



HAL
open science

On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces

Pascal Auscher, Hedong Hou

► **To cite this version:**

Pascal Auscher, Hedong Hou. On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces. 2023. hal-04276113v1

HAL Id: hal-04276113

<https://hal.science/hal-04276113v1>

Preprint submitted on 8 Nov 2023 (v1), last revised 29 Jul 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON WELL-POSEDNESS AND MAXIMAL REGULARITY FOR PARABOLIC CAUCHY PROBLEMS ON WEIGHTED TENT SPACES

PASCAL AUSCHER AND HEDONG HOU

ABSTRACT. We prove well-posedness in weighted tent spaces of weak solutions to the Cauchy problem $\partial_t u - \operatorname{div} A \nabla u = f$, $u(0) = 0$, where the source f also lies in (different) weighted tent spaces, provided the complex coefficient matrix A is bounded, measurable, time-independent, and uniformly elliptic. We rectify, streamline, and extend the theory of singular integral operators on tent spaces via off-diagonal estimates introduced by [AKMP12] to obtain estimates on solutions u , and also ∇u , $\partial_t u$, and $\operatorname{div} A \nabla u$ in weighted tent spaces, showing at the same time maximal regularity.

CONTENTS

1. Introduction	2
Organization	6
Notation	6
Acknowledgements	7
2. Singular integral operators	7
2.1. Singular kernels	7
2.2. Integrals associated with singular kernels	9
2.3. Singular integral operators	12
3. Extensions of SIOs to tent spaces	14
3.1. Basic properties of tent spaces	14
3.2. Main results	16
3.3. Proofs	18
4. Inhomogeneous Cauchy problems	23
4.1. Main results	24
4.2. <i>A priori</i> estimates	26
4.3. Existence	28
4.4. Uniqueness	29

Date: November 8, 2023.

2020 Mathematics Subject Classification. Primary 35K45; Secondary 42B37.

Key words and phrases. Parabolic Cauchy problems, maximal regularity, tent spaces, singular integral operators, off-diagonal estimates.

5. Maximal regularity on tent spaces	33
5.1. Maximal regularity operator	33
5.2. Boundary behavior	36
6. Further remarks	38
References	39

1. INTRODUCTION

In this work, we develop maximal regularity on tent spaces and prove well-posedness for the corresponding evolution equation.

To set the stage, let us start with the usual theory of maximal regularity. See [KW04] for a detailed exposition of this theme. Let X be a Banach space and L be a generator of a bounded analytic semigroup on X , denoted by $(e^{-tL})_{t \geq 0}$. Let $0 < T < \infty$. Consider the abstract inhomogeneous Cauchy problem on X as

$$(1.1) \quad \begin{cases} \partial_t u(t) + Lu(t) = f(t), & t \in [0, T] \\ u(0) = 0 \end{cases}.$$

Assume that f belongs to $L^1((0, T); X)$. We say $u \in L^1((0, T); X)$ is a *weak solution* of (1.1) in X , if for any $t \in [0, T]$ and $x^* \in D(L^*) \subset X^*$,

$$\langle x^*, u(t) \rangle + \int_0^t \langle L^* x^*, u(s) \rangle ds = \int_0^t \langle x^*, f(s) \rangle ds.$$

The pairing is understood as the action of the dual space X^* on X . It is known (*e.g.*, see [HvNVW17, Theorem G.3.2]) that there is a unique weak solution to (1.1) in X , given by Duhamel's formula

$$(1.2) \quad u(t) = \int_0^t e^{-(t-s)L} f(s) ds.$$

We say L has *maximal L^p -regularity* if for any $f \in L^p((0, T); X)$, this weak solution u satisfies that both $\partial_t u$ and Lu lie in $L^p((0, T); X)$ with

$$\|\partial_t u\|_{L^p((0, T); X)} + \|Lu\|_{L^p((0, T); X)} \lesssim \|f\|_{L^p((0, T); X)}.$$

This property turns out to be tied to the geometry of the underlying Banach space X . Such a property holds if and only if the *maximal regularity operator*

$$(1.3) \quad \mathcal{M}_L(f)(t) := \int_0^t L e^{-(t-s)L} f(s) ds,$$

initially defined on $L^1((0, T); D(L))$ extends to a bounded operator on $L^p((0, T); X)$. When X is a UMD Banach space, the characterization is given by L. Weis [Wei01] using R -boundedness of the semigroup.

When X is an abstract Banach space, this seems the only possible approach. Yet, if X is a space of functions or distributions and L is an unbounded differential operator, other types of solutions and solution spaces than $L^p((0, T); X)$ can also be taken into consideration, not using R -boundedness. For instance, tent spaces are natural candidates due to their strong connection with the harmonic analysis of boundary spaces. The extension of maximal regularity operators to tent spaces has been shown to be useful for studies on non-linear PDE's. For example, when L is the negative Laplacian $-\Delta$, it has been used by the first author and D. Frey [AF17] to provide another approach to the well-known work of H. Koch and D. Tataru [KT01] on Navier–Stokes equations. A recent work of R. Danchin and I. Vasilyev also implements tent spaces for global well-posedness of inhomogeneous incompressible Navier–Stokes equations [DV23]. We wish to extend our understanding by studying in what sense the corresponding evolution equation is considered, and establishing, at the same time, a notion of maximal regularity on tent spaces.

In order to have better understanding of the role of tent spaces, with goal to look at non-linear problems, let us consider more general linear parabolic equations than the heat equation. From now on, denote by

$$L := -\operatorname{div}(A\nabla)$$

with $A \in L^\infty(\mathbb{R}^n; \operatorname{Mat}_n(\mathbb{C}))$ uniformly elliptic. Using the theory of maximal-accretive operators, $-L$ generates an analytic contraction semigroup on $L^2(\mathbb{R}^n)$. Further considerations yield a limited range of q near 2, depending on the operator L , on which the above theory of maximal regularity applies on $X = L^q(\mathbb{R}^n)$ via R -boundedness [BK03].

A series of pioneering works of the first author *et al.* [AMP12, AKMP12] proves boundedness of maximal regularity operators associated with such L on a family of weighted tent spaces T_β^p , completely avoiding the notion of R -boundedness. Precise definitions of weighted tent spaces are referred to Section 3.1. For the time being, it suffices to know that p is an integrability index and $s = 2\beta + 1$ is a regularity index.

In this article, compared to [AKMP12], we streamline the presentation, in particular by adjusting the definition of singular integral operators, and extend the framework to include more operators, such as the one given by the Duhamel's formula, see Section 2.3. Furthermore, given $\beta > -1/2$, we also extend the range of p within which \mathcal{M}_L (already bounded on T_0^2) has a bounded extension on T_β^p . More

precisely, define

$$(1.4) \quad p_L(\beta) := \frac{np_-(L)}{n + (2\beta + 1)p_-(L)},$$

where $p_-(L)$ is introduced by [Aus07] as the infimum of $p \in [1, 2]$ so that the semigroup $(e^{-tL})_{t \geq 0}$ is uniformly bounded on $L^p(\mathbb{R}^n)$. For negative Laplacian $-\Delta$, $p_(-\Delta) = 1$. Observe that $\dot{H}^{2\beta+1, p_L(\beta)}(\mathbb{R}^n)$ embeds in $L^{p_-(L)}(\mathbb{R}^n)$ by Sobolev embedding, so $p_L(\beta)$ is in fact a Sobolev exponent of $p_-(L)$ at regularity $2\beta + 1$.

For the maximal regularity operators¹, we have the following streamlined result.

Proposition 1.1. *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. Then, \mathcal{M}_L can be extended to a bounded operator on T_β^p .*

This result yields a possible generalization of the notion of maximal regularity on tent spaces for the corresponding equation. However, due to the lack of semigroup theory, we need to answer the following questions.

- (a) Which notion of weak solution to (1.1) fits the tent spaces?
- (b) Does Duhamel's formula (1.2) or its extension provide a global weak solution?
- (c) In which class is this solution unique?
- (d) Does it satisfy an estimate $\|\partial_t u\|_{T_\beta^p} + \|Lu\|_{T_\beta^p} \lesssim \|f\|_{T_\beta^p}$, with proper interpretations of $\partial_t u$ and Lu ?

These questions arise naturally in the studies of the system (1.1) and its variants. Among these, let us mention three works towards stochastic analysis, one by the the first author, J. van Neerven, and P. Portal [AvNP14], one by P. Portal and M. Veraar [PV19], and one recent work of the first author and P. Portal [AP23].

Adopting the by now classical point of view of weak (energy) solutions for parabolic equations of type (1.1), we answer these questions in the following theorem. Precise definition of weak solutions is recalled in Section 4.

Theorem 1.2. *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. For any $f \in T_\beta^p$, there is a unique global weak solution $u \in T_{\beta+1}^p$ to the equation*

$$(1.5) \quad \partial_t u(t, x) - \operatorname{div}_x(A(x)\nabla_x u)(t, x) = f(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

¹We shall focus on Cauchy problems, but clearly our results on singular integral operators in weighted tent spaces apply to backward maximal regularity operators and equations. We shall not write precise statements here.

which is indeed given by (an extension of) Duhamel's formula (1.2). Moreover, it satisfies the estimate

$$\|u\|_{T_{\beta+1}^p} + \|\nabla u\|_{T_{\beta+\frac{1}{2}}^p} \lesssim \|f\|_{T_{\beta}^p},$$

and both of $\partial_t u$ and $\operatorname{div}(A\nabla u)$ belong to T_{β}^p with estimate

$$\|\partial_t u\|_{T_{\beta}^p} + \|\operatorname{div}(A\nabla u)\|_{T_{\beta}^p} \lesssim \|f\|_{T_{\beta}^p}.$$

Let us first explain the absence of initial conditions in the statement. Actually, there is a notion of trace for the space $T_{\beta+1}^p$ for $0 < p \leq \infty$ and $\beta > -1/2$ (or $s = 2\beta + 1 > 0$), which differs from the usual notion by taking completion from the usual trace of test functions. This notion, which we call the (*parabolic*) *Whitney trace*, consists in taking the limit of averages on parabolic Whitney cubes as $t \rightarrow 0$ for a.e. $x \in \mathbb{R}^n$, see Theorem 4.2 for a precise description. We shall show that all $T_{\beta+1}^p$ -functions in this range of indices have zero Whitney trace. In particular, the solution in the statement must have zero Whitney trace. It implies that there is no non-trivial global weak solution in $T_{\beta+1}^p$ of $\partial_t u - \operatorname{div}(A\nabla u) = 0$, and hence the initial value problem cannot be posed in $T_{\beta+1}^p$ when $\beta > -1/2$.

We briefly discuss the main ingredients of the proof of Theorem 1.2. Existence is based on the theory of singular integral operators mentioned above. Uniqueness relies on a method already used in [AMP19], which can be thought as an elaboration of Green's identity in the context of rough coefficients. It allows one to show that any global weak solution to (1.1) with null source term in a tent space satisfies an interior representation, which we may call "*homotopy identity*",

$$u(t) = e^{-(t-s)L}u(s) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)$$

when $0 < s < t$, in the sense that

$$\int_{\mathbb{R}^n} u(t, x) \bar{h}(x) dx = \int_{\mathbb{R}^n} u(s, x) \overline{((e^{-(t-s)L})^* h)}(x) dx$$

for any $h \in C_c^\infty(\mathbb{R}^n)$. With this identity, the problem is reduced to showing the convergence of $u(s)$ to 0 and of $(e^{-(t-s)L})^* h$ to $(e^{-tL})^* h$ in appropriate topology as $s \rightarrow 0$.

To finish the introduction, we mention further lines of exploration addressed in a forthcoming work [AH]. As we observed after Theorem 1.2, the weak solutions to this equation in $T_{\beta+1}^p$ have null Whitney trace when $2\beta + 1 > 0$ (and also zero distributional limit). However, if $2\beta + 1 = 0$, then the initial value problem can be posed (see [AMP19])

and [Zat20]), with non-zero Whitney trace, even for time-dependent coefficients. For time-independent coefficients, we will generalize it to $-1 \leq 2\beta + 1 < 0$, where the initial data can be taken with negative Sobolev regularity.

At the same time, other source terms in divergence form $\operatorname{div} F$ can be considered, in the spirit of the Lions' theory. The function F will be in $T_{\beta+\frac{1}{2}}^p$ if $2\beta + 1 > -1$. In particular, we will exhibit a range of (β, p) for which the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = u_0 \end{cases}$$

can be uniquely posed in weighted tent spaces provided $F \in T_{\beta+\frac{1}{2}}^p$ and $u_0 \in \dot{H}^{2\beta+1,p}$ when $-1 < 2\beta + 1 < 1$. And when $2\beta + 1 < 0$, the solution u will satisfy $u \in T_{\beta+1}^p$ and $\nabla u \in T_{\beta+\frac{1}{2}}^p$, but no maximal regularity estimates available as above.

Organization. The paper is organized as follows.

Section 2 is devoted to the definitions of two classes of singular integral operators. We show that they are *de facto* linked by duality.

Section 3 is concerned with the extension of these two classes of singular integral operators to tent spaces. Main results are given in Section 3.2.

Sections 4 and 5 are main sections of this paper. Section 4.1 outlines the proof of the main theorem, Theorem 1.2, about global well-posedness of (1.5), as a consequence of Theorems 4.2 and 4.3. Theorem 4.2 utilises the theory of singular integral operators developed in Section 3 to establish existence through a series of weighted tent space estimates on the Duhamel solution u , its derivatives ∇u , $\partial_t u$, and $\operatorname{div}(A\nabla u)$. The proof is provided in Section 4.3 and Section 5, where maximal regularity estimates are established, and boundary behavior of u is described. Theorem 4.3 states uniqueness of global weak solutions. Its proof is given in Section 4.4.

Notation. Throughout the paper, we write

$$[q, r] := \frac{1}{q} - \frac{1}{r}$$

for any $q, r \in (0, \infty]$, if there is no confusion with closed intervals. We write $X \lesssim Y$ (or $X \lesssim_A Y$, resp.) if $X \leq CY$ with an irrelevant constant C (or depending on A , resp.), and say $X \approx Y$ if $X \lesssim Y$ and $Y \lesssim X$. Write $\mathbb{R}_+^{1+n} := \mathbb{R}_+ \times \mathbb{R}^n = (0, \infty) \times \mathbb{R}^n$. For any (Euclidean) ball $B \subset \mathbb{R}^n$, write $r(B)$ for the radius of B .

Let (X, μ) be a measure space. For any measurable subset $E \subset X$ with finite measure and $f \in L^1(E, \mu)$, we write

$$\int_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu.$$

Write $\|\cdot\|_p$ as an abbreviation for the norm $\|\cdot\|_{L^p(X, \mu)}$.

We write $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, dx)$ where dx is the Lebesgue measure, and $L^2(\mathbb{R}^n)$ is equipped with the usual (complex) inner product. For any $\beta \in \mathbb{R}$, denote by $L^2_\beta(\mathbb{R}^{1+n}_+)$ the space $L^2(\mathbb{R}_+, t^{-2\beta} dt; L^2(\mathbb{R}^n))$.

We use the sans-serif font \mathfrak{c} in the scripts of function spaces in short of “with compact support” in the prescribed set, and \mathfrak{loc} if the prescribed property holds on all compact subsets of the prescribed set.

Acknowledgements. The authors were supported by the ANR project RAGE ANR-18-CE40-0012. We would like to thank Luca Haardt and Jonas Lenz for pointing out a gap in one of our arguments. The second author would also like to thank Jing'er JI for her long-lasting encouragements and discussions.

2. SINGULAR INTEGRAL OPERATORS

In this section, we rectify and generalize the abstract model for singular integral operators in [AKMP12] via off-diagonal decay.

Let π_1, π_2 be the projections of $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$ to \mathbb{R} and \mathbb{R}^n components, respectively. Denote by Δ the set $\{(t, s) \in (0, \infty) \times (0, \infty) : t = s\}$ and write $\Delta^c := ((0, \infty) \times (0, \infty)) \setminus \Delta$.

Definition 2.1 (Off-diagonal decay). Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq r \leq \infty$, and $M > 0$ be constants. Let $\{K(t, s)\}_{(t,s) \in \Delta^c}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$. We say that it has $L^q - L^r$ off-diagonal decay of type (κ, m, M) if there exists a constant $C > 0$ such that for any Borel sets $E, F \subset \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, and a.e. $(t, s) \in \Delta^c$,

$$\|\mathbb{1}_E K(t, s) \mathbb{1}_F f\|_r \leq C |t - s|^{-1 + \kappa - \frac{n}{m}[q, r]} \left(1 + \frac{\text{dist}(E, F)^m}{|t - s|}\right)^{-M} \|\mathbb{1}_F f\|_q. \quad (2.1)$$

2.1. Singular kernels. Let $\mathcal{L}(X)$ be the space of bounded linear operators on a (quasi-)Banach space X .

Definition 2.2 (Singular kernel). Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M > 0$ be constants. We say that an operator-valued function $K : \Delta^c \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))$ is a *singular kernel* (SK) of type (κ, m, q, M) , if

- (1) K is strongly measurable;

(2) There exists some constant $C > 0$ such that for a.e. $(t, s) \in \Delta^c$,

$$(2.2) \quad \|K(t, s)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C|t - s|^{-1+\kappa};$$

(3) The family $\{K(t, s)\}_{(t,s) \in \Delta^c}$ has off-diagonal $L^q - L^2$ decay of type (κ, m, M) if $1 \leq q \leq 2$, and off-diagonal $L^2 - L^q$ decay of type (κ, m, M) if $2 \leq q \leq \infty$.

Let $\text{SK}_{m,q,M}^\kappa$ be the set of singular kernels of type (κ, m, q, M) and $\text{SK}_{m,q,\infty}^\kappa$ be the intersection $\bigcap_{M>0} \text{SK}_{m,q,M}^\kappa$.

Remark 2.3. The class $\text{SK}_{m,q,M}^\kappa$ is stable with respect to truncation in the sense that, if $K \in \text{SK}_{m,q,M}^\kappa$, then $\mathbf{1}_A K \in \text{SK}_{m,q,M}^\kappa$ for any measurable set $A \subset \Delta^c$. The two cases, $A = \{t < s\}$ and $A = \{t > s\}$ are of particular interest.

Denote by T^* the adjoint of a linear operator T on $L^2(\mathbb{R}^n)$.

Lemma 2.4. *Let $1 \leq q \leq r \leq \infty$. Let $\{K(t, s)\}_{(t,s) \in \Delta^c}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$ with $L^q - L^r$ off-diagonal decay of type (κ, m, M) . Then $\{K(s, t)^*\}_{(t,s) \in \Delta^c}$ has $L^{r'} - L^{q'}$ off-diagonal decay of type (κ, m, M) .*

Proof. Let $E, F \subset \mathbb{R}^n$ be two Borel sets. For any $f \in L^2(\mathbb{R}^n) \cap L^{r'}(\mathbb{R}^n)$ and a.e. $(t, s) \in \Delta^c$,

$$\begin{aligned} \|\mathbf{1}_F K(s, t)^*(\mathbf{1}_E f)\|_{q'} &= \sup_{\phi} \langle \mathbf{1}_F \phi, K(s, t)^*(\mathbf{1}_E f) \rangle_{L^2(\mathbb{R}^n)} \\ &= \sup_{\phi} \langle K(s, t)(\mathbf{1}_F \phi), \mathbf{1}_E f \rangle_{L^2(\mathbb{R}^n)} \\ &\leq \sup_{\phi} \|\mathbf{1}_E K(s, t)(\mathbf{1}_F \phi)\|_r \|\mathbf{1}_E f\|_{r'} \\ &\lesssim |t - s|^{-1+\kappa - \frac{n}{m}[q,r]} \left(1 + \frac{\text{dist}(E, F)^m}{|t - s|}\right)^{-M} \|\mathbf{1}_E f\|_{r'}, \end{aligned}$$

where ϕ is taken in $L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with support in F and $\|\phi\|_q = 1$. Note that $[q, r] = [r', q']$ to conclude the proof. \square

Corollary 2.5. *If K belongs to $\text{SK}_{m,q,M}^\kappa$, then $K^* : (t, s) \mapsto K(s, t)^*$ belongs to $\text{SK}_{m,q',M}^\kappa$.*

Proof. The strong measurability of K^* follows from Pettis' measurability theorem (see [HvNVW16, Theorem 1.1.6]), since $L^2(\mathbb{R}^n)$ is separable. The boundedness condition is obvious, while the off-diagonal decay follows from Lemma 2.4. \square

2.2. Integrals associated with singular kernels. For any function f on \mathbb{R}_+^{1+n} , denote by $f(s)$ the function $x \mapsto f(s, x)$. Let us first consider a non-singular integral. Let K be in $\text{SK}_{m,q,M}^\kappa$ with $\kappa > 0$. For any $\beta > -1/2$ and $f \in L_\beta^2(\mathbb{R}_+^{1+n})$, we claim that the integral

$$(2.3) \quad \int_0^t K(t, s) f(s) ds$$

makes sense as a Bochner integral on $L^2(\mathbb{R}^n)$ for a.e. $t \in (0, \infty)$. Indeed, (2.2) implies

$$\begin{aligned} & \left(\int_0^\infty \left(t^{-\beta-\kappa} \int_0^t \|K(t, s) f(s)\|_2 ds \right)^2 dt \right)^{1/2} \\ & \lesssim \left(\int_0^\infty \left(\int_0^t s^{\beta+\frac{1}{2}} t^{-\beta-\kappa+\frac{1}{2}} (t-s)^{-1+\kappa} \|s^{-\beta+\frac{1}{2}} f(s)\|_2 \frac{ds}{s} \right)^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Write $k(t, s) = \mathbf{1}_{\{s < t\}}(t, s) s^{\beta+\frac{1}{2}} t^{-\beta-\kappa+\frac{1}{2}} (t-s)^{-1+\kappa}$. For a.e. $t > 0$,

$$\int_0^\infty k(t, s) \frac{ds}{s} = \int_0^1 \lambda^{\beta+\frac{1}{2}} (1-\lambda)^{\kappa-1} \frac{d\lambda}{\lambda} \leq C(\beta, \kappa).$$

With the same conditions, for a.e. $s > 0$,

$$\int_0^\infty k(t, s) \frac{dt}{t} = \int_0^\infty \lambda^{\kappa-1} (\lambda+1)^{-\beta-\kappa-\frac{1}{2}} d\lambda \leq C(\beta, \kappa).$$

Schur test hence implies

$$\left(\int_0^\infty \left(t^{-\beta-\kappa} \int_0^t \|K(t, s) f(s)\|_2 ds \right)^2 dt \right)^{1/2} \lesssim \|f\|_{L_\beta^2(\mathbb{R}_+^{1+n})} < \infty.$$

This proves our claim. It also shows that the function

$$(t, y) \mapsto \left(\int_0^t K(t, s) f(s) ds \right) (y)$$

belongs to $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$. Thanks to the strong measurability of K , we have the pointwise evaluation (see [HvNVW16, Proposition 1.2.25]) in the sense that for a.e. $(t, y) \in \mathbb{R}_+^{1+n}$,

$$(2.4) \quad \left(\int_0^t K(t, s) f(s) ds \right) (y) = \int_0^t (K(t, s) f(s))(y) ds.$$

For $f \in L_{-\beta-\kappa}^2(\mathbb{R}_+^{1+n})$, by a duality argument, we can also define the $L^2(\mathbb{R}^n)$ -valued Bochner integral

$$(2.5) \quad \int_t^\infty K(t, s) f(s) ds$$

for a.e. $t \in (0, \infty)$, and obtain similarly

$$(2.6) \quad \left(\int_t^\infty K(t, s) f(s) ds \right) (y) = \int_t^\infty (K(t, s) f(s))(y) ds$$

for a.e. $(t, y) \in \mathbb{R}_+^{1+n}$. We summarize the above discussion in the following lemma.

Lemma 2.6. *Let K be in $\text{SK}_{m,q,M}^\kappa$ with $\kappa > 0$. For any $\beta > -1/2$,*

- (1) (2.3) defines a bounded operator from $L_\beta^2(\mathbb{R}_+^{1+n})$ to $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$, and (2.4) holds;
- (2) (2.5) defines a bounded operator from $L_{-\beta-\kappa}^2(\mathbb{R}_+^{1+n})$ to $L_{-\beta}^2(\mathbb{R}_+^{1+n})$, and (2.6) holds.

We then turn to $\kappa \leq 0$, where a non-integrable singularity occurs at $t = s$. Still, as we are interested in handling functions of (t, x) , we adopt a refined strategy based on the following lemma. A subset $Q \subset \mathbb{R}_+^{1+n}$ is called a *rectangle* if it is of product form $Q = \pi_1(Q) \times \pi_2(Q)$. It is called a *bounded rectangle* if both components are bounded.

Lemma 2.7. *Let $K \in \text{SK}_{m,q,M}^\kappa$ with $\kappa \leq 0$ and $\frac{n}{m}|[q, 2] - \kappa < M \leq \infty$. Let Q_0, Q_1 be two bounded rectangles in \mathbb{R}_+^{1+n} with*

$$\max\{\text{dist}(\pi_i(Q_0), \pi_i(Q_1)) : i = 1, 2\} \geq \epsilon > 0.$$

For a.e. $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$, define

$$\omega_{Q_0, Q_1}(t, s) := \|\mathbf{1}_{Q_0}(t) K(t, s) \mathbf{1}_{Q_1}(s)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

Then, it holds that

$$\begin{aligned} \int_0^\infty \omega_{Q_0, Q_1}(t, s) ds &\leq C(\epsilon, Q_0, Q_1, \kappa, m, M, q), \\ \int_0^\infty \omega_{Q_0, Q_1}(t, s) dt &\leq C(\epsilon, Q_0, Q_1, \kappa, m, M, q). \end{aligned}$$

Proof. By symmetry, it suffices to prove the first inequality. Recall that $\mathbf{1}_{Q_0}(t)(x) = \mathbf{1}_{\pi_1(Q_0)}(t) \mathbf{1}_{\pi_2(Q_0)}(x)$ and $\mathbf{1}_{Q_1}(s)(y) = \mathbf{1}_{\pi_1(Q_1)}(s) \mathbf{1}_{\pi_2(Q_1)}(y)$.

If $\text{dist}(\pi_1(Q_0), \pi_1(Q_1)) \geq \epsilon/2$, for any $t \in \pi_1(Q_0)$, we have

$$\int_0^\infty \omega_{Q_0, Q_1}(t, s) ds \lesssim \int_{\pi_1(Q_1)} |t - s|^{-1+\kappa} ds \leq (\epsilon/2)^{\kappa-1} |\pi_1(Q_1)|.$$

Otherwise, $\text{dist}(\pi_1(Q_0), \pi_1(Q_1)) \leq \epsilon/2$ but then $\text{dist}(\pi_2(Q_0), \pi_2(Q_1)) \geq \epsilon$. Off-diagonal decay of $K(t, s)$ implies

$$\omega_{Q_0, Q_1}(t, s) \lesssim \begin{cases} \epsilon^{-mM} |t - s|^{-1+\kappa - \frac{n}{m}[q, 2] + M} |\pi_2(Q_1)|^{[q, 2]} & \text{if } 1 \leq q \leq 2 \\ \epsilon^{-mM} |t - s|^{-1+\kappa - \frac{n}{m}[2, q] + M} |\pi_2(Q_0)|^{[2, q]} & \text{if } 2 \leq q \leq \infty \end{cases}$$

by Hölder's inequality. In summary, we get

$$\int_0^\infty \omega_{Q_0, Q_1}(t, s) ds \lesssim_{Q_0, Q_1, q} \epsilon^{-mM} \int_{\pi_1(Q_1)} |t-s|^{-1+\kappa-\frac{n}{m}[q, 2]+M} ds,$$

which converges as $M > \frac{n}{m}[q, 2] - \kappa$. \square

Let $L_b^2(\mathbb{R}_+^{1+n})$ be the subspace of $L^2(\mathbb{R}_+^{1+n})$ consisting of functions with bounded support in $\overline{\mathbb{R}_+^{1+n}}$. For any $f \in L_b^2(\mathbb{R}_+^{1+n})$, define $\pi(f) := \pi_1(\text{supp } f) \times \pi_2(\text{supp } f)$ as a bounded rectangle. Let K, κ , and M be as in Lemma 2.7. For any bounded rectangle $Q \subset \mathbb{R}_+^{1+n}$ with $\text{dist}(Q, \pi(f)) > 0$ and a.e. $t \in \pi_1(Q)$,

$$\int_0^\infty \mathbf{1}_{\pi_2(Q)} K(t, s) f(s) ds$$

is defined as a Bochner integral valued in $L^2(\mathbb{R}^n)$, since

$$\int_0^\infty \|\mathbf{1}_{\pi_2(Q)}(K(t, s) f(s))\|_2 ds \leq \int_0^\infty \omega_{Q, \pi(f)}(t, s) \|f(s)\|_2 ds,$$

which is finite thanks to Lemma 2.7 and Schur test. Pointwise evaluation also holds, *i.e.*, for a.e. $(t, y) \in Q$,

$$\left(\int_0^\infty \mathbf{1}_{\pi_2(Q)} K(t, s) f(s) ds \right) (y) = \int_0^\infty \mathbf{1}_{\pi_2(Q)}(y) (K(t, s) f(s))(y) ds.$$

Furthermore, if $Q, Q' \subset \pi(f)^c$ are bounded rectangles, then for a.e. $(t, y) \in Q \cap Q'$, we have

$$(2.7) \quad \int_0^\infty \mathbf{1}_{\pi_2(Q)}(y) (K(t, s) f(s))(y) dy = \int_0^\infty \mathbf{1}_{\pi_2(Q')}(y) (K(t, s) f(s))(y) ds.$$

Indeed, as $t \in \pi_1(Q) \cap \pi_1(Q')$, we have when $x \in \pi_2(Q) \cap \pi_2(Q')$

$$\text{LHS} = \int_0^\infty \mathbf{1}_{\pi_2(Q')}(y) \mathbf{1}_{\pi_2(Q)}(y) (K(t, s) f(s))(y) ds = \text{RHS}.$$

Thus, we can define the function

$$(t, y) \mapsto \int_0^\infty (K(t, s) f(s))(y) ds$$

almost everywhere on $\pi(f)^c$ as follows. Pick an *exhaustion* of $\pi(f)^c$ by bounded rectangles $\{Q_i\}_{i \in \mathbb{N}}$, *i.e.*, $\bigcup_i Q_i = \pi(f)^c$. Then, we set almost everywhere

$$(2.8) \quad \int_0^\infty (K(t, s) f(s))(y) ds := \int_0^\infty \mathbf{1}_{\pi_2(Q_i)}(y) (K(t, s) f(s))(y) ds,$$

if $(t, y) \in Q_i$. This definition makes sense almost everywhere and is clearly independent of the choice of exhaustion $\{Q_i\}_{i \in \mathbb{N}}$ by (2.7). We summarize the above discussion in the following lemma.

Lemma 2.8. *Let $\kappa \leq 0$, $\frac{n}{m}||[q, 2]| - \kappa < M \leq \infty$, and $K \in \text{SK}_{m,q,M}^\kappa$. Then for any $f \in L_b^2(\mathbb{R}_+^{1+n})$,*

$$(t, x) \mapsto \int_0^\infty (K(t, s)f(s))(x)ds$$

defines a function in $L_{\text{loc}}^2(\overline{\mathbb{R}_+^{1+n}} \setminus \pi(f))$. Moreover, for any $g \in L_b^2(\mathbb{R}_+^{1+n})$ with $\pi(g) \cap \pi(f) = \emptyset$,

$$(t, s) \mapsto \langle K(t, s)f(s), g(t) \rangle_{L^2(\mathbb{R}^n)}$$

defines a function in $L^1((0, \infty) \times (0, \infty))$, and

$$\begin{aligned} & \int_0^\infty \int_0^\infty \langle K(t, s)f(s), g(t) \rangle_{L^2(\mathbb{R}^n)} ds dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left(\int_0^\infty (K(t, s)f(s))(x)ds \right) \cdot \bar{g}(t, x) dt dx. \end{aligned}$$

Proof. The first part follows from (2.8), and the second part follows from the first part by writing

$$\langle K(t, s)f(s), g(t) \rangle_{L^2(\mathbb{R}^n)} = \langle \mathbb{1}_{\pi(g)}(t)K(t, s)\mathbb{1}_{\pi(f)}(s)f(s), g(t) \rangle_{L^2(\mathbb{R}^n)}$$

and using pointwise evaluation and Fubini's theorem. \square

2.3. Singular integral operators. Define

$$(2.9) \quad M_{\kappa,q} := \begin{cases} 0 & \text{if } \kappa > 0 \\ \frac{n}{m}||[q, 2]| - \kappa & \text{if } \kappa \leq 0 \end{cases}.$$

Definition 2.9 (SIO of type $+(\kappa, m, q, M)$). Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M_{\kappa,q} < M \leq \infty$ be constants. An operator T is called a *singular integral operator* (SIO) of type $+(\kappa, m, q, M)$, if

- (1) T is a bounded operator from $L^2(\mathbb{R}_+^{1+n})$ to $L_\kappa^2(\mathbb{R}_+^{1+n})$;
- (2) There exists $K \in \text{SK}_{m,q,M}^\kappa$, called the *kernel* of T , such that the representation

$$(2.10) \quad Tf(t, y) = \int_0^t (K(t, s)f(s))(y)ds$$

holds

- for any $f \in L^2(\mathbb{R}_+^{1+n})$ and a.e. $(t, y) \in \mathbb{R}_+^{1+n}$, if $\kappa > 0$;
- for any $f \in L_b^2(\mathbb{R}_+^{1+n})$ and a.e. $(t, y) \in \pi(f)^c$, if $\kappa \leq 0$.

Denote by $\text{SIO}_{m,q,M}^{\kappa+}$ the set of all SIOs of type $+(\kappa, m, q, M)$.

There is a slight abuse of terminology since the operator is not singular when $\kappa > 0$.

Definition 2.10 (SIO of type $-(\kappa, m, q, M)$). Let $\kappa \in \mathbb{R}$, $m \in \mathbb{N}$, $1 \leq q \leq \infty$, and $M_{\kappa, q} < M \leq \infty$ be constants. An operator T is called a *singular integral operator of type $-(\kappa, m, q, M)$* , if

- (1) T is a bounded operator from $L^2_{-\kappa}(\mathbb{R}_+^{1+n})$ to $L^2(\mathbb{R}_+^{1+n})$;
- (2) There exists $K \in \text{SK}_{m, q, M}^\kappa$, called the *kernel* of T , such that the representation

$$(2.11) \quad Tf(t, y) = \int_t^\infty (K(t, s)f(s))(y)ds$$

holds

- for any $f \in L^2_{-\kappa}(\mathbb{R}_+^{1+n})$ and a.e. $(t, y) \in \mathbb{R}_+^{1+n}$, if $\kappa > 0$;
- for any $f \in L^2_{-\kappa}(\mathbb{R}_+^{1+n})$ with bounded support in $\overline{\mathbb{R}_+^{1+n}}$ and a.e. $(t, y) \in \pi(f)^c$, if $\kappa \leq 0$.

Denote by $\text{SIO}_{m, q, M}^{\kappa-}$ the set of all SIOs of type $-(\kappa, m, q, M)$.

For $\kappa > 0$, the integrals in (2.10) and (2.11) are well-defined by Lemma 2.6. For $\kappa \leq 0$, one uses Lemma 2.8. In the case $\kappa = 0$, compared with Definition 2.1 in [AKMP12], our definitions seem more restrictive, but in fact, (2.10) and (2.11) are all what is needed in their proofs to obtain tent space estimates.

These two types of SIOs are linked via $L^2(\mathbb{R}_+^{1+n})$ -duality given by

$$\langle f, g \rangle_{L^2(\mathbb{R}_+^{1+n})} := \int_{\mathbb{R}_+^{1+n}} f(t, y)\bar{g}(t, y)dtdy.$$

Proposition 2.11. *Let $\kappa \in \mathbb{R}$, $m \in \mathbb{N}$, $1 \leq q \leq \infty$, and $M_{\kappa, q} < M \leq \infty$. Let T be in $\text{SIO}_{m, q, M}^{\kappa\pm}$ with kernel K and T^* be the adjoint of T with respect to $L^2(\mathbb{R}_+^{1+n})$ -duality. Then T^* lies in $\text{SIO}_{m, q', M}^{\kappa\mp}$ with kernel K^* .*

Proof. We only prove the case assuming $T \in \text{SIO}_{m, q, M}^{\kappa+}$ with $\kappa \leq 0$. The other cases are left to the reader. The boundedness of T^* is clear, and Corollary 2.5 shows $K^* \in \text{SK}_{m, q', M}^\kappa$, so it suffices to prove the representation. For any $f \in L^2_{-\kappa}(\mathbb{R}_+^{1+n})$ with bounded support in $\overline{\mathbb{R}_+^{1+n}}$ and $g \in L^2_{\mathbb{b}}(\mathbb{R}_+^{1+n})$ so that $\pi(g)$ is disjoint with $\pi(f)$, we get by (2.10)

$$\langle Tg, f \rangle_{L^2(\mathbb{R}_+^{1+n})} = \int_{\mathbb{R}_+^{1+n}} \left(\int_0^s (K(s, t)g(t))(y)dt \right) \bar{f}(s, y)dsdy.$$

Thanks to Lemma 2.7, Schur test implies

$$\begin{aligned}
& \int_{\mathbb{R}_+^{1+n}} \int_0^s |(K(s,t)g(t))(y)| |f(s,y)| dt ds dy \\
&= \int_0^\infty ds \int_0^s dt \int_{\mathbb{R}^n} |\mathbb{1}_{\pi(f)}(s,y)(K(s,t)g(t))(y)| |f(s,y)| dy \\
&\leq \int_0^\infty ds \int_0^s \omega_{\pi(f),\pi(g)}(s,t) \|g(t)\|_2 \|f(s)\|_2 dt \\
&\lesssim \|g\|_{L^2(\mathbb{R}_+^{1+n})} \|f\|_{L^2(\mathbb{R}_+^{1+n})} \lesssim \|g\|_{L^2(\mathbb{R}_+^{1+n})} \|f\|_{L^2_{-\kappa}(\mathbb{R}_+^{1+n})} < \infty,
\end{aligned}$$

using that f has bounded support. Fubini's theorem ensures that

$$\begin{aligned}
\langle g, T^* f \rangle_{L^2(\mathbb{R}_+^{1+n})} &= \langle Tg, f \rangle_{L^2(\mathbb{R}_+^{1+n})} \\
&= \int_0^\infty ds \int_0^s \langle K(s,t)g(t), f(s) \rangle_{L^2(\mathbb{R}^n)} dt \\
&= \int_0^\infty dt \int_t^\infty \langle g(t), K(s,t)^* f(s) \rangle_{L^2(\mathbb{R}^n)} ds \\
&= \int_{\mathbb{R}_+^{1+n}} g(t,x) \left(\int_t^\infty \overline{(K(s,t)^* f(s))(x)} ds \right) dt dx.
\end{aligned}$$

We conclude by arbitrariness of g that for a.e. $(t,x) \in \pi(f)^c$,

$$T^*(f)(t,x) = \int_t^\infty (K(s,t)^* f(s))(x) ds.$$

Hence (2.11) holds for T^* with kernel K^* . \square

3. EXTENSIONS OF SIOS TO TENT SPACES

The bounded extensions of SIOs of type $\pm(0, m, q, M)$ on tent spaces have been studied in [AKMP12]. In this section, we improve their results in our framework, simplify some proofs, and generalize to SIOs of type $\pm(\kappa, m, q, M)$ for $\kappa \neq 0$.

3.1. Basic properties of tent spaces. Readers can refer to [CMS85] for the original definitions of tent spaces and proofs of the following facts. Some needed refinements can be found in [Aus11, Ame18].

For $0 < p < \infty$, $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, the *tent space* $T_\beta^{p;m}$ consists of (possibly vector-valued, strongly) measurable functions f on \mathbb{R}_+^{1+n} for which

$$\|f\|_{T_\beta^{p;m}} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t^{1/m})} |t^{-\beta} f(t,y)|^2 dt dy \right)^{p/2} dx \right)^{1/p} < \infty.$$

Our notation for weighted tent spaces is different from that in [AKMP12] and in fact corresponds to $T_{-\beta/2}^{p,2;m}$ there.

Note that $T_{\beta}^{2;m}$ can be identified with $L_{\beta}^2(\mathbb{R}_+^{1+n})$ for any $m \in \mathbb{N}$ by Fubini's theorem. Let $L_c^2(\mathbb{R}_+^{1+n})$ be the space of $L^2(\mathbb{R}_+^{1+n})$ -functions with compact support in \mathbb{R}_+^{1+n} . It is dense in $T_{\beta}^{p;m}$. In fact, for any $f \in T_{\beta}^{p;m}$ and compact subset $K \subset \mathbb{R}_+^{1+n}$, $\mathbf{1}_K f$ lies in $L^2(\mathbb{R}_+^{1+n})$ with

$$(3.1) \quad \|\mathbf{1}_K f\|_{L^2(\mathbb{R}_+^{1+n})} \widetilde{\asymp}_{p,m,\beta,K} \|\mathbf{1}_K f\|_{T_{\beta}^{p;m}} \leq \|f\|_{T_{\beta}^{p;m}}.$$

For any $p \in (1, \infty)$ and $\beta \in \mathbb{R}$, the dual of $T_{\beta}^{p;m}$ can be identified with $T_{-\beta}^{p';m}$ via $L^2(\mathbb{R}_+^{1+n})$ -duality. It also identifies the dual of $T_{\beta}^{p;m}$ with $T_{-\beta,([p,1])}^{\infty;m}$ when $p \leq 1$. The space $T_{-\beta,([p,1])}^{\infty;m}$ consists of measurable functions f on \mathbb{R}_+^{1+n} for which

$$\|f\|_{T_{-\beta,([p,1])}^{\infty;m}} := \sup_B \frac{1}{|B|^{[p,1]}} \left(\int_0^{r(B)^m} \int_B |t^{\beta} f(t, y)|^2 dt dy \right)^{1/2} < \infty,$$

where B describes all balls of \mathbb{R}^n .

Note that $L_c^2(\mathbb{R}_+^{1+n})$ is not dense in $T_{-\beta,([p,1])}^{\infty;m}$ for any $p \leq 1$ and $\beta \in \mathbb{R}$ with respect to the topology induced by its norm. However, it is dense in the weak* topology.

Lemma 3.1. *For any $0 < p \leq 1$ and $\beta \in \mathbb{R}$, $L_c^2(\mathbb{R}_+^{1+n})$ is dense in $T_{-\beta,([p,1])}^{\infty;m}$ with respect to the weak* topology as dual of $T_{\beta}^{p;m}$.*

Proof. Let f be in $T_{-\beta,([p,1])}^{\infty;m}$. For any $R > 1$, define $C_R := [1/R, R] \times B(0, R)$ and $f_R := f \mathbf{1}_{C_R}$. For any $g \in T_{\beta}^{p;m}$,

$$\left| \langle f - f_R, g \rangle_{L^2(\mathbb{R}_+^{1+n})} \right| \leq \int_{(C_R)^c} |f| |g| \lesssim \|f\|_{T_{-\beta,([p,1])}^{\infty;m}} \|\mathbf{1}_{(C_R)^c} g\|_{T_{\beta}^{p;m}},$$

which tends to 0 as $R \rightarrow \infty$ by dominated convergence. This proves the convergence of f_R to f as $R \rightarrow \infty$ for the weak* topology. \square

An important tool in the study of tent spaces is change of aperture. For any $\alpha > 0$ and $x \in \mathbb{R}^n$, the cone $\Gamma_{\alpha}^m(x)$ with aperture α and homogeneity m at x is given by

$$(3.2) \quad \Gamma_{\alpha}^m(x) := \{(t, y) \in \mathbb{R}_+^{1+n} : |x - y| < \alpha t^{1/m}\}.$$

Notice that the inner integral in the $T_{\beta}^{p;m}$ -norm is an L^2 -norm on $\Gamma_1^m(x)$. Changing aperture gives an equivalent norm. More precisely, define the conical square function as

$$(3.3) \quad \mathcal{A}_{\beta;m}^{(\alpha)}(f)(x) := \left(\int_0^{\infty} dt \int_{B(x, \alpha t^{1/m})} |t^{-\beta} f(t, y)|^2 \frac{dy}{t^{n/m}} \right)^{1/2}.$$

For $p \in (0, \infty)$, define

$$\|f\|_{T_{\alpha,\beta}^{p;m}} := \|\mathcal{A}_{\beta;m}^{(\alpha)}(f)\|_p.$$

For $\alpha > 0$ and $p \in (0, \infty)$, one has $T_{\beta}^{p;m} = T_{\alpha,\beta}^{p;m}$ and

$$(3.4) \quad \min\{\alpha^{\frac{n}{2}}, \alpha^{\frac{n}{p}}\} \|f\|_{T_{\beta}^{p;m}} \lesssim \|f\|_{T_{\alpha,\beta}^{p;m}} \lesssim \max\{\alpha^{\frac{n}{2}}, \alpha^{\frac{n}{p}}\} \|f\|_{T_{\beta}^{p;m}}.$$

We shall omit α if $\alpha = 1$, and also write $\mathcal{A}_{\beta;m} := \mathcal{A}_{\beta;m}^{(1)}$.

For $0 < p \leq 1$, a measurable function a on \mathbb{R}_+^{1+n} is called a $T_{\beta}^{p;m}$ -atom, if there exists a ball $B \subset \mathbb{R}^n$ such that $\text{supp}(a) \subset [0, r(B)^m] \times B$, and

$$\|a\|_{L_{\beta}^2(\mathbb{R}_+^{1+n})} \leq |B|^{-[p,2]}.$$

Such a ball B is said to be *associated* to a . Using the notion of atomic decomposition, the following result is a verbatim adaptation of [AMR08, Theorem 4.9], left to the reader.

Lemma 3.2. *Let $\beta, \gamma \in \mathbb{R}$ and $0 < p \leq 1$. Let T be a bounded linear operator from $L_{\beta}^2(\mathbb{R}_+^{1+n})$ to $L_{\gamma}^2(\mathbb{R}_+^{1+n})$. If T is uniformly bounded from $T_{\beta}^{p;m}$ -atoms to $T_{\gamma}^{p;m}$, then T can be extended to a bounded linear operator from $T_{\beta}^{p;m}$ to $T_{\gamma}^{p;m}$.*

3.2. Main results. In this section, we state our main results on the extension of SIOs and their adjoints to tent spaces $T_{\beta}^{p;m}$. This is divided into four statements depending on the type of SIOs and the range of p . Proofs are postponed to Section 3.3.

Throughout the section, $\kappa \in \mathbb{R}$ and $m \in \mathbb{N}$ are fixed constants.

The first statement concerns the extension of operators in $\text{SIO}_{m,q,M}^{\kappa+}$ to $T_{\beta}^{p;m}$ for $p \leq 2$. Define

$$(3.5) \quad p_M := \frac{2n}{n + 2mM}, \quad p_q(\beta) := \frac{2nq}{2n + (2\beta + 1)mq},$$

and

$$(3.6) \quad M_c(\kappa, q) := \max\left\{\frac{n}{2m}, M_{\kappa,q}\right\},$$

where $M_{\kappa,q}$ is defined in (2.9). When $M > M_c(\kappa, q)$ then $M > M_{\kappa,q}$ so the class $\text{SIO}_{m,q,M}^{\kappa+}$ is well-defined, and also $p_M < 1$.

Proposition 3.3. *Let $1 \leq q \leq 2$, $M_c(\kappa, q) < M \leq \infty$, and $\beta > -1/2$. Let T be in $\text{SIO}_{m,q,M}^{\kappa+}$, and assume that T is a bounded operator from $L_{\beta}^2(\mathbb{R}_+^{1+n})$ to $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$. Then T extends to a bounded operator from $T_{\beta}^{p;m}$ to $T_{\beta+\kappa}^{p;m}$ when $\max\{p_M, p_q(\beta)\} < p \leq 2$, if one of the following conditions holds:*

$$(1) \quad q' > \frac{2n}{m(2\beta+1)} \quad (\text{or equivalently } p_q(\beta) < 1);$$

(2) $2 \leq q' \leq \frac{2n}{m(2\beta+1)}$ (or equivalently $p_q(\beta) \geq 1$) and $M > \frac{n}{mq}$.

The next statement establishes the extension of operators in $\text{SIO}_{m,q,M}^{\kappa-}$ to $T_\beta^{p;m}$ when $p \leq 2$.

Proposition 3.4. *Let $1 \leq q \leq 2$, $M_c(\kappa, q) < M \leq \infty$, and $\beta < 1/2$. Let T be in $\text{SIO}_{m,q,M}^{\kappa-}$, and assume that T is a bounded operator from $L_{\beta-\kappa}^2(\mathbb{R}_+^{1+n})$ to $L_\beta^2(\mathbb{R}_+^{1+n})$. Then T extends to a bounded operator from $T_{\beta-\kappa}^{p;m}$ to $T_\beta^{p;m}$ when $p_M < p \leq 2$.*

The third statement describes the extension of operators in $\text{SIO}_{m,q,M}^{\kappa+}$ on $T_\beta^{p;m}$ when $p \geq 2$.

Corollary 3.5. *Let $2 \leq q \leq \infty$, $M_c(\kappa, q) < M \leq \infty$, and $\beta > -1/2$. Let T be in $\text{SIO}_{m,q,M}^{\kappa+}$, and assume that T is a bounded operator from $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$ to $L_\beta^2(\mathbb{R}_+^{1+n})$. Then T extends to a bounded operator*

- from $T_\beta^{p;m}$ to $T_{\beta+\kappa}^{p;m}$, when $2 \leq p \leq \infty$,
- from $T_{\beta,([p,1])}^{\infty;m}$ to $T_{\beta+\kappa,([p,1])}^{\infty;m}$, when $p_M < p \leq 1$.

The last statement is concerned with the extension of operators in $\text{SIO}_{m,q,M}^{\kappa-}$ on $T_\beta^{p;m}$ when $p \geq 2$.

Corollary 3.6. *Let $2 \leq q \leq \infty$, $M_c(\kappa, q) < M \leq \infty$, and $\beta < 1/2$. Let T be in $\text{SIO}_{m,q,M}^{\kappa-}$, and assume that T is a bounded operator from $L_{\beta-\kappa}^2(\mathbb{R}_+^{1+n})$ to $L_\beta^2(\mathbb{R}_+^{1+n})$.*

- (1) If $q > \frac{2n}{m(2\beta+1)}$ (or equivalently $p_{q'}(\beta) < 1$), then T extends to a bounded operator
 - from $T_{\beta-\kappa}^{p;m}$ to $T_\beta^{p;m}$, when $2 \leq p \leq \infty$,
 - from $T_{\beta-\kappa,([p,1])}^{\infty;m}$ to $T_{\beta,([p,1])}^{\infty;m}$, when $\max\{p_M, p_{q'}(\beta)\} < p \leq 1$.
- (2) If $2 \leq q \leq \frac{2n}{m(2\beta+1)}$ (or equivalently $p_{q'}(\beta) \geq 1$) and $M > \frac{n}{mq'}$, then T extends to a bounded operator from $T_{\beta-\kappa}^{p;m}$ to $T_\beta^{p;m}$ when $2 \leq p < (p_{q'}(\beta))'$.

Remark 3.7. Let us compare with [AKMP12].

- We treat the new cases $\kappa \neq 0$. The constant $M_c(\kappa, q)$ depends on κ , while the exponents p_M and $p_q(\beta)$ do not.
- There is an improvement in Proposition 3.3 (2) and hence Corollary 3.6 (2) when $q \neq 2$. The range of p is larger but at the expense of the extra decay on M . This range for p was obtained in the case of a maximal regularity operator when $\kappa = 0$ (unpublished communication of Y. Huang). This extra decay only comes from our argument and we do not know how to remove it. But it should be the case, as a discontinuity on the required

lower control for M with respect to q appears at $q' = \frac{2n}{m(2\beta+1)}$ by looking at the two cases.

Remark 3.8. For $\kappa > 0$, the bounded extensions of $\text{SIO}_{m,q,M}^{\kappa+}$ -operators from $L^2_\beta(\mathbb{R}_+^{1+n})$ to $L^2_{\beta+\kappa}(\mathbb{R}_+^{1+n})$ are automatic, see Lemma 2.6. For $\kappa = 0$, it was asserted in [AKMP12, Theorem 2.2] but in fact the proof has a gap. It is unclear whether the operators in $\text{SIO}_{m,q,M}^{0+}$ are bounded on $L^2_\beta(\mathbb{R}_+^{1+n})$, so it needs to be assumed, and also for $\kappa < 0$. However, it is true for maximal regularity operators, see [AA11, Theorem 1.3].

Remark 3.9. Both propositions are proved by density, while both corollaries are proved by duality arguments. See below. Usual arguments show that extension by duality agrees with extension by density in norm when $p < \infty$ and for the weak* topology when $p = \infty$.

3.3. Proofs. In this section, we provide the proof of results in Section 3.2. The arguments follow those of [AKMP12] to which we will refer, except for Proposition 3.3 (2). For this one, we still follow the setup there but introduce a new method based on extrapolation of weighted estimates to improve the range of the exponent p . We first settle the proof the last three results.

Proof of Proposition 3.4. The proof is a verbatim adaptation of that of [AKMP12, Proposition 3.7]. We leave it to the reader. \square

Proof of Corollary 3.5 and Corollary 3.6. The proofs are similar by a formal duality argument from Proposition 3.4 and Proposition 3.3, respectively. Therefore, we only prove the case $p_M < p \leq 1$ in Corollary 3.5, and the rest is left to the reader.

Proposition 2.11 says $T^* \in \text{SIO}_{m,q',M}^{\kappa-}$, and by duality, T^* extends to a bounded operator from $L^2_{-\beta-\kappa}(\mathbb{R}_+^{1+n})$ to $L^2_{-\beta}(\mathbb{R}_+^{1+n})$. Note that $M_c(\kappa, q) = M_c(\kappa, q')$, so for any $f \in T_{\beta+\kappa,([p,1])}^{\infty;m}$, Proposition 3.4 implies that $g \mapsto \langle f, T^*g \rangle$ is a bounded anti-linear functional on $T_{-\beta-\kappa}^{p;m}$. Define $Tf \in T_{\beta,([p,1])}^{\infty;m}$ as $Tf(g) := \langle f, T^*g \rangle$ via the identification, and we get

$$\|Tf\|_{T_{\beta+\kappa,([p,1])}^{\infty;m}} = \sup_{g: \|g\|_{T_{-\beta-\kappa}^{p;m}} = 1} |\langle f, T^*g \rangle| \leq \|T^*\|_{\mathcal{L}(T_{-\beta-\kappa}^{p;m}, T_{-\beta}^{p;m})} \|f\|_{T_{\beta,([p,1])}^{\infty;m}},$$

where $\mathcal{L}(X, Y)$ is the space of bounded linear operators from the (quasi-)Banach space X to the (quasi-)Banach space Y .

This completes the proof. \square

We now turn to the proof of Proposition 3.3. We split the operator T into two parts, the *regular part*

$$T_0(f)(t, y) = \int_0^{t/2} (K(t, s)f(s))(y)ds,$$

and the *singular part*

$$T_{\text{sing}}(f)(t, y) = \int_{t/2}^t (K(t, s)f(s))(y)ds.$$

The meaning of the integrals by taking the restricted singular kernels $\mathbb{1}_{\{s < t/2\}}(t, s)K(t, s)$ or $\mathbb{1}_{\{t/2 < s < t\}}(t, s)K(t, s)$ is that of (2.3) if $\kappa > 0$ or Lemma 2.8 if $\kappa \leq 0$.

Let us start with the singular part.

Lemma 3.10. *Let T be in $\text{SIO}_{m,q,M}^{\kappa+}$ with $M > M_c(\kappa, q)$ and assume that T is a bounded operator from $L_\beta^2(\mathbb{R}_+^{1+n})$ to $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$. Then, T_{sing} extends to a bounded operator from $T_\beta^{p;m}$ to $T_{\beta+\kappa}^{p;m}$ when $p_M < p \leq 2$.*

Proof. By interpolation, it suffices to consider the extension of T_{sing} to $T_\beta^{p;m}$ for $p_M < p \leq 1$. By Lemma 3.2, it suffices to prove that T is uniformly bounded from $T_\beta^{p;m}$ -atoms to $T_{\beta+\kappa}^{p;m}$. This is a verbatim adaptation of [AKMP12, Lemma 3.4] once we assume that T is bounded from $L_\beta^2(\mathbb{R}_+^{1+n})$ to $L_{\beta+\kappa}^2(\mathbb{R}_+^{1+n})$. We leave it to the reader. \square

We then move on to estimate the regular part T_0 . If $q' > \frac{2n}{m(2\beta+1)}$, cf. Proposition 3.3 (1), we also obtain the uniform boundedness of T_0 on $T_\beta^{p;m}$ -atoms for $p_q(\beta) < p \leq 1$ and conclude by Lemma 3.2. But it does not work if $q' \leq \frac{2n}{m(2\beta+1)}$, cf. Case (2). In this case, our strategy is to embed T_0 into the analytic family of operators $\{T_z\}_{z \in \mathbb{C}}$ given by

$$(3.7) \quad T_z(f)(t, y) := \int_0^{t/2} \left(\frac{s}{t}\right)^z (K(t, s)f(s))(y)ds.$$

If $f \in L_c^2(\mathbb{R}_+^{1+n})$, then the integral makes sense as a Bochner integral valued in $L^2(\mathbb{R}^n)$ for any $z \in \mathbb{C}$ and a.e. $(t, y) \in \mathbb{R}_+^{1+n}$.

We study the boundedness of T_0 by Stein's interpolation. We show that if $\Re(z)$ is sufficiently large, we can still obtain a uniform bound of T_z acting on atoms. The main task is to obtain boundedness when $\Re(z) < 0$. This is given by the following proposition.

Proposition 3.11. *Let K be in $\text{SK}_{m,q,M}^\kappa$ with $1 \leq q < 2$ and $M > \frac{n}{mq}$. Then, for any $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > -\beta - \frac{1}{2}$, T_z can be extended to a bounded operator from $T_\beta^{p;m}$ to $T_{\beta+\kappa}^{p;m}$ when $q < p \leq 2$.*

Compared to [AKMP12, Lemma 3.5], this improves the range of p from $p = 2$ to $q < p \leq 2$. This is achieved via an extrapolation of weighted estimates technique. Let us start with a lemma.

Lemma 3.12. *Let $\{K(t, s)\}_{(t,s) \in \Delta^c}$ be a strongly measurable family of bounded operators on $L^2(\mathbb{R}^n)$ such that (2.2) holds for some $\kappa \in \mathbb{R}$. Then, for any $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > -\beta - \frac{1}{2}$, the integral (3.7) converges for $f \in L^2_\beta(\mathbb{R}_+^{1+n})$. Moreover, it defines a bounded operator from $L^2_\beta(\mathbb{R}_+^{1+n})$ to $L^2_{\beta+\kappa}(\mathbb{R}_+^{1+n})$.*

Proof. The proof is the same as that of (2.3) but changing the kernel $k(t, s)$ to $\mathbf{1}_{\{0 < s < t/2\}}(t, s) \left(\frac{s}{t}\right)^{\Re(z)} (t-s)^{\kappa-1} s^{\beta+\frac{1}{2}} t^{-\beta-\kappa+\frac{1}{2}}$. \square

Remark 3.13. Lemma 3.12 says T_0 is always bounded from $L^2_\beta(\mathbb{R}_+^{1+n})$ to $L^2_{\beta+\kappa}(\mathbb{R}_+^{1+n})$ if $\beta > -1/2$. However, it is unclear for T_{sing} if $\kappa \leq 0$. This is why we need to assume it on T , or equivalently on T_{sing} .

Next, we bring in a pointwise estimate. For any $f \in L^2_{\text{loc}}(\mathbb{R}_+^{1+n})$ and $x \in \mathbb{R}^n$, define the *vertical square function* as

$$\mathcal{V}_\beta(f)(x) := \left(\int_0^\infty |t^{-\beta} f(t, x)|^2 dt \right)^{1/2}.$$

Let \mathcal{M} be the centred Hardy–Littlewood maximal function given by

$$\mathcal{M}(g)(x) := \sup_{\rho > 0} \int_{B(x, \rho)} |g(y)| dy$$

for any $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Lemma 3.14. *With the assumptions of Proposition 3.11, it holds that*

$$(3.8) \quad \mathcal{A}_{\beta+\kappa; m}(T_z(f))(x) \lesssim \mathcal{V}_\beta(\mathcal{M}_q(f))(x)$$

for any $f \in L^2_\beta(\mathbb{R}_+^{1+n})$ and $x \in \mathbb{R}^n$, where

$$\mathcal{M}_q(f)(t, x) := \sup_{\rho > 0} \left(\int_{B(x, \rho)} |f(t, y)|^q dy \right)^{1/q} = \mathcal{M}(|f(t)|^q)(x)^{1/q}.$$

Proof. From the definition (3.3), we get

$$(3.9) \quad \mathcal{A}_{\beta+\kappa; m}(T_z(f))(x) \approx \left\| t^{-\frac{n}{2m}-\beta-\kappa} \|T_z(f)(t)\|_{L^2(B(x, t^{1/m}))} \right\|_{L^2(\mathbb{R}_+, dt)}.$$

Fix $(t, x) \in \mathbb{R}_+^{1+n}$. Define $B_0 := B(x, t^{1/m})$, $C_0 := 2B_0$, and $C_j := 2^{j+1}B_0 \setminus 2^j B_0$ for $j \geq 1$, so that

$$\|T_z(f)(t)\|_{L^2(B_0)} \leq \sum_{j \geq 0} \|\mathbf{1}_{B_0} T_z(\mathbf{1}_{C_j} f)(t)\|_2.$$

Using $L^q - L^2$ off-diagonal decay and $t - s \sim t$, we get

$$\begin{aligned} \|\mathbf{1}_{B_0} T_z(\mathbf{1}_{C_j} f)(t)\|_2 &\leq \int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)} \|\mathbf{1}_{B_0} K(t, s)(\mathbf{1}_{C_j} f(s))\|_2 ds \\ &\lesssim 2^{-jmM} t^{-\Re(z)-1+\kappa-\frac{n}{m}[q,2]} \int_0^{t/2} s^{\Re(z)} \|\mathbf{1}_{C_j} f(s)\|_q ds \\ &\lesssim 2^{-j(mM-\frac{n}{q})} t^{-\Re(z)-1+\kappa+\frac{n}{2m}} \int_0^{t/2} s^{\Re(z)} \mathcal{M}_q(f(s))(x) ds, \end{aligned}$$

due to the fact that

$$\|\mathbf{1}_{C_j} f(s)\|_q \leq \|\mathbf{1}_{2^{j+1}B_0} f(s)\|_q \lesssim_q (2^j t^{\frac{1}{m}})^{\frac{n}{q}} \mathcal{M}_q(f(s))(x).$$

Since $M > \frac{n}{mq}$, taking the sum on j , we get

$$\|T_z(f)(t)\|_{L^2(B_0)} \lesssim t^{-\Re(z)-1+\kappa+\frac{n}{2m}} \int_0^{t/2} s^{\Re(z)} \mathcal{M}_q(f(s))(x) ds.$$

By (3.9), we derive

$$\begin{aligned} \mathcal{A}_{\beta+\kappa; m}(T_z(f))(x)^2 &\lesssim \int_0^\infty \left(\int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \mathcal{M}_q(s^{-\beta+\frac{1}{2}} f(s))(x) \frac{ds}{s} \right)^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty (s^{-\beta} \mathcal{M}_q(f(s))(x))^2 ds = \mathcal{V}_\beta(\mathcal{M}_q(f))(x)^2. \end{aligned}$$

The last inequality follows by Schur test, as

$$\int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \frac{ds}{s} \leq C(z, \beta), \quad \int_{2s}^\infty \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \frac{dt}{t} \leq C(z, \beta),$$

when $\Re(z) > -\beta - \frac{1}{2}$. \square

We can now prove Proposition 3.11. Terminologies and facts related to weights w can be found in [CUMP11].

Proof of Proposition 3.11. The conclusion when $p = 2$ has been shown in Lemma 3.12. We turn to proving it for $p < 2$.

For any weight $w \in A_{2/q}$ and $f \in L_c^2(\mathbb{R}_+^{1+n})$, Lemma 3.14 and the $L^{2/q}(w)$ -boundedness of \mathcal{M} imply

$$\begin{aligned} \|\mathcal{A}_{\beta+\kappa; m}(T_z(f))\|_{L^2(w)}^2 &\lesssim \|\mathcal{V}_\beta(\mathcal{M}_q(f))\|_{L^2(w)}^2 \\ &= \int_0^\infty t^{-2\beta} dt \int_{\mathbb{R}^n} \mathcal{M}(|f(t)|^q)(x)^{\frac{2}{q}} w(x) dx \\ &\lesssim \int_0^\infty t^{-2\beta} dt \int_{\mathbb{R}^n} |f(t, x)|^2 w(x) dx = \|\mathcal{V}_\beta(f)\|_{L^2(w)}^2. \end{aligned}$$

Limited-range extrapolation (see [CUMP11, Theorem 3.31]) implies for any $p \in (q, 2)$, $w \in A_{p/q} \cap \text{RH}_{(2/p)'}$, and $f \in L_c^2(\mathbb{R}_+^{1+n})$,

$$\|\mathcal{A}_{\beta+\kappa;m}(T_z(f))\|_{L^p(w)} \lesssim \|\mathcal{V}_\beta(f)\|_{L^p(w)}.$$

As $p < 2$, we invoke [AHM12, Proposition 2.3(b)] and get

$$\|\mathcal{V}_\beta(f)\|_{L^p(w)} \lesssim \|\mathcal{A}_{\beta;m}(f)\|_{L^p(w)}$$

by change of scaling $t \rightarrow t^{1/m}$. Taking $w = 1$, we obtain that for any $p \in (q, 2)$ and $f \in L_c^2(\mathbb{R}_+^{1+n})$,

$$\|T_z(f)\|_{T_{\beta+\kappa}^{p;m}} \approx \|\mathcal{A}_{\beta+\kappa;m}(T_z(f))\|_p \lesssim \|\mathcal{A}_{\beta;m}(f)\|_p = \|f\|_{T_\beta^{p;m}}.$$

We conclude by a density argument. \square

Let us finish the proof of Proposition 3.3.

Proof of Proposition 3.3. The bounded extension of T_{sing} from $T_\beta^{p;m}$ to $T_{\beta+\kappa}^{p;m}$ for $p_M < p \leq 2$ has been shown in Lemma 3.10. Consider the regular part T_0 .

Case (1): $q' > \frac{2n}{m(2\beta+1)}$. We obtain the uniform $T_{\beta+\kappa}^{p;m}$ -bound of T_0 on $T_\beta^{p;m}$ -atoms when $p_q(\beta) < p \leq 1$ by adapting the calculation in [AKMP12, Lemma 3.5]. As the range of q corresponds to $p_q(\beta) < 1$, we conclude using Lemma 3.2.

Case (2): $q' \leq \frac{2n}{m(2\beta+1)}$. We first claim that T_z extends to a bounded operator from $T_\beta^{1;m}$ to $T_{\beta+\kappa}^{1;m}$ if $\Re(z) > \frac{n}{mq'} - \beta - \frac{1}{2} \geq 0$. Indeed, the adaptation of [AKMP12, Lemma 3.5] also implies the uniform $T_{\beta+\kappa}^{1;m}$ -bound of T_z on $T_\beta^{1;m}$ -atoms, so the claim follows by Lemma 3.2.

Then the discussion is organised into two cases.

- If $q' > 2$, Proposition 3.11 says T_z extends to a bounded operator from $T_\beta^{p;m}$ to $T_{\beta+\kappa}^{p;m}$ if $\Re(z) > -\beta - \frac{1}{2}$ and $q < p \leq 2$.

Then for any $p \in (p_q(\beta), 2]$, one can find $t_0 \in (-\beta - \frac{1}{2}, 0)$, $t_1 > \frac{n}{mq'} - \beta - \frac{1}{2}$, and $q_0 \in (p, 2]$ so that

$$(1 - \theta)t_0 + \theta t_1 = 0, \quad \frac{1 - \theta}{q_0} + \frac{\theta}{1} = \frac{1}{p}$$

for some $\theta \in (0, 1)$. For any $y \in \mathbb{R}$, the above discussion implies $T_{t_0+iy} \in \mathcal{L}(T_\beta^{q_0;m}, T_{\beta+\kappa}^{q_0;m})$ and $T_{t_1+iy} \in \mathcal{L}(T_\beta^{1;m}, T_{\beta+\kappa}^{1;m})$ with their operator norms independent of y . Then T_0 extends to a bounded operator from T_β^p to $T_{\beta+\kappa}^p$ as desired, thanks to Stein's interpolation theorem on tent spaces (see [HTV91]).

- If $q' = 2$, the adaptation of [AKMP12, Lemma 3.5] shows that T_z extends to a bounded operator from $T_\beta^{2;m}$ to $T_{\beta+\kappa}^{2;m}$ if $\Re(z) > -\beta - \frac{1}{2}$. We hence also get the bounded extension of T_0 from T_β^p to $T_{\beta+\kappa}^p$ when $p \in (p_2(\beta), 2]$ by the same argument as above.

This completes the proof. \square

4. INHOMOGENEOUS CAUCHY PROBLEMS

We may now begin the study of well-posedness in weighted tent spaces of Cauchy problems for the inhomogeneous parabolic equation

$$(IC) \quad \partial_t u(t, x) - \operatorname{div}_x(A(x)\nabla_x u)(t, x) = f(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

where $f \in \mathcal{D}'(\mathbb{R}_+^{1+n})$ is called the *source term*. Assume that $A \in L^\infty(\mathbb{R}^n; \operatorname{Mat}_n(\mathbb{C}))$ is *uniformly elliptic*, i.e., there exist $\Lambda_0, \Lambda_1 > 0$ such that, for a.e. $x \in \mathbb{R}^n$ and any $\xi, \eta \in \mathbb{C}^n$,

$$(4.1) \quad \Re(\langle A(x)\xi, \xi \rangle) \geq \Lambda_0 |\xi|^2, \quad |\langle A(x)\xi, \eta \rangle| \leq \Lambda_1 |\xi| |\eta|.$$

Let us recall the classical definition of weak solutions to (IC). Let $0 \leq a < b \leq \infty$ be constants, Ω be an open subset of \mathbb{R}^n , and $Q := (a, b) \times \Omega$. Let f be in $\mathcal{D}'(Q)$. We say that a function $u \in L_{\text{loc}}^2((a, b); H_{\text{loc}}^1(\Omega))$ is a *weak solution to (IC) with source term f* , if

$$(4.2) \quad - \int_Q u(t, x) \overline{\partial_t \phi}(t, x) + \int_Q (A(x)\nabla u(t, x)) \cdot \overline{\nabla \phi}(t, x) = (f, \overline{\phi})$$

holds for any $\phi \in C_c^\infty(Q)$, where the right-hand side is understood as the pairing between distributions and test functions on Q . Here and often in the sequel, when unspecified, the measures in the integrals are unweighted Lebesgue measures. We say u is a *global weak solution to (IC) with source term f* when $Q = \mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n$.

We shall use an extension of Duhamel's formula to build weak solutions. Define the operator $L := -\operatorname{div}(A(x)\nabla)$ on $L^2(\mathbb{R}^n)$ with domain

$$D(L) := \{f \in H^1(\mathbb{R}^n) : \operatorname{div}(A\nabla f) \in L^2(\mathbb{R}^n)\}.$$

It is well-known that $-L$ is a densely-defined closed operator on $L^2(\mathbb{R}^n)$ that generates a bounded analytic semigroup $(e^{-tL})_{t \geq 0}$ on $L^2(\mathbb{R}^n)$.

Let $1 \leq p \leq q \leq \infty$ and $M > 0$ be constants. We say that a family of bounded operators $(T_t)_{t \geq 0}$ on $L^2(\mathbb{R}^n)$ has (*parabolic*) $L^p - L^q$ *off-diagonal estimates of order M* , if there is a constant $C > 0$ so that, for any Borel sets $E, F \subset \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and a.e. $t > 0$,

$$(4.3) \quad \|\mathbf{1}_E T_t \mathbf{1}_F f\|_q \leq C t^{-\frac{n}{2}[p,q]} \left(1 + \frac{\operatorname{dist}(E, F)^2}{t}\right)^{-M} \|\mathbf{1}_F f\|_p.$$

Let $p_{\pm}(L), q_{\pm}(L) \in [1, \infty]$ be the critical numbers introduced by [Aus07]. The semigroup $(e^{-tL})_{t \geq 0}$ has $L^p - L^q$ off-diagonal estimates of any order M when $p_-(L) < p \leq q < p_+(L)$, and the same holds for the family $(t^{1/2} \nabla e^{-tL})_{t > 0}$ when $q_-(L) < p \leq q < q_+(L)$. Moreover, $p_-(L) = q_-(L) < 2$ and $p_+(L) \geq q_+(L) > 2$.

Let us also recall a family of slice spaces. Readers can refer to [AM19, §3] for the proof of the following facts. For any $p \in [1, \infty]$ and $\delta > 0$, the (parabolic) slice space E_{δ}^p consists of measurable functions f for which

$$\|f\|_{E_{\delta}^p} := \left\| \left(\int_{B(\cdot, \delta^{1/2})} |f(y)|^2 dy \right)^{1/2} \right\|_p < \infty.$$

Let $\delta, \delta' > 0$ be constants. Then, for any measurable function f on \mathbb{R}^n ,

$$(4.4) \quad \|f\|_{E_{\delta}^p} \lesssim \max \left\{ 1, \left(\frac{\delta'}{\delta} \right)^{\frac{n}{2}[2,p]} \right\} \|f\|_{E_{\delta'}^p}.$$

Fix $\delta > 0$. For any $p \in [1, \infty)$, the dual of E_{δ}^p can be identified with $E_{\delta}^{p'}$ via $L^2(\mathbb{R}^n)$ -duality. Moreover, let $1 \leq p_0 < p < p_1 \leq \infty$ be with $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta > 0$. Then, the complex interpolation space $[E_{\delta}^{p_0}, E_{\delta}^{p_1}]_{\theta}$ can be identified with E_{δ}^p .

4.1. Main results. Define the operator \mathcal{L}_1 on $L^2(\mathbb{R}_+^{1+n})$ by

$$(4.5) \quad \mathcal{L}_1(f)(t, y) := \int_0^t (e^{-(t-s)L} f(s))(y) ds.$$

For any $f \in L^2(\mathbb{R}_+^{1+n})$, $\mathcal{L}_1(f)$ and $\nabla \mathcal{L}_1(f)$ are all well-defined in $L_{\text{loc}}^2(\mathbb{R}_+^{1+n})$. The first proposition concerns *a priori* estimates in tent spaces. Introduce

$$(4.6) \quad p_L(\beta) := \frac{np_-(L)}{n + (2\beta + 1)p_-(L)},$$

which agrees with $p_q(\beta)$ defined in (3.5) for $q = p_-(L)$ and $m = 2$. From now on, we fix $m = 2$ and write $T_{\beta}^p := T_{\beta}^{p;2}$.

Proposition 4.1 (*A priori estimates*). *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. Let f be in $L_c^2(\mathbb{R}_+^{1+n})$.*

(i) $\mathcal{L}_1(f)$ lies in $T_{\beta+1}^p$ and $\nabla \mathcal{L}_1(f)$ lies in $T_{\beta+\frac{1}{2}}^p$ with

$$\|\mathcal{L}_1(f)\|_{T_{\beta+1}^p} \lesssim \|f\|_{T_{\beta}^p}, \quad \|\nabla \mathcal{L}_1(f)\|_{T_{\beta+\frac{1}{2}}^p} \lesssim \|f\|_{T_{\beta}^p}.$$

(ii) For any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{1+n}})$,

$$(4.7) \quad \int_{\mathbb{R}_+^{1+n}} |\mathcal{L}_1(f)| |\partial_t \phi| + \int_{\mathbb{R}_+^{1+n}} |A \nabla \mathcal{L}_1(f)| |\nabla \phi| + \int_{\mathbb{R}_+^{1+n}} |f| |\phi| \lesssim_\phi \|f\|_{T_\beta^p},$$

and

$$(4.8) \quad - \int_{\mathbb{R}_+^{1+n}} \mathcal{L}_1(f) \overline{\partial_t \phi} + \int_{\mathbb{R}_+^{1+n}} (A \nabla \mathcal{L}_1(f)) \cdot \overline{\nabla \phi} = \int_{\mathbb{R}_+^{1+n}} f \overline{\phi}.$$

The proof is postponed to Section 4.2. Obviously, the proposition implies that \mathcal{L}_1 can be extended to an operator acting on T_β^p . We summarize the properties of the extension in the following theorem. For any $(t, x) \in \mathbb{R}_+^{1+n}$, let $W(t, x) := (t, 2t) \times B(x, t^{1/2})$ be a (parabolic) Whitney cube at (t, x) .

Theorem 4.2 (Existence). *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. There is an extension of \mathcal{L}_1 defined by (4.5) on $L_c^2(\mathbb{R}_+^{1+n})$ to a bounded operator from T_β^p to $T_{\beta+1}^p$, also denoted by \mathcal{L}_1 , such that the following properties hold when $f \in T_\beta^p$ and $u := \mathcal{L}_1(f)$.*

(a) (Regularity) u lies in $T_{\beta+1}^p$ and ∇u lies in $T_{\beta+\frac{1}{2}}^p$ with

$$\|u\|_{T_{\beta+1}^p} \lesssim \|f\|_{T_\beta^p}, \quad \|\nabla u\|_{T_{\beta+\frac{1}{2}}^p} \lesssim \|f\|_{T_\beta^p}.$$

(b) For any $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{1+n}})$,

$$(4.9) \quad \int_{\mathbb{R}_+^{1+n}} |u| |\partial_t \phi| + \int_{\mathbb{R}_+^{1+n}} |A \nabla u| |\nabla \phi| + \int_{\mathbb{R}_+^{1+n}} |f| |\phi| \lesssim_\phi \|f\|_{T_\beta^p},$$

and

$$(4.10) \quad - \int_{\mathbb{R}_+^{1+n}} u \overline{\partial_t \phi} + \int_{\mathbb{R}_+^{1+n}} (A \nabla u) \cdot \overline{\nabla \phi} = \int_{\mathbb{R}_+^{1+n}} f \overline{\phi}.$$

Therefore, u is a global weak solution to (IC) with source term f .

(c) (Maximal regularity) $\partial_t u$ and $\operatorname{div}(A \nabla u)$ belong to T_β^p with

$$\|\partial_t u\|_{T_\beta^p} + \|\operatorname{div}(A \nabla u)\|_{T_\beta^p} \lesssim \|f\|_{T_\beta^p}.$$

(d) (Whitney trace) For a.e. $x \in \mathbb{R}^n$,

$$\lim_{t \rightarrow 0} \left(\int_{W(t, x)} |u(s, y)|^2 ds dy \right)^{1/2} = 0.$$

(e) (Distributional trace) $u(t)$ converges to 0 as $t \rightarrow 0$ in $L^p(\mathbb{R}^n)$ if $p_L(\beta) < p \leq 2$, and in E_δ^q for any $\delta > 0, q \in [p, \infty]$ if $2 < p \leq \infty$.

Statements (a) and (b) are proved in Section 4.3, and the others are proved in Section 5.2. Let us first make some remarks.

In (b), not only do we prove that $\mathcal{L}_1(f)$ is a global weak solution to (IC) with source term f , but we also implicitly imply $\mathcal{L}_1(f)(0) = 0$ as the test functions are arbitrary at $t = 0$. In turn, (d) and (e) explicitly show the null *boundary behavior* of $\mathcal{L}_1(f)(t)$ as $t \rightarrow 0$. As we shall see in Lemma 5.3, (d) in fact holds for any function $u \in T_{\beta+1}^p$.

Furthermore, (b) yields the equality $\partial_t u - \operatorname{div}(A\nabla u) = f$ holds in $\mathcal{D}'(\mathbb{R}_+^{1+n})$, but (c) implies it actually holds in T_β^p . In fact, $\partial_t u$ and $\operatorname{div}(A\nabla u)$ cannot lie in a better space simultaneously, otherwise the regularity of f could be raised. It is exactly the spirit of maximal regularity although the interpretation is no longer via semigroup theory.

Then, we turn to uniqueness.

Theorem 4.3 (Uniqueness). *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. Then, for any $f \in T_\beta^p$, there exists at most one global weak solution $u \in T_{\beta+1}^p$ to (IC) with source term f .*

The proof is deferred to Section 4.4. Altogether, we have demonstrated the main theorem, Theorem 1.2.

Remark 4.4. Local well-posedness also holds, as the proofs will show that one can work on any given strip $[0, T] \times \mathbb{R}^n$ instead.

4.2. A priori estimates. In this section, we prove Proposition 4.1. Define the operator $\mathcal{L}_{1/2} : L^2(\mathbb{R}_+^{1+n}) \rightarrow L_{\text{loc}}^1((0, \infty); L^2(\mathbb{R}^n))$ by

$$(4.11) \quad \mathcal{L}_{1/2}(f)(t, y) := \int_0^t (\nabla e^{-(t-s)L} f(s))(y) ds, \quad \text{a.e. } (t, y) \in \mathbb{R}_+^{1+n}.$$

The integral exists as an $L^2(\mathbb{R}^n)$ -valued Bochner integral.

Lemma 4.5. *For any $f \in L_c^2(\mathbb{R}_+^{1+n})$,*

$$(4.12) \quad \mathcal{L}_{1/2}(f) = \nabla \mathcal{L}_1(f) \quad \text{in } \mathcal{D}'(\mathbb{R}_+^{1+n}).$$

In particular, (4.8) holds if and only if for all $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{1+n}})$,

$$(4.13) \quad - \int_{\mathbb{R}_+^{1+n}} \mathcal{L}_1(f) \overline{\partial_t \phi} + \int_{\mathbb{R}_+^{1+n}} A \mathcal{L}_{1/2}(f) \cdot \overline{\nabla \phi} = \int_{\mathbb{R}_+^{1+n}} f \overline{\phi}.$$

Proof. It suffices to prove (4.12). Indeed, if it holds, then $\nabla \mathcal{L}_1(f) = \mathcal{L}_{1/2}(f)$ in $L_{\text{loc}}^1((0, \infty); L^2(\mathbb{R}^n))$, so the equivalence follows obviously.

Let us prove (4.12). Let $\psi \in C_c^\infty(\mathbb{R}_+^{1+n}; \mathbb{C}^n)$. Fubini's theorem yields

$$\begin{aligned} \int_{\mathbb{R}_+^{1+n}} \mathcal{L}_{1/2}(f) \cdot \bar{\psi} &= \int_0^\infty dt \int_0^t \langle \nabla e^{-(t-s)L} f(s), \psi(t) \rangle_{L^2(\mathbb{R}^n; \mathbb{C}^n)} ds \\ &= - \int_0^\infty dt \int_0^t \langle e^{-(t-s)L} f(s), \operatorname{div} \psi(t) \rangle_{L^2(\mathbb{R}^n)} ds \\ &= - \int_{\mathbb{R}_+^{1+n}} \mathcal{L}_1(f) \operatorname{div} \bar{\psi}. \end{aligned}$$

This completes the proof. \square

We also need an embedding lemma.

Lemma 4.6. *Let $m \in \mathbb{N}$ and $\beta > -1/2$. Then, $L_c^\infty(\overline{\mathbb{R}_+^{1+n}})$ embeds into $T_{-\beta}^{p;m}$ and $T_{-\beta;(\sigma)}^{\infty;m}$ for $p \in (0, \infty)$ and $0 \leq \sigma \leq \frac{m}{n}(\beta + \frac{1}{2})$.*

Proof. Let ϕ be an $L_c^\infty(\overline{\mathbb{R}_+^{1+n}})$ -function supported in $[0, R] \times B(0, R^{1/m})$ for some $R > 0$. Then $\mathcal{A}_{-\beta;m}(\phi)$ is supported on $B(0, 2R^{1/m})$, and for any $x \in B(0, 2R^{1/m})$,

$$\mathcal{A}_{-\beta;m}(\phi)(x)^2 = \int_0^R dt \int_{B(x, t^{1/m})} |t^\beta \phi(t, y)|^2 dy \leq \|\phi\|_\infty^2 \int_0^R t^{2\beta} dt \lesssim \|\phi\|_\infty^2$$

as $\beta > -1/2$. Thus, for any $p \in (0, \infty)$,

$$\|\phi\|_{T_{-\beta}^{p;m}} = \|\mathcal{A}_{-\beta;m}(\phi)\|_p \lesssim_{R,\beta} \|\phi\|_\infty.$$

Similarly, for any ball $B = B(x_0, r^{1/m}) \subset \mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{|B|^\sigma} \left(\int_0^r \mathbf{1}_{[0,R]}(t) dt \int_B |t^\beta \phi(t, y)|^2 dy \right)^{1/2} \\ &\lesssim r^{-\frac{n}{m}\sigma} \left(\int_0^{\min\{r,R\}} t^{2\beta} dt \right)^{1/2} \|\phi\|_\infty \lesssim R^{\beta + \frac{1}{2} - \frac{n}{m}\sigma} \|\phi\|_\infty. \end{aligned}$$

We hence conclude by taking the supremum over all balls B , since the controlling constant is independent of B . \square

We now prove Proposition 4.1.

Proof of Proposition 4.1. First, observe that

$$(4.14) \quad \begin{cases} \mathcal{L}_1 \in \operatorname{SIO}_{2,q,\infty}^{1+} & \text{if } p_-(L) < q < p_+(L), \\ \mathcal{L}_{1/2} \in \operatorname{SIO}_{2,q,\infty}^{\frac{1}{2}+} & \text{if } q_-(L) < q < q_+(L). \end{cases}$$

Indeed, it suffices to consider the former assertion, and the latter follows similarly. When $p_-(L) < q \leq r < p_+(L)$, the family $(e^{-(t-s)L})_{t>s}$ has $L^q - L^r$ off-diagonal decay of type $(1, 2, M)$ for any $M > 0$. Thus, for

$q \in (p_-(L), p_+(L))$, the function $(t, s) \mapsto \mathbf{1}_{\{t>s\}}(t, s)e^{-(t-s)L}$ belongs to $\text{SK}_{2,q,\infty}^1$. In consequence, Lemma 2.6 implies $\mathcal{L}_1 \in \text{SIO}_{2,q,\infty}^{1+}$.

Next, we prove (i). Since $L_c^2(\mathbb{R}_+^{1+n})$ is contained in T_β^p , applying (4.14), Proposition 3.3, Corollary 3.5, and Lemma 4.5, we get (i) as

$$\|\mathcal{L}_1(f)\|_{T_{\beta+1}^p} \lesssim \|f\|_{T_\beta^p}, \quad \|\nabla \mathcal{L}_1(f)\|_{T_{\beta+\frac{1}{2}}^p} = \|\mathcal{L}_{1/2}(f)\|_{T_{\beta+\frac{1}{2}}^p} \lesssim \|f\|_{T_\beta^p}.$$

Then consider (ii). Fix $\phi \in C_c^\infty(\overline{\mathbb{R}_+^{1+n}})$. For (4.7), Lemma 4.6 yields $\phi \in (T_\beta^p)'$, $\nabla \phi \in (T_{\beta+\frac{1}{2}}^p)'$, and $\partial_t \phi \in (T_{\beta+1}^p)'$. Using duality of tent spaces and (i), we get

$$\text{LHS (4.7)} \lesssim_\phi \|\mathcal{L}_1(f)\|_{T_{\beta+1}^p} + \|\nabla \mathcal{L}_1(f)\|_{T_{\beta+\frac{1}{2}}^p} + \|f\|_{T_\beta^p} \lesssim \|f\|_{T_\beta^p}.$$

This proves (4.7). For (4.8), the proof is divided into two cases.

Case 1: $f \in L_c^2((0, \infty); D(L))$. By definition of \mathcal{L}_1 (cf. (4.5)), we have $\mathcal{L}_1(f) \in C([0, \infty); L^2(\mathbb{R}^n))$ with $\mathcal{L}_1(f)(0) = 0$, and $\mathcal{L}_1(f)(t) \in D(L)$ for all $t > 0$. Thus, we infer $L\mathcal{L}_1(f) \in L^2(\mathbb{R}_+^{1+n})$, so

$$(4.15) \quad \partial_t \mathcal{L}_1(f) = f + \text{div}(A\nabla \mathcal{L}_1(f)) = f - L\mathcal{L}_1(f) \in L^2(\mathbb{R}_+^{1+n}).$$

Then (4.8) holds by testing $\phi \in C_c^\infty(\mathbb{R}_+^{1+n})$ against (4.15) and using integration by parts.

Case 2: $f \in L_c^2(\mathbb{R}_+^{1+n})$. Lemma 4.5 says that it is equivalent to prove (4.13). Lemma 4.6 implies $\phi \in L^2(\mathbb{R}_+^{1+n})$, $\nabla \phi \in L_{-1/2}^2(\mathbb{R}_+^{1+n})$, and $\partial_t \phi \in L_{-1}^2(\mathbb{R}_+^{1+n})$. Thus, (4.13) readily follows by a standard density argument using Case 1 and continuity of $\mathcal{L}_1 : L^2(\mathbb{R}_+^{1+n}) \rightarrow L_1^2(\mathbb{R}_+^{1+n})$ and $\mathcal{L}_{1/2} : L^2(\mathbb{R}_+^{1+n}) \rightarrow L_{1/2}^2(\mathbb{R}_+^{1+n})$. This completes the proof. \square

4.3. Existence. In this section, we construct the extension of \mathcal{L}_1 , and prove Theorem 4.2 (a) and (b).

Lemma 4.7. *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. Then, for $\kappa = 1$ and $\kappa = 1/2$, \mathcal{L}_κ extends to bounded operators from T_β^p to $T_{\beta+\kappa}^p$. Furthermore, (4.12) holds for the extensions applied to $f \in T_\beta^p$.*

Proof. Using (4.14), the extension follows by applying Proposition 3.3 and Corollary 3.5. In the following, the extension of \mathcal{L}_κ to tent spaces is still denoted by \mathcal{L}_κ . We next divide the proof of (4.12) in two cases.

Case 1: $p_L(\beta) < p < \infty$. Lemma 4.5 says that it holds when $f \in L_c^2(\mathbb{R}_+^{1+n})$. A standard density argument extends it to $f \in T_\beta^p$, ensured by three ingredients: the density of $L_c^2(\mathbb{R}_+^{1+n})$ in T_β^p ; the continuity of \mathcal{L}_κ from T_β^p to $T_{\beta+\kappa}^p$ for $\kappa = 1, 1/2$; and the fact that any $C_c^\infty(\mathbb{R}_+^{1+n})$ -function belongs to $(T_{\beta+1}^p)' \cap (T_{\beta+\frac{1}{2}}^p)'$ (cf. Lemma 4.6).

Case 2: $p = \infty$. Let \mathcal{L}_κ^* be the adjoint of \mathcal{L}_κ with respect to $L^2(\mathbb{R}_+^{1+n})$ -duality for $\kappa = 1, 1/2$. Lemma 3.1 shows $L_c^2(\mathbb{R}_+^{1+n})$ is weak* dense in T_β^∞ , so the desired distribution equality follows by the continuity of $\mathcal{L}_\kappa^* : T_{-\beta-\kappa}^1 \rightarrow T_{-\beta}^1$ for $\kappa = 1, 1/2$, and the fact that $C_c^\infty(\mathbb{R}_+^{1+n})$ is contained in $T_{-\beta-1}^1 \cap T_{-\beta-\frac{1}{2}}^1$. This completes the proof. \square

We can now prove Theorem 4.2 (a) and (b).

Proof of Theorem 4.2 (a) and (b). The extension is given by Lemma 4.7. It also implies that if $u = \mathcal{L}_1(f)$ with $f \in T_\beta^p$, then $\nabla u = \mathcal{L}_{1/2}(f)$, so (a) follows by the boundedness of \mathcal{L}_1 and $\mathcal{L}_{1/2}$.

Then consider (b). By (3.1), it is clear that both of u and ∇u belong to $L_{\text{loc}}^2(\mathbb{R}_+^{1+n})$, so $u \in L_{\text{loc}}^2((0, \infty); H_{\text{loc}}^1(\mathbb{R}^n))$. The inequality (4.9) is also clear by Lemma 4.6 and (a), so it only remains to prove (4.10). Here, we can follow the proof of Proposition 4.1 (ii), that is to say,

- Interpreting (4.10) as (4.13) using duality of tent spaces, since $\nabla \mathcal{L}_1(f)$ agrees with $\mathcal{L}_{1/2}(f)$ in $T_{\beta+\frac{1}{2}}^p$ and $\nabla \phi \in (T_{\beta+\frac{1}{2}}^p)'$;
- Showing (4.13) by density with a sequence in $L_c^2(\mathbb{R}_+^{1+n})$ that approximates f in T_β^p (or weak* if $p = \infty$, see Case 2 of Lemma 4.7).

All the ingredients have been provided in Lemma 4.6 and 4.7 and details are left to the reader. \square

4.4. Uniqueness. In this section, we demonstrate Theorem 4.3 using the homotopy identity. We start with several preliminary lemmas.

Lemma 4.8. *Let $0 < p \leq 2$ and $\beta \in \mathbb{R}$. For $0 < a < b < \infty$, T_β^p can be embedded into $L_\beta^2((a, b) \times \mathbb{R}^n) := L^2((a, b), t^{-2\beta} dt; L^2(\mathbb{R}^n))$ with*

$$\|u\|_{L_\beta^2((a,b) \times \mathbb{R}^n)} \lesssim_p a^{-\frac{n}{2p}} b^{\frac{n}{4}} \|u\|_{T_\beta^p}.$$

Proof. It suffices to prove the case $\beta = 0$. For any $R > a^{1/2}$, observe that $(a, b) \times B(0, R) \subset \Gamma_\lambda(x)$ for any $x \in B(0, R)$ with $\lambda := 2Ra^{-1/2} > 1$, where $\Gamma_\lambda(x)$ is the cone at x of aperture λ and homogeneity $m = 2$,

given by (3.2). Then applying (3.4) for $p \leq 2$ yields

$$\begin{aligned}
\|u\|_{L^2((a,b) \times B(0,R))} &\leq b^{n/4} \left(\int_a^b \int_{B(0,R)} |u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2} \\
&\leq b^{n/4} \inf_{x \in B(0,R)} \mathcal{A}_{0;2}^{(\lambda)}(u)(x) \\
&\leq b^{n/4} \left(\int_{B(0,R)} \mathcal{A}_{0;2}^{(\lambda)}(u)(x)^p dx \right)^{1/p} \\
&\lesssim b^{\frac{n}{4}} R^{-\frac{n}{p}} \|u\|_{T_{\lambda,0}^{p;2}} \lesssim b^{\frac{n}{4}} R^{-\frac{n}{p}} \lambda^{\frac{n}{p}} \|u\|_{T_0^p} \lesssim a^{-\frac{n}{2p}} b^{\frac{n}{4}} \|u\|_{T_0^p},
\end{aligned}$$

where $\mathcal{A}_{0;2}^{(\lambda)}$ is defined in (3.3). The controlling constant is independent of R , so we conclude by letting R tend to infinity. \square

Lemma 4.9. *Let $0 < p \leq \infty$ and $\beta \in \mathbb{R}$. Let u be in T_β^p . Then for $0 < a < b < \infty$, the function*

$$F(x) := \left(\int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dt dy \right)^{1/2}$$

belongs to $L^p(\mathbb{R}^n)$ with $\|F\|_p \lesssim_{a,b,\beta} \|u\|_{T_\beta^p}$.

Proof. First consider $0 < p < \infty$. Pick $(z_k)_{1 \leq k \leq N}$ in $B(0, b^{1/2})$ such that $B(0, b^{1/2}) \subset \bigcup_{k=1}^N B(z_k, a^{1/2})$. Thus, for any $x \in \mathbb{R}^n$, $B(x, b^{1/2}) \subset \bigcup_{k=1}^N B(x+z_k, a^{1/2})$. We hence get

$$\begin{aligned}
F(x) &\leq \sum_{k=1}^N b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \left(\int_a^b \int_{B(x+z_k, a^{1/2})} |t^{-\beta} u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2} \\
&\leq b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \sum_{k=1}^N \left(\int_a^b \int_{B(x+z_k, t^{1/2})} |t^{-\beta} u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2}.
\end{aligned}$$

Therefore, by taking L^p -norms, we conclude that

$$(4.16) \quad \|F\|_p \lesssim_N b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \|u\|_{T_\beta^p}.$$

For $p = \infty$, note that $(a, b) \times B(x, b^{1/2}) \subset (0, b) \times B(x, b^{1/2})$, so

$$\begin{aligned}
F(x) &\lesssim b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \left(\int_a^b \int_{B(x, b^{1/2})} |t^{-\beta} u(t,y)|^2 dt dy \right)^{1/2} \\
&\leq b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \sup_{B:x \in B} \left(\int_0^{r(B)^2} \int_B |t^{-\beta} u(t,y)|^2 dt dy \right)^{1/2}.
\end{aligned}$$

Taking L^∞ -norm on both sides implies (4.16). \square

The next lemma is a standard result for such parabolic equations, and we use the form given in [AMP19, Proposition 3.6].

Lemma 4.10. *Let $0 < a < b < \infty$, $B \subset \mathbb{R}^n$ be a ball, and f be in $L^2((a, b) \times 2B)$. Let $u \in L^2((a, b); H^1(2B))$ be a weak solution to (IC) with source term f in $(a, b) \times 2B$. Then, u lies in $C([a, b]; L^2(B))$ and*

$$\begin{aligned} \|u(b)\|_{L^2(B)}^2 &\lesssim \left(\frac{\Lambda_1^2}{\Lambda_0 r(B)^2} + \frac{1}{b-a} \right) \int_a^b \|u(s)\|_{L^2(2B)}^2 ds \\ &\quad + (b-a) \int_a^b \|f(s)\|_{L^2(2B)}^2 ds. \end{aligned}$$

We also recall a standard corollary of Banach–Steinhaus theorem.

Lemma 4.11. *Let X be a Banach space and X^* be its dual. Let (x_j) be a sequence in X that converges to x in X and (y_j) be a sequence in X^* that weakly* converges to y in X^* . Then, the sequence of pairing $(\langle y_j, x_j \rangle)$ converges to $\langle y, x \rangle$.*

We now prove Theorem 4.3.

Proof of Theorem 4.3. Let u and \tilde{u} be two global weak solutions in $T_{\beta+1}^p$ to (IC) with same source term $f \in T_{\beta}^p$. Define $v := u - \tilde{u}$. It lies in $L_{\text{loc}}^2((0, \infty); H_{\text{loc}}^1(\mathbb{R}^n)) \cap T_{\beta+1}^p$ and is a global weak solution to (IC) with null source term. For $0 < a < b < \infty$ and $\gamma > 0$, we have

$$\int_{\mathbb{R}^n} \left(\int_a^b \int_{B(x, b^{1/2})} |v(t, y)|^2 dt dy \right)^{1/2} e^{-\gamma|x|^2} dx \lesssim_{\gamma, p, a, b, \beta} \|v\|_{T_{\beta}^p} < \infty$$

by Lemma 4.8 for $p_L(\beta) < p \leq 2$ and Lemma 4.9 for $2 \leq p \leq \infty$. Thus, [AMP19, Theorem 5.1] ensures that the homotopy identity holds. More precisely, for $0 < s < t < \infty$ and $h \in C_c^\infty(\mathbb{R}^n)$, we have

$$(4.17) \quad \int_{\mathbb{R}^n} v(t, x) \bar{h}(x) dx = \int_{\mathbb{R}^n} v(s, x) \overline{((e^{-(t-s)L})^* h)}(x) dx.$$

We now take limits as $s \rightarrow 0$ on both sides to show $v(t) = 0$ in $\mathcal{D}'(\mathbb{R}^n)$, hence proving $v \equiv 0$. The proof is divided into three cases.

Case 1: $2 \leq p < \infty$. Let $\delta > 0$ be a constant. Using (4.4) for $t \in (0, \delta]$, we have $\|v(t)\|_{E_{\delta}^p} \lesssim \|v(t)\|_{E_{t/16}^p}$. Since v lies in $L_{\text{loc}}^2((0, \infty); H_{\text{loc}}^1(\mathbb{R}^n))$,

Lemma 4.10 yields for any $t \in (0, \delta]$,

$$\begin{aligned}
(4.18) \quad & \left(\int_{\mathbb{R}^n} \left(\int_{B(x, \frac{\sqrt{t}}{4})} |v(t, y)|^2 dy \right)^{p/2} dx \right)^{1/p} \\
& \lesssim \left(\int_{\mathbb{R}^n} \left(\int_{t/2}^t \int_{B(x, \frac{\sqrt{t}}{2})} |v(s, y)|^2 ds dy \right)^{p/2} dx \right)^{1/p} \\
& \lesssim t^{\beta + \frac{1}{2}} \|v\|_{T_{\beta+1}^p}.
\end{aligned}$$

Hence $\|v(t)\|_{E_{t/16}^p} \lesssim t^{\beta + \frac{1}{2}} \|v\|_{T_{\beta+1}^p}$. Thus, $v(t)$ tends to 0 in E_δ^p as $t \rightarrow 0$. Thanks to [AMP19, Lemma 4.7], we have $(e^{-(t-s)L})^* h$ tends to $(e^{-tL})^* h$ in $E_\delta^{p'}$ as $s \rightarrow 0$. By Lemma 4.11 with $X = E_\delta^p$, the right-hand side of (4.17) converges to 0 as $s \rightarrow 0$, and we obtain $v(t) = 0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Case 2: $p = \infty$. Let $\delta > 0$ be a constant. For any $t \in (0, \delta]$,

$$\begin{aligned}
\|v(t)\|_{E_\delta^\infty} & \lesssim \|v(t)\|_{E_{t/16}^\infty} = \operatorname{esssup}_{x \in \mathbb{R}^n} \left(\int_{B(x, \frac{\sqrt{t}}{4})} |v(t, y)|^2 dy \right)^{1/2} \\
& \lesssim \operatorname{esssup}_{x \in \mathbb{R}^n} \left(\int_{t/2}^t \int_{B(x, \frac{\sqrt{t}}{2})} |v(s, y)|^2 ds dy \right)^{1/2} \lesssim t^{\beta + \frac{1}{2}} \|v\|_{T_{\beta+1}^\infty}.
\end{aligned}$$

Thus, $v(t)$ converges to 0 in E_δ^∞ as $t \rightarrow 0$. It is also shown in [AMP19, Lemma 4.7] that $(e^{-(t-s)L})^* h$ tends to $(e^{-tL})^* h$ in E_δ^1 as $s \rightarrow 0$. Thus, we get $v(t) = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ by the same argument as in Case 1.

Case 3: $p_L(\beta) < p < 2$. We claim that,

$$(4.19) \quad \|v(t)\|_p \lesssim t^{\beta + \frac{1}{2}} \|v\|_{T_{\beta+1}^p}, \quad \forall 0 < p \leq 2, \quad \forall t > 0.$$

Indeed, using Hölder's inequality and (4.18), we have

$$\|v(t)\|_p = \left(\int_{\mathbb{R}^n} \int_{B(x, \frac{\sqrt{t}}{4})} |v(t, y)|^p dy dx \right)^{1/p} \lesssim t^{\beta + \frac{1}{2}} \|v\|_{T_{\beta+1}^p}.$$

The claim hence follows. Moreover, Lemma 4.10 and Lemma 4.8 imply

$$\begin{aligned}
\|v(t)\|_{L^2(B(x, R))}^2 & \lesssim \left(\frac{\Lambda_1^2}{\Lambda_0 R^2} + \frac{1}{t} \right) t^{2(\beta+1)} \int_{t/2}^t \|s^{-(\beta+1)} v(s)\|_{L^2(B(x, 2R))}^2 ds \\
& \lesssim \left(\frac{\Lambda_1^2}{\Lambda_0 R^2} + \frac{1}{t} \right) t^{2(\beta+1) + n[2, p]} \|v\|_{T_{\beta+1}^p}^2.
\end{aligned}$$

By letting $R \rightarrow \infty$, we get

$$(4.20) \quad \|v(t)\|_2 \lesssim t^{\beta+\frac{1}{2}-\frac{n}{2}[p,2]} \|v\|_{T_{\beta+1}^p}, \quad \forall 0 < p \leq 2, \quad \forall t > 0.$$

As $p_L(\beta) < p < 2$ implies

$$\frac{1}{p} - \frac{2\beta+1}{n} < \frac{1}{p_-(L)},$$

there exists $q \in (p, 2) \cap (p_-(L), 2)$ such that

$$\frac{1}{p} - \frac{2\beta+1}{n} < \frac{1}{q}.$$

Interpolating (4.19) and (4.20) yields

$$\|v(t)\|_q \lesssim t^{\beta+\frac{1}{2}-\frac{n}{2}[p,q]} \|v\|_{T_{\beta+1}^p}, \quad \forall q \in (p, 2), \quad \forall t > 0.$$

Thus, $v(s)$ tends to 0 in $L^q(\mathbb{R}^n)$ as $s \rightarrow 0$. Thanks to [Aus07, Proposition 3.15], we also know that $(e^{-(t-s)L})^*h$ tends to $(e^{-tL})^*h$ in $L^{q'}(\mathbb{R}^n)$ as $s \rightarrow 0$ since $2 < q' < p_+(L^*) = (p_-(L))'$. Using (4.17) and Lemma 4.11 with $X = L^q(\mathbb{R}^n)$, we conclude that $v(t) = 0$ in $\mathcal{D}'(\mathbb{R}^n)$.

This completes the proof. \square

5. MAXIMAL REGULARITY ON TENT SPACES

In this section, we prove maximal regularity of the system (IC), *i.e.*, Theorem 4.2 (c), and apply it to investigate the boundary behavior of global weak solutions to (IC), see (d) and (e). Throughout the section, the notation is the same as in Section 4 without special mention.

5.1. Maximal regularity operator. Define the *maximal regularity operator* \mathcal{L}_0 on $L^2((0, \infty); D(L))$ by

$$(5.1) \quad \mathcal{L}_0(f)(t, y) := \int_0^t (Le^{-(t-s)L}f(s))(y)ds, \quad \text{a.e. } (t, y) \in \mathbb{R}_+^{1+n}.$$

The integral makes sense as an $L^2(\mathbb{R}^n)$ -valued Bochner integral. Recall that the celebrated de Simon's theorem [dS64] asserts \mathcal{L}_0 can be extended by density to a bounded operator on $L^2(\mathbb{R}_+^{1+n})$. Denote by $\tilde{\mathcal{L}}_0$ this extension.

Lemma 5.1. $\tilde{\mathcal{L}}_0$ belongs to $\text{SIO}_{2,q,\infty}^{0+}$ for any $q \in (p_-(L), p_+(L))$.

Proof. The boundedness of $\tilde{\mathcal{L}}_0$ on $L^2(\mathbb{R}_+^{1+n})$ is clear by construction, and the function $(t, s) \mapsto \mathbf{1}_{\{t>s\}}(t, s)Le^{-(t-s)L}$ belongs to $\text{SK}_{2,q,\infty}^0$ for

any $q \in (p_-(L), p_+(L))$ by [Aus07, Proposition 3.15]. Thus, Lemma 2.8 yields for any $f \in L^2_{\mathbf{b}}(\mathbb{R}_+^{1+n})$,

$$(t, x) \mapsto \int_0^t (Le^{-(t-s)L}f)(x)ds$$

defines a function in $L^2_{\text{loc}}(\overline{\mathbb{R}_+^{1+n}} \setminus \pi(f))$. It only remains to prove the representation (2.10), or equivalently, for any $f \in L^2_{\mathbf{b}}(\mathbb{R}_+^{1+n})$ and $g \in C_c^\infty(\mathbb{R}_+^{1+n})$ with $\pi(g) \cap \pi(f) = \emptyset$,

(5.2)

$$\left\langle \tilde{\mathcal{L}}_0(f), g \right\rangle_{L^2(\mathbb{R}_+^{1+n})} = \int_0^\infty \int_{\mathbb{R}^n} \left(\int_0^t (Le^{-(t-s)L}f(s))(x)ds \right) \bar{g}(t, x) dt dx.$$

To prove (5.2), we proceed by putting the following two observations together. Fix $g \in C_c^\infty(\mathbb{R}_+^{1+n})$.

The first observation is that for any $f \in L^2((0, \infty); D(L))$,

$$(5.3) \quad \langle \mathcal{L}_0(f), g \rangle_{L^2(\mathbb{R}_+^{1+n})} = \langle A\mathcal{L}_{1/2}(f), \nabla g \rangle_{L^2(\mathbb{R}_+^{1+n})}.$$

Indeed, since $\mathcal{L}_{1/2}(f)$ is given by an $L^2(\mathbb{R}^n)$ -valued Bochner integral, all the following integrals converge absolutely, so Fubini's theorem ensures

$$\begin{aligned} \langle \mathcal{L}_0(f), g \rangle_{L^2(\mathbb{R}_+^{1+n})} &= \int_0^\infty \int_0^t \langle Le^{-(t-s)L}f(s), g(t) \rangle_{L^2(\mathbb{R}^n)} ds dt \\ &= \int_0^\infty \int_0^t \langle A\nabla e^{-(t-s)L}f(s), \nabla g(t) \rangle_{L^2(\mathbb{R}^n)} ds dt \\ &= \langle A\mathcal{L}_{1/2}(f), \nabla g \rangle_{L^2(\mathbb{R}_+^{1+n})}. \end{aligned}$$

The second observation is that when f lies in $L^2_{\mathbf{b}}(\mathbb{R}_+^{1+n})$ with $\pi(f)$ disjoint with $\pi(g)$, we have the identity

$$(5.4) \quad \begin{aligned} &\langle A\mathcal{L}_{1/2}(f), \nabla g \rangle_{L^2(\mathbb{R}_+^{1+n})} \\ &= \int_0^\infty \int_0^t \int_{\mathbb{R}^n} (A\nabla e^{-(t-s)L}f(s))(x) \cdot \overline{\nabla g}(t, x) dx ds dt \\ &= - \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \text{div}(A\nabla e^{-(t-s)L}f(s))(x) \bar{g}(t, x) dx ds dt =: I \end{aligned}$$

Indeed, all the integrals converge absolutely as for fixed (t, s) , both $x \mapsto (A\nabla e^{-(t-s)L}f(s))(x)$ and $x \mapsto \text{div}(A\nabla e^{-(t-s)L}f(s))(x)$ are integrable on the support of $g(t)$, using Lemma 2.6 with $\kappa = 1/2$ and Lemma 2.8 with $\kappa = 0$. Thus, the second equality follows by integration by parts for the divergence.

We now prove (5.2) as follows. By a limiting argument from the first observation, we have that for any $f \in L^2(\mathbb{R}_+^{1+n})$, (5.3) becomes

$$(5.5) \quad \left\langle \tilde{\mathcal{L}}_0(f), g \right\rangle_{L^2(\mathbb{R}_+^{1+n})} = \left\langle A\mathcal{L}_{1/2}(f), \nabla g \right\rangle_{L^2(\mathbb{R}_+^{1+n})}$$

Indeed, the argument is similar to Lemma 4.5, but using the continuity of $\mathcal{L}_{1/2}$ and $\tilde{\mathcal{L}}_0$. We hence leave details to the reader.

Next, for any $f \in L^2_{\mathfrak{b}}(\mathbb{R}_+^{1+n})$ with $\pi(f)$ disjoint with $\pi(g)$, (5.5) and (5.4) yield

$$\left\langle \tilde{\mathcal{L}}_0(f), g \right\rangle_{L^2(\mathbb{R}_+^{1+n})} = I.$$

So it only remains to observe that for a.e. $s < t$ and $x \in \mathbb{R}^n$,

$$\operatorname{div}(A\nabla e^{-(t-s)L}f(s))(x) = (Le^{-(t-s)L}f(s))(x)$$

by definition of L , as $f(s)$ belongs to $L^2(\mathbb{R}^n)$. \square

We then consider a further extension of $\tilde{\mathcal{L}}_0$ to tent spaces. The following proposition corresponds to Proposition 1.1.

Proposition 5.2. *Let $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$. Then, $\tilde{\mathcal{L}}_0$ can be extended to a bounded operator on T^p_{β} .*

Proof. Lemma 5.1 says $\tilde{\mathcal{L}}_0$ lies in $\text{SIO}_{2,q,\infty}^{0+}$ when $p_-(L) < q < p_+(L)$, and [AA11, Theorem 1.3] shows $\tilde{\mathcal{L}}_0$ is bounded on $L^2_{\beta}(\mathbb{R}_+^{1+n})$ for any $\beta > -1/2$. We hence conclude by invoking Proposition 3.3 and Corollary 3.5. \square

The extension of $\tilde{\mathcal{L}}_0$ to T^p_{β} is still denoted by $\tilde{\mathcal{L}}_0$ in the sequel. Write \mathcal{L}_1 and $\mathcal{L}_{1/2}$ for their extensions to T^p_{β} . Let us prove Theorem 4.2 (c).

Proof of Theorem 4.2 (c). Let f be in T^p_{β} and $u = \mathcal{L}_1(f)$. It suffices to identify $-\operatorname{div}(A\nabla u)$ with $\tilde{\mathcal{L}}_0(f)$ in $\mathcal{D}'(\mathbb{R}_+^{1+n})$. Indeed, if it holds, then Proposition 5.2 yields

$$\|\operatorname{div}(A\nabla u)\|_{T^p_{\beta}} = \|\tilde{\mathcal{L}}_0(f)\|_{T^p_{\beta}} \lesssim \|f\|_{T^p_{\beta}}.$$

Moreover, (b) implies $\partial_t u = \operatorname{div}(A\nabla u) + f$ in $\mathcal{D}'(\mathbb{R}_+^{1+n})$. Thus, it follows directly that $\partial_t u$ lies in T^p_{β} with

$$\|\partial_t u\|_{T^p_{\beta}} \lesssim \|\operatorname{div}(A\nabla u)\|_{T^p_{\beta}} + \|f\|_{T^p_{\beta}} \lesssim \|f\|_{T^p_{\beta}}.$$

To prove the identification, we first observe that

$$(5.6) \quad \tilde{\mathcal{L}}_0(f) = -\operatorname{div}(A\mathcal{L}_{1/2}(f)) \quad \text{in } \mathcal{D}'(\mathbb{R}_+^{1+n}).$$

In fact, it follows from (5.5) with a limiting argument applied to a sequence in $T_\beta^p \cap L^2(\mathbb{R}_+^{1+n})$ approximating f (weak* if $p = \infty$, see Case 2 of Lemma 4.7). The second observation is that

$$(5.7) \quad -\operatorname{div}(A\mathcal{L}_{1/2}(f)) = -\operatorname{div}(A\nabla\mathcal{L}_1(f)) \quad \text{in } \mathcal{D}'(\mathbb{R}_+^{1+n}).$$

Indeed, Lemma 4.7 says $\mathcal{L}_{1/2}(f) = \nabla\mathcal{L}_1(f)$ in $T_{\beta+\frac{1}{2}}^p$, hence in $L_{\text{loc}}^2(\mathbb{R}_+^{1+n})$, so (5.7) follows by multiplying by A and applying the divergence. Combining (5.6) and (5.7) yields $\tilde{\mathcal{L}}_0(f) = -\operatorname{div}(A\nabla u)$ as desired. \square

5.2. Boundary behavior. In this section, we prove Theorem 4.2 (d) and (e). The first result is valid for arbitrary function in $T_{\beta+1}^p$.

Lemma 5.3. *Let $0 < p \leq \infty$ and $\beta > -1/2$. Let u be in $T_{\beta+1}^p$. Then, for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{t \rightarrow 0} \left(\int_{W(t,x)} |u(s,y)|^2 ds dy \right)^{1/2} = 0.$$

Proof. First consider $0 < p < \infty$. For any $t > 0$ and $x \in \mathbb{R}^n$, we have

$$\int_{W(t,x)} |u(s,y)|^2 ds dy \lesssim \int_t^{2t} ds \int_{B(x,s^{1/2})} |u(s,y)|^2 dy \lesssim t^{2\beta+1} \mathcal{A}_{\beta+1;2}(u)(x),$$

where $\mathcal{A}_{\beta+1;2}$ is defined in (3.3). Since $\|\mathcal{A}_{\beta+1;2}(u)\|_p \approx \|u\|_{T_{\beta+1}^p} < \infty$, $\mathcal{A}_{\beta+1}(u)(x)$ is finite for a.e. $x \in \mathbb{R}^n$, so the right-hand side tends to 0 when $t \rightarrow 0$ as $\beta > -1/2$.

Then for $p = \infty$, note that $W(t,x) \subset (0,2t) \times B(x,(2t)^{1/2})$, so

$$\int_{W(t,x)} |u(s,y)|^2 ds dy \lesssim t^{2\beta+1} \int_t^{2t} ds \int_{B(x,(2t)^{1/2})} |s^{-(\beta+1)} u(s,y)|^2 dy.$$

Taking esssup on $x \in \mathbb{R}^n$, we get

$$\operatorname{esssup}_{x \in \mathbb{R}^n} \int_{W(t,x)} |u(s,y)|^2 ds dy \lesssim t^{2\beta+1} \|u\|_{T_{\beta+1}^\infty}.$$

This completes the proof. \square

Lemma 5.4. *For any $p \in [2, \infty]$ and $\beta \geq -1$, $T_{\beta+1}^p$ is contained in $L_{\text{loc}}^2(\overline{\mathbb{R}_+^{1+n}})$. More precisely, for ball $B \subset \mathbb{R}^n$ and $0 < T < r(B)^2$,*

$$(5.8) \quad \left(\int_0^T \int_B |u(t,y)|^2 dt dy \right)^{1/2} \lesssim_p T^{\beta+1} |B|^{[2,p]} \|u\|_{T_{\beta+1}^p}.$$

Proof. By definition, it is trivial if $p = \infty$. We thus assume $2 \leq p < \infty$. For any $\alpha > 0$ and ball $B \subset \mathbb{R}^n$, define the *tent* on B of *aperture* α (and *homogeneity* $m = 2$) as

$$T_\alpha(B) := \{(t,y) \in \mathbb{R}_+^{1+n} : 0 < \alpha t^{1/2} \leq \operatorname{dist}(y, B^c)\}.$$

Observe that $(0, T) \times B \subset T_\alpha(2B)$ with $\alpha := T^{-1/2}r(B) > 1$. Moreover, for any $(t, y) \in T_\alpha(2B)$, since $\alpha t^{1/2} < r(B) < \text{dist}(y, (2B)^c)$, we know that $B(y, \alpha t^{1/2}) \subset 2B$. In other words,

$$\int_{2B} \mathbf{1}_{B(0,1)} \left(\frac{x-y}{\alpha t^{1/2}} \right) dx \approx (\alpha t^{1/2})^n.$$

With the help of these two observations, we get

$$\begin{aligned} \|u\|_{L^2((0,T) \times B)}^2 &\lesssim \alpha^{-n} \int_{T_\alpha(2B)} |u(t, y)|^2 dt \frac{dy}{t^{n/2}} \int_{2B} \mathbf{1}_{B(0,1)} \left(\frac{x-y}{\alpha t^{1/2}} \right) dx \\ &\leq \alpha^{-n} \int_{2B} dx \int_0^\infty \int_{B(x, \alpha t^{1/2})} |u(t, y)|^2 dt \frac{dy}{t^{n/2}} \\ &\leq T^{2(\beta+1)} \alpha^{-n} \int_{2B} \mathcal{A}_{\beta+1;2}^{(\alpha)}(u)(x)^2 dx \\ &\lesssim_p T^{2(\beta+1)} \alpha^{-n} |B|^{[2,p]} \|\mathcal{A}_{\beta+1;2}^{(\alpha)}(u)\|_p^2 \lesssim T^{2(\beta+1)} |B|^{[2,p]} \|u\|_{T_{\beta+1}^p}^2. \end{aligned}$$

The last inequality follows from the change of aperture, cf. (3.4). \square

Corollary 5.5. *Let $2 \leq p \leq \infty$ and $\beta \geq -1$. Let $B \subset \mathbb{R}^n$ be a ball and $0 < T < r(B)^2$. For any $u \in T_{\beta+1}^p$,*

$$(5.9) \quad \int_0^T \|u(t)\|_{L^2(B)} dt \lesssim_\beta T^{\beta+\frac{1}{2}} |B|^{[2,p]} \|u\|_{T_{\beta+1}^p}.$$

In particular, if $\beta > -1/2$, $u(t)$ tends to 0 in a Cesàro mean in $L_{\text{loc}}^2(\mathbb{R}^n)$ as $t \rightarrow 0$.

Proof. It suffices to prove (5.9), which is a direct consequence of Lemma 5.4 by Jensen's inequality as

$$\int_0^T \|u(t)\|_{L^2(B)} dt \leq \left(\int_0^T \|u(t)\|_{L^2(B)}^2 dt \right)^{1/2} \lesssim T^{\beta+\frac{1}{2}} |B|^{[2,p]} \|u\|_{T_{\beta+1}^p}.$$

This completes the proof. \square

Corollary 5.6. *Let $2 \leq p \leq \infty$ and $\beta \geq -1$. Let u be in $T_{\beta+1}^p$ with $\partial_t u \in T_\beta^p$. Then, for any $\delta > 0$ and $t \in (0, \delta)$,*

$$(5.10) \quad \|u(t)\|_{E_\delta^\infty} \lesssim \delta^{-\frac{n}{2p}} t^{\beta+\frac{1}{2}} (\|u\|_{T_{\beta+1}^p} + \|\partial_t u\|_{T_\beta^p}).$$

Proof. Fix a ball $B \subset \mathbb{R}^n$. For any $T > 0$, by (3.1), both of u and $\partial_t u$ belong to $L^2((T/8, 2T) \times B)$. Thus, for any $T' \in [T/4, T/2]$, we have

$$u(T) = u(T') + \int_{T'}^T \partial_t u(t) dt$$

valued in $L^2(B)$. Taking average with respect to T' , we get

$$u(T) = \int_{T/4}^{T/2} u(T') dT' + \int_{T/4}^{T/2} dT' \int_{T'}^T \partial_t u(t) dt.$$

Since $T/4 \leq T' \leq \min\{t, T/2\}$, Fubini's theorem yields

$$\|u(T)\|_{L^2(B)} \lesssim \int_{T/4}^{T/2} \|u(T')\|_{L^2(B)} dT' + \int_{T/4}^T t \|\partial_t u(t)\|_{L^2(B)} dt.$$

Note that both of u and $t\partial_t u$ lie in $T_{\beta+1}^p$. Using Corollary 5.5, we deduce

$$\|u(t)\|_{E_\delta^\infty} \approx \delta^{-\frac{n}{2}} \sup_B \|u(t)\|_{L^2(B)} \lesssim \delta^{-\frac{n}{2p}} t^{\beta+\frac{1}{2}} (\|u\|_{T_{\beta+1}^p} + \|\partial_t u\|_{T_\beta^p}),$$

where B describes all balls in \mathbb{R}^n with $r(B) = \delta^{1/2}$. \square

We may finish the proof of Theorem 4.2.

Proof of Theorem 4.2 (d) and (e). Statement (d) is clear by Lemma 5.3. For (e), first note that the same deduction as for (4.18) implies

$$(5.11) \quad \|u(t)\|_{E_{t/16}^p} \lesssim t^{\beta+\frac{1}{2}} (\|u\|_{T_{\beta+1}^p} + \|f\|_{T_\beta^p}).$$

Then the discussion is divided into two cases.

Case 1: $p_L(\beta) < p \leq 2$. Observe that $E_{t/16}^p$ embeds into $L^p(\mathbb{R}^n)$ by Hölder's inequality, so $u(t)$ tends to 0 in $L^p(\mathbb{R}^n)$ as $t \rightarrow 0$.

Case 2: $2 < p \leq \infty$. Fix $\delta > 0$ and $t \in (0, \delta)$. Applying (4.4) and (5.11) yields

$$\|u(t)\|_{E_\delta^p} \lesssim \|u(t)\|_{E_{t/16}^p} \lesssim t^{\beta+\frac{1}{2}} (\|u\|_{T_{\beta+1}^p} + \|f\|_{T_\beta^p}).$$

Moreover, (5.10) also holds for u by Corollary 5.6 as $u \in T_{\beta+1}^p$ (by (a)) and $\partial_t u \in T_\beta^p$ (by (c)). Interpolation of slice spaces hence implies $u(t)$ tends to 0 as $t \rightarrow 0$ in E_δ^q for any $q \in [p, \infty]$.

This completes the proof. \square

6. FURTHER REMARKS

In fact, our results can be generalized in several directions. Let us briefly mention two possible results.

(1) For any $\beta \in \mathbb{R}$, $T \in (0, \infty]$, and $u \in L_{\text{loc}}^2((0, T) \times \mathbb{R}^n)$, the (local) weighted Kenig–Pipher non-tangential maximal function $\mathcal{N}_{\beta, T}$ is defined by

$$\mathcal{N}_{\beta, T}(u)(x) := \sup_{0 < t < T} \left(\int_{t/2}^t \int_{B(x, t^{1/2})} |s^{-\beta} u(s, y)|^2 ds dy \right)^{1/2}.$$

For any $p \in (0, \infty]$, the (local) weighted Kenig–Pipher space $X_{\beta,T}^p$ consists of measurable functions $u \in L_{\text{loc}}^2((0, T) \times \mathbb{R}^n)$ for which

$$\|u\|_{X_{\beta,T}^p} := \|\mathcal{N}_{\beta,T}(u)\|_p < \infty.$$

Simple calculation shows that for any $p \in (0, \infty]$, $\beta \in \mathbb{R}$, and $T \in (0, \infty]$, $T_{\beta+\frac{1}{2}}^p \hookrightarrow X_{\beta,\infty}^p \hookrightarrow X_{\beta,T}^p$. Thus, when $\beta > -1/2$ and $p_L(\beta) < p \leq \infty$, for any $f \in T_{\beta}^p$, the global weak solution u given by Theorem 4.2 lies in $X_{\beta+\frac{1}{2},T}^p$. In fact, only knowing that $\mathbf{1}_{(0,T]}f \in T_{\beta}^p$ suffices to conclude that $u \in X_{\beta+\frac{1}{2},T}^p$. Also, there is uniqueness in $X_{\beta+\frac{1}{2},T}^p$.

Proposition 6.1. *Let $\beta > -1/2$, $T \in (0, \infty]$, and $p_L(\beta) < p \leq \infty$. Then any weak solution $u \in X_{\beta+\frac{1}{2},T}^p$ to $\partial_t u + \text{div}(A\nabla u) = 0$ vanishes.*

It suffices to follow our argument of Theorem 4.3, with little modifications. The only noticeable one is that we use (twice) Lemma 4.8 on strips $(\frac{t}{2}, t) \times \mathbb{R}^n$ for $0 < t < T$ with the remark that

$$\|\mathbf{1}_{(\frac{t}{2}, t)}u\|_{T_{\beta+1}^p} \approx \|\mathbf{1}_{(\frac{t}{2}, t)}u\|_{X_{\beta+\frac{1}{2},T}^p}.$$

(2) Observe that Corollary 3.5 and (4.14) imply \mathcal{L}_1 extends to a bounded operator from $T_{\beta,([p,1])}^{\infty}$ to $T_{\beta+1,([p,1])}^{\infty}$ for any $0 < p \leq 1$. In fact, we can also establish well-posedness within this range using the same method as what we did for $f \in T_{\beta}^{\infty}$.

Proposition 6.2. *Let $\beta > -1/2$ and $0 < p \leq 1$. Then for any $f \in T_{\beta,([p,1])}^{\infty}$, there exists a unique global weak solution $u \in T_{\beta+1,([p,1])}^{\infty}$ to (IC) with source term f .*

Similarly, all the properties of Theorem 4.2 can be accordingly established. We only mention that (e) turns out to be that:

(e') $u(t)$ converges to 0 as $t \rightarrow 0$ in E_{δ}^{∞} for any $\delta > 0$.

REFERENCES

- [AA11] P. Auscher and A. Axelsson. Remarks on maximal regularity. In *Parabolic Problems: The Herbert Amann Festschrift*, volume 80 of *Progress in Nonlinear Differential Equations and Their Applications*, pages 45–55. Springer Basel, Basel, 2011.
- [AF17] P. Auscher and D. Frey. On the well-posedness of parabolic equations of Navier–Stokes type with BMO^{-1} data. *Journal of the Institute of Mathematics of Jussieu*, 16(5):947–985, 2017.

- [AH] P. Auscher and H. Hou. On well-posedness for parabolic Cauchy problems of Lions type with very rough initial data. In preparation.
- [AHM12] P. Auscher, S. Hofmann, and J.M. Martell. Vertical versus conical square functions. *Transactions of the American Mathematical Society*, 364(10):5469–5489, 2012.
- [AKMP12] P. Auscher, C. Kriegler, S. Monniaux, and P. Portal. Singular integral operator on tent spaces. *Journal of Evolution Equations*, 12:741–765, 2012.
- [AM19] P. Auscher and M. Mourgoglou. Representation and uniqueness for boundary value elliptic problems via first order systems. *Revista Matemática Iberoamericana*, 35(1):241–315, 2019.
- [Ame18] A. Amenta. Interpolation and embeddings of weighted tent spaces. *Journal of Fourier Analysis and Applications*, 24:108–140, 2018.
- [AMP12] P. Auscher, S. Monniaux, and P. Portal. The maximal regularity operator on tent spaces. *Communications on Pure and Applied Analysis*, 11(6):2213–2219, 2012.
- [AMP19] P. Auscher, S. Monniaux, and P. Portal. On existence and uniqueness for non-autonomous parabolic Cauchy problems with rough coefficients. *Annali Scuola Normale Superiore – Classe di Scienze*, pages 387–471, 2019.
- [AMR08] P. Auscher, A. McIntosh, and E. Russ. Hardy spaces of differential forms on Riemannian manifolds. *Journal of Geometric Analysis*, 18:192–248, 2008.
- [AP23] P. Auscher and P. Portal. Stochastic and deterministic parabolic equations with bounded measurable coefficients in space and time: well-posedness and maximal regularity, 2023. [arXiv:2310.06460](https://arxiv.org/abs/2310.06460).
- [Aus07] P. Auscher. *On necessary and sufficient conditions for L^p -estimates of Riesz transforms associated to elliptic operators on \mathbb{R}^n and related estimates*. Memoirs of the American Mathematical Society, Volume 186, Number 871. American Mathematical Society, Providence, Rhode Island, 2007.
- [Aus11] P. Auscher. Change of angle in tent spaces. *Comptes Rendus Mathématique*, 349(5):297–301, 2011.
- [AvNP14] P. Auscher, J. van Neerven, and P. Portal. Conical stochastic maximal L^p -regularity for $1 \leq p < \infty$. *Mathematische Annalen*, 359:863–889, 2014.

- [BK03] S. Blunck and P.C. Kunstmann. Calderón–Zygmund theory for non-integral operators and the H^∞ -functional calculus. *Revista Matemática Iberoamericana*, 19(3):919–942, 2003.
- [CMS85] R.R. Coifman, Y. Meyer, and E.M. Stein. Some new function spaces and their applications to harmonic analysis. *Journal of Functional Analysis*, 62(2):304–335, 1985.
- [CUMP11] D.V. Cruz-Uribe, J.M. Martell, and C. Pérez. *Weights, Extrapolation and the Theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser Basel, 2011.
- [dS64] L. de Simon. Un’applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rendiconti del Seminario Matematico della Università di Padova*, 34:205–223, 1964.
- [DV23] R. Danchin and I. Vasilyev. Density-dependent incompressible Navier–Stokes equations in critical tent spaces, 2023. [arXiv:2305.09027](https://arxiv.org/abs/2305.09027).
- [HTV91] E. Harboure, J.L. Torrea, and B.E. Viviani. A vector-valued approach to tent spaces. *Journal d’Analyse Mathématique*, 56:125–140, 1991.
- [HvNVW16] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer, 2016.
- [HvNVW17] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer, 2017.
- [KT01] H. Koch and D. Tataru. Well-posedness for the Navier–Stokes Equations. *Advances in Mathematics*, 157(1):22–35, 2001.
- [KW04] P. Kunstmann and L. Weis. *Maximal L_p -regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞ -functional Calculus*, pages 65–311. Springer Berlin Heidelberg, 2004.
- [PV19] P. Portal and M. Veraar. Stochastic maximal regularity for rough time-dependent problems. *Stochastics and Partial Differential Equations: Analysis and Computations*, 7:541–597, 2019.

- [Wei01] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p -regularity. *Mathematische Annalen*, 319:735–758, 2001.
- [Zat20] W. Zatoń. Tent space well-posedness for parabolic Cauchy problems with rough coefficients. *Journal of Differential Equations*, 269(12):11086–11164, 2020.

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY,
91405 ORSAY, FRANCE

Email address: `pascal.auscher@universite-paris-saclay.fr`

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY,
91405 ORSAY, FRANCE

Email address: `hedong.hou@universite-paris-saclay.fr`