

From the Rössler attractor to the Templex

**Gisela D. Charó, Christophe Letellier &
Denisse Sciamarella**

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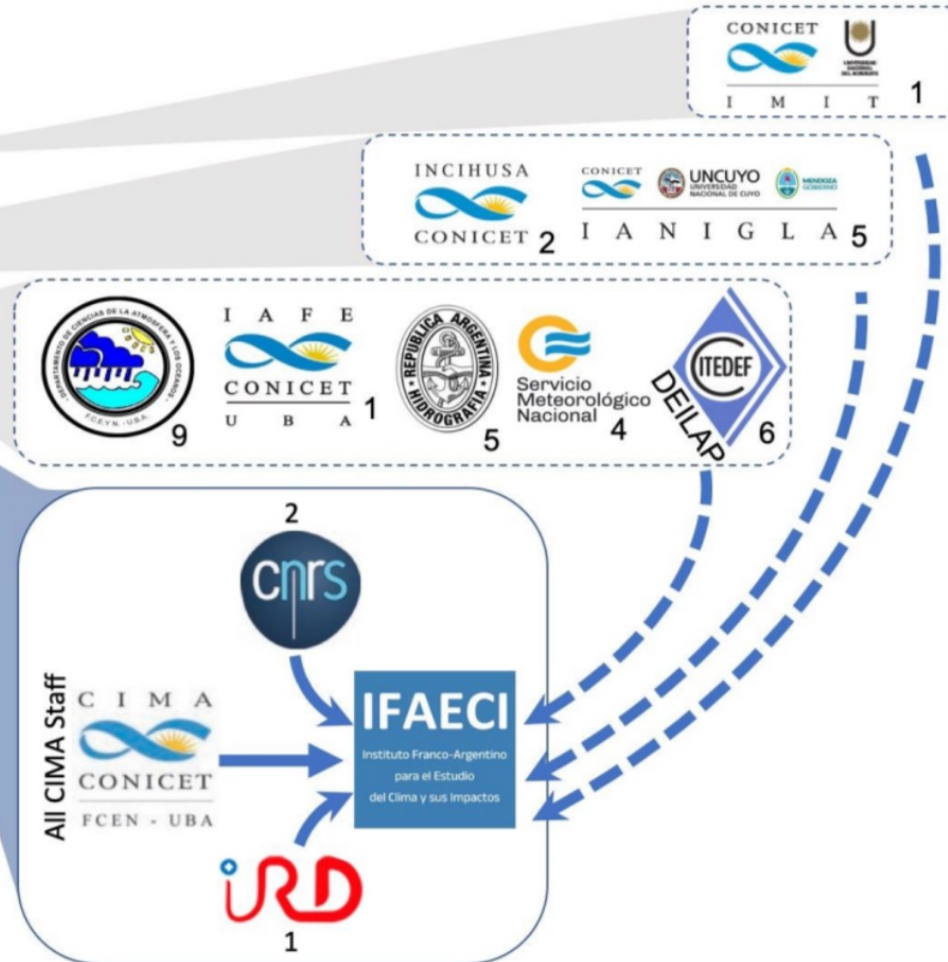
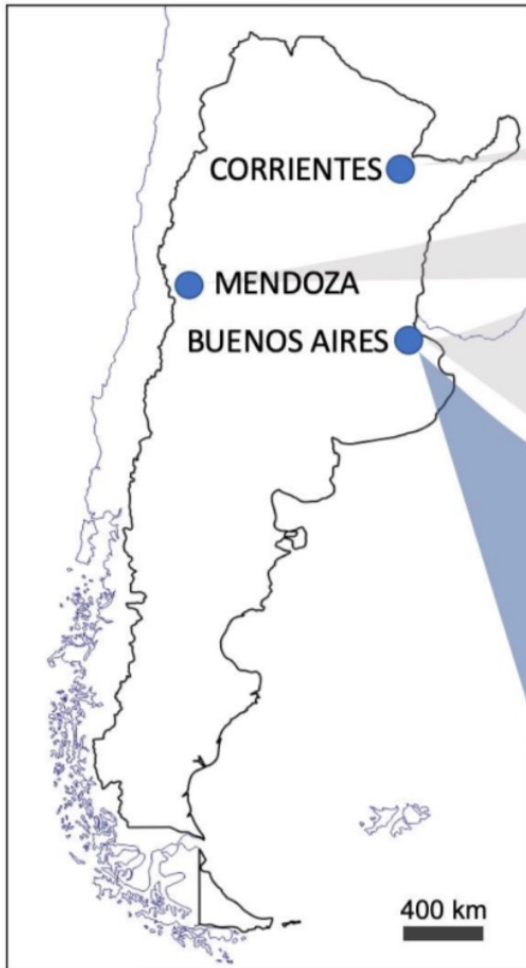
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**OTTOCHAOS: FROM THE NONLINEAR DYNAMICAL SYSTEMS THEORY TO OBSERVATIONAL CHAOS,
OCTOBER 9-11, 2023 – TOULOUSE, FRANCE**

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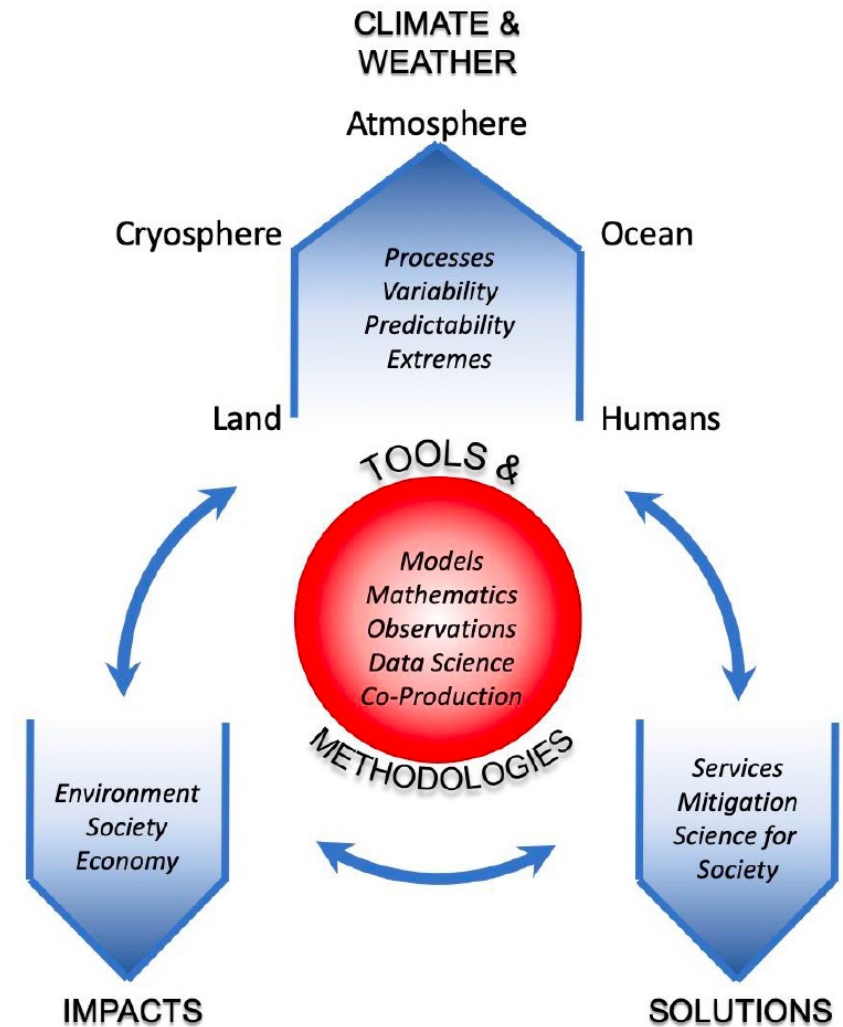
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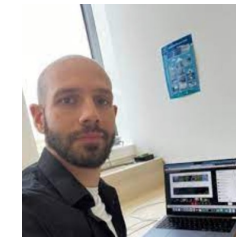
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AAPG2023	TeMPlex	LEFE-MANU MITI (CNRS)
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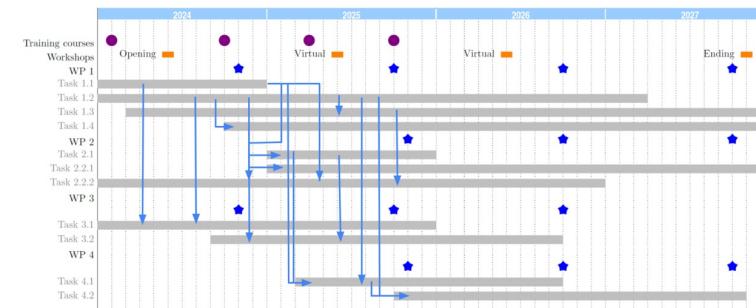
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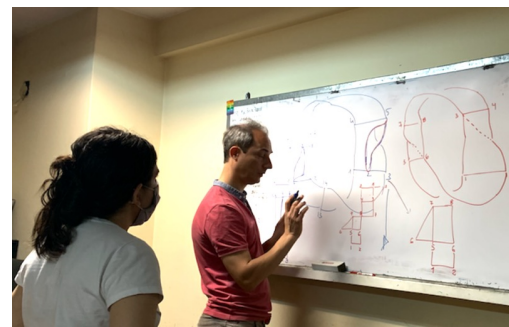
A. Sorensson



B. Dewitte

Collaborators

Christophe Letellier, CORIA, Rouen &
Sylvain Mangiarotti, CESBIO, Toulouse



CLIMAT-AmSud 2022 workshop with collaborators from the south of Argentina (Río Negro), **Chile** (Axel Osses) & **Uruguay** (Rafael Terra).



Michael Ghil,
Ecole Normale
Supérieure, Paris
&
Mickaël
Chekroun,
Weizmann Institute
of Science, Israël



Presentation Outline

Templates, the knot content of attractors

- Knots and braids to build a template

Algebraic topology

- Using homology group theory

The Rössler attractor

- Spiral versus funnel

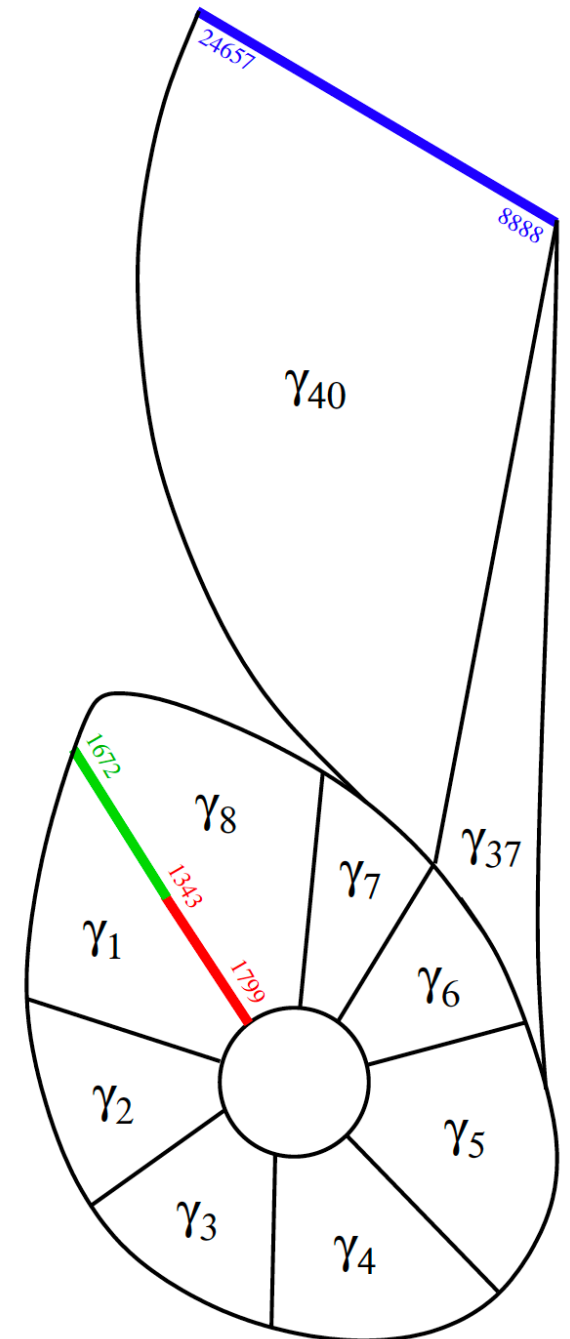
The birth of the templex

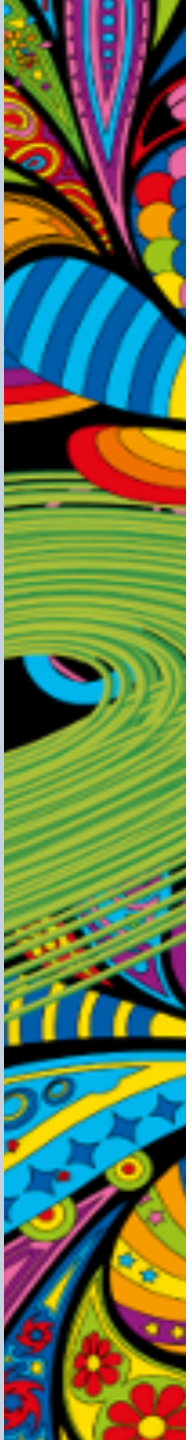
- How was it conceived? How is it computed?

Templex examples

- Lorenz 63 and a four-dimensional case

Conclusions





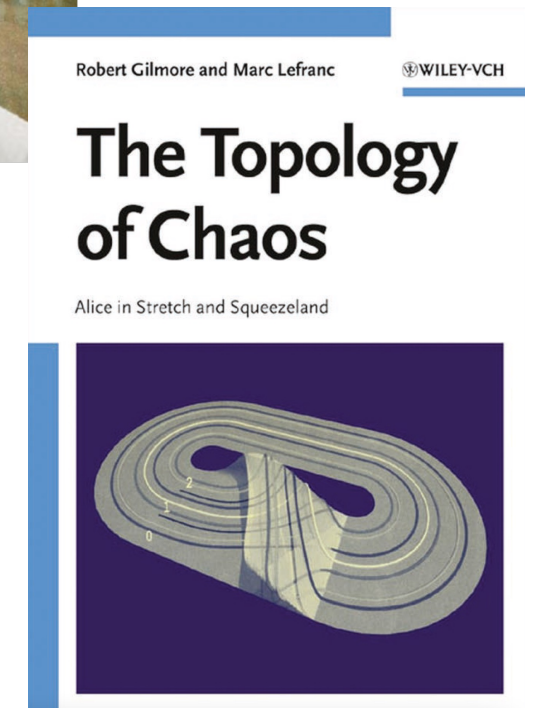
The first sentence of Leo Tolstoy's novel Anna Karenina is:

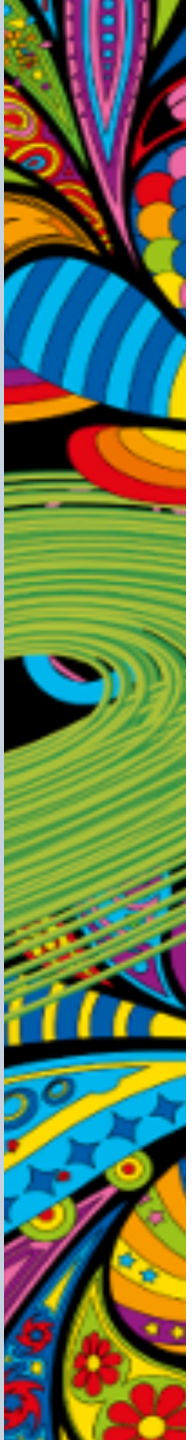
"All happy families are alike; each unhappy family is unhappy in its own way".

Following the famous writer, Robert Gilmore and Marc Lefranc, tell us:

"All linear systems are alike; each nonlinear system is nonlinear in its own way".

"It was a very happy and shocking discovery that there were structures in nonlinear systems that are always the same if you looked at them the right way."





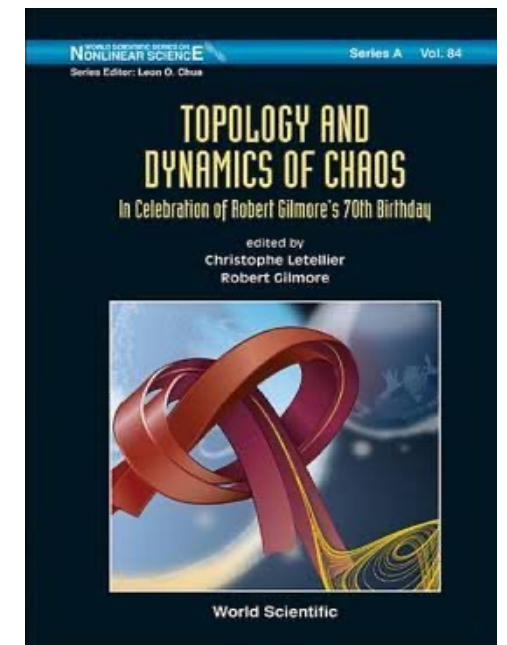
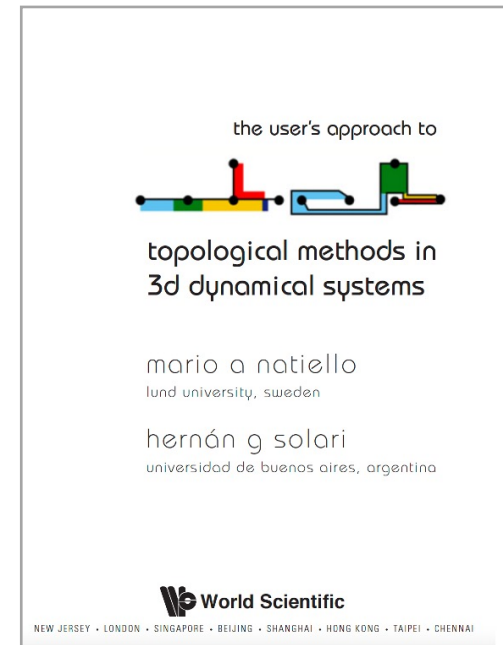
Topological analysis in phase space is the correct way of looking at dynamical systems.

For deterministic systems, the topological structure of a flow is an invariant that provides information on the mechanisms acting in phase space to shape the flow.

A topological analysis involves finding a topological representation of the underlying structure and obtaining an algebraic description allowing for the computation of topological invariants.

It was through the study of two variants of the Rössler attractor that we realized that it was necessary to define a new mathematical object in algebraic topology, now called templex.

The aim of this talk is to tell the story of the birth of templex starting with the Rössler attractor.



AIP Chaos: An Interdisciplinary Journal of Nonlinear Science

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Featured

Templex: A bridge between homologies and templates for chaotic attractors


Gisela D. Charó, Christophe Letellier and Denisse Sciamarella

Charó, G. D., Letellier, C., & Sciamarella, D. (2022). Templex: A bridge between homologies and templates for chaotic attractors. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(8), 083108.


- **Templates: the knot content of attractors**

- 1) Approximate trajectories by closed curves.
- 2) Find a topological representation for the orbit structure.
- 3) Obtain an algebraic description for the topological structure.

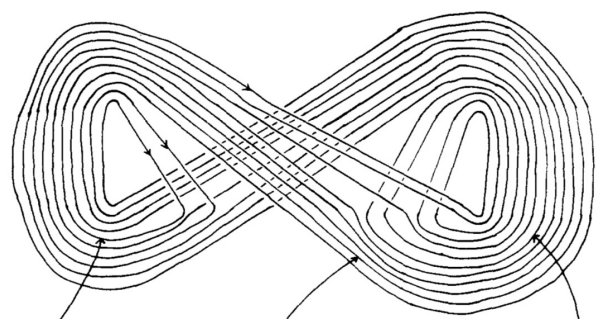
Computing topological invariants using **knots**...



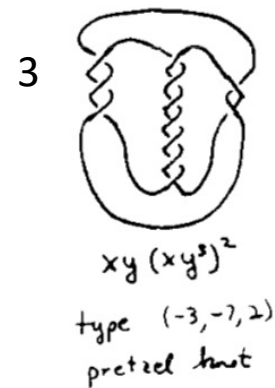
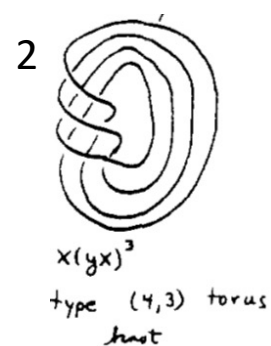
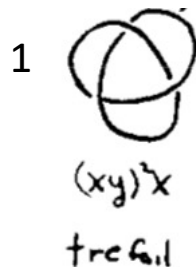
3D trajectory set



Knot invariants

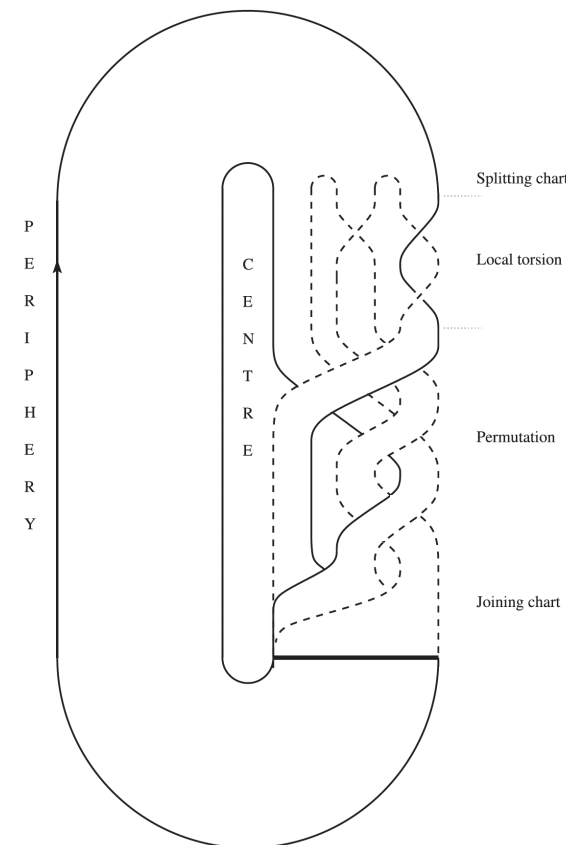


1 2 3



- 1) Close-return method – time series, though, must be **long and noise free**.
- 2) Knot theory – knot \equiv orbit **in three dimensions**.
- 3) Knot invariants – *e.g.*, linking numbers, Conway polynomials

Knot information for a chaotic attractor can be condensed in a **knot-holder** or **template**.



- **Templates: the knot content of attractors**

What's wrong with templates? They are very difficult to construct... they imply the reconstruction of Unstable Periodic Orbits (UPOs) and besides, they are no longer meaningful if a fourth dimension in phase space is involved, since they unknot....

Chapter 7

A braided view of a knotty story

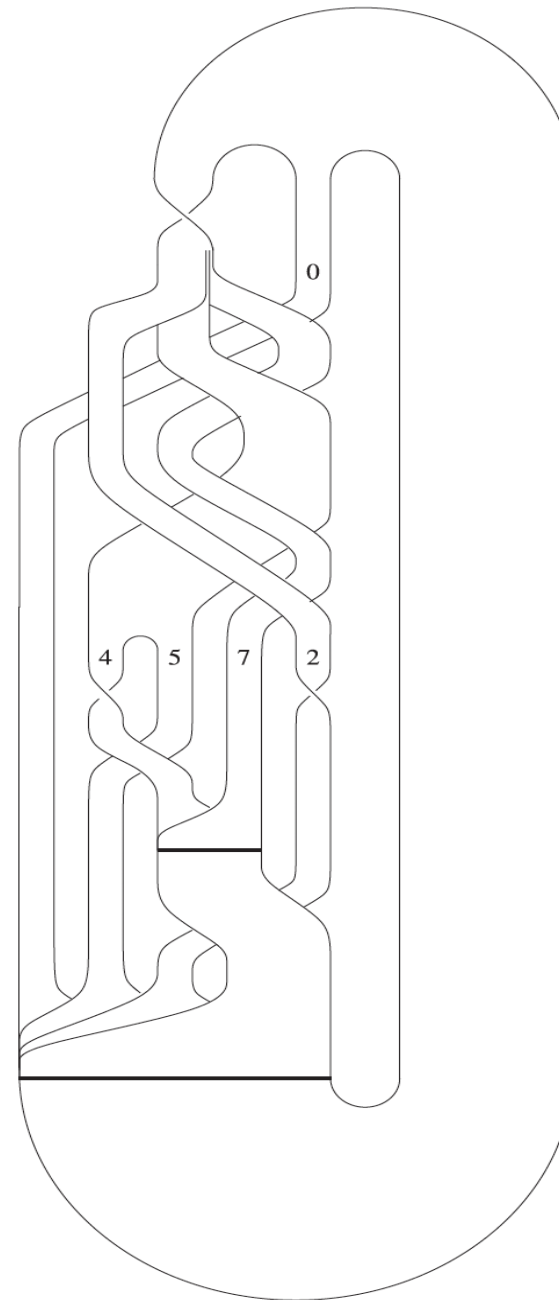
Mario Natiello

*Matematikcentrum-LTH, Lunds Universitet
Box 118, 221 00 Lund, Sverige*

Hernán Solari

*Departamento de Física, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Argentina*

Periodic orbits of 3-d dynamical systems admitting a Poincaré section can be described as braids. This characterisation can be transported to the Poincaré section and Poincaré map, resulting in the braid type. Information from braid types allows to estimate bounds for the topological entropy of the map while revealing detailed orbit information from the original system, such as the orbits that are necessarily present along with the given one(s) and their organisation. We review this characterisation with some examples –from a user-friendly perspective–, focusing on systems whose Poincaré section is homotopic to a disc.



- Algebraic topology: using homology groups

Computing topology
using knot theory



3-D trajectory set



Knot invariants

Computing topology
using homologies



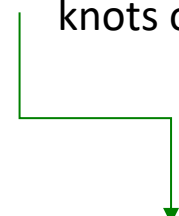
n -D point-cloud



Homologies

RESTRICTIONS

- Precision and length of time series must be good enough for orbits in phase space to be reconstructed accurately ...
- Phase space dimension cannot be higher than three, since knots or braids unknot ...



HOMOMOLOGY GROUPS

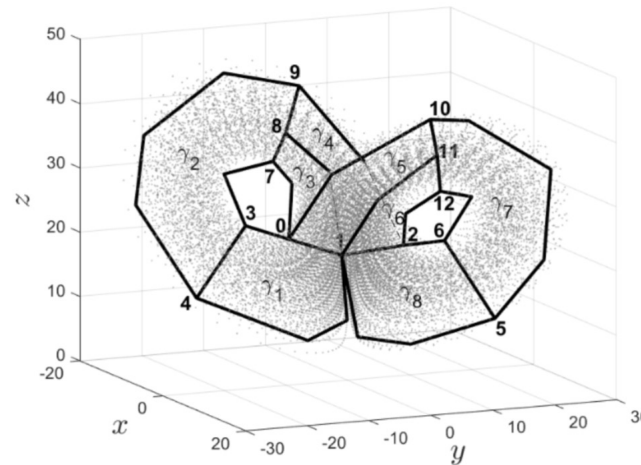
- Time series can be shorter and noisy since the method is independent of the reconstruction of trajectories in phase space (no periodic orbit).
- Applicable in n dimensions: the method is knotless and braidless.

- Algebraic topology: using homology groups

- 1) Approximate points as lying on a branched manifold.
- 2) Find a topological representation for the branched manifold.
- 3) Obtain an algebraic description of the topological structure.

BraMAH
COMPLEX
construction

BRAMAH



Computing topological invariants
using **homologies**



nD point-cloud



Homologies

HOMOLOGY
GROUP
computation

$$H_0 \approx \mathbb{Z}^1;$$

$$H_1 \approx \mathbb{Z}^2;$$

$$H_2 \approx 0$$

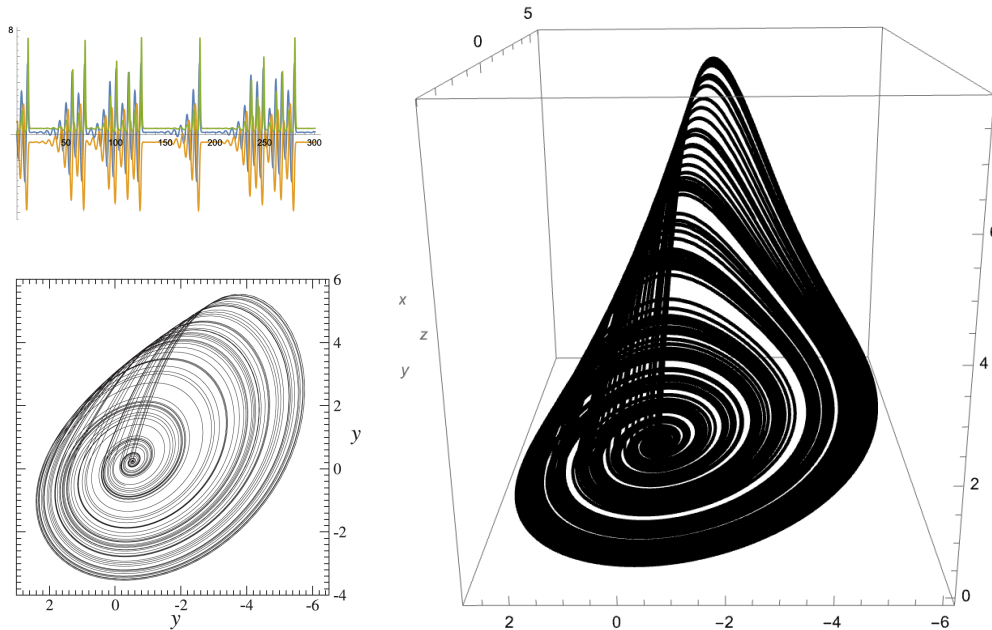
- 1) Local approximation by d -disks using SVD => short and noisy time series *can* be handled.
- 2) Build a cell complex keeping track of the gluing prescriptions => 3D+ *can* be handled.
- 3) Compute homologies and orientability properties of the cell complex => the structure *can* be identified.

- **Rössler attractor: spiral versus funnel**

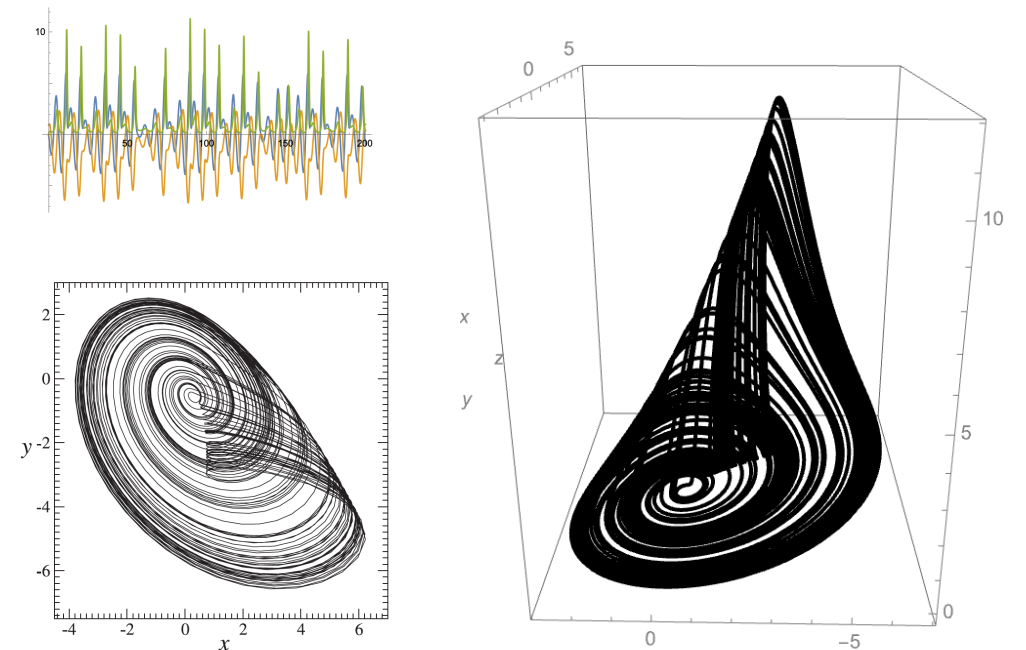
Let us consider two different attractors produced by the Rössler dynamical system for different parameter values (spiral Rössler attractor with $a = 0.343295$ on the left and the funnel Rössler attractor with $a = 0.492$ on the right).

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases}$$

Spiral Rössler attractor



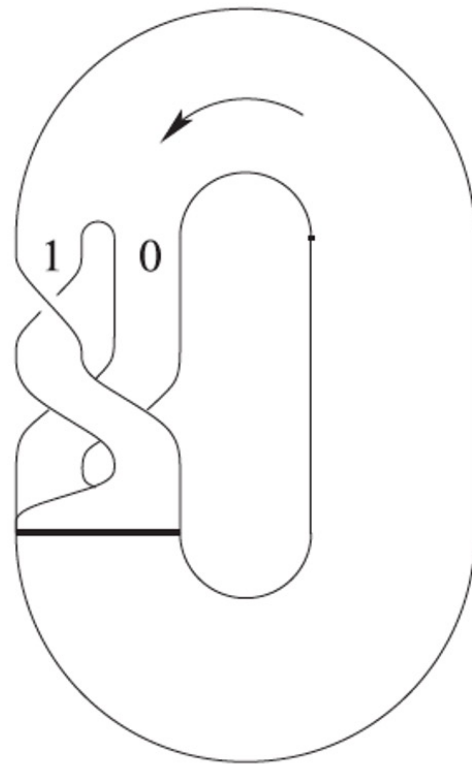
Funnel Rössler attractor



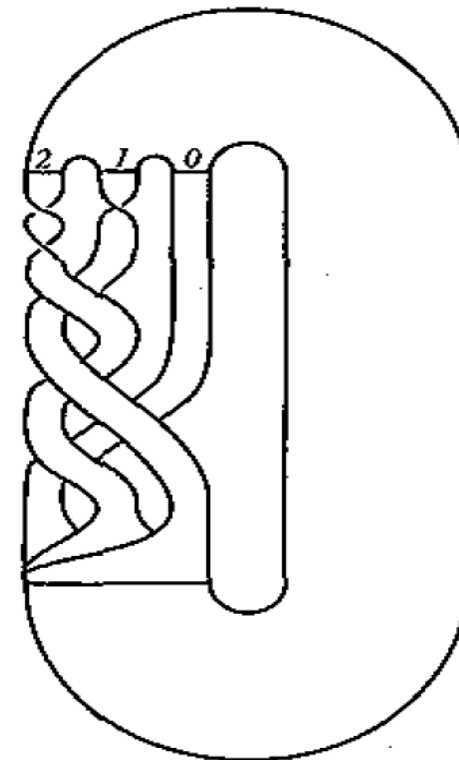
- **Rössler attractor: spiral versus funnel**

The spiral Rössler attractor has two strips (0, 1) while the funnel Rössler attractor has three strips (0,1,2).

Spiral Rössler attractor

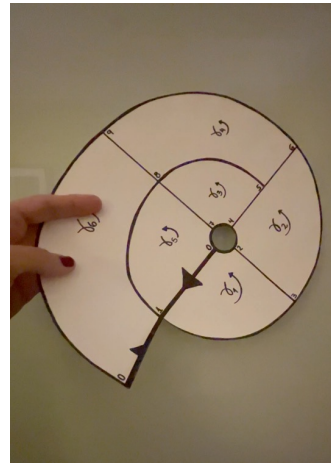
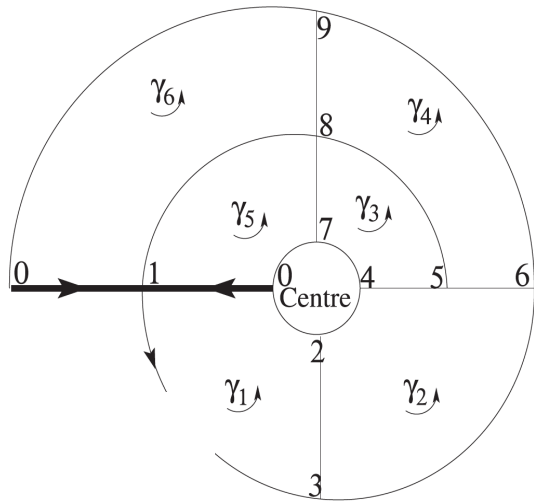


Funnel Rössler attractor

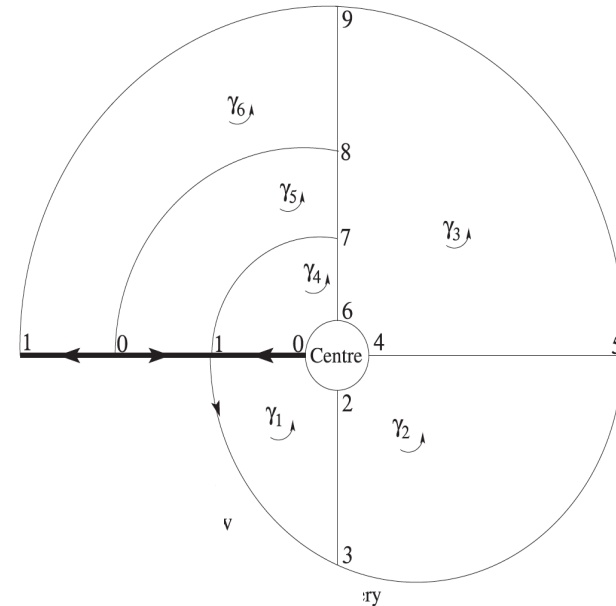


- **Rössler attractor: spiral versus funnel**

The spiral and funnel Rössler attractors are homologically equivalent: **they have both one hole in the centre** ($H_1 = \mathbb{Z}^1$).



$$\mathcal{H}_1(K(R)) = [[\langle 0, 2 \rangle - \langle 0, 7 \rangle + \langle 2, 4 \rangle + \langle 4, 7 \rangle]]$$



$$\mathcal{H}_1(K(R_3)) = [[\langle 0, 2 \rangle - \langle 0, 6 \rangle + \langle 2, 4 \rangle + \langle 4, 6 \rangle]]$$

But there is more information in a cell complex than the one contained in its homology groups... for instance, the **joining lines**! They can be detected as the 1-cells shared by at least three 2-cells (thick lines). Notice that the recipe to **scotch** the cell complexes is different.

- **Templex: template + complex**

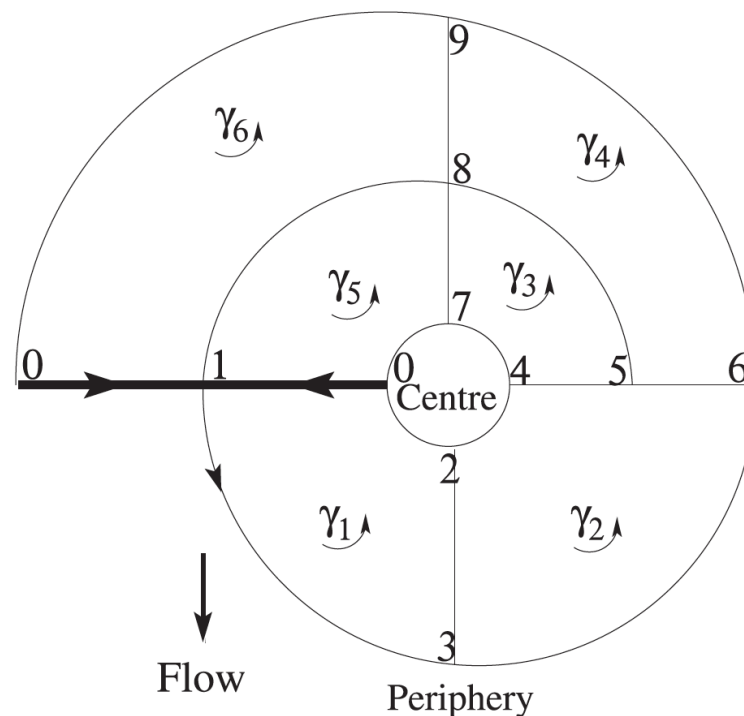
But there is something else that is very important and that is missing in a cell complex representing a branched manifold.

This is the *flow* or, in other words, *time*!

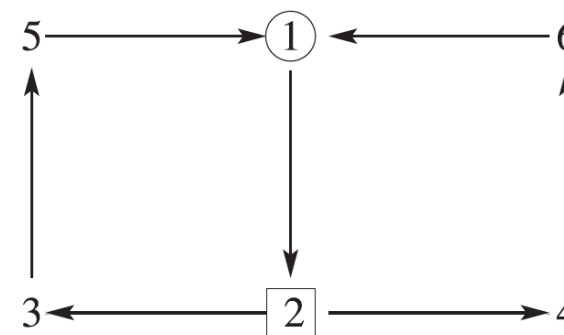
Homologies do not consider the action of the flow on the cell complex...

In order to take the flow on the complex into account, the cell complex will be endowed with a directed graph that prescribes the flow direction between its highest-dimensional cells.

Definition 1. A **templex** $T \equiv (K, G)$ is made of a complex K of dimension $\dim(K) = \kappa$ and a digraph $G = (N, E)$ whose underlying space is a branched κ -manifold associated with a dynamical system, such that (i) the nodes N are the κ -cells of K and (ii) the edges E are the connections between the κ -cells allowed by the flow.



(a)

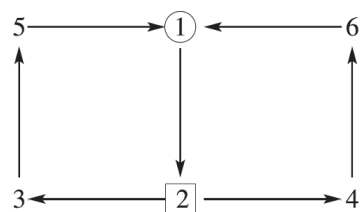
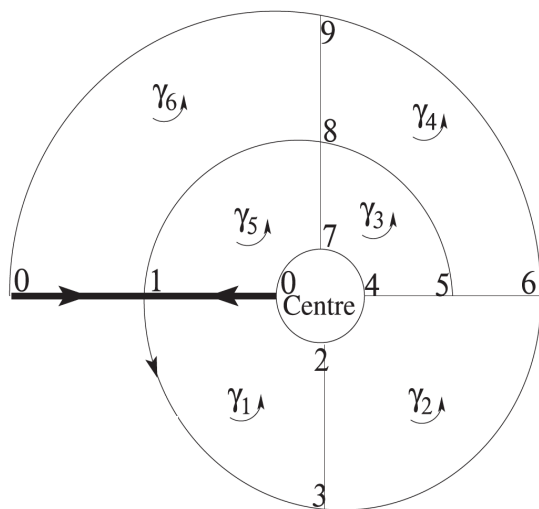
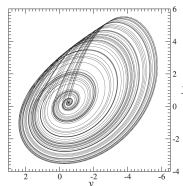


(b)

- **Templex: template + complex**

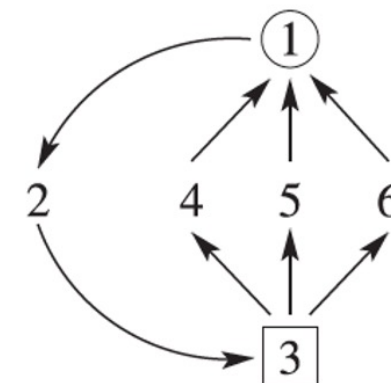
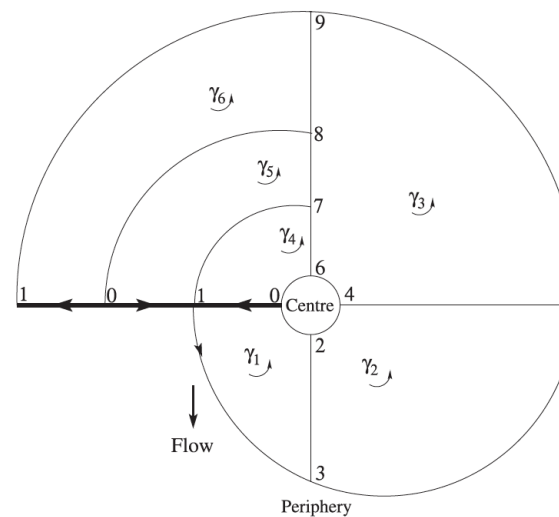
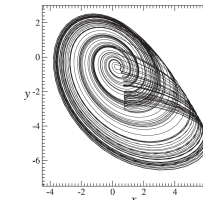
Spiral Rössler attractor

$$T(R)=(K(R),G(R))$$



Funnel Rössler attractor

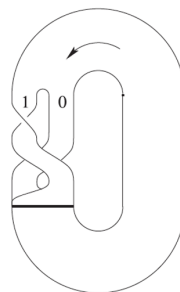
$$T(R_3)=(K(R_3),G(R_3))$$



Stripexes in $T(R)$:

twisted $\underline{1} \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow \underline{1}$,

$\underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow \underline{1}$.

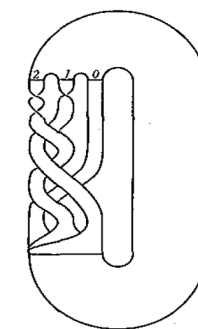


Stripexes in $T(R_3)$:

$\underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \underline{1}$,

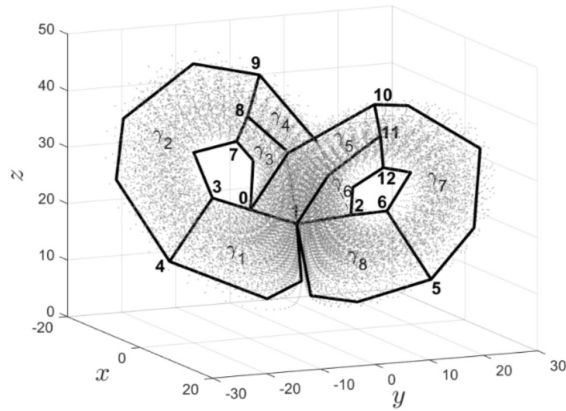
twisted $\underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow \underline{1}$,

$\underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 6 \rightarrow \underline{1}$,



- **Templex examples**

Lorenz 63 attractor



Four stripexes in $T(L)=(K(L),G(L))$:

$$c_1 \equiv \underline{1} \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow \underline{1},$$

$$c_2 \equiv \underline{8} \rightarrow 9 \rightarrow 11 \rightarrow 13 \rightarrow \underline{8},$$

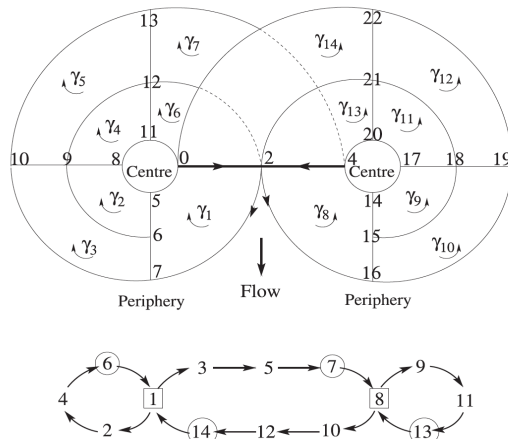
$$c_{3_1} \equiv \underline{1} \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8$$

$$c_{3_2} \equiv \underline{8} \rightarrow 10 \rightarrow 12 \rightarrow 14 \rightarrow \underline{1}.$$

} strong cycles

} weak cycles

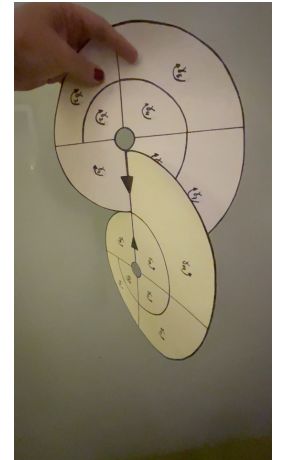
The weak cycles that form the **two twisted stripexes** correspond to a single generatex of order 2.



<https://git.cima.fcen.uba.ar/sciamarella/templex-properties>
Forthcoming version in Python.

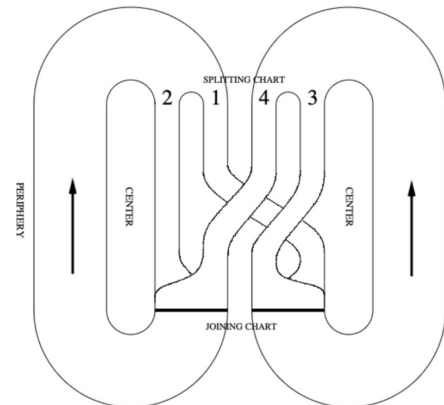
$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = Rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases}$$

$r = 28, s = 10, \text{ and } b = 8/3 :$



Kirigami L63

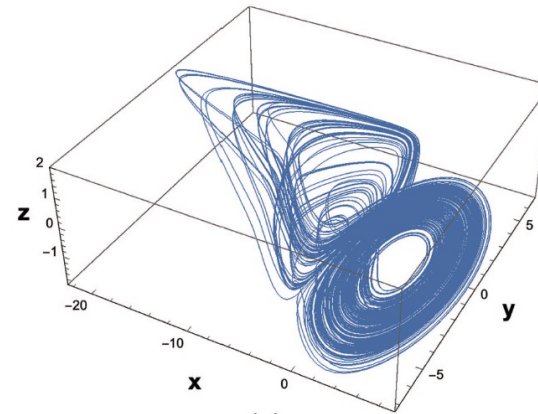
This is consistent with the L63 template that has four strips.



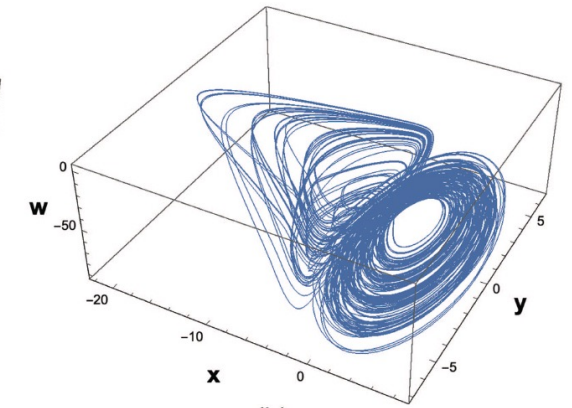
- **Templex examples**

A four-dimensional system designed from a three-dimensional system proposed by Deng.

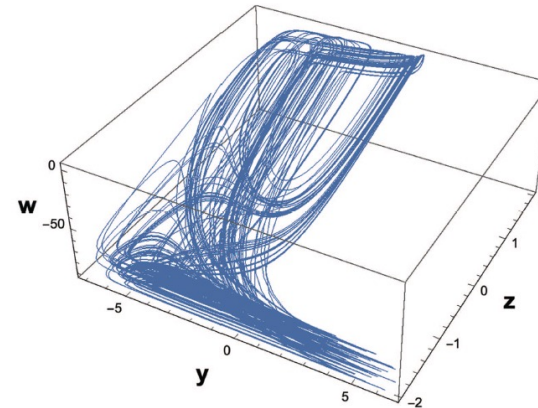
$$\begin{cases} \dot{x} = -(z+2)d(x - [a + \epsilon_3(2+w)]) + (2-z) \\ \quad \left[\alpha(x-2) - \beta y - \alpha(x-2) \frac{(x-2)^2 + y^2}{R^2} \right], \\ \dot{y} = -(z+2)(y-b) + (2-z) \\ \quad \left[\beta(x-2) + \alpha y - \alpha y \frac{(x-2)^2 + y^2}{R^2} \right], \\ \dot{z} = (4-z^2) \frac{z+2 - \mu(x+2)}{\epsilon_1} - cz, \\ \dot{w} = (4-z^2) \frac{z+2 - \mu(x+2)}{\epsilon_2} - cz. \end{cases}$$



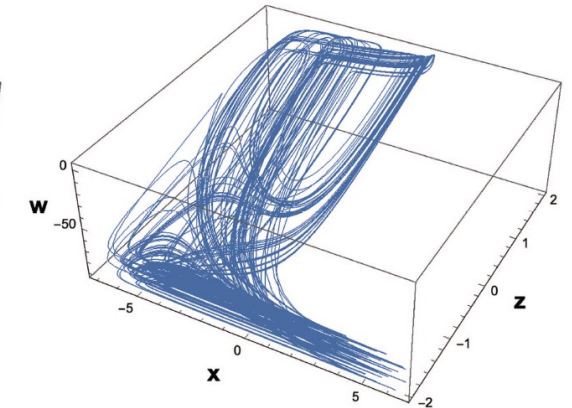
(a)



(b)



(c)



(d)

A solution to this system was already investigated with a BraMAH cell complex (but not with a templex) in Sciamarella & Mindlin, 2001.

- **Templex examples**

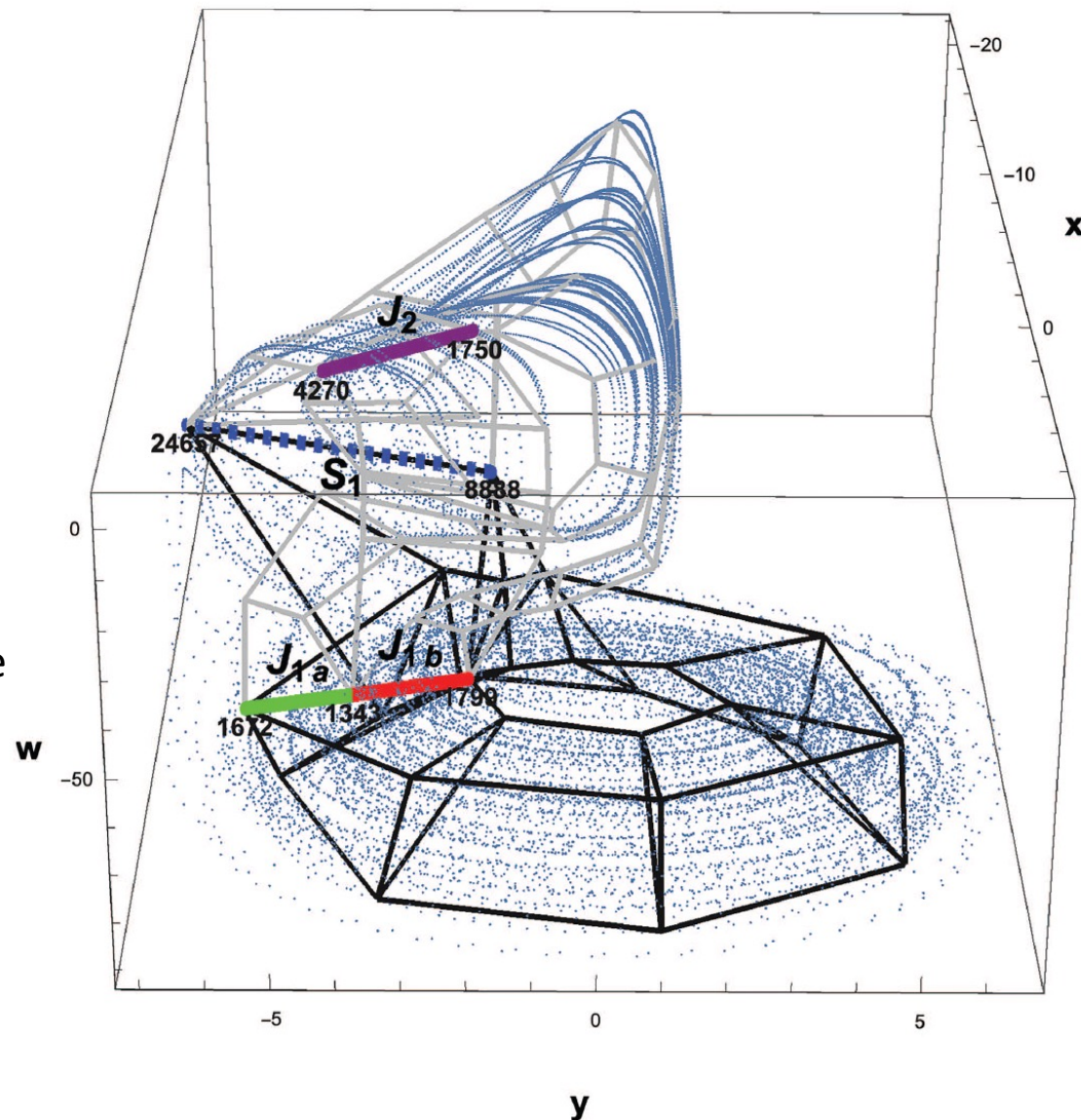
The BRAMAH complex $K(4D)$ is built from a cloud of 24999 points (once the transient regime is completed).

The highest-dimensional cells of this complex are the 3-cells γ_i , and the 2-cells σ_i which are not the border of any 3-cell.

The complex has one 0-hole, five 1-holes and no 2-holes (no enclosed empty cavity).

The joining lines are highlighted in colour. They are the 1-cells shared by a set of at least three k -cells, with $k=2,3$.

The **green** and **red** lines form the main joining locus in the bottom scroll and the second one (**violet**) is in the upper part of the attractor according to our plot.



- **Templex examples**

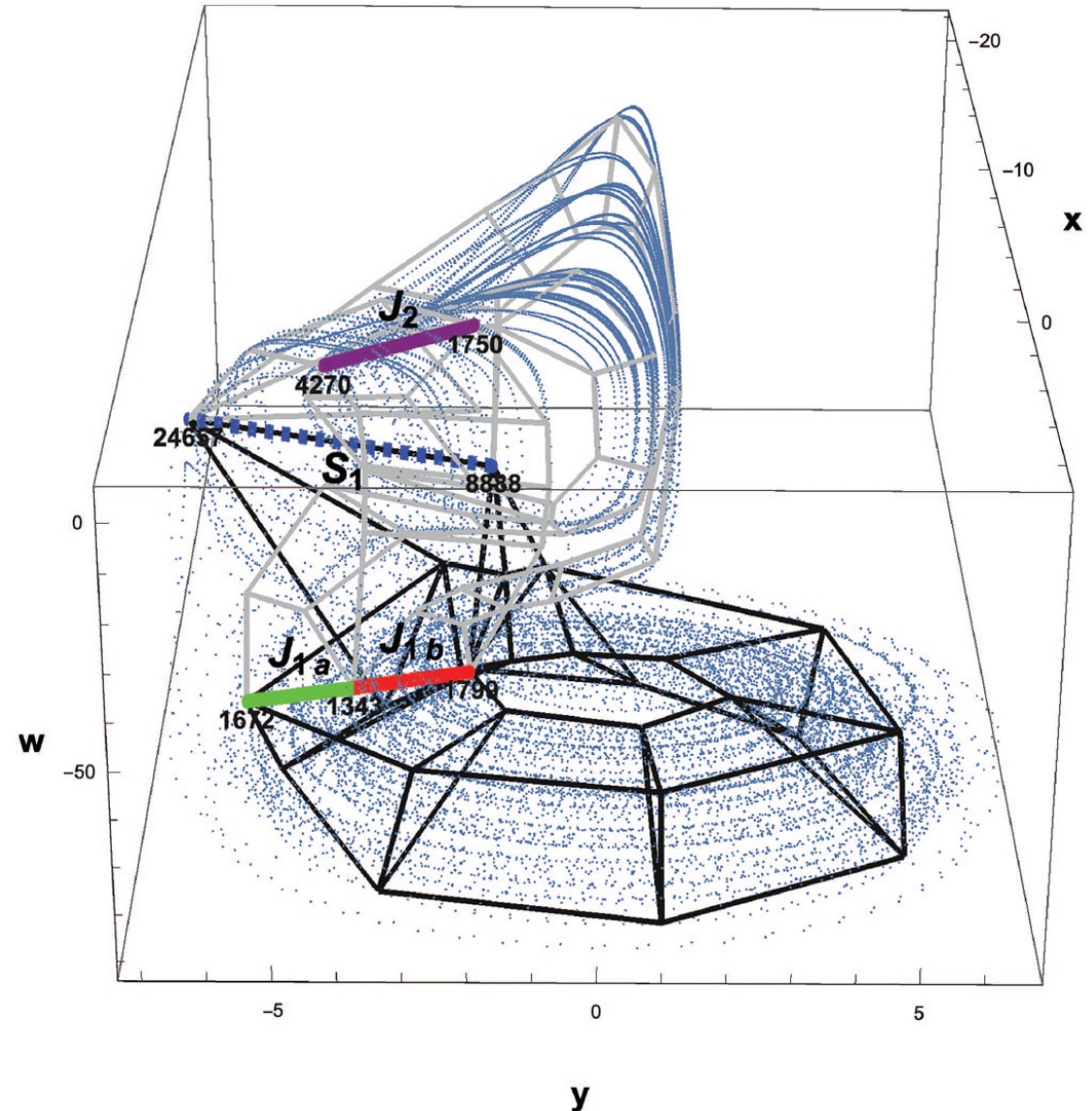
The dotted blue line is a splitting line between one 3-cell and three 2-cells.

The fact that the ingoing cell is of dimension 3 allows us to preserve the determinism at this **splitting locus**.

This is a feature that we had not observed for three-dimensional systems.

The simplest deterministic representation of the flow around this splitting locus is to consider that three strips—close to each other as in a foliation within the 3-cell— are sent away from each other.

This specificity will require a novel element for drawing a template.



- **Templex examples**

Digraph on the BRAMAH complex constituting the templex. Ingoing and outgoing nodes are squared and circled, respectively. Splitting loci are not considered for stripex computation.

Three order-1 cycles:

$$c_1 \equiv \underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow \underline{1},$$

$$c_2 \equiv \underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 37 \rightarrow 40 \rightarrow 36 \\ \rightarrow 35 \rightarrow 25 \rightarrow 26 \rightarrow 27 \rightarrow 28 \rightarrow 29 \rightarrow 30 \rightarrow 31 \\ \rightarrow 32 \rightarrow 33 \rightarrow 34 \rightarrow \underline{1},$$

$$c_3 \equiv \underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 37 \rightarrow 40 \rightarrow 38 \\ \rightarrow 20 \rightarrow 19 \rightarrow 18 \rightarrow 17 \rightarrow 21 \rightarrow 22 \rightarrow 23 \rightarrow 24 \rightarrow \underline{1},$$

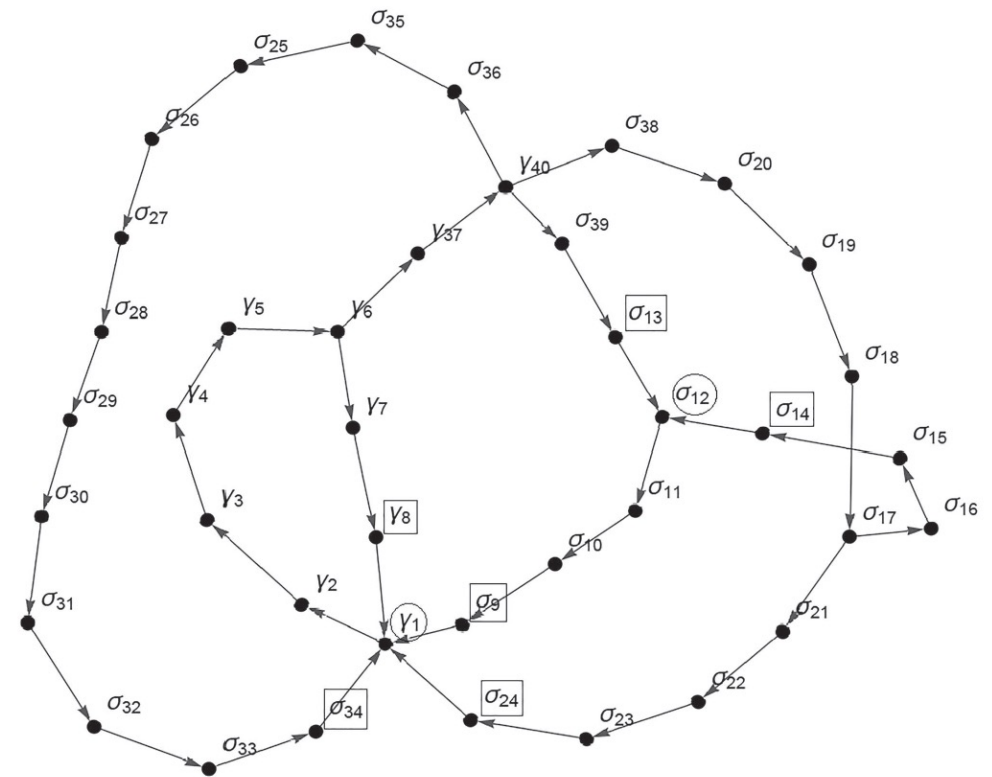
Two order-2 cycles:

$$c_{4_1} \equiv \underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 37 \rightarrow 40 \rightarrow 38 \\ \rightarrow 20 \rightarrow 19 \rightarrow 18 \rightarrow 17 \rightarrow 16 \rightarrow 15 \rightarrow 14 \rightarrow \underline{12},$$

$$c_{4_2} \equiv \underline{12} \rightarrow 11 \rightarrow 10 \rightarrow 9 \rightarrow \underline{1}.$$

$$c_{5_1} \equiv \underline{1} \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 37 \rightarrow 40 \rightarrow 39 \rightarrow 13 \rightarrow \underline{12},$$

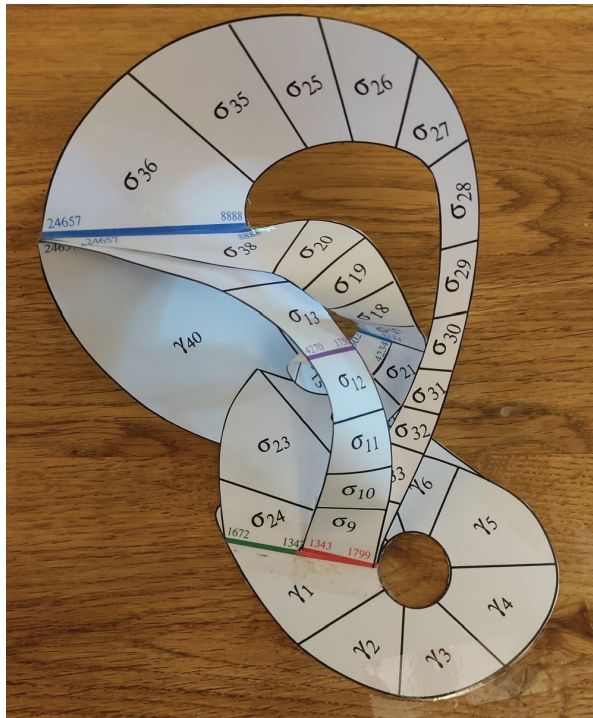
Five stripexes in $T(4D)=(K(4D),G(4D))$:



- **Templex examples**

The four-dimensional attractor can be described with five stripexes, that is, with a five symbol dynamics.

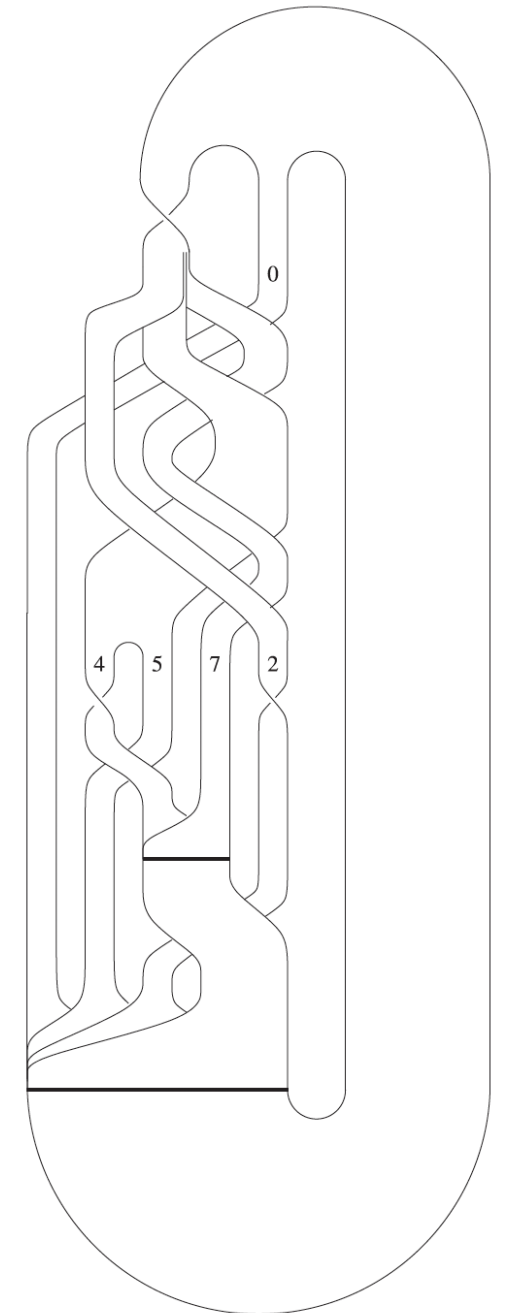
A paper model of the attractor is constructed, similar to the xyz projection of the system.



The volume formed by the ten 3-cells of $K(4D)$ is topologically equivalent to a solid (filled) torus.

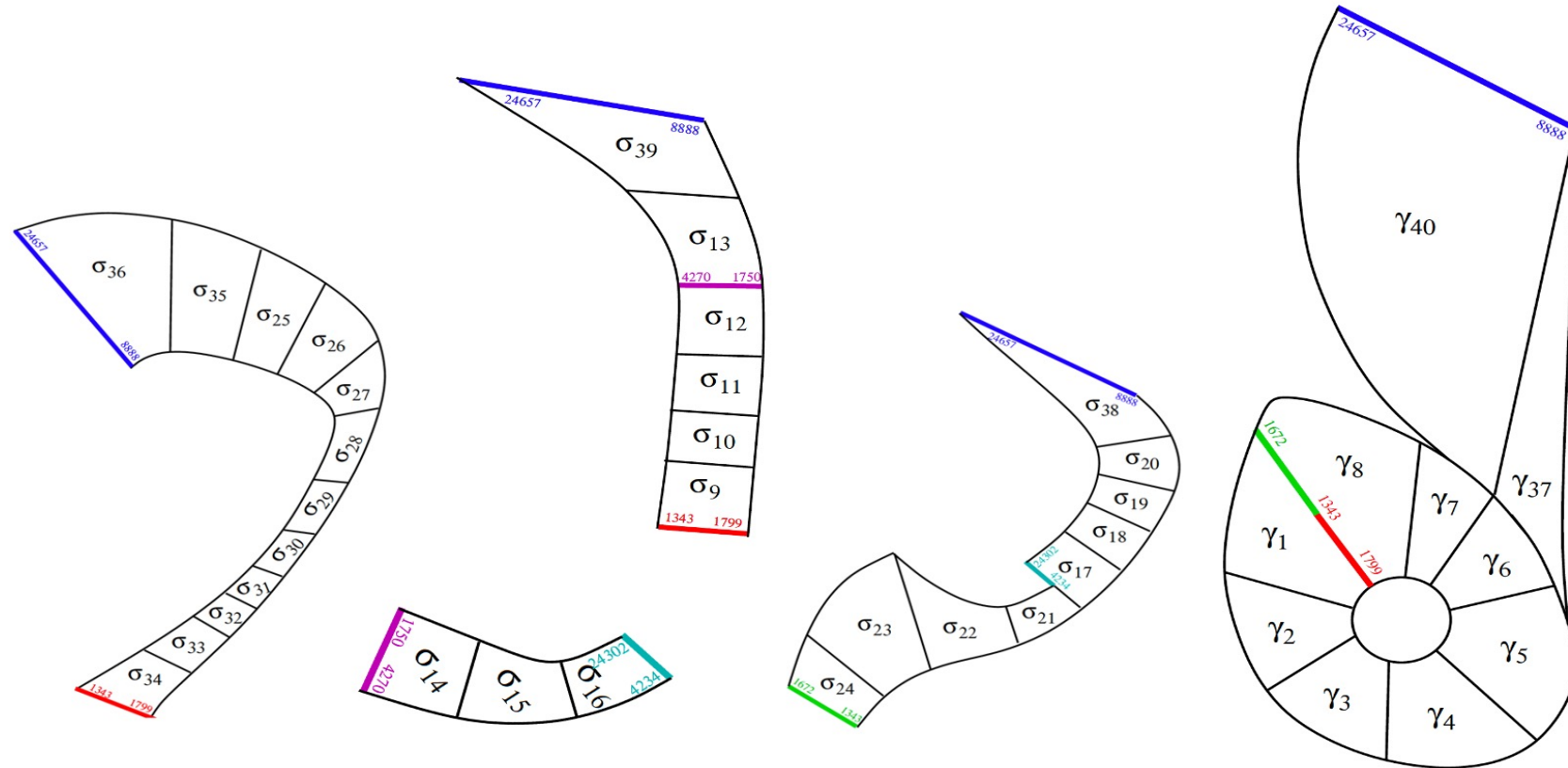
Homologies of a filled torus are equivalent to those of a cylinder, and so, the paper model is homologically equivalent to $K(4D)$.

A direct template can be drawn for the four-dimensional system from the templex analysis.



• Conclusions

The templex dissects the phase-space structure into several identifiable components, connected at certain joints. Closed non-redundant pathways are cell sequences encoded by the digraph.



The templex properties: some describe the spatial structure alone (homology groups, torsion groups, weak boundaries) and a subset for the topology of flow along the structure (generatex, stripex, twists).



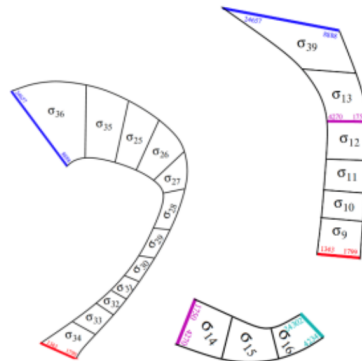
- **Conclusions**

With the initial aim of distinguishing between the spiral and funnel **Rössler** attractors, a new concept, named **templex**, was introduced.

- ✓ Knot-headers distinguish these two flows, but knot theory is not extensible to higher dimensions. Reconstructing orbits is not always easy. Searching for an orbitless / knotless method.
- ✓ Cell complexes can be built from clouds of points in state space, but homologies will not distinguish between these two attractors.
- ✓ The cell complex describes the structure but does not contain the **action of the flow on the cell complex**. In phase space, we are not only interested in the topology of the structure but also in the non-equivalent paths around the structure.
- ✓ The goal is achieved endowing the **cell complex** with a **directed graph** that prescribes the flow direction between its highest-dimensional cells.
- ✓ The tandem of cell complex and directed graph is shown to allow for a sophisticated characterization of topology of flows in phase space.

*"All linear systems are alike; each nonlinear system is nonlinear according to its **templex** properties".*

Thank you for your attention!

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Templex : une nouvelle mathématique pour comprendre le chaos

11 octobre 2022

RÉSULTAT SCIENTIFIQUE OCÉAN ATMOSPHÈRE

A - / A+

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Le "templex" est un concept novateur qui émerge au sein d'une équipe de recherche comprenant des scientifiques du CNRS-INSU (voir encadré) dédiée à l'étude du climat et de ses impacts. Le principe repose sur des méthodes physico-mathématiques avancées pour comprendre, décrire et prédire les effets à haut risque des changements anthropiques. Dans le domaine des sciences du climat, il existe plus de vingt modèles qui sont utilisés pour les simulations et les projections futures. Bien qu'ils aient tous une formulation basée sur les équations qui gouvernent les fluides, ils diffèrent sur plusieurs aspects : la modélisation des aérosols, la dynamique glaciaire ou des océans, entre autres processus. On parle d'une Babel des modèles. Comment savoir quel modèle choisir ? Dans quelle mesure peut-on dire qu'un modèle est une bonne représentation des observations ? Le 'templex' est l'outil mathématique avec le potentiel pour apporter une réponse à ces questions.

<https://www.insu.cnrs.fr/fr/cnrsinfo/templex-une-nouvelle-mathematique-pour-comprendre-le-chaos>