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# Homogeneous Output-Feedback Control with Disturbance-Observer for a class of Nonlinear Systems

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## Abstract

In this paper we propose a finite-time output-feedback control scheme for a class of nonlinear systems. The dynamic part of the controller consists in an extended order observer which is based on the higher-order sliding-mode exact differentiator. With such an observer, the states of the system are exactly estimated in finite-time, moreover, the additional state in the observer estimates exactly and in finite-time Lipschitz disturbances in the system. Such an estimation is used by the static part of the controller to compensate the disturbances. The static part of the controller can be chosen from a class of homogeneous controllers. The whole control scheme allows to recover in finite-time several useful properties of homogeneous systems despite the additional and uncertain nonlinear terms. The effect of the noise in the measurement is also studied.

Keywords: Nonlinear systems, Homogeneous systems, Sliding mode control, Disturbance rejection, Lyapunov methods.

## 1 Introduction

In control systems theory, there are several techniques to improve the robustness properties of a controlled dynamic system [32, 18, 31, 21], for example, Lyapunov redesign, integral control, and disturbance observer-based control<sup>1</sup> (DOBC). The DOBC technique, also known as active disturbance rejection control, consists in identifying the disturbance and compensating it by means of the control action.

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<sup>1</sup>For recent surveys of this technique see e.g. [24, 7]

An interesting DOBC scheme is that proposed in [13], where one of the most important advantages of the DOBC technique is emphasized: *A good disturbance estimation allows to recover the performance of the nominal controlled system.* In that work, an extended-order high-gain observer is used to estimate the states of the system and the disturbance, which is assumed to be bounded with bounded derivative. The static part of the control consists in an exact linearizing state-feedback. The trajectories of the closed-loop system are ultimately bounded with a bound depending on the high-gain parameter of the observer. Thus, to reduce the ultimate bound, the high-gain must be increased.

In [10], another DOBC scheme was proposed. There, the states and the disturbance are estimated in finite-time by means of a sliding-mode-based observer. The disturbance is assumed bounded and sufficiently differentiable. The static controller is linear, therefore, only exponential stability can be guaranteed (see, e.g., [11] for a recount of input-output linearization based on sliding-mode techniques). In [17] and [11], the higher-order sliding-mode exact differentiator [22] is used to exactly estimate the states of the system and the uncertain terms (in finite-time). However, the assumptions on the uncertain terms considerably restrict the class of systems that can be controlled.

In this paper we propose a DOBC scheme for a class of nonlinear systems. As in [17, 11], we use an extended-order higher-order sliding-mode observer to estimate exactly and in finite-time the states of the system and the matched external disturbances. Nonetheless, our proposal incorporates significant improvements with respect to the existent methods:

- the assumptions on the uncertain nonlinear terms of the system are less restrictive than in [17, 10, 11], and this allows us to deal with a wider set of uncertainties;
- we prove that the static part of the controller can be chosen from a class of nonlinear homogeneous controllers, which in the state-feedback and the undisturbed case guarantee a robust closed-loop system with finite-time convergence of the trajectories to the origin by means of a continuous control action;
- in contrast with [13, 10, 17, 11] the proposed output-feedback controller guarantees zero steady-state error in finite-time despite uncertainties and disturbances. Hence, this scheme guarantees the recovery of the finite-time convergence provided by the state-feedback controllers mentioned in the previous item;
- unlike most of the existent DOBC schemes that use high-order sliding-mode differentiators, we provide a methodology that verifies Lyapunov stability of the origin of the closed-loop system;
- we investigate the robustness properties of the closed-loop system in presence of noise in the measured output;
- unlike [10, 11] we analyse the case of uncertain control coefficient. Although the uncertain control coefficient is considered in [17], its assumptions may be quite restrictive;
- one interesting advantage of the proposed scheme is that the domain of attraction and the convergence rate can be adjusted by only one parameter.

*Paper organization:* In Section 2 we describe the class of systems considered in this paper. Section 3 contains the main result whose proof is given in Section 4. In Section 5, the case of uncertain control coefficient is analysed. Two simulation examples are given in Section 6, one of them consists of a comparison example between our proposed controller and the scheme from [13]. Some final remarks are stated in Section 7. The definition of homogeneity and some auxiliary results are collected in the appendix.

*Notation:* Real numbers are denoted by  $\mathbb{R}$ .  $\mathbb{R}_{>0}$  denotes the set  $\{x \in \mathbb{R} : x > 0\}$ , and analogously for the sign  $\geq$ . For  $x \in \mathbb{R}^n$ , the Euclidean norm is denoted by  $|x|$ , and an  $\mathbf{r}$ -homogeneous norm (see Definition 3 in Appendix) by  $\|x\|_{\mathbf{r}}$ . For  $x, \rho \in \mathbb{R}$  we denote  $[x]^\rho = |x|^\rho \text{sign}(x)$ . The set of strictly increasing continuous functions  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  with  $\alpha(0) = 0$  is denoted by  $\mathcal{K}$ . The set of functions in  $\mathcal{K}$  such that  $\alpha(t) \rightarrow \infty$  as  $t \rightarrow \infty$  is denoted by  $\mathcal{K}_\infty$ . A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is in the set of functions  $\mathcal{KL}$  if for each fixed  $s \in \mathbb{R}_{\geq 0}$ ,  $\beta(\cdot, s) \in \mathcal{K}$ , and for each fixed  $r \in \mathbb{R}_{\geq 0}$ ,  $\beta(r, \cdot)$  is continuous, strictly decreasing, and  $\lim_{s \rightarrow \infty} \beta(r, s) = 0$ .  $\triangleleft$

## 2 Problem statement

In this paper we consider the class of nonlinear systems that can be described by

$$\dot{x}_i = x_{i+1}, \quad \dot{x}_n = f(t, x) + u + \delta(t), \quad y = x_1, \quad i = 1, \dots, n-1, \quad (1)$$

where  $x(t) = [x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the control input,  $y(t) \in \mathbb{R}$  is the measured output,  $f(t, x) \in \mathbb{R}$  is an uncertain function, and  $\delta(t) \in \mathbb{R}$  is an unknown exogenous disturbance.

In a second stage we consider the presence of noise in the measured output, namely  $y = x_1 + \nu$ , where  $\nu$  is a Lebesgue-measurable and essentially bounded function. In Section 5, we study the case of uncertain control coefficient.

**Assumption 1.** Consider (1), and define the set  $D = \{x \in \mathbb{R}^n : |x| \leq d\}$  for some  $d \in \mathbb{R}_{>0}$ .

- (i) The uncertain function  $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the following:  $f(t, 0) = 0$  for all  $t \in \mathbb{R}_{\geq 0}$ , it is piecewise-continuous and bounded in  $t$ , and it is locally Lipschitz in  $x$ , i.e. for any  $d \in \mathbb{R}_{\geq 0}$  there exists a constant  $d_1 = d_1(d) \in \mathbb{R}_{\geq 0}$  such that

$$|f(t, x) - f(t, x')| \leq d_1 |x - x'|, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall x, x' \in D.$$

- (ii) A model  $\bar{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$  for  $f$  is known, and it satisfies the following: it is a piecewise-continuous and bounded function in  $t$ , it is locally Lipschitz in  $x$ , and for any  $d \in \mathbb{R}_{\geq 0}$  there exists a constant  $d_2 = d_2(d) \in \mathbb{R}_{\geq 0}$  such that

$$|f(t, x) - \bar{f}(t, x)| \leq d_2 |x|, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall x \in D.$$

- (iii) The disturbance  $\delta$  is such that  $|\dot{\delta}(t)| \leq \Delta$  for all  $t \in \mathbb{R}_{\geq 0}$  and a known constant  $\Delta \in \mathbb{R}_{\geq 0}$ .

**Remark 1.** Note that as in [13, 17, 11], the considered class of systems (1) is in the control-lability canonical form. However, two of the main differences are: 1) we are not requiring the

uncertain term  $f(t, x)$  to be continuously differentiable with Lipschitz derivatives, but only to be Lipschitz continuous; 2) the external disturbance  $\delta$  is not assumed to be bounded.

In Assumption 1, the disturbance  $\delta$  is assumed to be a function of time, nonetheless, this can be considered as a function of the state as well, if it can be guaranteed that for any gain  $\gamma \in \mathbb{R}_{>0}$  of the controller designed in Section 3.2 the following holds: for any  $d \in \mathbb{R}_{>0}$  there exists a known constant  $\Delta = \Delta(d) \in \mathbb{R}_{\geq 0}$  such that  $|\frac{\partial \delta(t, x)}{\partial t} + \frac{\partial \delta(t, x)}{\partial x} \dot{x}| \leq \Delta$  for all  $t \in \mathbb{R}_{\geq 0}$  and all  $x \in D$ , with  $D$  as in Assumption 1.  $\triangleleft$

The control objective is to stabilize in finite-time the origin of the system despite the disturbance  $\delta$  and the uncertainty in  $f$  by means of a continuous control signal and by using only the output  $y$  as information for the controller.

## 3 Controller

### 3.1 State feedback nominal controller

Since homogeneity offers several advantages in dynamic systems, in this paper we consider homogeneous controllers. See Appendix for the definition and some properties of homogeneity. To construct the output-feedback controller we assume that a homogeneous controller is already known as stated below.

**Assumption 2.** A state feedback controller  $u_0(x)$  is known and satisfies the following:

- $u_0$  is a locally Hölder continuous<sup>2</sup> function;
- $u_0$  is an  $\mathbf{r}_c$ -homogeneous function such that the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dots, \quad \dot{x}_n = u_0(x), \quad (2)$$

is  $\mathbf{r}_c$ -homogeneous of degree  $\kappa = -1$  with weights  $\mathbf{r}_c = [n + 1, n, \dots, 2]^\top$ ;

- the origin of (2) is an asymptotically stable equilibrium point.  $\triangleleft$

Note that the asymptotic stability and the negative homogeneity degree of (2) guarantee that its origin is globally finite-time stable [5, 16]. Assumption 2 is not restrictive, in the sense that, a homogeneous continuous controller providing finite-time stability to (2) does exist (see, e.g., [2, Theorem 5.12]). There exist in the literature several explicit controllers satisfying Assumption 2, for example:

- for  $n = 2$ , the following controller was proposed in [6],

$$u_0(x) = -[x_2]^\rho - \left[ x_1 + \frac{1}{2-\rho} [x_2]^{2-\rho} \right]^{\frac{\rho}{2-\rho}},$$

the homogeneity condition in Assumption 2 is satisfied with  $\rho = 1/2$ ;

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<sup>2</sup>A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called Hölder continuous with exponent  $\alpha \in (0, 1]$ , if for every compact set  $I \subset \mathbb{R}^n$  there exists  $L_I \in \mathbb{R}_{>0}$  such that  $|V(x) - V(y)| \leq L_I |x - y|^\alpha$  for all  $x, y \in I$ , see e.g. [14, p. 52].

- in [8], a wide class of homogeneous controllers was proposed, a particular family of such controllers is

$$u_0 = -l_n [\sigma_n]^{\frac{1}{n+1}}, \quad \sigma_i = [x_i]^{\frac{n+1}{r_i}} + l_{i-1} \sigma_{i-1}, \quad (3)$$

with  $i = 1, \dots, n-1$  and  $l_0 = 0$  (for the selection of the gains  $l_i \in \mathbb{R}_{>0}$ , see the method given in [8]).

Now, define the controller

$$u = u_0(x) - f(t, x) - \delta, \quad (4)$$

that can be considered as an exact state-feedback homogenization of (1). This way, the closed-loop (1), (4) equals (2) and therefore its origin is finite-time stable. However, we have the following obvious restrictions:

- R1 the term  $\delta(t)$  is unknown, hence we will opt for an identification mechanism to compensate for it;
- R2 since the only available measurement is the output signal  $y$  (and the control signal  $u$ ), then a state observer is required;
- R3 the term  $f(t, x)$  is uncertain, thus, although a model  $\bar{f}(t, x)$  is known, the control scheme must be able to deal with the uncertainty produced by the term  $f(t, x) - \bar{f}(t, \hat{x})$ .

To overcome these problems, we propose in the following section an output-feedback controller.

### 3.2 Output-feedback controller

According to restrictions *R1* – *R3*, we propose the following modification of controller (4),

$$u = \gamma u_0(\gamma^{-1} \hat{x}) - \bar{f}(t, \hat{x}) - \hat{\delta}, \quad \gamma \in \mathbb{R}_{>0}, \quad (5)$$

where  $\hat{x}$  is the observed state,  $\hat{\delta}$  is an estimation of the disturbance  $\delta$ , and  $\gamma$  is a gain to be adjusted. For state observation and disturbance identification we propose the following extended-order observer

$$\begin{aligned} \dot{\hat{x}}_i &= -\gamma^{p_i} k_i [\hat{x}_1 - y]^{q_i} + \hat{x}_{i+1}, \\ \dot{\hat{x}}_n &= -\gamma^{p_n} k_n [\hat{x}_1 - y]^{q_n} + \gamma u_0(\gamma^{-1} \hat{x}), \\ \dot{\hat{\delta}} &= -\gamma k_{n+1} [\hat{x}_1 - y]^0, \end{aligned} \quad (6)$$

where  $p_i = \frac{i}{n+1}$  and  $q_i = \frac{n-i+1}{n+1}$  for  $i = 1, \dots, n$ . Such an observer is based on the sliding mode differentiator proposed in [22]. Thus, the gains  $k_i$  have to be chosen as in the following assumption.

**Assumption 3.** For (6), the gains  $k_i$ ,  $i = 1, \dots, n+1$ , are such that the origin of the system

$$\begin{aligned} \dot{w}_i &= -k_i [w_1]^{q_i} + w_{i+1}, \quad i = 1, \dots, n, \\ \dot{w}_{n+1} &= -k_{n+1} [w_1]^0 - \dot{\bar{\delta}}(t), \end{aligned} \quad (7)$$

is asymptotically stable for any function  $\bar{\delta} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  which satisfies that  $|\dot{\bar{\delta}}(t)| \leq 1$  for all  $t \in \mathbb{R}_{\geq 0}$ . ◁

Again, Assumption 3 is not restrictive since there exist the gains  $k_i \in \mathbb{R}_{>0}$  (and procedures to compute them) that accomplish such an assumption, see e.g. [22, 30, 9].

Before stating the main result, let us establish some notation. The closed-loop (1), (5) is given by

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= \gamma u_0(\gamma^{-1}\hat{x}) + \tilde{f}(t, x, \hat{x}) + \delta - \hat{\delta},\end{aligned}\tag{8}$$

where  $\tilde{f}(t, x, \hat{x}) := f(t, x) - \bar{f}(t, \hat{x})$ . Defining the following observation errors

$$e_i = \hat{x}_i - x_i, \quad i = 1, \dots, n, \quad e_{n+1} = \hat{\delta} - \delta,\tag{9}$$

system (8) can be rewritten as follows

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= \gamma u_0(\gamma^{-1}[x + \bar{e}]) + \tilde{f}(t, x, x + \bar{e}) - e_{n+1},\end{aligned}\tag{10}$$

where  $\bar{e} := [e_1, \dots, e_n]^\top$ . The observation error dynamics is given by

$$\begin{aligned}\dot{e}_i &= -\gamma^{p_i} k_i [e_1]^{q_i} + e_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{e}_n &= -\gamma^{p_n} k_n [e_1]^{q_n} + e_{n+1} - \tilde{f}(t, x, x + \bar{e}), \\ \dot{e}_{n+1} &= -\gamma k_{n+1} [e_1]^0 - \dot{\delta}(t).\end{aligned}\tag{11}$$

Finally, define  $X := [x^\top, e^\top]^\top$  with  $e := [e_1, \dots, e_{n+1}]^\top$ , and  $\mathbf{r} = [n+1, n, \dots, 2, n+1, n, \dots, 1]^\top$ . Now we are ready to give the main result.

**Theorem 1.** *Consider (1) in closed-loop with the output-feedback controller (5)-(6). Under Assumptions 1-3, for any  $a, \Delta \in \mathbb{R}_{\geq 0}$  there exists  $\gamma \in \mathbb{R}_{>0}$  such that the origin of (10)-(11) is finite-time stable with a domain of attraction  $\mathcal{D}$  such that  $\{X \in \mathbb{R}^{2n+1} : \|X\|_{\mathbf{r}} \leq a\} \subset \mathcal{D}$ .*

*Proof.* See Section 4. □

**Remark 2.** *It is clear from Theorem 1 that output-feedback controller (5)-(6) guarantees semi-global stabilization of the origin, i.e., the domain of attraction can be enlarged as desired if the gain  $\gamma$  is suitably chosen. Indeed, what is verified in the proof of Theorem 1, is that there exist two functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that the origin of (10)-(11) is finite-time stable for all  $x \in \mathbb{R}^n$  and all  $e \in \mathbb{R}^{n+1}$  such that*

$$\|X\|_{\mathbf{r}} \leq \frac{\alpha_1(\gamma)}{\alpha_2(d_1 + d_2)}, \quad \gamma \geq \Delta,$$

with  $\Delta$ ,  $d_1$ , and  $d_2$  as in Assumption 1. Hence, the estimated domain of attraction can be increased by increasing the gain  $\gamma$ . ◁

**Corollary 1.** *Under assumptions of Theorem 1, there exist two functions  $\theta_1, \theta_2 \in \mathcal{KL}$  and a constant  $c_0 \in \mathbb{R}_{>0}$  such that, for all  $X(0) \in \{X \in \mathbb{R}^{2n+1} : \|X\|_{\mathbf{r}} \leq a\}$ , the inequality  $c_0 > (d_1 + d_2)\theta_2(a, \gamma)$  holds, and the settling-time function  $T : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$  of (10)-(11) satisfies*

$$0 \leq T(X(0)) \leq \frac{\theta_1(a, \gamma)}{c_0 - (d_1 + d_2)\theta_2(a, \gamma)},\tag{12}$$

with  $d_1$  and  $d_2$  as in Assumption 1. ◁

**Remark 3.** Observe that, since  $\theta_1$  and  $\theta_2$  are class  $\mathcal{KL}$  functions (see Section 1 for notation), they are decreasing in the second argument. Hence, the right-hand side of (12) is a decreasing function of  $\gamma$ . Therefore, the convergence time to the origin can be reduced by increasing the gain  $\gamma$ .  $\triangleleft$

**Corollary 2.** If in addition to the assumptions of Theorem 1, there exist  $\bar{d}_1, \bar{d}_2 \in \mathbb{R}_{\geq 0}$  such that

$$|f(t, x)| \leq \bar{d}_1, \quad |\bar{f}(t, x)| \leq \bar{d}_2, \quad \forall t \in \mathbb{R}_{\geq 0}, \forall x \in \mathbb{R}^n,$$

then there exists  $\gamma \in \mathbb{R}_{> 0}$  such that the origin of (10)-(11) is globally finite-time stable.  $\triangleleft$

### 3.3 Noise in the output

Now suppose that the measured output is contaminated with noise, i.e.

$$y(t) = x_1(t) + \nu(t), \tag{13}$$

where  $\nu : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a Lebesgue-measurable function such that  $\text{ess sup}_{t \in \mathbb{R}_{\geq 0}} |\nu(t)| \leq N$  for some  $N \in \mathbb{R}_{> 0}$ .

**Corollary 3.** Under assumptions of Theorem 1, system (10)-(11), with  $y$  as in (13), is locally input-to-state stable with respect to the input  $\nu$ , i.e. there exist some functions  $\beta \in \mathcal{KL}$ ,  $\eta \in \mathcal{K}$ , and a constant  $\Gamma \in \mathbb{R}_{> 0}$ , such that for any  $\nu$  and each  $X(0) \in \mathbb{R}^{2n+1}$  satisfying  $N \leq \Gamma$  and  $|X(0)| \leq \Gamma$ , the solution  $\phi(t; X(0), \nu)$  of (10)-(11) satisfies

$$|\phi(t; X(0), \nu)| \leq \beta(|X(0)|, t) + \eta(N), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

### 3.4 Performance recovery

In this section we verify that the closed-loop (8) is able to recover the performance of the nominal homogeneous system

$$\dot{\bar{x}}_1 = \bar{x}_2, \quad \dot{\bar{x}}_2 = \bar{x}_3, \quad \dots, \quad \dot{\bar{x}}_n = \gamma u_0(\gamma^{-1} \bar{x}). \tag{14}$$

To describe the ISS robustness of (14), let us consider its disturbed version given by

$$\dot{\bar{x}}_1 = \bar{x}_2, \quad \dot{\bar{x}}_2 = \bar{x}_3, \quad \dots, \quad \dot{\bar{x}}_n = \gamma u_0(\gamma^{-1} \bar{x}) + g(t), \tag{15}$$

where  $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a Lebesgue-measurable function such that  $|g(t)| \leq \bar{g}$  for all  $t \in \mathbb{R}_{\geq 0}$  for some constant  $\bar{g} \in \mathbb{R}_{\geq 0}$ . The following result describes the ISS property of (15) with respect to the input  $g$ .

**Lemma 1.** Consider (15) with  $u_0$  as in Assumption 2. There exist two functions  $\beta_1, \beta_2 \in \mathcal{KL}$  such that for any  $\gamma \in \mathbb{R}_{> 0}$ ,

$$|\bar{x}(t)| \leq \beta_1(|\bar{x}(0)|, t) + \beta_2(\bar{g}, \gamma), \quad \forall \bar{x}(0) \in \mathbb{R}^n, \forall t \in \mathbb{R}_{\geq 0}. \tag{16}$$



The proof of this lemma is given in Section 4. From this lemma we can see that, the origin of (14) is globally asymptotically stable. Moreover, since (14) is  $\mathbf{r}_c$ -homogeneous of negative degree (see Assumption 2), its origin is finite-time stable. From these facts and from Theorem 1 we conclude that, for the solution  $\bar{x}(t)$  of (14) and the solution  $x(t)$  of (8), there exists  $T \in \mathbb{R}_{\geq 0}$  such that  $|\bar{x}(t) - x(t)| = 0$  for all  $t \geq T$ . However, this trivial conclusion is only for the steady-state, and it does not give information about the performance recovery during the transient phase.

On the other hand, Lemma 1 shows that, for any  $\bar{g} \in \mathbb{R}_{\geq 0}$ ,  $\beta_2(\bar{g}, \gamma) \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Thus, the effect of the disturbance  $g$  on the solution  $\bar{x}(t)$  is reduced by increasing the gain  $\gamma$ . In this sense, we can say that (15) recovers the performance of (14) by increasing the gain  $\gamma$ . Now, we show that (8) also has this property.

**Remark 4.** *According to Theorem 1 (see also Remark 2), for each  $a \in \mathbb{R}_{>0}$  there exists  $\gamma^* \in \mathbb{R}_{>0}$  such that for all  $\gamma \geq \gamma^*$  the origin of (8) is finite-time stable. Hence, for each  $\gamma \geq \gamma^*$  and for each  $X_0 = [x^\top(0), e^\top(0)]^\top \in \mathcal{D}$  there exists  $t_0(\gamma, X_0) \in \mathbb{R}_{>0}$  such that  $x(t) \in \mathcal{D}$  and  $e(t) = 0$  for all  $t \geq t_0$ .*  $\triangleleft$

**Theorem 2.** *Consider (8) and Remark 4. Under the assumptions of Theorem 1, for each  $\gamma \geq \gamma^*$  there exists  $t_0 \in \mathbb{R}_{>0}$  such that*

$$|x(t)| \leq \beta_1(|x(t_0)|, t) + \beta_2(d_2 a, \gamma), \quad \forall x(0) \in \mathcal{D}, \quad \forall t \geq t_0, \quad (17)$$

where  $d_2$  is as in Assumption 1 and the functions  $\beta_1, \beta_2$  as in Lemma 1.  $\triangleleft$

*Proof.* The proof of this theorem is analogous to the proof of Lemma 1. We only have to clarify the following:

1. According to Remark 4,  $e(t) = 0$  for all  $t \geq t_0$ . Thus, we can rewrite (8) (or, equivalently (10)) as follows

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= \gamma u_0(\gamma^{-1} x) + \tilde{f}(t, x, x), \quad \forall t \geq t_0. \end{aligned} \quad (18)$$

2. Finally, from Assumption 1,  $|\tilde{f}(t, x, x)| \leq d_2|x|$ , and from Remark 4,  $|x(t)| \leq a$  for all  $t \geq t_0$ . Hence,  $|\tilde{f}(t, x, x)| \leq d_2 a$  for all  $t \geq t_0$ .

□

## 4 Proofs

### 4.1 Proof of Theorem 1

Let us consider the auxiliary systems (7) and

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{z}_n &= u_0(z + \bar{w}) - w_{n+1}, \end{aligned} \quad (19)$$

where  $\bar{w} = [w_1, \dots, w_n]^\top$ . First note that (7) is discontinuous, thus its solutions and its associated differential inclusion are considered as defined in [12]. Also note that the differential inclusion associated to (7) is  $\mathbf{r}_o$ -homogeneous of degree  $\kappa = -1$  with  $\mathbf{r}_o = [n+1, n, \dots, 1]^\top$  [23]. Hence, under Assumption 3, the origin of (7) is finite-time stable.

Define the vector  $w = [w_1, \dots, w_n, w_{n+1}]^\top$ . Observe that Assumption 2 guarantees that (19) is also finite-time stable for  $w \equiv 0$ , moreover, it is input-to-state stable (ISS) with respect to the input  $w$  [4] (the ISS concept is recalled in Definition 4 in Appendix). Thus, according to [15], the origin of the interconnected system (19), (7) is asymptotically stable.

Now, it is easy to see that the differential inclusion associated to (19), (7) is  $\mathbf{r}$ -homogeneous of degree  $\kappa = -1$  with  $\mathbf{r} = [\mathbf{r}_c^\top, \mathbf{r}_o^\top]^\top$ . Therefore, its origin is finite-time stable [26, 28, 23].

Thus, according to [26, 3] there exists a smooth Lyapunov function  $V_0 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$  for (19), (7) such that it is  $\mathbf{r}$ -homogeneous of some degree<sup>3</sup>  $m \in \mathbb{R}_{>0}$  and its derivative along (19), (7) satisfies<sup>4</sup>  $\dot{V}_0 \leq -W_0(z, w)$ , where the function  $W_0 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$  is  $\mathbf{r}$ -homogeneous of degree  $m + \kappa$ , strictly positive for all  $[z^\top, w^\top]^\top \in \mathbb{R}^{2n+1} \setminus \{0\}$  and continuous for all  $[z^\top, w^\top]^\top \in \mathbb{R}^{2n+1}$ .

Now, consider the change of coordinates

$$\begin{aligned} x_i &= \gamma z_i, & i &= 1, \dots, n, \\ e_j &= \gamma w_j, & j &= 1, \dots, n+1. \end{aligned} \quad (20)$$

For these new variables we obtain from (7) and (19) the dynamics

$$\begin{aligned} \dot{x}_i &= x_{i+1}, & i &= 1, \dots, n-1, \\ \dot{x}_n &= \gamma u_0(\gamma^{-1}[x + e^n]) - e_{n+1}, \end{aligned} \quad (21)$$

$$\begin{aligned} \dot{e}_i &= -\gamma^{p_i} k_i [e_1]^{q_i} + e_{i+1}, & i &= 1, \dots, n, \\ \dot{e}_{n+1} &= -\gamma k_{n+1} [e_1]^0 - \gamma \dot{\delta}(t). \end{aligned} \quad (22)$$

Note that for any  $\gamma \in \mathbb{R}_{>0}$ , the origin of the interconnected systems (21)-(22) is finite-time stable, and the function  $V : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$ , given by

$$V(x, e) = V_0(\gamma^{-1}x, \gamma^{-1}e),$$

is a Lyapunov function for (21)-(22). Indeed, the derivative of  $V$  along (21)-(22) is such that  $\dot{V} = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial e} \dot{e} = \frac{\partial V_0}{\partial z} \frac{1}{\gamma} \gamma \dot{z} + \frac{\partial V_0}{\partial w} \frac{1}{\gamma} \gamma \dot{w}$ . Thus,  $\dot{V} = \dot{V}_0$ , and  $\dot{V} \leq -W(x, e)$  where  $W(x, e) = W_0(\gamma^{-1}x, \gamma^{-1}e)$ .

Now, we consider  $V$  as a Lyapunov function candidate for (10)-(11). The derivative of  $V$  along (10)-(11) is such that

$$\dot{V} \leq -W(x, e) + \left( \frac{\partial V}{\partial x_n} - \frac{\partial V}{\partial e_n} \right) \tilde{f}(t, x, x + \bar{e}).$$

<sup>3</sup>Indeed,  $m$  must satisfy:  $m > \max_{j \in \{1, \dots, 2n+1\}} r_j$  to guarantee differentiability of  $V_0$  and;  $m > \max\{-\kappa, 0\}$  to have  $W_0$  with a positive homogeneity degree. We assume that these conditions hold.

<sup>4</sup>Since the right-hand side of (7) is discontinuous, the Lyapunov analysis for (19), (7) has to be done by considering the differential inclusion associated to (19), (7). Thus, for each  $[z^\top, w^\top]^\top \in \mathbb{R}^{2n+1}$ ,  $\dot{V}_0 = \left\{ \frac{\partial V_0(z, w)}{\partial(z, w)} h \in \mathbb{R} : h \in H(z, w) \right\}$ , where  $H$  denotes the set-valued vector field of the differential inclusion associated to (19), (7). Hence,  $\dot{V}_0 \leq -W_0(z, w)$  means  $\max_{h \in H(z, w)} \frac{\partial V_0(z, w)}{\partial(z, w)} h \leq -W_0(z, w)$ , see [2].

Observe that for any  $\gamma \geq \Delta$ ,  $W(x, e)$  is positive definite. By considering again (20) we have that (let us denote  $V_0(z, w)$  with  $V_0$ )

$$\begin{aligned} \dot{V} &\leq -W_0(z, w) + \left( \frac{\partial V_0}{\partial z_n} \frac{1}{\gamma} - \frac{\partial V_0}{\partial w_n} \frac{1}{\gamma} \right) \tilde{f}(t, x, x + \bar{e}), \\ &\leq -W_0(z, w) + \frac{1}{\gamma} \left| \frac{\partial V_0}{\partial z_n} - \frac{\partial V_0}{\partial w_n} \right| |\tilde{f}(t, x, x + \bar{e})|. \end{aligned} \quad (23)$$

Since  $W$  and the function  $(z, w) \mapsto \left| \frac{\partial V_0}{\partial z_n} - \frac{\partial V_0}{\partial w_n} \right|$  are homogeneous of degrees  $m - 1$  and  $m - r_n = m - 2$ , respectively, Lemma 3 (see Appendix) ensures the existence of  $c_0, c_1 \in \mathbb{R}_{>0}$  such that, from (23),

$$\dot{V} \leq -c_0 V_0^{\frac{m-1}{m}} + \frac{1}{\gamma} c_1 V_0^{\frac{m-2}{m}} |\tilde{f}(t, x, x + \bar{e})|. \quad (24)$$

Let us now analyse the term  $|\tilde{f}(t, x, x + \bar{e})|$ . Note that we can rewrite  $\tilde{f}$  as follows

$$\begin{aligned} \tilde{f}(t, x, x + \bar{e}) &= f(t, x) - \bar{f}(t, x + \bar{e}), \\ &= [f(t, x) - f(t, x + \bar{e})] + [f(t, x + \bar{e}) - \bar{f}(t, x + \bar{e})]. \end{aligned} \quad (25)$$

Thus, according to Assumption 1, for any  $a \in \mathbb{R}_{>0}$  there exists  $d_1, d_2 \in \mathbb{R}_{\geq 0}$  such that

$$|\tilde{f}(t, x, x + \bar{e})| \leq d_1 |\bar{e}| + d_2 |x + \bar{e}| = \gamma d_1 |\bar{w}| + \gamma d_2 |z + \bar{w}|.$$

Now, according to Lemma 3 (see Appendix), there exist  $c_2, \bar{c}_2 \in \mathbb{R}_{>0}$  such that

$$\begin{aligned} |z + \bar{w}|^2 &= \sum_{i=1}^n |z_i + w_i|^2 \leq \sum_{j=1}^n (c_2 V_0^{(j+1)/m})^2 \leq n (c_2 V_0^{\rho/m})^2, \\ |\bar{w}|^2 &= \sum_{i=1}^n |w_i|^2 \leq \sum_{j=1}^n (\bar{c}_2 V_0^{(j+1)/m})^2 \leq n (\bar{c}_2 V_0^{\rho/m})^2, \end{aligned}$$

where  $\rho = (n + 1)$  if  $V_0(z, w) \geq 1$ , and  $\rho = 2$  if  $V_0(z, w) < 1$ . Thus,

$$|\tilde{f}(t, x, x + \bar{e})| \leq \gamma \sqrt{n} (d_1 \bar{c}_2 V_0^{\rho/m} + d_2 c_2 V_0^{\rho/m}) = \gamma (d_1 + d_2) c_3 V_0^{\rho/m},$$

with  $c_3 = \sqrt{n} \max\{c_2, \bar{c}_2\}$ . Hence, from (24) we obtain

$$\dot{V} \leq -c_0 V_0^{\frac{m-1}{m}} + c_1 c_3 (d_1 + d_2) V_0^{\frac{m-2+\rho}{m}}. \quad (26)$$

Note that no constant in (26) depends on  $\gamma$ . Also note that in any case  $m - 2 + \rho > m - 1$ . Thus, since

$$-c_0 V_0^{\frac{m-1}{m}} + c_1 c_3 (d_1 + d_2) V_0^{\frac{m-2+\rho}{m}} = -\frac{1}{2} c_0 V_0^{\frac{m-1}{m}} + \left( c_1 c_3 (d_1 + d_2) V_0^{\frac{m-2+\rho}{m}} - \frac{1}{2} c_0 V_0^{\frac{m-1}{m}} \right),$$

it is clear that  $\dot{V} \leq -\frac{1}{2} c_0 V_0^{\frac{m-1}{m}}(z, w) = -\frac{1}{2} c_0 V^{\frac{m-1}{m}}(x, e)$  if

$$\frac{1}{2} c_0 V_0^{\frac{m-1}{m}}(z, w) \geq c_1 c_3 (d_1 + d_2) V_0^{\frac{m-2+\rho}{m}}(z, w). \quad (27)$$

Let us analyse (27). First note that (for  $d_1 \neq 0$  or  $d_2 \neq 0$ ) (27) is equivalent to

$$\beta^{-1}\left(\frac{c_0}{2c_1c_3(d_1+d_2)}\right) \geq V_0^{\frac{1}{m}}(z, w), \quad (28)$$

where  $\beta$  is a function of class  $\mathcal{K}_\infty$  given by

$$\beta(s) = \begin{cases} s^n, & s \geq 1, \\ s, & s < 1. \end{cases} \quad (29)$$

Now, Lemma 3 (see Appendix) ensures the existence of  $c_4 \in \mathbb{R}_{>0}$  such that

$$V_0^{\frac{1}{m}}(z, w) \leq c_4 \|[z^\top, w^\top]^\top\|_{\mathbf{r}},$$

but Lemma 4 (see Appendix) ensures that (by using (20))

$$\|[z^\top, w^\top]^\top\|_{\mathbf{r}} \leq \frac{1}{\alpha(\gamma)} \|[x^\top, e^\top]^\top\|_{\mathbf{r}},$$

where  $\alpha$  is a function of class  $\mathcal{K}_\infty$  given by

$$\alpha(\gamma) = \begin{cases} \gamma, & \gamma \leq 1, \\ \gamma^{\frac{1}{n+1}}, & \gamma > 1. \end{cases}$$

Thus,

$$V_0^{\frac{1}{m}}(z, w) \leq \frac{c_4}{\alpha(\gamma)} \|[x^\top, e^\top]^\top\|_{\mathbf{r}}, \quad (30)$$

and from (28), we can see that (27) is satisfied for all  $x \in \mathbb{R}^n$  and all  $e \in \mathbb{R}^{n+1}$  such that

$$\|[x^\top, e^\top]^\top\|_{\mathbf{r}} \leq \frac{\alpha(\gamma)}{c_4} \beta^{-1}\left(\frac{c_0}{2c_1c_3(d_1+d_2)}\right).$$

Therefore, (since  $\alpha \in \mathcal{K}_\infty$ ) for any  $d_1, d_2, a \in \mathbb{R}_{>0}$ ,  $\gamma$  can always be chosen such that (27) holds.

## 4.2 Proof of Corollary 1

From (26) and (30) we have that

$$\begin{aligned} \dot{V} &\leq -[c_0 - c_1c_3(d_1+d_2)V_0^{\frac{\rho-1}{m}}]V_0^{\frac{m-1}{m}}(z, w) \\ &\leq -[c_0 - c_1c_3(d_1+d_2)\beta\left(\frac{c_4}{\alpha(\gamma)}\|[x^\top, e^\top]^\top\|_{\mathbf{r}}\right)]V^{\frac{m-1}{m}}(x, e), \end{aligned}$$

with  $\beta$  as in (29). Thus, in the set  $\{X \in \mathbb{R}^{2n+1} : \|X\|_{\mathbf{r}} \leq a\}$ ,

$$\dot{V} \leq -[c_0 - c_1c_3(d_1+d_2)\beta\left(\frac{c_4a}{\alpha(\gamma)}\right)]V^{\frac{m-1}{m}}(x, e),$$

where  $c_0 - c_1c_3(d_1+d_2)\beta\left(\frac{c_4a}{\alpha(\gamma)}\right) > 0$ . By direct integration over the interval  $[0, T(x(0), e(0))]$  (where  $T$  is the settling-time function) we obtain

$$\begin{aligned} T(x(0), e(0)) &\leq \frac{m}{c_0 - c_1c_3(d_1+d_2)\beta\left(\frac{c_4a}{\alpha(\gamma)}\right)} V^{\frac{1}{m}}(x(0), e(0)), \\ &\leq \frac{m \frac{c_4a}{\alpha(\gamma)}}{c_0 - c_1c_3(d_1+d_2)\beta\left(\frac{c_4a}{\alpha(\gamma)}\right)}. \end{aligned}$$

The proof is completed by taking  $\theta_1(a, \gamma) = mc_4 \frac{a}{\alpha(\gamma)}$ , and  $\theta_2(a, \gamma) = c_1c_3\beta\left(c_4 \frac{a}{\alpha(\gamma)}\right)$ .

### 4.3 Proof of Corollary 2

The proof of this corollary consists in verifying that a neighbourhood of the origin is globally attractive. The proof is completed by proving that the radius of such a neighbourhood is inversely proportional to the the gain  $\gamma$ . This property and the local asymptotic stability from Theorem 1 guarantee the result.

We use the Lyapunov function from the proof of Theorem 1. According to the assumption of the corollary, from (25) we obtain  $|\tilde{f}(t, x, x + \bar{e})| \leq \bar{d} := \bar{d}_1 + \bar{d}_2$ . Thus, from (24) we have that

$$\dot{V} \leq -c_0 V_0^{\frac{m-1}{m}} + \frac{\bar{d}}{\gamma} c_1 V_0^{\frac{m-2}{m}}.$$

Since  $-c_0 V_0^{\frac{m-1}{m}} + c_1 V_0^{\frac{m-2}{m}} \frac{\bar{d}}{\gamma} = -\frac{1}{2} c_0 V_0^{\frac{m-1}{m}} + c_1 V_0^{\frac{m-2}{m}} \frac{\bar{d}}{\gamma} - \frac{1}{2} c_0 V_0^{\frac{m-1}{m}}$ , we have that,  $\dot{V} \leq -\frac{1}{2} c_0 V_0^{\frac{m-1}{m}}$  if  $\frac{2c_1 \bar{d}}{\gamma c_0} \leq V_0^{\frac{1}{m}}(z, w)$ . Now, Lemma 3 (see Appendix) ensures the existence of  $\bar{c}_4 \in \mathbb{R}_{>0}$  such that  $V_0^{\frac{1}{m}}(z, w) \geq \bar{c}_4 \|[z^\top, w^\top]^\top\|_{\mathbf{r}}$ , but Lemma 4 (see Appendix) ensures that  $\|[z^\top, w^\top]^\top\|_{\mathbf{r}} = \|\gamma^{-1}[x^\top, e^\top]^\top\|_{\mathbf{r}} \geq \frac{1}{\underline{\alpha}(\gamma)} \|[x^\top, e^\top]^\top\|_{\mathbf{r}}$ , where the function  $\underline{\alpha} \in \mathcal{K}_\infty$  is given by

$$\underline{\alpha}(\gamma) = \begin{cases} \gamma^{\frac{1}{n+1}}, & \gamma \leq 1, \\ \gamma^{\frac{1}{2}}, & \gamma > 1. \end{cases} \quad (31)$$

Thus,  $V_0^{\frac{1}{m}}(z, w) \geq \frac{c_2}{\underline{\alpha}(\gamma)} \|[x^\top, e^\top]^\top\|_{\mathbf{r}}$ . Hence, since  $V_0(z, w) = V(x, e)$ ,

$$\dot{V} \leq -\frac{1}{2} c_0 V^{\frac{m-1}{m}}(x, e) \quad \text{if} \quad \|[x^\top, e^\top]^\top\|_{\mathbf{r}} \geq \frac{2c_1 \bar{d} \underline{\alpha}(\gamma)}{c_0 \bar{c}_4 \gamma} = \frac{2c_1 \bar{d}}{c_0 \bar{c}_4} \frac{1}{\hat{\alpha}(\gamma)}, \quad (32)$$

where  $\hat{\alpha}$  is a function of class  $\mathcal{K}_\infty$  given by

$$\hat{\alpha}(\gamma) = \begin{cases} \gamma^{\frac{n}{n+1}}, & \gamma \leq 1, \\ \gamma^{\frac{1}{2}}, & \gamma > 1. \end{cases} \quad (33)$$

Therefore, there exists a globally attractive neighbourhood of the origin  $B$ , whose radius depends in inverse proportion to  $\gamma$ . Moreover, there is a  $\gamma \in \mathbb{R}_{>0}$  such that  $B \subset \mathcal{D}$ , with  $\mathcal{D}$  as in Theorem 1.

### 4.4 Proof of Corollary 3

Since we are now considering the noisy output (13), the observation error dynamics is given by

$$\begin{aligned} \dot{e}_i &= -\gamma^{p_i} k_i [e_1 - \nu]^{q_i} + e_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{e}_n &= -\gamma^{p_n} k_n [e_1 - \nu]^{q_n} + e_{n+1} - \tilde{f}(t, x, x + \bar{e}), \\ \dot{e}_{n+1} &= -\gamma k_{n+1} [e_1 - \nu]^0 - \dot{\delta}(t). \end{aligned} \quad (34)$$

Thus, the proof of the corollary consists in verifying the existence of a local ISS-Lyapunov function for the closed-loop (10), (34). First, let us define the auxiliary system

$$\begin{aligned} \dot{w}_i &= -k_i [w_1 - \nu]^{q_i} + w_{i+1}, \quad i = 1, \dots, n, \\ \dot{w}_{n+1} &= -k_{n+1} [w_1 - \nu]^0 - \dot{\delta}(t). \end{aligned} \quad (35)$$

Note that for  $\nu \equiv 0$ , (35) equals (7). Now, we analyse the interconnection of the auxiliary systems (19), (35).

We have already mentioned in the proof of Theorem 1 that (19) is ISS with respect to the input  $w$ . Hence, according to [4], there exists an  $\mathbf{r}_c$ -homogeneous function  $V_1 : \mathbb{R}^n \rightarrow \mathbb{R}$  of some degree  $m_c \in \mathbb{R}_{>0}$ , which is an ISS-Lyapunov function for (19). Thus, there exists  $\alpha_1 \in \mathcal{K}$  such that

$$|z| \geq \alpha_1(|w|) \quad \Rightarrow \quad \dot{V}_1 \leq -W_1(z), \quad (36)$$

for some positive definite function  $W_1 : \mathbb{R}^n \rightarrow \mathbb{R}$  which is  $\mathbf{r}_c$ -homogeneous of degree  $m_c + \kappa$ .

In the same manner, according to [30], there exists an  $\mathbf{r}_o$ -homogeneous function  $V_2 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  of some degree  $m_o \in \mathbb{R}_{>0}$ , which is an ISS-Lyapunov function for (35). Hence, there exists  $\alpha_2 \in \mathcal{K}$  such that

$$|w| \geq \alpha_2(N) \quad \Rightarrow \quad \dot{V}_2 \leq -W_2(w), \quad (37)$$

for some positive definite function  $W_2 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  which is  $\mathbf{r}_o$ -homogeneous of degree  $m_o + \kappa$ . Observe that  $m_c$  and  $m_o$  can be assumed such that  $m_c = m_o = m$ , for some  $m \in \mathbb{R}_{>0}$ .

Now, define the function  $V_3 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$  given by  $V_3(z, w) = V_1(z) + V_2(w)$ . Note that  $V_3$  is positive definite and  $\mathbf{r}$ -homogeneous of degree  $m$  with  $\mathbf{r} = [\mathbf{r}_c^\top, \mathbf{r}_o^\top]^\top$ . Also note that  $\dot{V}_3 = \dot{V}_1 + \dot{V}_2$ .

From (36) and (37) we can see that  $|z| \geq \alpha_1(|w|) \geq \alpha_1(\alpha_2(N))$ . Thus, from the properties of class  $\mathcal{K}$  functions (see, e.g. [20]), there exists  $\alpha_3 \in \mathcal{K}$  such that

$$|[z^\top, w^\top]^\top| \geq \alpha_3(N) \quad \Rightarrow \quad \dot{V}_3 \leq -W_1(z) - W_2(w). \quad (38)$$

Observe that the function  $W_3 := W_1 + W_2$  is positive definite and  $\mathbf{r}$ -homogeneous of degree  $m + \kappa$ . Now we can repeat the procedure of the proof of Theorem 1 to deal with the term  $\tilde{f}$  by replacing  $W_0$  with  $W_3$ .

Finally, by using (20), we obtain from (38)

$$|[x^\top, e^\top]^\top| \geq \gamma \alpha_3(N). \quad (39)$$

Hence, for any  $\gamma \in \mathbb{R}_{>0}$  there exists a sufficiently small level of noise  $N$  such that the closed-loop (10)-(11) is ISS inside the domain determined by  $\gamma$ .

## 4.5 Proof of Lemma 1

Let us consider the auxiliary system

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dots, \quad \dot{z}_n = u_0(z). \quad (40)$$

Observe that Assumption 2 guarantees that (40) is finite-time stable, moreover, there exists a smooth Lyapunov function  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that it is  $\mathbf{r}$ -homogeneous of some degree  $m \in \mathbb{R}_{>0}$  and its derivative along (40) satisfies  $\dot{V}_0 = -W_0(z)$ , where the function  $W_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is  $\mathbf{r}$ -homogeneous of degree  $m + \kappa$ , strictly positive for all  $z \in \mathbb{R}^n \setminus \{0\}$  and continuous for all  $z \in \mathbb{R}^n$  [29]. Now, consider the change of coordinates  $\bar{x}_i = \gamma z_i$ ,  $i = 1, \dots, n$ , and the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , given by  $V(\bar{x}) = V_0(\gamma^{-1}\bar{x})$ . The derivative of  $V$  along (14) is such

that  $\dot{V} = \frac{\partial V(\bar{x})}{\partial \bar{x}} \dot{\bar{x}} = \frac{\partial V_0(z)}{\partial z} \frac{1}{\gamma} \dot{z} = -W(\bar{x})$  where  $W(\bar{x}) = W_0(\gamma^{-1}\bar{x})$ . Hence, the derivative of  $V$  along (15) is given by

$$\dot{V} = -W(\bar{x}) + \frac{\partial V(\bar{x})}{\partial \bar{x}_n} g(t) = -W_0(z) + \frac{\partial V_0(z)}{\partial z_n} \frac{1}{\gamma} g(t) \leq -W_0(z) + \left| \frac{\partial V_0(z)}{\partial z_n} \right| \frac{1}{\gamma} \bar{g}.$$

Since  $W_0$  and the function  $z \mapsto \left| \frac{\partial V_0(z)}{\partial z_n} \right|$  are homogeneous of degrees  $m-1$  and  $m-r_n = m-2$ , respectively, Lemma 3 (see Appendix) ensures the existence of  $c_0, c_1 \in \mathbb{R}_{>0}$  such that

$$\dot{V} \leq -c_0 V_0^{\frac{m-1}{m}}(z) + c_1 V_0^{\frac{m-2}{m}}(z) \frac{1}{\gamma} \bar{g}.$$

Since  $-c_0 V_0^{\frac{m-1}{m}}(z) + c_1 V_0^{\frac{m-2}{m}}(z) \frac{\bar{g}}{\gamma} = -(1-\mu)c_0 V_0^{\frac{m-1}{m}}(z) + c_1 V_0^{\frac{m-2}{m}}(z) \frac{\bar{g}}{\gamma} - \mu c_0 V_0^{\frac{m-1}{m}}(z)$  for any  $\mu \in (0, 1)$ , we have that,  $\dot{V} \leq -(1-\mu)c_0 V_0^{\frac{m-1}{m}}(z)$  if  $c_1 V_0^{\frac{m-2}{m}}(z) \frac{\bar{g}}{\gamma} \leq \mu c_0 V_0^{\frac{m-1}{m}}(z)$ . This condition is equivalent to  $\frac{c_1 \bar{g}}{\gamma \mu c_0} \leq V_0^{\frac{1}{m}}(z)$ .

Now, Lemma 3 (see Appendix) ensures the existence of  $c_2 \in \mathbb{R}_{>0}$  such that  $V_0^{\frac{1}{m}}(z) \geq c_2 \|z\|_{\mathbf{r}}$ , but Lemma 4 (see Appendix) ensures that

$$\|z\|_{\mathbf{r}} = \|\gamma^{-1}\bar{x}\|_{\mathbf{r}} \geq \frac{1}{\underline{\alpha}(\gamma)} \|\bar{x}\|_{\mathbf{r}},$$

where the function  $\underline{\alpha} \in \mathcal{K}_{\infty}$  is given by (31). Thus,  $V_0^{\frac{1}{m}}(z) \geq \frac{c_2}{\underline{\alpha}(\gamma)} \|\bar{x}\|_{\mathbf{r}}$ . Hence, a sufficient condition to guarantee the inequality  $\frac{c_1 \bar{g}}{\gamma \mu c_0} \leq V_0^{\frac{1}{m}}(z)$  is that  $\frac{c_1 \bar{g}}{\gamma \mu c_0} \leq \frac{c_2}{\underline{\alpha}(\gamma)} \|\bar{x}\|_{\mathbf{r}}$ . Equivalently (since  $V_0(\gamma^{-1}\bar{x}) = V(x)$ ),

$$\dot{V} \leq -(1-\mu)c_0 V^{\frac{m-1}{m}}(\bar{x}) \quad \text{if} \quad \|\bar{x}\|_{\mathbf{r}} \geq \frac{c_1 \bar{g}}{\mu c_0 c_2} \frac{\underline{\alpha}(\gamma)}{\gamma} = \frac{c_1}{\mu c_0 c_2} \frac{\bar{g}}{\hat{\alpha}(\gamma)}, \quad (41)$$

where  $\hat{\alpha}$  is a function of class  $\mathcal{K}_{\infty}$  given by (33). According to [4], (41) guarantees the existence of a function  $\beta_1 \in \mathcal{KL}$  and a function  $\bar{\beta}_2 \in \mathcal{K}_{\infty}$  such that

$$|\bar{x}(t)| \leq \beta_1(|\bar{x}(0)|, t) + \bar{\beta}_2\left(\frac{\bar{g}}{\hat{\alpha}(\gamma)}\right), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Finally, we take  $\beta_2(\bar{g}, \gamma) = \bar{\beta}_2\left(\frac{\bar{g}}{\hat{\alpha}(\gamma)}\right)$ .

## 5 Uncertain control coefficient

In this section we consider (1) with a time-varying uncertain control coefficient, i.e.

$$\dot{x}_i = x_{i+1}, \quad \dot{x}_n = f(t, x) + b(t)u + \delta(t), \quad b(t) \in \mathbb{R}_{>0}. \quad (42)$$

We assume that a model  $\bar{b} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$  for  $b$  is known. We also assume that  $|b(t)| \geq \underline{b}$  and  $|\bar{b}(t)| \geq \underline{b}$  for all  $t \in \mathbb{R}_{\geq 0}$  for some constant  $\underline{b} \in \mathbb{R}_{>0}$ . The output-feedback controller (5) is now modified as follows

$$u = (1/\bar{b}) \left( \gamma u_0(\gamma^{-1}\hat{x}) - \bar{f}(t, \hat{x}) - \hat{\delta} \right), \quad (43)$$

where the observed state  $\hat{x}$  and the disturbance estimation  $\hat{\delta}$  are provided by (6).

**Remark 5.** Note that in (43), two important cases are covered:

(i) the case of constant and uncertain coefficient  $b$ ;

(ii) the case of disturbances coupled to the input, i.e.  $\dot{x}_n = f(t, x) + b(t)(u + \bar{\delta}(t))$ , or equivalently,  $\delta(t) = b(t)\bar{\delta}(t)$ .  $\triangleleft$

Let us introduce the following auxiliary system

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{z}_n &= u_0(z + \bar{w}) - w_{n+1} + \tilde{c}(u_0(z + \bar{w}) - w_{n+1}), \\ \dot{w}_i &= -k_i [w_1]^{q_i} + w_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{w}_n &= -k_n [w_1]^{q_n} + w_{n+1} - \tilde{c}(u_0(z + \bar{w}) - w_{n+1}), \\ \dot{w}_{n+1} &= -k_{n+1} [w_1]^0 - \dot{\tilde{\delta}}(t), \end{aligned} \tag{44}$$

where  $\tilde{c} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a continuous function, and  $\bar{w} := [w_1, \dots, w_n]^\top$ . Note that for  $\tilde{c} = 0$ , (44) equals the nominal closed-loop (19), (7). The following lemma establishes a robustness property of (19), (7).

**Lemma 2.** *If assumptions 2 and 3 hold, then there exists  $\tilde{C}_0 \in \mathbb{R}_{>0}$  such that the origin of (44) is asymptotically stable for all  $\tilde{c}$  satisfying  $|\tilde{c}(t)| \leq \tilde{C}_0$  for all  $t \in \mathbb{R}_{\geq 0}$ .*  $\triangleleft$

*Proof.* We use the same arguments given in the proof of Theorem 1 to assure that there exists a smooth Lyapunov function  $V_0 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$  for (19), (7) such that it is  $\mathbf{r}$ -homogeneous of some degree  $m \in \mathbb{R}_{>0}$  and its derivative along (44) satisfies  $\dot{V}_0 \leq -W_0(z, w)$ , where  $W_0 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}_{\geq 0}$  is  $\mathbf{r}$ -homogeneous of degree  $m + \kappa$ , strictly positive for all  $[z^\top, w^\top]^\top \in \mathbb{R}^{2n+1} \setminus \{0\}$  and continuous for all  $[z^\top, w^\top]^\top \in \mathbb{R}^{2n+1}$ . Therefore, according to Lemma 5 (see Appendix), there exists  $\tilde{C}_0 \in \mathbb{R}_{>0}$  such that  $V_0$  is a Lyapunov function for (44) for all  $\tilde{c}$  such that  $|\tilde{c}(t)| \leq \tilde{C}_0$ .  $\square$

Note that the method to prove Lemma 2 provides, in general, a conservative bound  $\tilde{C}_0$  to verify asymptotic stability of the origin of (44): the proof is based on Lemma 5 (see Appendix) whose proof uses a quite restrictive bound for the derivative of the Lyapunov function; moreover, in such a general setting, it is not possible to choose the Lyapunov function that provides the largest bound  $\tilde{C}_0$ .

The main result of this section is that (43) does not destroy the robustness property of the nominal closed-loop stated in Lemma 2. Nonetheless, (43) retains the same stability properties described in Theorem 1. To state this result, let us introduce an assumption and some notation.

**Assumption 4.** *Consider (42) and (43) and define  $c(t) := b(t)/\bar{b}(t)$ . Functions  $b$  and  $\bar{b}$  are such that  $\tilde{c}(t) := c(t) - 1$ , satisfies  $|\tilde{c}(t)| \leq \tilde{C}_0$  for all  $t \in \mathbb{R}_{\geq 0}$ , with  $\tilde{C}_0$  as given in the proof of Lemma 2.*  $\triangleleft$

Observe that Assumption 4 can be intuitively understood as a requirement of how well the model  $\bar{b}$  must approximate  $b$  in order to maintain the stability properties of the auxiliary system (44).



The closed-loop of (42) with (43) is given by

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= c\gamma u_0(\gamma^{-1}\hat{x}) + f(t, x) - c\bar{f}(t, \hat{x}) + \delta - c\hat{\delta}.\end{aligned}\quad (45)$$

Note that the last equation of (45) can be rewritten as follows

$$\dot{x}_n = \gamma u_0(\gamma^{-1}\hat{x}) - [\hat{\delta} - \delta/c] + \tilde{f}(t, x, \hat{x}) + \tilde{c}(\gamma u_0(\gamma^{-1}\hat{x}) - [\hat{\delta} - \delta/c] - \bar{f}(t, \hat{x})), \quad (46)$$

where  $\tilde{f}(t, x, \hat{x}) := f(t, x) - \bar{f}(t, \hat{x})$ . Consider the observation errors defined in (9) with the following modification

$$e_{n+1} = \hat{\delta} - \delta/c. \quad (47)$$

Thus, (45) can be rewritten as

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{x}_n &= \gamma u_0(\gamma^{-1}[x + \bar{e}]) - e_{n+1} + \tilde{f}(t, x, x + \bar{e}) + \\ &\quad \tilde{c}(\gamma u_0(\gamma^{-1}[x + \bar{e}]) - e_{n+1} - \bar{f}(t, x + \bar{e})),\end{aligned}\quad (48)$$

where  $\bar{e} := [e_1, \dots, e_n]^\top$ , and the observation error dynamics is given by

$$\begin{aligned}\dot{e}_i &= -\gamma^{p_i} k_i [e_1]^{q_i} + e_{i+1}, \quad i = 1, \dots, n-1, \\ \dot{e}_n &= -\gamma^{p_n} k_n [e_1]^{q_n} + e_{n+1} - \tilde{f}(t, x, x + \bar{e}) - \\ &\quad \tilde{c}(\gamma u_0(\gamma^{-1}[x + \bar{e}]) - e_{n+1} - \bar{f}(t, x + \bar{e})), \\ \dot{e}_{n+1} &= -\gamma k_{n+1} [e_1]^0 - \frac{d}{dt}(\delta/c).\end{aligned}\quad (49)$$

**Theorem 3.** Consider (42) in closed-loop with the output-feedback controller (43), (6). Assume that  $|\frac{d}{dt}(\delta(t)/c(t))| \leq \Delta_c$  for all  $t \in \mathbb{R}_{\geq 0}$  for a constant  $\Delta_c \in \mathbb{R}_{\geq 0}$ . Under assumptions 1-4, for any  $a, \Delta_c \in \mathbb{R}_{\geq 0}$  there exists  $\gamma \in \mathbb{R}_{> 0}$  such that the origin of (48)-(49) is finite-time stable with a domain of attraction  $\mathcal{D}$  such that  $\{X \in \mathbb{R}^{2n+1} : \|X\|_{\mathbf{r}} \leq a\} \subset \mathcal{D}$ .

*Proof.* The proof of this theorem is analogous to the proof of Theorem 1. It is only required to consider the proof of Lemma 2.  $\square$

**Remark 6.** Recall that, for the case of perfectly known control coefficient, the signal  $\delta$  is allowed to be unbounded with bounded derivative (see item (iii) in Assumption 1). However, for the case of uncertain coefficient, the condition  $|\frac{d}{dt}(\delta(t)/c(t))| \leq \Delta_c$  in Theorem 3, in general requires boundedness of the disturbance  $\delta$ , but observe that this restriction still allows a wide class of disturbances  $\delta$ , for example, constant and sinusoid signals. Nonetheless, a very interesting case is when the disturbance is coupled to the input (see Remark 5). For such a case the external disturbance is the signal  $\bar{\delta}$ , and  $\delta(t) = b(t)\bar{\delta}(t)$ . Hence, the condition  $|\frac{d}{dt}(\delta(t)/c(t))| \leq \Delta_c$  is rewritten as  $|\frac{d}{dt}(\delta(t)/c(t))| = |\frac{d}{dt}(\bar{b}(t)\bar{\delta}(t))| \leq \Delta_c$ , which is equivalent to ask the product  $\bar{b}\bar{\delta}$  being of the form  $\bar{b}(t)\bar{\delta}(t) = a + \int_0^t \alpha(\tau) d\tau$  for some constant  $a \in \mathbb{R}$  and some integrable function  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $|\alpha(t)| \leq \Delta_c$  for all  $t \in \mathbb{R}_{\geq 0}$  [25]. Thus, for this case, the disturbance  $\bar{\delta}$  is allowed to be unbounded with bounded derivative, and the controller exactly compensates it despite the uncertainty in the control coefficient, this property is illustrated in Example 1.  $\triangleleft$

$k_1$	$k_2$	$k_3$	$x_1(0)$	$x_2(0)$	$\hat{x}_1(0)$	$\hat{x}_2(0)$	$\hat{\delta}(0)$
3.1	5.7	1.1	5	5	0	0	0

Table 1: Observer gains and initial conditions.

## 6 Examples

In this section we show some examples to illustrate the results provided in previous sections. For the simulations we use the explicit Euler method with an integration step of 1ms.

**Example 1.** Consider the following second order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(t, x) + b(t)(u + \bar{\delta}), \quad f(t, x) = 1.1(\sin(x_1) + |x_2|), \quad (50)$$

The model for  $f$  considered in this example is  $\bar{f}(t, x) = \sin(x_1) + x_2$ , and the external disturbance  $\bar{\delta}(t) = 1 + \cos(t) + t/5$ . First, we consider the control coefficient  $b(t) = 1$ , hence,  $\delta(t) = \bar{\delta}(t)$ . For this case, controller (5), (6) is as follows

$$\begin{aligned} u &= \gamma u_0(\gamma^{-1} \hat{x}) - \bar{f}(t, \hat{x}) - \hat{\delta}, \\ \dot{\hat{x}}_1 &= -\gamma^{\frac{1}{3}} k_1 [\hat{x}_1 - x_1]^{\frac{2}{3}} + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\gamma^{\frac{2}{3}} k_2 [\hat{x}_1 - x_1]^{\frac{1}{3}} + \gamma u_0(\gamma^{-1} \hat{x}), \\ \dot{\hat{\delta}} &= -\gamma k_3 [\hat{x}_1 - x_1]^0. \end{aligned}$$

For this example we choose  $u_0$  given by (3), namely,  $u_0(x) = -2[|x_2|^{\frac{3}{2}} + x_1]^{\frac{1}{3}}$ . For the simulation we use the parameters shown in Table 1. First we set  $\gamma = 2$ . In Fig. 1 we observe that the states of the observer converge in finite-time to the states of plant. In Fig. 2 we can see that the extended state of the observer identifies exactly the disturbance  $\delta$ .

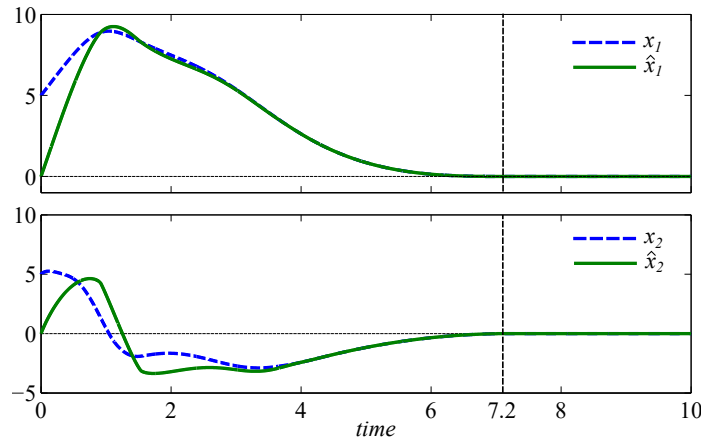


Figure 1: Original and observed states of (50) with  $\gamma = 2$ .

Now we set  $\gamma = 4$ . In Fig. 3 and Fig. 4 we observe that the convergence time is reduced in comparison with the simulation with  $\gamma = 2$ .

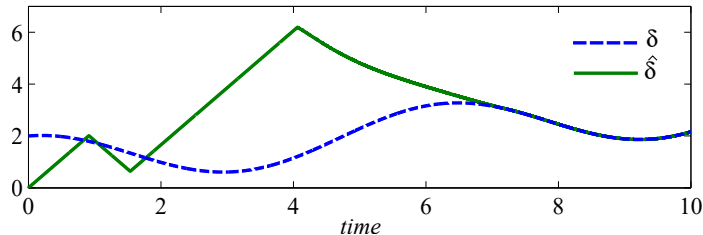


Figure 2: Extended state of the observer with  $\gamma = 2$ .

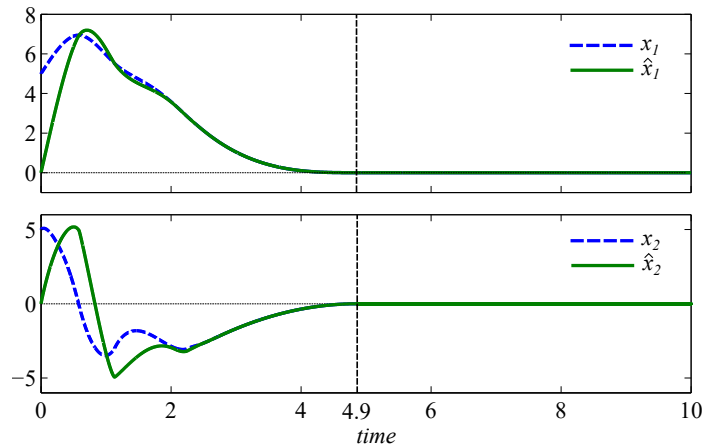


Figure 3: Original and observed states of (50) with  $\gamma = 4$ .

Now, we consider the case with twenty percent of uncertainty in the control coefficient, namely  $b(t) = 1 + \frac{1}{5} \sin(5t)$ . Recall that  $\delta(t) = b(t)\bar{\delta}(t)$ . The model for the control coefficient is taken as  $\bar{b} = 1$ , thus  $c(t) = b(t)$ . Hence,  $|\frac{d}{dt} \frac{\delta(t)}{c(t)}| < \frac{6}{5}$ . We use the same controller as above, with  $\gamma = 2$ .

In Fig. 5 we observe that the states of the observer converge in finite-time to the states of the plant, and they converge to zero also in finite-time despite the disturbance and the uncertainties. It is important to highlight the fact that the disturbance  $\bar{\delta}$  is unbounded, but it is still compensated by the controller even when the control coefficient is uncertain, as guaranteed by Theorem 3.

To finalise this example, we consider the presence of noise in the measured output. Thus, for (13) we use  $\nu(t) = \frac{1}{20} \text{sign}(\sin(120\pi t))$ . We use the same disturbance, and the same control uncertain control coefficient  $b(t) = 1 + \frac{1}{5} \sin(5t)$ . In Fig. 6 the states of plant are shown. Observe that the states remain bounded close to the origin, making evident the robustness properties of the closed-loop. Note that this example was very challenging since we are considering an unstable plant with uncertainty in the nonlinear terms of the model, uncertainty in the control coefficient, external disturbances, and the presence of noise in the measurement.  $\triangleleft$

**Example 2.** In this example we consider the disturbed double integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u + \delta, \quad (51)$$

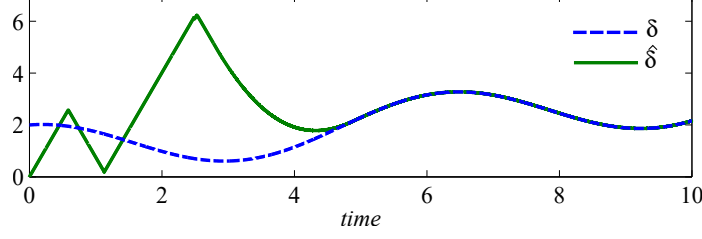


Figure 4: Extended state of the observer with  $\gamma = 4$ .

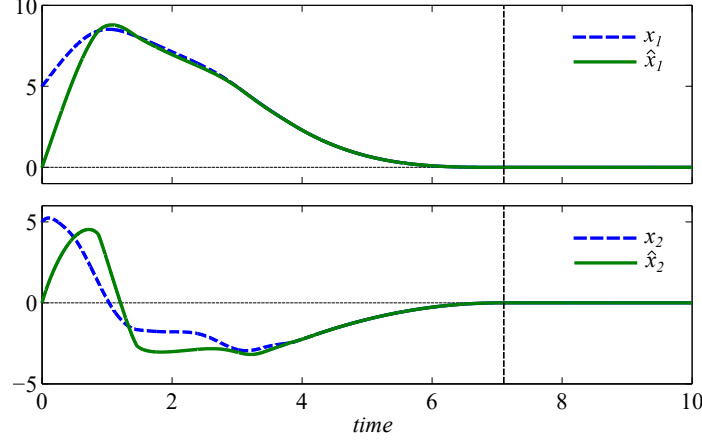


Figure 5: Original and observed states of (50) with uncertain control coefficient.

With the output  $y$  contaminated with noise, i.e.  $y = x_1 + \nu$ . The disturbance is chosen as  $\delta(t) = 1 + \cos(3t)$ . Our proposed controller (nonlinear-homogeneous) is

$$\begin{aligned}
 u &= \gamma u_0(\gamma^{-1} \hat{x}) - \hat{\delta}, \\
 \dot{\hat{x}}_1 &= -\gamma^{\frac{1}{3}} k_1 [\hat{x}_1 - x_1]^{\frac{2}{3}} + \hat{x}_2, \\
 \dot{\hat{x}}_2 &= -\gamma^{\frac{2}{3}} k_2 [\hat{x}_1 - x_1]^{\frac{1}{3}} + \gamma u_0(\gamma^{-1} \hat{x}), \\
 \dot{\hat{\delta}} &= -\gamma k_3 [\hat{x}_1 - x_1]^0,
 \end{aligned} \tag{52}$$

with  $u_0(x) = -[[x_2]^{\frac{3}{2}} + x_1]^{\frac{1}{3}}$  and  $\gamma = 4$ . The observer gains are in Table 1.

We also apply the control scheme proposed in [13] which is based on an extended high-gain observer and is given by  $u = M \text{sat}(\bar{u}/M)$ , where  $\text{sat}$  is the standard saturation function and

$$\begin{aligned}
 \bar{u} &= u_0(\hat{x}) - \hat{\delta}, \\
 \dot{\hat{x}}_1 &= -\gamma k_1 (\hat{x}_1 - x_1) + \hat{x}_2, \\
 \dot{\hat{x}}_2 &= -\gamma^2 k_2 (\hat{x}_1 - x_1) + u_0(\hat{x}), \\
 \dot{\hat{\delta}} &= -\gamma^3 k_3 (\hat{x}_1 - x_1),
 \end{aligned}$$

with  $u_0(x) = -x_1 - x_2$  and  $M = 10$ . To achieve a steady-state observation error (for  $\nu = 0$ ) such that  $|\delta(t) - \hat{\delta}(t)| \leq 0.5$ , we use the gain  $\gamma = 60$ .

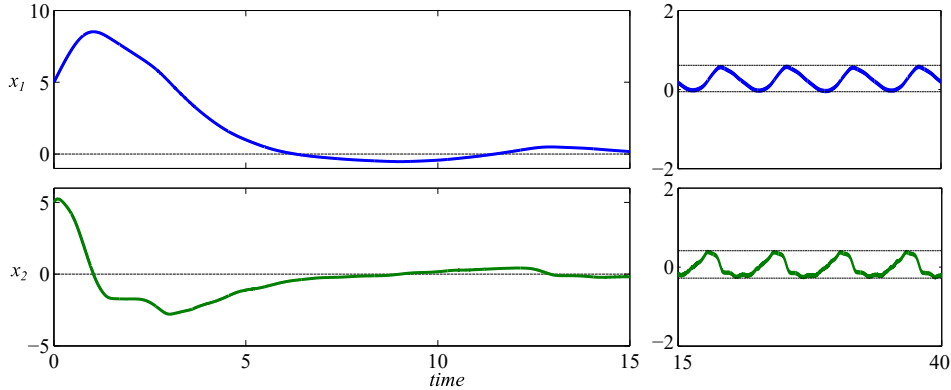


Figure 6: Transient and steady-state phases of (50) with noise in the output.

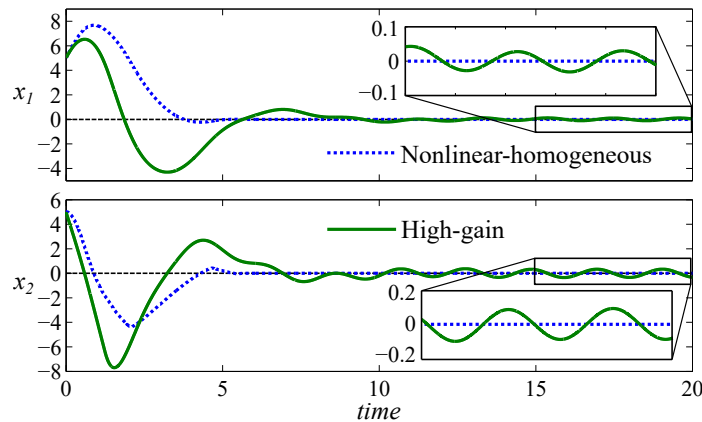


Figure 7: States of (51) with the nonlinear-homogeneous and the high-gain output-feedback controllers ( $\nu = 0$ ).

For the simulation we use the parameters shown in Table 1. Fig. 7 shows the states of (51) with the nonlinear and the linear controllers, respectively, for  $\nu = 0$ . The clear advantage of the nonlinear scheme is that the states are driven to zero in finite-time. The control signals are shown in Fig. 8. There, we can see that the nonlinear homogeneous controller has less amplitude than the high-gain output-feedback, which reaches its saturation level.

Now we simulate the two different closed-loops by using a noise signal  $\nu = (1 + \sin(\omega t))/10$  with  $\omega = 5$ . In this case the steady-state performance of the linear scheme seems to be better than the nonlinear one as it can be appreciated in Fig. 9 and Fig. 10. However, with  $\omega = 50$ , although the amplitude of the steady-state errors are comparable with both controllers, see Fig. 11, the transient behaviour with the linear scheme is clearly deteriorated. Nonetheless, a major issue with the linear scheme is the control signal that exhibits a high-frequency oscillation between the saturation values of the controller, see Fig. 12.

It is important to mention that this is a simple comparison example and no general conclusion can be established from it, thus, a deeper study of comparison between these two control techniques is required.  $\triangleleft$

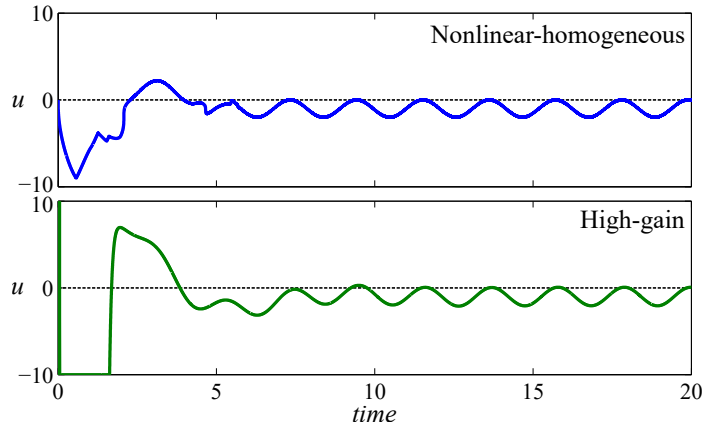


Figure 8: Control signals of the nonlinear-homogeneous and the high-gain output-feedback controllers ( $\nu = 0$ ).

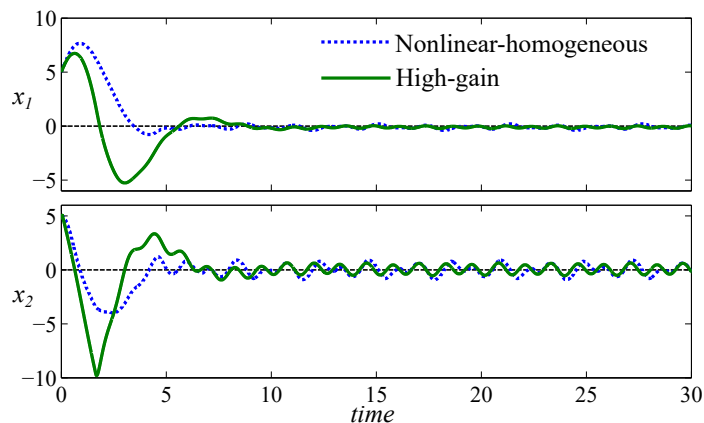


Figure 9: States of the system with the nonlinear-homogeneous and the high-gain output-feedback controllers in presence of noise ( $\omega = 5$ ).

## 7 Conclusions

Through the DOBC strategy, we have proposed an output-feedback controller for a class of uncertain nonlinear systems. Such a controller provides an exact compensation of the disturbance in finite-time despite the uncertainty in the model. The scheme is a simple but fruitful application of: 1) ISS robustness properties of homogeneous controllers and; 2) finite-time and exactness of the sliding mode differentiator. The control scheme allows to recover in finite-time the nominal controlled system. From this point of view, many of the performance features of the nominal state feedback controller are unaffected by the output-feedback. The scheme allows us to design the gains of the controller and the observer independently, and only one parameter is used to adjust the closed-loop. Since the attraction domain of the closed-loop is not the whole state space, a direction of future work is to try to improve the scheme to obtain a global controller.

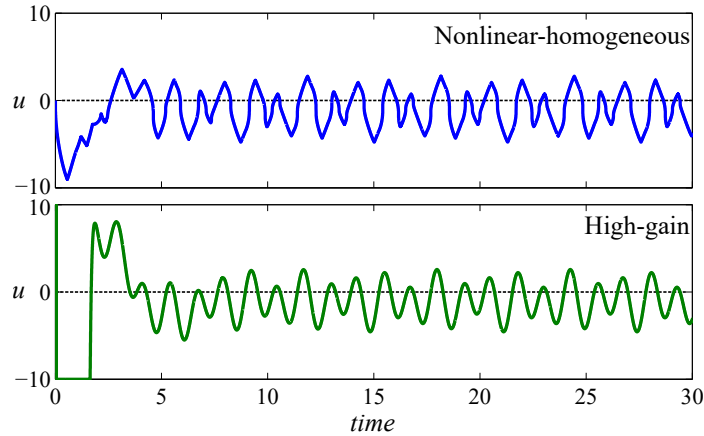


Figure 10: Control signals of the nonlinear-homogeneous and the high-gain output-feedback controllers in presence of noise ( $\omega = 5$ ).

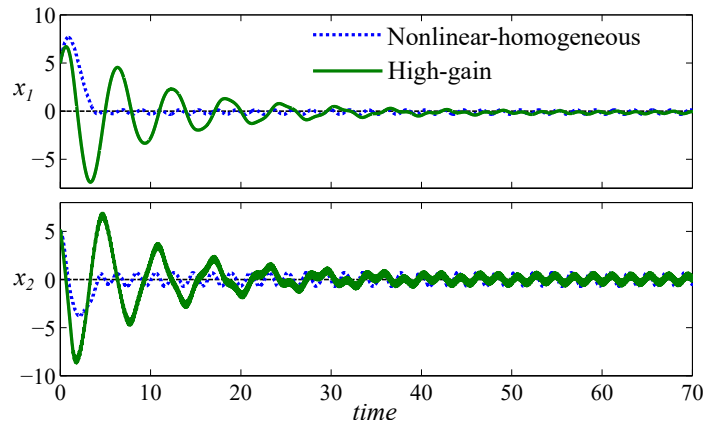


Figure 11: States of the system with the nonlinear-homogeneous and the high-gain output-feedback controllers in presence of noise ( $\omega = 50$ ).

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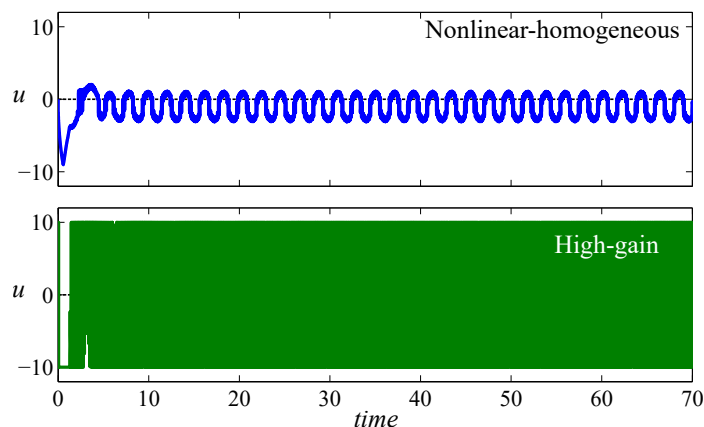


Figure 12: Control signals of the nonlinear-homogeneous and the high-gain output-feedback controllers in presence of noise ( $\omega = 50$ ).

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## A Homogeneity

We recall briefly the definitions of Weighted Homogeneity, see e.g. [2].

**Definition 1.** Let  $\Lambda_\epsilon^{\mathbf{r}} = \text{diag}(\epsilon^{r_1}, \dots, \epsilon^{r_n})$  be a square diagonal matrix where  $\mathbf{r} = [r_1, \dots, r_n]^\top$ ,  $r_i \in \mathbb{R}_{>0}$ , and  $\epsilon \in \mathbb{R}_{>0}$ . The components of  $\mathbf{r}$  are called the weights of the coordinates. A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\mathbf{r}$ -homogeneous of degree  $m \in \mathbb{R}$  if  $f(\Lambda_\epsilon^{\mathbf{r}}x) = \epsilon^m f(x)$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall \epsilon \in \mathbb{R}_{>0}$ . The vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $f = [f_1(x), \dots, f_n(x)]^\top$ , is  $\mathbf{r}$ -homogeneous of degree  $\kappa \in \mathbb{R}$  if  $f_i(\Lambda_\epsilon^{\mathbf{r}}x) = \epsilon^{\kappa+r_i} f_i(x)$ ,  $i = 1, 2, \dots, n$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall \epsilon \in \mathbb{R}_{>0}$ . A dynamic system  $\dot{x} = f(x)$ ,  $x \in \mathbb{R}^n$ , is  $\mathbf{r}$ -homogeneous of degree  $\kappa$  if  $f$  is  $\mathbf{r}$ -homogeneous of degree  $\kappa$ .  $\triangleleft$

**Definition 2.** (See e.g. [27, 23, 3]) A vector-set field  $F \subset \mathbb{R}^n$  is  $\mathbf{r}$ -homogeneous of degree  $\kappa$  if the identity  $F(\Lambda_\epsilon^{\mathbf{r}}x) = \epsilon^\kappa \Lambda_\epsilon^{\mathbf{r}} F(x)$  holds for any  $\epsilon \in \mathbb{R}_{>0}$ , this is equivalent to the invariance of the differential inclusion  $\dot{x} \in F(x)$  with respect to the transformation  $G_\epsilon : (t, x) \mapsto (\epsilon^{-\kappa}t, \Lambda_\epsilon^{\mathbf{r}}x)$ . A differential inclusion is  $\mathbf{r}$ -homogeneous of degree  $\kappa$  if its vector-set field  $F$  is  $\mathbf{r}$ -homogeneous of degree  $\kappa$ .  $\triangleleft$

**Definition 3** (See e.g. [2]). Given a vector of weights  $\mathbf{r}$ , a  $\mathbf{r}$ -homogeneous norm is defined as a function from  $\mathbb{R}^n$  to  $\mathbb{R}$ , and given by  $\|x\|_{\mathbf{r},p} = \left( \sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}$ ,  $\forall x \in \mathbb{R}^n$ , for any  $p \geq 1$ .  $\triangleleft$

Note that any  $\mathbf{r}$ -homogeneous norm is a  $\mathbf{r}$ -homogeneous function of degree  $m = 1$ . Since, for a given  $\mathbf{r}$ , the  $\mathbf{r}$ -homogeneous norms are equivalent [19], they are usually denoted as  $\|\cdot\|_{\mathbf{r}}$ , without the specification of  $p$ .

**Lemma 3** (See, e.g. [16]). *Let  $v_1$  and  $v_2$  be real functions, continuous in  $\mathbb{R}^n$  and  $\mathbf{r}$ -homogeneous of degree  $m_1, m_2$ , respectively, and  $v_1$  be positive definite. Then, for any  $x \in \mathbb{R}^n$ ,*

$$c_1 v_1^{\frac{m_2}{m_1}}(x) \leq v_2(x) \leq c_2 v_1^{\frac{m_2}{m_1}}(x),$$

where  $c_1 = \min_{x \in E} v_2(x)$ ,  $c_2 = \max_{x \in E} v_2(x)$ , and  $E = \{x \in \mathbb{R}^n : v_1(x) = 1\}$ .  $\triangleleft$

**Lemma 4.** *For  $s \in \mathbb{R}^m$ , and  $a \in \mathbb{R}_{>0}$ ,*

$$\frac{1}{\underline{\alpha}(a)} \|s\|_{\mathbf{r}} \leq \|\frac{1}{a}s\|_{\mathbf{r}} \leq \frac{1}{\alpha(a)} \|s\|_{\mathbf{r}},$$

where  $\alpha, \underline{\alpha}$  are functions of class  $\mathcal{K}_\infty$  given by

$$\alpha(a) = \begin{cases} a^{\frac{1}{\min_i r_i}}, & a \leq 1, \\ a^{\frac{1}{\max_i r_i}}, & a > 1, \end{cases} \quad \underline{\alpha}(a) = \begin{cases} a^{\frac{1}{\max_i r_i}}, & a \leq 1, \\ a^{\frac{1}{\min_i r_i}}, & a > 1. \end{cases}$$

$\triangleleft$

*Proof.* The proof is straightforward from the definition of homogeneous norm, namely,

$$\|\frac{1}{a}s\|_{\mathbf{r}} = \left( \sum_{i=1}^m |\frac{1}{a}s_i|^{p/r_i} \right)^{1/p} \leq \begin{cases} (1/a)^{\frac{1}{\min_i r_i}} \|s\|_{\mathbf{r}}, & 1/a \geq 1, \\ (1/a)^{\frac{1}{\max_i r_i}} \|s\|_{\mathbf{r}}, & 1/a < 1, \end{cases}$$

hence,

$$\|\frac{1}{a}s\|_{\mathbf{r}} \leq \begin{cases} (1/a)^{\frac{1}{\min_i r_i}} \|s\|_{\mathbf{r}}, & a \leq 1, \\ (1/a)^{\frac{1}{\max_i r_i}} \|s\|_{\mathbf{r}}, & a > 1. \end{cases}$$

The proof for the lower bound is analogous.  $\square$

**Lemma 5.** *Consider the  $\mathbf{r}$ -homogeneous system  $\dot{x} = f(x)$  of degree  $\kappa \in \mathbb{R}$  whose origin is asymptotically stable. If  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an  $\mathbf{r}$ -homogeneous vector field of degree  $\kappa \in \mathbb{R}$ , then there exists  $C \in \mathbb{R}_{>0}$  such that the origin of the system  $\dot{x} = f(x) + c(t)g(x)$  is asymptotically stable for any continuous function  $c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  such that  $|c(t)| \leq C$  for all  $t \in \mathbb{R}_{\geq 0}$ . Moreover, if  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is an  $\mathbf{r}$ -homogeneous Lyapunov function (of some degree  $m \in \mathbb{R}_{>0}$ ) for  $\dot{x} = f(x)$ , then  $V$  is also a Lyapunov function for  $\dot{x} = f(x) + c(t)g(x)$ , and there exists an  $\mathbf{r}$ -homogeneous function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  of degree  $m_w := m + \kappa$ , which is continuous and positive definite, such that  $\frac{\partial V(x)}{\partial x}(f(x) + c(t)g(x)) \leq -W(x)$ .  $\triangleleft$*

*Proof.* The derivative of  $V$  along the solutions of  $\dot{x} = f(x)$  is such that  $\dot{V} \leq -W_0(x)$  for some continuous and positive definite function  $W_0$  which is  $\mathbf{r}$ -homogeneous of degree  $m_w$  [3]. Now, the derivative of  $V$  along the solutions of  $\dot{x} = f(x) + c(t)g(x)$  is such that

$$\dot{V} \leq -W_0(x) + c(t) \frac{\partial V(x)}{\partial x} g(x), \tag{53}$$

hence,

$$\dot{V} \leq -W_0(x) + |c(t)| \tilde{W}(x), \quad \tilde{W}(x) := \left| \frac{\partial V(x)}{\partial x} g(x) \right|. \tag{54}$$

Note that  $\tilde{W}$  is positive semidefinite and  $\mathbf{r}$ -homogeneous of degree  $m_w$ . From Lemma 3 there exist  $\epsilon_0, \epsilon_1 \in \mathbb{R}_{>0}$  such that

$$\dot{V} \leq -\epsilon_0 \|x\|_{\mathbf{r}}^{m_w} + |c(t)| \epsilon_1 \|x\|_{\mathbf{r}}^{m_w} \leq -(\epsilon_0 - |c(t)| \epsilon_1) \|x\|_{\mathbf{r}}^{m_w}. \quad (55)$$

Thus,

$$\dot{V} \leq -\epsilon_2 \|x\|_{\mathbf{r}}^{m_w}, \quad (56)$$

for all  $\epsilon_2 \in (0, \epsilon_0)$  such that  $|c(t)| \leq \frac{\epsilon_0 - \epsilon_2}{\epsilon_1}$ . The result follows by taking  $C = \frac{\epsilon_0 - \epsilon_2}{\epsilon_1}$ .  $\square$

Consider the system  $\dot{x} = f(x, d)$ ,  $x(t) \in \mathbb{R}^n$ ,  $d(t) \in \mathbb{R}^m$ , where  $f$  is locally Hölder continuous. Let  $X(t; x(0), d)$  denote the solution of the system for the initial condition  $x(0)$  and for all  $t \in \mathbb{R}_{\geq 0}$ .

**Definition 4** ([33, 1]). *The system is called input-to-state stable (ISS), if for any Lebesgue-measurable and essentially bounded input  $d$  and any  $x(0) \in \mathbb{R}^n$  there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that, for all  $t \in \mathbb{R}_{\geq 0}$ ,*

$$|X(t; x(0), d)| \leq \beta(|x(0)|, t) + \gamma(\text{ess sup}_{\tau \in [0, t]} |d(\tau)|).$$

$\triangleleft$