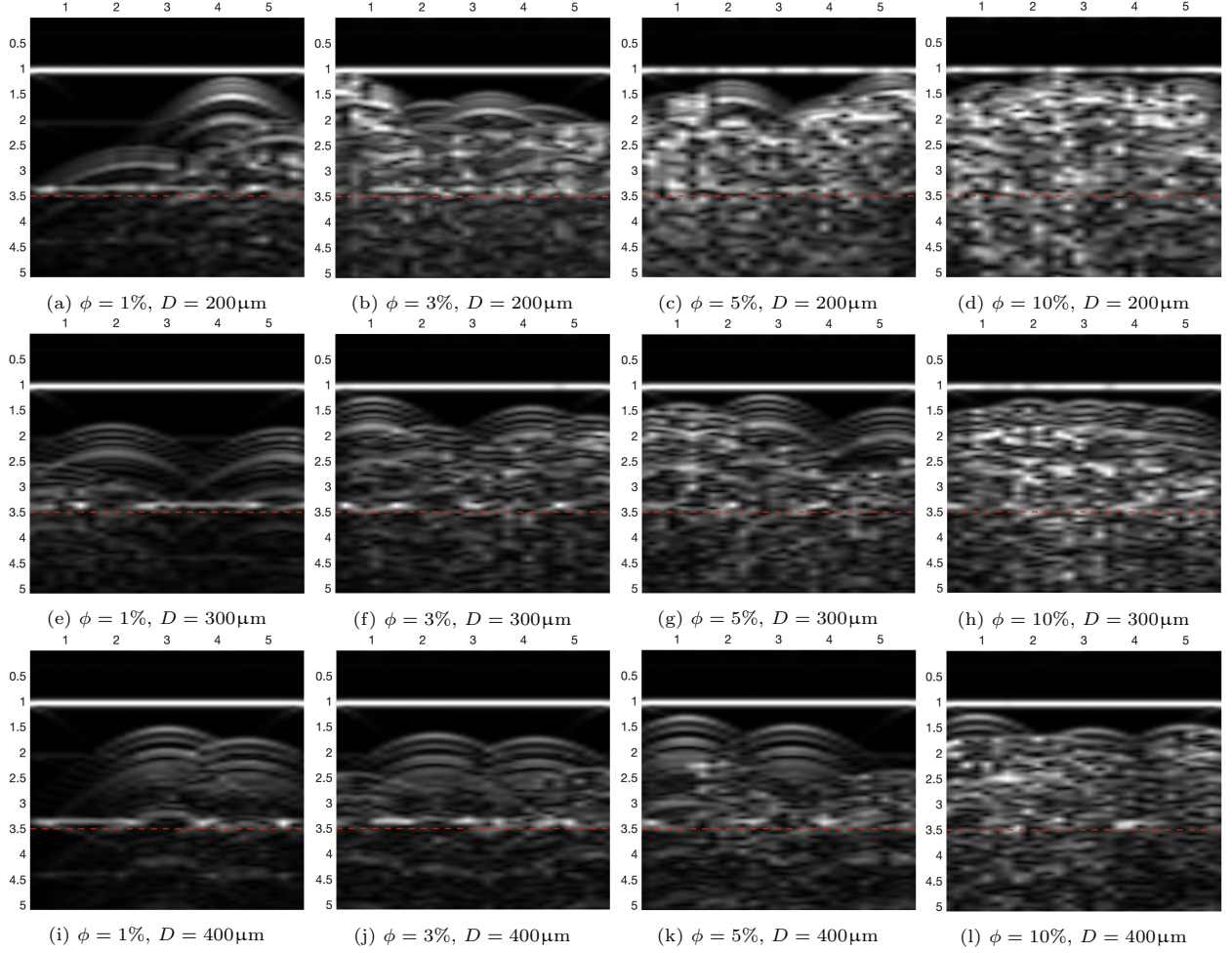


Figure 14. Reconstructed B-mode images for area fractions of inclusions 1%, 5%, 5%, and 10%

of interest (ROI), which corresponds to the actual thickness of the material, is chosen as a parameter to measure the reflectivity of the material to ultrasound waves, which is defined as follows:

$$\bar{G}_{ROI} = \frac{1}{N_{piezo} \times (d_{bottom} - d_{top})} \sum_{i=1}^{N_{piezo}} \int_{d_{top}}^{d_{bottom}} G(x_i, y) dy \quad (64)$$

where $d_{top} = 1\text{mm}$ and $d_{bottom} = 3.5\text{mm}$ are respectively the distance from the transducer to the top water-matrix interface and the bottom matrix-water interface. Moreover, the average gray level of the region below the bottom matrix-water interface, hereafter named the region of the residual signal (RORS), is calculated as follows:

$$\bar{G}_{RORS} = \frac{1}{N_{piezo} \times (d_{total} - d_{bottom})} \sum_{i=1}^{N_{piezo}} \int_{d_{bottom}}^{d_{total}} G(x_i, y) dy \quad (65)$$

where the total distance $d_{total} = T_{total} \times c_{p,water}/2$ which can be observed in Figure 12d. The ratio of \bar{G}_{RORS} to \bar{G}_{ROI} is used as another important metric to indicate how well the thickness of the material can be distinguished.

Figure 15a and Figure 15b show the average of the gray levels \bar{G}_{ROI} with respect to the area fraction of the inclusions and the diameter size, respectively. These figures show an increasing trend in the average gray

453 level when the area fraction is increased for all three sizes. It also indicates that for a given area fraction of
 454 inclusion, the average gray level increases as the size of the inclusions decreases. However, the same behavior
 455 is observed for the \overline{G}_{RORS} as illustrated in Figure 15c and Figure 15d, which is not desirable.

456 Looking at the ratio of the \overline{G}_{RORS} to the \overline{G}_{ROI} , we notice that it increases for $D = 200\mu\text{m}$ and
 457 $D = 300\mu\text{m}$ as the area fraction is raised. On the other hand, for $D = 400\mu\text{m}$, the ratio decreases when
 458 we increase the area fraction from 5 to 10, which means that the actual thickness of the material is more
 459 recognizable. This behavior could be observed in Figure 15e and Figure 15f.

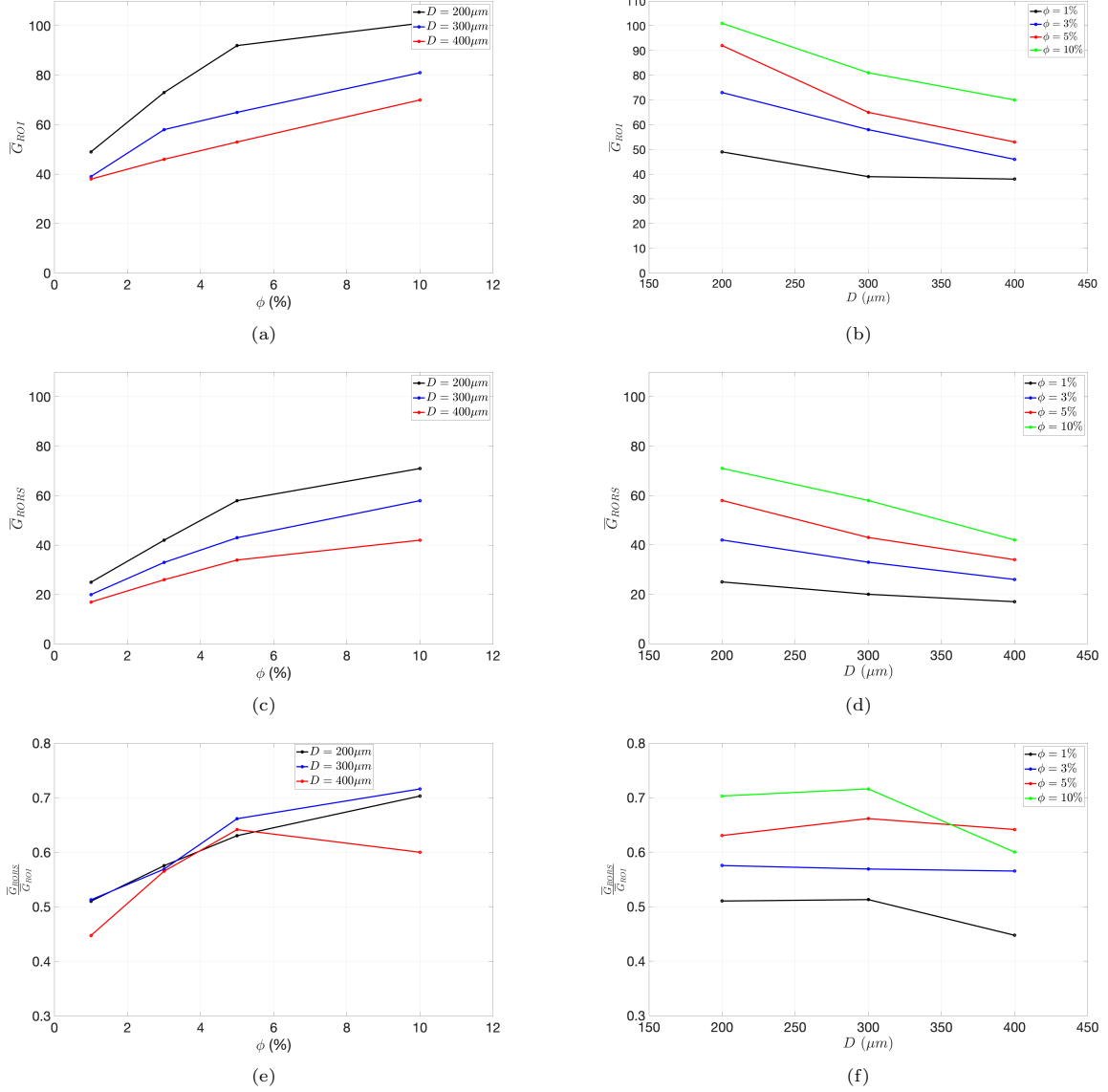


Figure 15. Quantitative analysis of the average gray levels. (a) \overline{G}_{ROI} , (c) \overline{G}_{RORS} , and (e) $\frac{\overline{G}_{RORS}}{\overline{G}_{ROI}}$ with respect to the area fraction ϕ ; (b) \overline{G}_{ROI} , (d) \overline{G}_{RORS} , and (f) $\frac{\overline{G}_{RORS}}{\overline{G}_{ROI}}$ with respect to the inclusion's diameter D

460 The same calculations are repeated for another batch of samples with $\phi = 10\%$, and the results are
 461 presented in Table III. According to this table, the average gray levels are similar for two different batches,
 462 independent of the inclusions' distribution. However, a more rigorous statistical study is required to study

463 different random distributions of inclusions for a given area fraction and size of inclusions.

Table III. Comparison of average gray levels for two different batch of samples with $\phi = 10\%$

| Diameter Size | $D = 200\mu\text{m}$ | | $D = 300\mu\text{m}$ | | 400 μm | |
|--|----------------------|-----------|----------------------|-----------|-------------------|-----------|
| | 1st batch | 2nd batch | 1st batch | 2nd batch | 1st batch | 2nd batch |
| \overline{G}_{ROI} | 101 | 100 | 81 | 81 | 70 | 73 |
| \overline{G}_{RORS} | 71 | 72 | 58 | 58 | 42 | 43 |
| $\frac{\overline{G}_{RORS}}{\overline{G}_{ROI}}$ | 0.70 | 0.72 | 0.72 | 0.72 | 0.60 | 0.59 |

464 Conclusions

465 The derivation of the upwind numerical fluxes for the space discontinuous Galerkin finite element method
466 was first proposed for the numerical modeling of the coupled acoustic/elastic wave propagation in multidimensional
467 media with arbitrary anisotropic solid and acoustic fluid. The upwind numerical fluxes derived
468 at the acoustic/elastic interfaces are original within the frameworks of the first-order velocity-pressure and
469 the first-order velocity-stress formulation governing the domains composed of acoustic and elastic subdomains.
470 Developed by analytically solving the Riemann problem, they were expressed in explicit closed forms
471 presented within a compact and intrinsic tensorial framework. An analytical/numerical comparison was
472 performed for a problem with a circular acoustic/elastic interface, and it was shown that the developed
473 numerical fluxes provided accurate results. Indeed, according to local misfit criteria, which decompose the
474 difference between two signals into the envelope (amplitude) and phase misfits, the mismatch between the
475 numerical and analytical solutions was quantified, and numerical errors were evaluated. However, it is necessary
476 to compare the performance of the proposed method to other methods, such as those developing
477 the numerical flux (e.g., the penalty flux method), in terms of accuracy and convergence rate. This needs
478 yet to be studied in detail in future works. Finally, the developed coupled solver was applied to simulate
479 ultrasonic wave propagation in a simplified model of a 3D-printed matrix-inclusion composite microstructure
480 intended to mimic the acoustic behavior of the biological tissues under the ultrasonic imaging procedure. A
481 parametric study of the effects of the area fraction and the size of the inclusions on the average gray level
482 of the reconstructed B-images was performed. It was shown that increasing the area fraction of the circular
483 inclusions and decreasing their size improved the echogenicity of the 3D-printed composite layer. However,
484 it also makes it more challenging to determine the thickness of the layer accurately. Modification of the
485 microstructure is therefore necessary and should be investigated in future work.

486 Acknowledgments

487 The authors would like to thank Dr. Bruno Lombard (research director at LMA, Marseille, France) for
488 kindly providing the analytical solution required for validating the numerical results. This work was granted
489 by (Île-de-France Region, DIM Respole, Biomodex) Computations were performed using HPC resources
490 from the “Mésocentre” computing center of CentraleSupélec and ENS Paris-Saclay.

491 CRediT authorship contribution statement

492 **Hossein Kamalinia:** Conceptualization, Software, Validation, Formal analysis, Writing - Original
493 Draft, Writing - Review & Editing, Visualization. **Andrea Barbarulo:** Validation, Writing - Review &
494 Editing, Funding acquisition. **Bing Tie:** Conceptualization, Methodology, Software, Validation, Writing -
495 Original Draft, Writing - Review & Editing, Supervision, Funding acquisition.

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602 Appendix A1: Acoustic-acoustic interface

603 By adding the three equations of (32), the unknown states $\{U^a, U^{a'}\}$ are eliminated and one obtains:

$$A_n(U_h) - \alpha \lambda_n^- M(R_n^-) + A'_n(U'_h) - \alpha' \lambda_n'^- M'(R_n'^-) = 0 \quad (66)$$

Regarding the definition of M (2), the Jacobian operator (4) and the eigenmodes (22) and (24), one can obtain the following equations by applying the \wp_{vect} and \wp_{scalar} on (66):

$$-p_h \mathbf{n} - p'_h \mathbf{n}' = \alpha z_n^- \frac{1}{\sqrt{2}} \mathbf{n} + \alpha' z_n'^- \frac{1}{\sqrt{2}} \mathbf{n}' \quad (67a)$$

$$-\mathbf{n} \cdot \mathbf{v}_h - \mathbf{n}' \cdot \mathbf{v}'_h = -\frac{1}{\sqrt{2}} \alpha - \frac{1}{\sqrt{2}} \alpha' \quad (67b)$$

604 Hence, one obtains a linear system of two equations for two unknowns $\{\alpha, \alpha'\}$. Its solution gives rise to
605 the expressions introduced in (34) by performing the following manipulations:

- 606 • To obtain α
 607 $-\frac{z_n^-}{z_n^-} \frac{1}{\sqrt{2}} \left(\frac{\mathbf{n}'}{z_{\mathbf{n}'}} \cdot \text{Eq.}(67a) + \text{Eq.}(67b) \right)$ leads to:

$$\tilde{\mathbf{L}}_{\mathbf{n}'}^- \cdot (\mathbf{U}_h - \mathbf{U}'_h) = \alpha \quad (68)$$

- 608 • To obtain α'
 609 $-\frac{z_n^-}{z_{\mathbf{n}'}} \frac{1}{\sqrt{2}} \left(\frac{\mathbf{n}}{z_n^-} \cdot \text{Eq.}(67a) + \text{Eq.}(67b) \right)$ leads to:

$$\tilde{\mathbf{L}}_{\mathbf{n}'}'^- \cdot (\mathbf{U}'_h - \mathbf{U}_h) = \alpha' \quad (69)$$

Finally, using the (38b) and (68), the term in the variational formulation (19) and linked to the numerical flux on the acoustic-acoustic interface EE' is calculated as follows:

$$\langle \hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}_{\mathbf{n}}(\mathbf{U}), \mathbf{W} \rangle_{EE'} = \langle -\alpha z_n^- \tilde{\mathbf{L}}_{\mathbf{n}'}^-, \mathbf{W} \rangle_{EE'} \quad (70a)$$

$$= z_n^- \langle -\tilde{\mathbf{L}}_{\mathbf{n}'}^- \cdot \mathbf{U}_h, \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{W} \rangle_{EE'} + z_n^- \langle \tilde{\mathbf{L}}_{\mathbf{n}'}^- \cdot \mathbf{U}'_h, \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{W} \rangle_{EE'} \quad (70b)$$

610 Appendix A2: Acoustic-elastic interface

611 In this appendix, we consider the numerical flux in an acoustic element E having an interface with an
 612 adjacent elastic element E' .

613 By applying the operator $\mathbf{\Pi}_{\mathbf{n}'}$ to (43c) and then adding it to the other two equations of (43), the unknown
 614 states $\{\mathbf{U}^a, \mathbf{U}^{c'}\}$ are eliminated and one obtains:
 615

$$\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h) - \alpha z_n^- \tilde{\mathbf{L}}_{\mathbf{n}'}^- + \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{A}'_{\mathbf{n}'}(\mathbf{U}'_h)) - \alpha'_k z_{\mathbf{n}',k}^- \tilde{\mathbf{L}}_{\mathbf{n}',k}^- \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{L}_{\mathbf{n}',k}^-) = \mathbf{0} \quad (71)$$

Regarding the definition of the Jacobian operator by (4) and (16), the left eigenvectors by (24) and (26), one can obtain the following equations by applying the \mathfrak{p}_{vect} and \mathfrak{p}_{scalar} on (71):

$$-p_h \mathbf{n} - \boldsymbol{\sigma}'_h \cdot \mathbf{n}' = \alpha z_n^- \frac{1}{\sqrt{2}} \mathbf{n} + \sum_l \alpha'_l z_{\mathbf{n}',l}^- \frac{\boldsymbol{\gamma}'_{\mathbf{n}',l}}{\sqrt{2}} \quad (72a)$$

$$-\mathbf{n} \cdot \mathbf{v}_h - \mathbf{n}' \cdot \mathbf{v}'_h = -\alpha \frac{1}{\sqrt{2}} - \sum_l \alpha'_l (\mathbf{n}' \cdot \frac{\boldsymbol{\gamma}'_{\mathbf{n}',l}}{\sqrt{2}}) \quad (72b)$$

616 Hence, one obtains a linear system of four equations for four unknowns $\{\alpha, \{\alpha'_k\}\}$. Its solution gives rise
 617 to the expressions introduced in (44) by performing the following manipulations:

- 618 • First equation (for α):
 619 $\frac{1}{\sqrt{2}} \left(-\frac{\mathbf{n}}{z_n^-} \cdot \text{Eq.}(72a) + \text{Eq.}(72b) \right)$ leads to:

$$\tilde{\mathbf{L}}_{\mathbf{n}'}^- \cdot (\mathbf{U}_h - \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{U}'_h)) = \alpha - \sum_l \frac{z_n^- - z_{\mathbf{n}',l}^-}{2z_n^-} (\mathbf{n} \cdot \boldsymbol{\gamma}'_{\mathbf{n}',l}) \alpha'_l \quad (73)$$

- 620 • Equations for $\{\alpha'_k\}$:
 621 For each k , $\sqrt{2} \frac{\boldsymbol{\gamma}'_{\mathbf{n}',k}}{z_{\mathbf{n}',k}^-} \cdot (\text{Eq.}(72a))$ leads to:

$$2\mathfrak{p}_{tens}(\tilde{\mathbf{L}}_{\mathbf{n}',k}^-) : \mathfrak{p}_{tens}(\mathbf{U}'_h - \mathbf{\Psi}_{\mathbf{n}}(\mathbf{U}_h)) = \frac{z_n^-}{z_{\mathbf{n}',k}^-} \mathbf{n} \cdot \boldsymbol{\gamma}'_{\mathbf{n}',k} \alpha + \alpha'_k \quad (74)$$

In order to calculate the numerical flux on the acoustic-elastic interface for the acoustic element E , we recall that the equation (38b) is used. So, one needs to find α by solving the system of equations (44), which gives rise to:

$$\begin{pmatrix} \alpha \\ \{\alpha'_k\} \end{pmatrix} = [R^{ae}]^{-1} \cdot \left[\begin{array}{c} \mathbf{L}_{\mathbf{n}'}^- \cdot (\mathbf{U}_h - \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{U}'_h)) \\ \{2\wp_{tens}(\mathbf{L}_{\mathbf{n}',k}^-) : \wp_{tens}(\mathbf{U}'_h - \mathbf{\Psi}_{\mathbf{n}}(\mathbf{U}_h))\} \end{array} \right] \quad (75)$$

622 with $[R^{ae}]^{-1}$ defined by (48). (47) is finally proved.

Finally, the term in the variational formulation (19) for the acoustic element E and linked to the numerical flux on the acoustic-elastic interface EE' with the element E' is calculated using the following equation:

$$\begin{aligned} < \hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}_{\mathbf{n}}(\mathbf{U}), \mathbf{W} >_{EE'} = < -\alpha z_{\mathbf{n}}^- \mathbf{L}_{\mathbf{n}}^-, \mathbf{W} >_{EE'} = -z_{\mathbf{n}}^- < (D_{11}^{ae} \mathbf{L}_{\mathbf{n}}^-) \cdot (\mathbf{U}_h - \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{U}'_h)), \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} \\ & + z_{\mathbf{n}}^- < 2(H_{1k}^{ae} \wp_{tens}(\mathbf{L}_{\mathbf{n}',k}^-)) : \wp_{tens}(\mathbf{U}'_h - \mathbf{\Psi}_{\mathbf{n}}(\mathbf{U}_h)), \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} \\ & = -z_{\mathbf{n}}^- < (D_{11}^{ae} \mathbf{L}_{\mathbf{n}}^-) \cdot \mathbf{U}_h, \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} + z_{\mathbf{n}}^- < (D_{11}^{ae} \mathbf{L}_{\mathbf{n}}^-) \cdot \mathbf{\Pi}_{\mathbf{n}'}(\mathbf{U}'_h), \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} \\ & + z_{\mathbf{n}}^- < 2(H_{1k}^{ae} \wp_{tens}(\mathbf{L}_{\mathbf{n}',k}^-)) : \wp_{tens}(\mathbf{U}'_h), \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} - z_{\mathbf{n}}^- < 2(H_{1k}^{ae} \wp_{tens}(\mathbf{L}_{\mathbf{n}',k}^-)) : \wp_{tens}(\mathbf{\Psi}_{\mathbf{n}}(\mathbf{U}_h)), \mathbf{L}_{\mathbf{n}}^- \cdot \mathbf{W} >_{EE'} \end{aligned} \quad (76)$$

623 Appendix A3: Elastic-acoustic interface

624 To calculate the numerical flux on the acoustic-elastic interface for the elastic element E , we recall that
625 the equation (54) is used. So, one needs to find α_k by solving the system of equations (50), which gives rise to:
626

$$\begin{pmatrix} \{\alpha_k\} \\ \alpha' \end{pmatrix} = [R^{ea}]^{-1} \cdot \left[\begin{array}{c} \{2\wp_{tens}(\mathbf{L}_{\mathbf{n},k}^-) : \wp_{tens}(\mathbf{U}_h - \mathbf{\Psi}_{\mathbf{n}'}(\mathbf{U}'_h))\} \\ \mathbf{L}_{\mathbf{n}'}^- \cdot (\mathbf{U}'_h - \mathbf{\Pi}_{\mathbf{n}}(\mathbf{U}_h)) \end{array} \right] \quad (77)$$

627 with $[R^{ae}]^{-1}$ is defined in (53). Hence, (52) is proved.

628 Finally, the term in the variational formulation (19) for the elastic element E and linked to the numerical
629 flux on the elastic-acoustic interface EE' with the element E' is calculated using the following equation:

$$\begin{aligned} < \hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}_{\mathbf{n}}(\mathbf{U}), \mathbf{W} >_{EE'} = < -\alpha_k z_{\mathbf{n},k}^- \mathbf{L}_{\mathbf{n},k}^-, \mathbf{W} >_{EE'} = z_{\mathbf{n},k}^- < H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot (\mathbf{U}'_h - \mathbf{\Pi}_{\mathbf{n}}(\mathbf{U}_h)), \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} \\ & + z_{\mathbf{n},k}^- < 2D_{kl}^{ea} \wp_{tens}(\mathbf{L}_{\mathbf{n},k}^-) : \wp_{tens}(\mathbf{U}_h - \mathbf{\Psi}_{\mathbf{n}}(\mathbf{U}'_h)), \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} \\ & = +z_{\mathbf{n},k}^- < H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{U}'_h, \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} - z_{\mathbf{n},k}^- < H_{k1}^{ea} \mathbf{L}_{\mathbf{n}'}^- \cdot \mathbf{\Pi}_{\mathbf{n}}(\mathbf{U}_h), \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} \\ & - z_{\mathbf{n},k}^- < 2D_{kl}^{ea} \wp_{tens}(\mathbf{L}_{\mathbf{n},k}^-) : \wp_{tens}(\mathbf{U}_h), \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} + z_{\mathbf{n},k}^- < 2D_{kl}^{ea} \wp_{tens}(\mathbf{L}_{\mathbf{n},k}^-) : \wp_{tens}(\mathbf{\Psi}_{\mathbf{n}'}(\mathbf{U}'_h)), \mathbf{L}_{\mathbf{n},k}^- \cdot \mathbf{W} >_{EE'} \end{aligned} \quad (78)$$

630 Appendix A4: Boundary Conditions Acoustic Solver

$$\hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) = \frac{1}{2} (\mathbf{A}_{\mathbf{n}}(\mathbf{U}_h) - \mathbf{A}'_{\mathbf{n}'}(\mathbf{U}'_h) - \alpha z_{\mathbf{n}}^- \mathbf{L}_{\mathbf{n}}^- + \alpha' z_{\mathbf{n}}'^- \mathbf{L}_{\mathbf{n}}'^-) \quad (79)$$

$$\hat{\mathbf{F}}_{\mathbf{n}}(\mathbf{U}_h, \mathbf{U}'_h) = \frac{1}{2} \begin{pmatrix} -(p_h + p'_h) \mathbf{n} \\ -(\mathbf{v}_h + \mathbf{v}'_h) \cdot \mathbf{n} \end{pmatrix} - \frac{1}{4} \begin{pmatrix} z_{\mathbf{n}}^- \mathbf{n} \cdot (\mathbf{v}_h - \mathbf{v}'_h) \mathbf{n} \\ \frac{1}{z_{\mathbf{n}}^-} (p_h - p'_h) \end{pmatrix} \quad (80)$$

631 As for the numerical flux on an external element boundary, a neighbor element E' having the same acoustic
632 behavior as E is introduced. In E' , it should be imposed that: $p'_h \mathbf{n} = 2\mathbf{g} - p_h \mathbf{n}$ and $\mathbf{v}'_h = \mathbf{v}_h$ for the

633 Neumann boundary conditions on $\partial\Omega_N$, and $p_h = p'_h$ and $\mathbf{v}'_h = 2\partial\mathbf{u}_D - \mathbf{v}_h$ for the Dirichlet boundary
 634 conditions on $\partial\Omega_D$.

635 For the Dirichlet boundary conditions, the flux is calculated as follows:

$$\hat{\mathbf{F}}_D = \hat{\mathbf{F}}_n(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}_n(\mathbf{U}) = \begin{pmatrix} \mathbf{0} \\ (v_h - v_D) \cdot \mathbf{n} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} z_n^- \mathbf{n} \cdot (v_h - v_D) \mathbf{n} \\ \mathbf{0} \end{pmatrix} \quad (81)$$

636 and for the Neumann boundary condition:

$$\hat{\mathbf{F}}_N = \hat{\mathbf{F}}_n(\mathbf{U}_h, \mathbf{U}'_h) - \mathbf{F}_n(\mathbf{U}) = \begin{pmatrix} p_h \mathbf{n} - \mathbf{g} \\ \mathbf{0} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \mathbf{0} \\ \frac{1}{z_n^-} (p_h - \mathbf{g} \cdot \mathbf{n}) \end{pmatrix} \quad (82)$$

637 So the flux on the external boundaries where $\partial\Omega_{Ext} = \partial\Omega_D \cup \partial\Omega_N$ and $\partial\Omega_D \cup \partial\Omega_N = \phi$, is calculated as
 638 follows:

$$\begin{aligned} \langle \hat{\mathbf{F}}_{ext}, \mathbf{W} \rangle_{\partial\Omega} &= \langle \hat{\mathbf{F}}_D, \mathbf{W} \rangle_{\partial\Omega_D} + \langle \hat{\mathbf{F}}_N, \mathbf{W} \rangle_{\partial\Omega_N} = \frac{1}{2} \langle z_n^- \mathbf{n} \cdot (v_h) \mathbf{n}, \mathbf{w} \rangle_{\partial\Omega_D} \\ &\quad - \frac{1}{2} \langle z_n^- \mathbf{n} \cdot (v_D) \mathbf{n}, \mathbf{w} \rangle_{\partial\Omega_D} + \langle \mathbf{n} \cdot v_h, q \rangle_{\partial\Omega_D} - \langle \mathbf{n} \cdot v_D, q \rangle_{\partial\Omega_D} \\ &\quad + \langle p_h \mathbf{n}, \mathbf{w} \rangle_{\partial\Omega_N} - \langle \mathbf{g}, \mathbf{w} \rangle_{\partial\Omega_N} + \frac{1}{2} \langle \frac{1}{z_n^-} p_h, q \rangle_{\partial\Omega_N} - \frac{1}{2} \langle \frac{1}{z_n^-} \mathbf{g} \cdot \mathbf{n}, q \rangle_{\partial\Omega_N} \quad (83) \end{aligned}$$