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Denisse Sciamarella, Gisela D Charó, Christophe Letellier. From the Rössler attractor to the templex. OTTOCHAOS: FROM THE NONLINEAR DYNAMICAL SYSTEMS THEORY TO OBSERVATIONAL CHAOS, Oct 2023, Toulouse, France. hal-04276337

HAL Id: hal-04276337

<https://hal.science/hal-04276337v1>

Submitted on 8 Nov 2023

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From the Rössler attractor to the tempex

Gisela D. Charó,^{1,2} Christophe Letellier³ & Denisse Sciamarella^{2,4}

¹ CONICET – Universidad de Buenos Aires. Centro de Investigaciones del Mar y la Atmósfera (CIMA), C1428EGA CABA, Argentina.

² CNRS – IRD – CONICET – UBA. Institut Franco-Argentin d’Études sur le Climat et ses Impacts (IRL 3351 IFAECI), C1428EGA CABA, Argentina.

³ Rouen Normandie University — CORIA, Campus Universitaire du Madrillet, F-76800 Saint-Etienne du Rouvray, France.

⁴ CNRS – Centre National de la Recherche Scientifique, 75016 Paris, France.

denisse.sciamarella@cnr.fr

Session: Nonlinear dynamical systems theory

Methods to accomplish a topological description of the structure of a flow in high-dimensional state space (in more than three dimensions) have a long history, full of partially fruitful attempts, but the ideal mathematical object to achieve this task seems to be what we now call tempex. [1] Cell complexes can be traced back to Poincaré’s papers during 1900 [5] and the study of chaotic attractors using cell complexes to the 1990s. [6,8] Since then, algebraic topology has been regarded as the most promising mathematical formalism to describe chaos beyond three dimensions, overcoming the restrictions that templates, based on the knot content of attractors, cannot represent. In this talk, we present the road leading to the tempex starting from the Rössler attractor [14] and ending with a four-dimensional system designed in [8] on the basis of a set of equations proposed by Deng. [11]

1 Introduction

Henri Poincaré (1854-1912) was the first to understand how a dynamical system depends on its topological structure in phase space. In contrast to geometric properties, topological properties in phase space provide information about the stretching, folding, tearing and squeezing mechanisms shaping a flow. In 1983, the mathematician R. F. Williams introduced the concept of branched manifold and used it to classify chaotic attractors in terms of the way in which their branches are knotted [13,12]. Some of these features can be captured without dimensionality restrictions using cell complexes and computing their homology groups [6]. Cell complexes can be constructed from a cloud of points in an arbitrary number of dimensions: the cloud is segmented into subsets of points forming a set of glued patches.[18]

Before attacking high-dimensional attractors, it is necessary to develop an approach based on complexes and homology groups which provides a description of chaotic attractors at the accuracy offered by templates. This involves two steps: orienting the cells of the complex according to the flow direction and associating to the complex a directed graph whose nodes are the cells of the highest local dimension. To our knowledge, there is no work prior to Charó et al, 2022, using a cell complex linked to a digraph, carrying the information of the flow direction in terms of allowed cell connections. The properties of this new mathematical object called ‘tempex’ will allow us to extract all the characteristics that are relevant to the description of the topological structure of a flow in phase space. In order to derive this set of characteristics from calculations performed by combining the digraph and the complex, we will review the relevant definitions. In particular, certain parts of the original tempex will play the role of strips in templates.

A brief introduction to templates and homology groups is provided in Section 2. In Section 3 we introduce the specific concepts of the tempex and two attractors produced by strongly dissipative systems, namely the spiral Rössler attractor, and a four-dimensional system, are extensively treated. Section 4 gives a conclusion.

2 Topological background

2.1 Templates as knot-holders for chaotic attractors

The study of the topological structure of flows in general began naturally with the analysis of paradigmatic chaotic attractors, such as the one in Edward Lorenz's 1963 paper. Otto Rössler, in turn, sought to reproduce the Lorenz attractor in a chemical reaction, discovering that chaotic behavior could be achieved with a system even simpler than the Lorenz [7] attractor. This is a single-spiral tractor named after him and used as a paradigmatic example of a simple topological structure. How to characterize this structure, that of some more complex variants, and that of a flow in phase space in general?

Chaotic attractors can be characterized through the invariant structure underlying a set of trajectories that solve the system of differential equations and parameter values that define them. The invariant set under the action of the flow ϕ_t can be bounded by genus- g tori whose holes are most often associated with singular points circled by the flow.[15,16] We must be careful in the use of the term "hole", since holes in algebraic topology are associated with the generators of the homology groups of a cell complex. For now we are using the term informally, but we will see later that these holes in tori do not necessarily constitute holes in the sense of homology theory. These holes can be of the focus type or be associated with a flow tear, dividing the attractor into 'strips'. The term is used in a rather loose way here, but it will be accurately defined as a templex element. The bounding torus helps identify the periphery of the attractor, which can be naturally oriented according to the flow. The next step required to construct a template is to compute a first-return map to a Poincaré section.

Because the first-return map is one-dimensional for strongly dissipative systems, the number of monotone branches provides the number N_s of strips.[20] The map is thus partitioned into N_s branches discriminating the different paths followed by the flow and differentiating the different strips. In template theory, a strip is typically defined between a splitting chart and a joining chart [Fig. 1(b)] in order to provide a spick and span representation of the different paths. The total number of non-trivial dynamical processes are supposed to be captured between the splitting chart and the joining chart, ended by a joining line which corresponds to a Poincaré section [thick line in Fig. 1(b)].

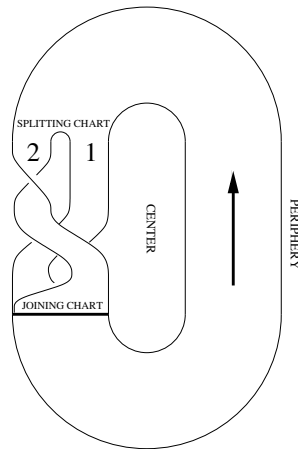


Fig. 1: Twists and crossings of the two strips in the template obtained for the Rössler spiral attractor. Source: Charó & Sciamarella, 2023.

What can happen to strips in a template? They can be permuted and they can also present local torsion. The connection between the joining and splitting charts 'close' the template with a trivial strip, as shown in Fig. 1. As we shall see, the joining chart will be related to the joining locus in a templex [9]. The joint identifies the hinge where the flow cannot be reversed without violating uniqueness. Charts and strips are the elements of a templex that enable describing a specific dynamics.

The next section discusses how the definitions of cell complexes and homology groups used in algebraic topology provide, not a definitive procedure, but a first step in generalizing the templates to higher dimensions.

2.2 Homology groups

Topological data analysis by means of homologies can be applied to multidimensional spaces of all types, without specific reference to the existence of a flow in that space. One of the most popular procedures consists in the construction of a complex from a set of points with n coordinates. In the field of dynamical systems theory, this set of points can come, for example, from the integration of a system of ordinary differential equations, or from an embedding of a time series dataset. There are a variety of rules for forming different types of complexes, depending on how the points are grouped in cells. Most of these rules are such that cells are added every time the distance between points exceeds a certain threshold.

A specific methodology was proposed by Sciamarella & Mindlin in 1999 for the construction of a cell complex from a set of points corresponding a manifold with branches in phase space. This type of construction is now called a BRAMAH complex. The method assumes that subsets of points in such a point cloud lie locally on a branched manifold, i.e. on a manifold that is not required to be Hausdorff and that may have boundaries [10]. The attractors we are going to consider are strongly dissipative dynamical systems whose data points are recorded from the invariant set and lie on a branched 2-manifold. With this assumption, it is natural to group subsets of points into 2-cells (polygons) which approximate discs or half-discs. The dimension of a BRAMAH complex coincides, by construction, with the local dimension of the manifold on which an attractor lies. An algorithm to construct a BRAMAH complex from data is described in Refs. [6,8,2,3,4].

Let us now consider a few classical definitions of the theory of homology groups, which are used to compute the properties of a cell complex, independently of the rules used to construct it. A k -cell is defined as a set whose interior is homeomorphic to a k -dimensional disc with the additional property that its boundary must be divided into a finite number of $(k - 1)$ -cells, called the faces of the k -cell. A 0-dimensional cell is thus a point, a 1-cell is a line, a 2-cell is a polygon, a 3-cell is a solid polyhedron with polygons, edges, and vertices as faces. Thus, the endpoints of a 1-cell are 0-cells, the boundary of a 2-cell consists of 1-cells, and so forth. A finite number of cells glued together is said to form a cell complex K provided two conditions are satisfied. The first condition establishes that if σ and τ are k -cells in K , then all $(k - 1)$ -cells of σ and τ are elements of K . The second condition requires that $\text{Int}(\sigma) \cap \text{Int}(\tau) = \emptyset$, where $\text{Int}(A)$ denotes the interior of A . The dimension κ of a cell complex is defined by the dimension of its highest-dimensional cells.

Let us bear in mind that a cell complex is not a simple grid approximating a surface, for example, on which lies a cloud of points with a certain shape. Paths can be defined on a cell complex, leading to an algebra of chains. This requires orienting the complex, which means defining the order in which the vertices of an edge are read and whether the vertices of a polygon are listed in a counterclockwise or clockwise direction. Let us recall that a cell complex is defined for a generic point cloud which is not necessarily associated to a flow in phase space: the choices of orientations or directions for 1-cells and 2-cells are *a priori* arbitrary. For oriented complexes K of dimension κ , a k -chain C_k is defined as the linear combination of k -cells. To illustrate how the algebra of chains work, consider two k -cells which are adjacent, i.e. they share a $(k - 1)$ -cell. The boundary operator $\partial_k : \mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$ is defined so that the boundary of an directed 2-cell is the chain formed by the 1-cells on its border, with a positive sign if the orientation of an edge is consistent with the direction of the 2-cell, and with a negative sign otherwise. The image of the boundary operator is formed by the elements of the group $\mathcal{B}_k(K) = \text{im}(\partial_{k+1})$. A special role will be assigned to the elements of $\mathcal{Z}_k(K) = \ker(\partial_k)$, called k -cycles – here $\ker(\partial_k)$ denotes the kernel of the boundary operator. Note that a k -cycle C_k is such that $\partial_k(C_k) = 0$, and therefore, a k -cycle has no boundary. The most ‘important’ k -cycles are those that are not the boundaries of some $(k + 1)$ -cells, assembled in \mathcal{Z}_k .

The quotient group $\mathcal{H}_k := \ker(\partial_k)/\text{im}(\partial_{k+1}) = \mathcal{Z}_k/\mathcal{B}_k$ leads to the k th homology group of K . A homology group can be expressed in terms of its generators, that is, as $\mathcal{H}_k = [g_1, \dots, g_q]$, with $q \in \mathbb{N}$. The cardinal q of \mathcal{H}_k corresponds to the k th Betti number β_k . Notice that the k -generators g_i of a homology group \mathcal{H}_k are homologically independent, that is, they cannot be deformed into each other by a continuous transformation (an isotopy). The interest in computing homology groups is that unnecessary information cancels out, providing the layered invariants that describe the underlying branched manifold. The layered information is given by $\mathcal{H}_k(K)$, so that the rank of $\mathcal{H}_0(K)$ yields the number of connected components of the cell complex and the group $\mathcal{H}_1(K)$ identifies non-trivial

cycles or holes around the complex. Last but not least, if $\mathcal{H}_2(K) \neq \emptyset$, the generators encircle the enclosed cavities in K .

Notice that the k -holes encircled by the k -generators of the homology groups do not depend on the particular complex that is built, and allow to easily distinguish some topologically non equivalent manifolds. Identifying and locating the k -generators (which one calculates and obtains written in terms of the labeled cells of the complex) is equivalent to locating the holes of dimension k in the complex. The relative entanglement of the generators provides information about how the underlying branched manifold is structured. This is why homology theory is of interest here not only for its ability to detect the number of k -holes, but also their relative arrangement.

The generators of the homology groups and the torsion groups of a complex, which we do not define here for the sake of brevity, allows us to obtain the properties describing the topology of the structure itself, but not of the non-equivalent ways of flowing around this structure. The theory of homologies was not conceived to specifically study trajectories in phase space, and does not contain any element that captures the existence of a flow over the branched manifold. In order to incorporate this information to our study, a new mathematical object is required. This leads us to the concept of *templex* as introduced in the next section.

3 Templex

As seen in the previous section, the algebra of chains enables defining paths in a cell complex. These paths will enable detecting cycles in the complex, and therefore holes. But in principle, any path is possible. This is of course true for a set of points in a space without time. But phase space is a particular kind of space, with time playing the role of an implicit variable. Of all the paths around a complex, only a few will correspond to flow-compatible path. However, the possibility of endowing the complex with this information is possible if we define a directed graph whose nodes represent the highest dimensional cells of the complex and whose arrows represent the transitions between cells that are compatible with the flow of the dynamical system represented by the point cloud. This double-function object is called *templex*, a compound word that suggests that we are now working with a temporal complex, or with a complex that will lead us to a template, computing k -holes but also strips, which will be called *stripexes*.

Such a digraph can be computed directly from the attractor dataset using the sequential information given by a time series. Let K_R be the complex of the Rössler attractor. In K_R the flow path from the 2-cell γ_1 connects it with γ_2 . Instead, γ_2 may lead either to γ_3 or to γ_4 ; γ_3 is connected with γ_1 and with γ_4 . Consider Fig. 2 for further details.

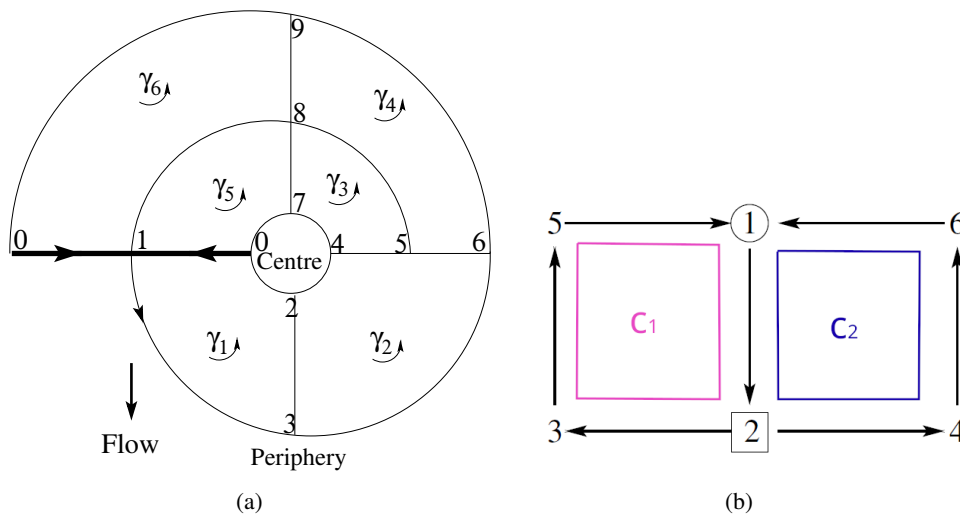


Fig. 2: Templex $T(R)$ for the Rössler attractor constituted by complex $K(R)$ shown as a planar diagram in (a) and a digraph $G(R)$ with two cycles c_1 and c_2 shown in (b). The thick line indicates the joining 1-cell $\langle 0, 1 \rangle$. Source: [1].

The advantage of a templex (over a complex) is clear, since it provides more information than the one related to its central 1-hole, associated to the focus-type fixed point of the Rössler attractor. The templex unveils the existence of two different closed paths (cycles) along the cell complex encoded by the digraph. These two non-equivalent cycles around the cell complex are related to the two strips in the template shown in Fig. 1. The advantage of the templex (over the template) is methodological (the templex is based on a cell complex that can be constructed for all dimensions) but also theoretical (the templex contains all the information needed to describe a flow in such a multidimensional space). Let us define it precisely:

Definition 1 A **templex** $T \equiv (K, G)$ is made of a complex K of dimension $\dim(K) = d$ and a digraph $G = (N, E)$ whose underlying space is a branched d -manifold associated with a dynamical system, such that (i) the nodes N are the d -cells of K and (ii) the edges E are the connections between the d -cells allowed by the flow.

The particularity of a branched 2-manifold as underlying a chaotic attractor is to present a joining chart. In terms of a templex, the joining locus can be identified seeking the locations at which more than two 2-cells share a 1-cell. Let G_s be the sub-graph of G of the complex K , whose nodes are the 2-cells that share a 1-cell.

Definition 2 Let $T = (K, G)$ be a templex, a **joining node** in the subgraph G_s is defined as the node whose number of edges leading into that node (in-degree) is greater or equal than 2. If K has dimension 2, the joining nodes are called joining 2-cells, and the 1-cells shared by at least three joining 2-cells are called joining 1-cells. The 1-chain of these 1-cells is called the joining locus.

Figure 2 shows K_R and G_R . In order to orient the complex according to the flow a 2-cell with one 1-cell at the periphery of the complex is arbitrarily chosen: this 1-cell is oriented according to the flow. For K_R , γ_1 is chosen: the peripheral 1-cell is thus oriented along the flow and, consequently, the 2-cell is anticlockwise. The other 2-cells are oriented by a simple propagation of this orientation up to the joining locus.

Computing the properties of the templex from complexes and the digraphs requires some extra work if the cells of the complex participating in the joining loci are not minimal. We will not describe this procedure here since K_R already satisfies this condition. For details on this procedure, the reader is referred to [?], where the templex is introduced for the first time.

In the joining locus we will identify the *ingoing* and *outgoing* 2-cells, i.e. the 2-cells which are visited by the flow just before or just after crossing one of the joining 1-cells respectively. The only outgoing 2-cell in $T_R \equiv (K_R, G_R)$ is γ_1 , and its ingoing 2-cells are γ_4 and γ_3 . The joining-locus is said to have one component: $J_1 = -\langle 0, 1 \rangle$ (thick arrow). In a template, the strips indicate the number of non equivalent paths around the branched manifold. In a templex, this information can be extracted looking for cycles in the digraph that start and end in the outgoing 2-cells of the joining-locus. As the nodes of the digraph are 2-cells of the cell complex, such cycles will immediately denote tracks of 2-cells in the complex K .

Definition 3 A **generatex** $\mathcal{G} = (K^g, G^g)$ is a subtemplex in $T = (K, G)$, where G^g is a cycle of the digraph G , and $K^g \subset K$. A generatex is said to be of order p with $p \in \mathbb{N}$, $p \geq 1$, if the cycle has p distinct ingoing nodes. A generatex with $p = 1$ will be said to be simple, and degenerated otherwise.

The number of generatexes will depend on the number of cells in the complex, but the tracks of cells that start and finish in the same outgoing cell (cycles) will be considered equivalent or redundant. The union of all equivalent generatexes (associated with equivalent cycles) plays a fundamental role in the characterization of branched manifolds. We will thus retain a single representative for each group of equivalent generatexes and we will refer to them as representative generatexes. In the examples we are using here, the structure of the complexes is simplified and there are therefore no redundant cycles.

The order p of a cycle in the digraph G^g is given by the number of ingoing nodes which correspond to the ingoing 2-cell of the subcomplex K^g . A generatex of order p can be considered the union of p weak cycles.

Definition 4 Let \mathcal{G}_i be the i -th representative generatex of templex $T = (K, D)$. If \mathcal{G}_i is of order one, it is said to be a strong **stripex** \mathcal{S}_i . If \mathcal{G}_i is degenerated, with order $p > 1$, $p \in \mathbb{N}$, it is said to be composed by p weak stripexes $\mathcal{S}_{i,j}$ with $1 \leq j \leq p$.

The free edges of a stripex \mathcal{S}_i are composed by the 1-chains of the associated sub-complex K_i that result from applying the boundary operator to the sum of all the cells in K_i with the exception of those composing the joining locus. The stripex will be said to have a local twist if the free edges of K_i change their relative positions with respect to the orientation from the center to the periphery. In the analogy with templates, uneven local torsions in a strip correspond to a local twist in a stripex. The cycles c_1 and c_2 (Fig.2 (b)) lead to two stripexes:

$$\begin{aligned}\mathcal{S}_1 &\equiv \gamma_1 - \gamma_2 - \gamma_3 - \gamma_5 \\ \mathcal{S}_2 &\equiv \gamma_1 - \gamma_2 - \gamma_4 - \gamma_6\end{aligned}$$

The stripex \mathcal{S}_2 has a local twist.

3.1 A four-dimensional system

Let us restart from a four-dimensional system designed by Mindlin and Sciamarella from a three-dimensional system proposed by Deng.[11] The extended system reads [8]

$$\begin{cases} \dot{x} = -(z+2)d(x - [a + \epsilon_3(2+w)]) + (2-z) \\ \quad \left[\alpha(x-2) - \beta y - \alpha(x-2) \frac{(x-2)^2 + y^2}{R^2} \right] \\ \dot{y} = -(z+2)(y-b) + (2-z) \\ \quad \left[\beta(x-2) + \alpha y - \alpha y \frac{(x-2)^2 + y^2}{R^2} \right] \\ \dot{z} = (4-z^2) \frac{z+2 - \mu(x+2)}{\epsilon_1} - cz \\ \dot{w} = (4-z^2) \frac{z+2 - \mu(x+2)}{\epsilon_2} - cz \end{cases} \quad (1)$$

This system produces a chaotic attractor for the parameter value $b = 1.45$. We obtain a cell complex as shown in Fig. 3 (left). The highest dimensional cells of this complex are the 3-cells γ_i , and the 2-cells σ_i which are not the border of any 3-cell. The homologies of the $K(4D)$ are: $\mathcal{H}_0 \approx \mathbb{Z}_1$ (there is one connected component), $\mathcal{H}_1 \approx \mathbb{Z}_5$ (there are five non-trivial 1-loops), $\mathcal{H}_2 \approx \emptyset$ (there are no enclosed empty cavities) and $\mathcal{H}_3 \approx \emptyset$. Let $T((4D) = (K(4D), G(4D)))$ be the templex for this system. In this particular case the nodes of $G(4D)$ are 3-cells and 2-cells.

As shown in Fig. 3, the first two loci form the main joining locus $J_1 = J_{1a} \cup J_{1b}$ (Fig. 3) and J_2 is a second joining locus in the upper part of the attractor according to our plot. The third locus is a splitting line between the 3-cell γ_{40} and the three 2-cells σ_{36} , σ_{38} and σ_{39} .

All the 1-cells of the joining lines J_{1a} and J_{1b} are oriented in the same directions and, consequently, The 2-cells γ_1 and σ_{12} are the outgoing joining 2-cells, and $\gamma_8, \sigma_9, \sigma_{13}, \sigma_{14}, \sigma_{24}$, and σ_{34} are the six ingoing joining 2-cells. The splitting locus has one ingoing 3-cell and three outgoing 2-cells: the fact that the ingoing cell is of dimension 3 allows to preserve the determinism at this splitting locus. This is a feature that we had not observed for three-dimensional systems. The simplest deterministic representation of flow around this splitting locus is to consider that three strips — close to each other as in a foliation within the 3-cell γ_{40} — are sent away from each other.

From the digraph, we found three order-1 cycles c_1, c_2 and c_3 , and two order-2 cycles c_4 and c_5 which are the union of two weak cycles $c_4 = c_{4_1} \cup c_{4_2}$ and $c_5 = c_{5_1} \cup c_{4_2}$ (Figure 4). To extract the stripexes from the cycles, we only consider J_1 and J_2 (splitting loci are not taken into account for stripex computation). The first three cycles lead to three stripexes:

$$\begin{aligned}\mathcal{S}_1 &\equiv \gamma_1 \dots \gamma_8 \\ \mathcal{S}_2 &\equiv \gamma_1 \dots \gamma_6 - \gamma_{37} - \gamma_{40} - \sigma_{36} - \sigma_{35} - \sigma_{25} \dots \sigma_{34} \\ \mathcal{S}_3 &\equiv \gamma_1 \dots \gamma_6 - \gamma_{37} - \gamma_{40} - \sigma_{38} - \sigma_{20} - \sigma_{19} - \sigma_{18} - \\ &\quad \sigma_{17} - \sigma_{21} \dots \sigma_{24}\end{aligned}$$

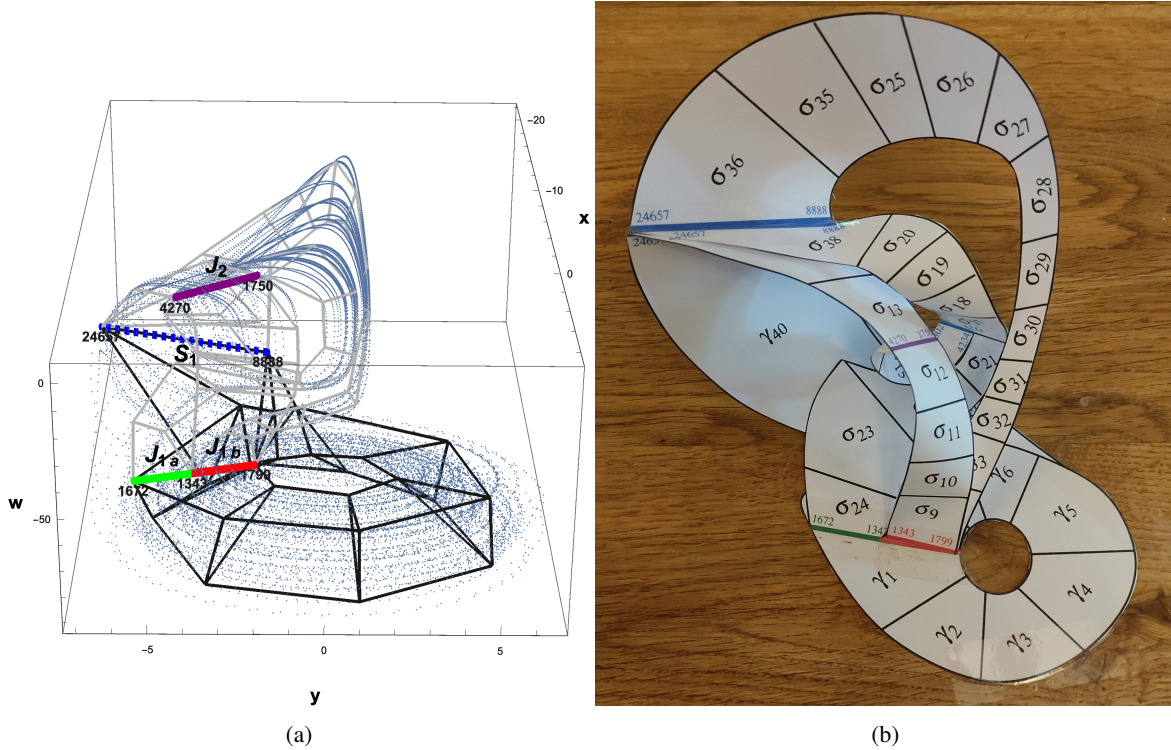


Fig. 3: (a) Complex $K(4D)$ in the x - y - w subspace from the cloud of points produced by the four-dimensional system (1). Parameter values: $a = 7$, $b = 1.45$, $c = 1$, $d = 0.5$, $R = 6$, $\alpha = 0.3$, $\beta = 7$, $\epsilon_1 = 0.165$, $\epsilon_2 = 0.01$, $\epsilon_3 = 2$, and $\mu = 1.543$. Labeled lines (in color) correspond to the joining loci (solid) or splitting loci (dashed). (b) Paper model of a 2-complex where the 3-cells of $K(4D)$ have been contracted to equivalent 2-cells. It is used as a guide to construct the template of the attractor produced by the four-dimensional system (1). The flow is counterclockwise in this paper model. The main joining locus where four strips are merged is located at the bottom left of the construction (green and red); the splitting locus (blue) is located slightly above the left middle of the paper model. Between these two loci, there is the second joining locus (purple). Source: [1].

The cycle c_{4_2} leads to the stripex $\hat{S} \equiv \sigma_{12} - \sigma_{11} - \sigma_{10} - \sigma_9$, which corresponds to a trivial strip, and the joining line J_2 can be identified with the joining line J_1 under an isotopy.

The cycles c_4 and c_5 leads to stripexes S_4 and S_5 :

$$\begin{aligned} S_4 &\equiv \gamma_1 \dots \gamma_6 - \gamma_{37} - \gamma_{40} - \sigma_{38} - \sigma_{20} - \sigma_{19} - \sigma_{18} - \sigma_{17} - \\ &\quad \sigma_{16} - \sigma_{15} - \sigma_{14} - \sigma_{12} - \sigma_{11} - \sigma_{10} - \sigma_9 \\ S_5 &\equiv \gamma_1 \dots \gamma_6 - \gamma_{37} - \gamma_{40} - \sigma_{39} - \sigma_{13} - \sigma_{12} - \sigma_{11} - \sigma_{10} - \sigma_9 \end{aligned}$$

Local twists are found in S_3 and S_5 .

Thus, the four-dimensional attractor can be described with five stripexes, that is, with a five symbol dynamics. The volume formed by the ten 3-cells of $K(4D)$, namely $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_{37}$ and γ_{40} , is topologically equivalent to a solid (filled) torus. Notice that the homologies of the solid torus ($\mathcal{H}_0 \approx \mathbb{Z}_1$, $\mathcal{H}_1 \approx \mathbb{Z}_1$, $\mathcal{H}_2 \approx \emptyset$, $\mathcal{H}_3 \approx \emptyset$) are equivalent to those of the cylinder or normal band. Because of this, the paper model is homologically equivalent to $K(4D)$.

The strips visiting the 3-cell γ_{40} required to sketch them as one foliated strip The strip associated with the 3-cell γ_{40} is drawn as a three-foliated strip which is then split into three distinct strips corresponding to the 2-cells σ_{36} , σ_{38} and σ_{39} , respectively. The determinism is not broken since the 3-cell is in fact a volume. The three-foliated strip is distinguished in tripling one of its boundary. In doing this, we lose the non-ambiguous description of the relative position between periodic orbits. This would have been crucial for three-dimensional links and the computation of

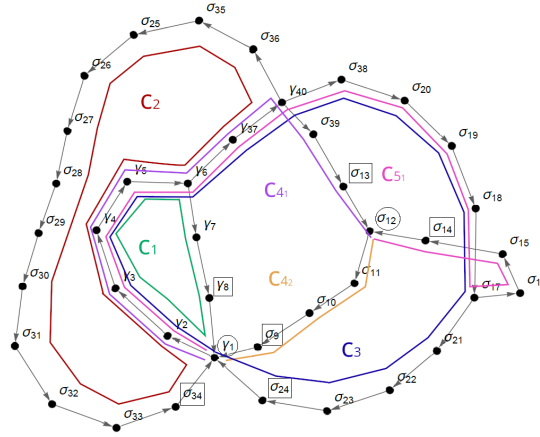


Fig. 4: Digraph on the complex constituting the templex for the analysis of the four-dimensional attractor. Ingoing and outgoing nodes are squared and circled, respectively

their linking numbers, for instance, but in a four-dimensional space, all knots are trivial and unknotted and this is no longer relevant. What persists with this description is the possibility to have a ‘knot-holder’ associated with a generating symbolic dynamics, that is, with a symbolic dynamics with a minimum number of symbols allowing a non-ambiguous description of all the periodic orbits.

Once this key point overcome, it was rather easy to construct a representation in the fashion of the direct template, drawn in Fig. 5. Among its specificities, we have the previously described splitting chart, and the joining line J_2 . Typically, this is a template bounded by a genus-one torus. Fig 5 shows the template with the common convention according to which all possible transitions between the strips are made possible. From this direct template, it can be easily understood why the joining line J_2 can be identified with the joining line J_1 under an isotopy.

It was built by counting the local torsion in each strip and the permutations between pairs of them. From this linking matrix, it was possible to draw a reduced template (Fig. 5) from which the unusual nature of the attractor here investigated is better exhibited in its upper part with the foliated splitting chart. Another particularity lies in the bottom part of the permutation between strips 5 and 7. From this representation, the strips can be univocally labelled according to the parity of the local torsion of each strip, and increasing integers, from the centre of the attractor to its periphery as required.[19]

Although characterized by a genus-1 torus, to unfold the foliated structure for computing a readable first-return map, the Poincaré section should have four components, one corresponding to the joining locus J_1 , one in each of the strips 2 and 7, and one before the splitting of the two strips 4 and 5.

Notice that such a labelling is obvious neither from the direct template nor from the cell complex. The fourth dimension, even for an attractor that is well-described by a two-dimensional manifold, offers more possibilities to intricate the different strips, a complexity which would have not been able to extract without our BRAMAH complex and our analysis in terms of homologies.

4 Conclusion

Here we summarize the main properties of a new mathematical concept, called templex, introduced by Charó et al., 2022, to describe not only the shape of a structure in phase space, but also the non-equivalent ways of flowing along it. The templex is here illustrated with two examples showing how this object is particularly useful to describe and classify chaotic attractors. The new elements that we added to the BRAMAH complex are (i) the joining/splitting loci (thus allowing for an appropriate description of branched manifolds), (ii) the direction of the flow for the orientation of the cells, and (iii) the non-equivalent cycles from a digraph now associated with the complex. Indeed, the underlying flow implies that there are only some transitions between cells which are possible. This information

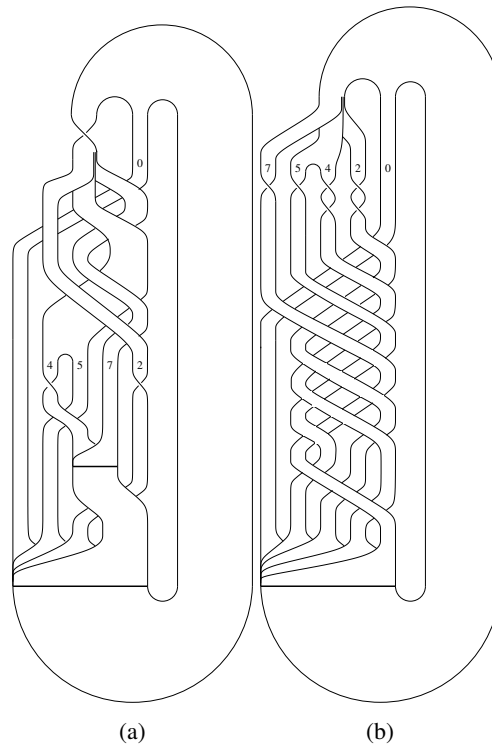


Fig. 5: (a) Direct template and (b) reduced template for the attractor produced by the four-dimensional. Source: [1].

is encoded with a directed graph whose nodes are the highest dimensional cells of the complex. The digraph is the key companion of the cell complex. The templex contains all the information that is necessary to dissect the original complex into an invariant number of smaller units, the analogs of the strips in a template.

Studying the topological structure of a system through a templex involves a few steps as follows.

- Constructing a BRAMAH complex from the cloud of points;
- locating the joining/splitting loci;
- re-orienting the cells according to the flow;
- endowing the complex with a digraph describing how the cells are visited by the flow;
- extracting the characteristics of the templex (generatex, stripex, twists) from the cell complex and the digraph.
- drawing a direct template, writing the linking matrix and drawing the reduced template.

The first step was already used in previous works. All the next steps correspond to the more recent developments. The attractor produced by the four-dimensional system is pertinent to show that a template can be drawn out of the properties of the generating templex. The templex approach opens the gate to a topological characterization of chaotic attractors produced by dynamical systems whose dimension is greater than three: other cases are currently under study and will be discussed in future articles.

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