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Flat pushforwards of Chern classes and the smoothing problem for cycles in the Whitney range

Claire Voisin*

Abstract

We investigate the smoothability of cycles in the Whitney range, that is, when the dimension is strictly smaller than the codimension. Introducing the notion of "flat pushforwards of Chern classes" and studying its properties, we prove that cycles of dimension $d \leq 7$ are smoothable in a smooth variety of dimension $> 2d$, and more generally that $(d-6)!Z$, $d = \dim Z$, is smoothable in a variety of dimension $> 2d$. We also prove that cycles of any dimension on homogeneous varieties are smoothable in the Whitney range. We prove more generally in all these cases that the considered cycles are flat pushforwards of intersections of divisors on a smooth projective variety.

1 Introduction

Let X be a smooth projective variety of dimension n. Following [8], we will say that a cycle class $z \in \mathrm{CH}_d(X)$ is smoothable if it belongs to the subgroup of $\mathrm{CH_d(X)}$ generated by the classes of d-dimensional smooth subvarieties of X. A number of nonsmoothability results have been proved outside of the Whitney range, that is, when $2d \geq n$, since the question of smoothability was first raised by Borel and Haefliger [2] for cohomology classes. Hartshorne, Rees and Thomas [8] proved the nonsmoothability of the second Chern class $c_2(\mathcal{E})$ of the tautological subbundle $\mathcal{E}$ on a Grassmannian $G(3,n)$, $n \geq 6$. Debarre [5] proved the nonssmoothability of the minimal class $\frac{\theta^2}{2}$ on a very general Jacobian of curve of genus ≥ 7, where θ is the class of a Theta-divisor. This class of examples has been greatly expanded in [3]. Benoist [1] exhibits examples of nonsmoothable d-cycles on varieties of dimension n for many possible pairs (d,n) outside of the Whitney range, including the case where $2d = n$ under some condition on the codimension $c = n - d$. In the Whitney range where $2d < n$, Hironaka [9] proves that for any cycle $z \in \mathrm{CH}_d(X)$ of codimension c, the cycle $(c-1)!z$ is smoothable, a result that has been improved later on by Kleiman [10], who also includes the case where $2d = n$.

For the cycles themselves (as opposed to a multiple), Hironaka proved in 1968 the following result.

Theorem 1.1. *(Hironaka [9]) Cycles of dimension $d \leq 3$ are smoothable on smooth varieties of dimension $n > 2d$.*

Our goal in this paper is to study the smoothability problem for cycles in the Whitney range, which is mentioned in the introduction of [1] and that we state as

Conjecture 1.2. *Let X be a smooth projective variety of dimension n, defined over a field of characteristic 0. Then for any integer d such that $2d < n$, any cycle $z \in \mathrm{CH_d(X)}$ is smoothable.*

For any smooth variety X, we denote by $\mathrm{CH}^*(X)_{\mathrm{Ch}} \subset \mathrm{CH}^*(X)$ the subring generated by Chern classes of vector bundles on X. The standard formula (7) combined with locally

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free resolutions shows that

$$(c-1)!\mathrm{CH}^c(X) \subset \mathrm{CH}^c(X)_{\mathrm{Ch}}.$$

However, it is well-known that $\mathrm{CH}^*(X)_{\mathrm{Ch}}$ can be a proper subring of $\mathrm{CH}^*(X)$. We refer to [4] for an explicit example and to Lemma 3.7 for another example. As discussed in Section 3.2, further examples are provided by homogeneous varieties G/H, where G is a semi-simple, simply connected group, H is a Borel subgroup and the torsion order of G is not 1 (I am grateful to Burt Totaro for explaining this to me). Cycles in $\mathrm{CH}_d(X)_{\mathrm{Ch}}$ are relevant for our subject, since, as proved by Hironaka in [9], they are smoothable under the Whitney condition $2d < \dim X$. (We will give in Section 2 an argument which involves Segre classes and seems slightly different from Hironaka's argument in [9].)

Let us now introduce further definitions.

Definition 1.3. *Let X be smooth. We will denote by $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ (for "flat pushforward of Chern classes"), resp. $\mathrm{CH}(X)_{\mathrm{smpshCh}}$ (for "smooth pushforward of Chern classes") the subgroup generated by cycles of the form $\pi_* z'$ for a flat, resp. smooth, proper morphism $\pi : P \to X$ and for some cycle $z' \in \mathrm{CH}(P)_{\mathrm{Ch}}$.*

Lemma 2.4 proved in Section 3.1 says that $\mathrm{CH}^*(X)_{\mathrm{Ch}} \subset \mathrm{CH}^*(X)$ is generated by smooth pushforwards of classes of complete intersections of divisors, so that, in Definition 1.3, we could replace "cycle in $\mathrm{CH}(P)_{\mathrm{Ch}}$" by "cycle in the subring of $\mathrm{CH}^*(P)$ generated by divisors". Further easy properties are discussed in Section 3.1.

We prove in Section 2 the following basic

Proposition 1.4. *(Cf. Proposition 2.9) Cycles in $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ are smoothable in the Whitney range $2d < \dim X$.*

The analogous result for cycles in $\mathrm{CH}_d(X)_{\mathrm{smpshCh}}$ is a standard Whitney type statement. Proposition 1.4 is our motivation to introduce Definition 1.3 and study the following

Conjecture 1.5. *For any smooth projective variety X, we have*

$$\mathrm{CH}(X)_{\mathrm{flpshCh}} = \mathrm{CH}(X),$$

that is, any cycle on X is the push-forward, under a flat morphism $f : Y \to X$ from a smooth projective variety Y, of a class $z' \in \mathrm{CH}(Y)_{\mathrm{Ch}}$.

We prove in this paper Conjecture 1.5 for cycles of dimension ≤ 7. More generally, we prove in Section 3.2 the following

Theorem 1.6. *Let X be a smooth projective variety. Then for any integer $d \geq 0$,*

$$(d-6)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}}, \tag{1}$$

(where $(d-6)! := 1$ if $d \leq 6$). In particular $\mathrm{CH}_d(X)_{\mathrm{flpshCh}} = \mathrm{CH}_d(X)$ for $d \leq 7$.

One gets from Theorem 1.6 and Proposition 2.9 the following improvement and generalization of Hironaka's Theorem 1.1.

Corollary 1.7. *For any smooth variety X of dimension n, cycles in $(d-6)!\mathrm{CH}_d(X)$ are smoothable when $2d < n$. In particular, cycles of dimension $d \leq 7$ are smoothable if $2d < n$.*

Remark 1.8. In [1, Theorem 0.3], Benoist produces examples of cycles of dimension 6 and codimension 6 that are not smoothable. Corollary 1.7 shows that, at least for dimension 6 and codimension 6, his examples are optimal and the Whitney condition is necessary for smoothability.

Theorem 1.6 is obtained as a consequence of the following Theorem 1.9. We will use here and in the rest of the paper the terminology of *generalized complete intersection in X*, for "closed algebraic subset of codimension c which is the zero-set of a section of a vector bundle of rank c on X".

Theorem 1.9. *(i) (Cf. Proposition 3.9.) Given a finite morphism $j : Y \to X$, where Y and X are smooth projective and $\dim Y = \dim X - 1$, one has*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

(ii) (Cf. Proposition 3.12.) Let X be a smooth projective variety and let $j : Y \hookrightarrow X$ be the inclusion of a smooth subvariety which is a generalized complete intersection in X. Then $j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$.

(iii) (Cf. Proposition 3.14.) Let X be smooth projective and let $Y \subset X$ be a smooth generalized complete intersection in X. Let $\tau : \widetilde{X} \to X$ be the blow-up of X along Y. Then

$$\tau_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

This theorem (together with the trivial Lemmas 3.3 and 3.4) shows that the subgroup $\mathrm{CH}(X)_{\mathrm{flpshCh}} \subset \mathrm{CH}(X)$ is stable under many operations and this is a strong evidence for Conjecture 1.5. For example, we have the following

Corollary 1.10. *Let X be smooth and let $Z \subset X$ be a smooth subvariety which is a connected component of a smooth generalized complete intersection W in X. Then $[Z] \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$.*

Proof. Indeed, we blow-up X along W, getting $\tau : X' \to X$. Let $E_Z \subset X'$ be the exceptional divisor over Z. Theorem 1.9(iii) tells us that $\tau_*(\mathrm{CH}(X')_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$. As we have $[Z] = \pm\tau_*(E_Z^c)$, where $c := \operatorname{codim} Z$, it follows that $[Z] \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$. □

The following corollary shows that the conjectures 1.5 and 1.2 for d-cycles are implied by the following statement.

(*) *Let X be a smooth projective variety and $Z \subset X$ be a smooth projective subvariety of codimension ≥ 3. For a general hypersurface $Y \subset X$ containing Z, there exist an integer r, a smooth projective variety X' and a morphism $\tau : X' \to X_\sharp := X \times \mathbb{P}^r$ constructed by successive blow-ups along smooth generalized complete intersections centers, such that the proper transform Y' of $Y = Y \times \mathrm{pt} \subset X_\sharp$ is smooth and contains a smooth subvariety Z' birational to $Z \times \mathrm{pt}$ via τ.*

Indeed, we will prove

Corollary 1.11. *(See also Proposition 3.15) If the statement (*) is true for any X and Z of dimension d, Conjecture 1.5 holds for cycles of dimension d.*

The proof of Theorem 1.6 will be obtained by establishing statement (*) when the codimension of Z in X is at least $d - 4$. In this case, the singular locus of Y has dimension at most 4 and this is used in our proof.

The methods used to prove Theorems 1.6 and 1.9 do not allow us to prove the stronger result that $\mathrm{CH}_d(X) = \mathrm{CH}_d(X)_{\mathrm{smpshCh}}$ for $d \leq 7$. In particular, the proof of the main Proposition 3.9 (Theorem 1.9(i) above) does not work if we replace the groups $\mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$ and $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ respectively by $\mathrm{CH}_d(Y)_{\mathrm{smpshCh}}$ and $\mathrm{CH}_d(X)_{\mathrm{smpshCh}}$. This leaves open the following

Question 1.12. *Are there smooth projective varieties X such that $\mathrm{CH}(X) \neq \mathrm{CH}(X)_{\mathrm{smpshCh}}$?*

As follows from Theorem 1.13 below, the equality $\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{smpshCh}}$ is satisfied by abelian varieties but it could be that for the example treated in Lemma 3.7, or for some homogeneous varieties, we have $\mathrm{CH}_0(X) \neq \mathrm{CH}(X)_{\mathrm{smpshCh}}$.

Our second main result is a proof of Conjectures 1.2 and 1.5 for homogeneous varieties.

Theorem 1.13. *Let X be a homogeneous projective variety. Then*

(1) We have $\mathrm{CH}_d(X) = \mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ for any d. In particular

(2) Any cycle $z \in \mathrm{CH}_d(X)$ with $2d < \dim X$ is smoothable.

(3) If A is an abelian variety, then $\mathrm{CH}_d(A) = \mathrm{CH}_d(A)_{\mathrm{smpshCh}}$ for any d.

As above, item (2) follows from (1) by Proposition 2.9 since it says that cycles in $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ are smoothable when $2d < \dim X$.

Theorem 1.13(1) is proved as part of Theorem 4.1. We also prove that, if there exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies $\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{smpshCh}}$, then

$$\mathrm{CH}_i(X)_{\mathrm{smpshCh}} = \mathrm{CH}_i(X) \tag{2}$$

for all i. We do not know if the condition above is always satisfied. The stronger version that there always exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies

$$\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{Ch}} \tag{3}$$

is wrong for abelian varieties. For simply connected groups, the condition (3) seems to be closely related to the torsion order of G being 1 (see [6] and Section 3.2), but the precise relation is not obvious to me.

Theorem 1.13(3) will be proved in Section 4.1 and the proof of Theorem 1.13(1), which uses Theorem 1.6 for 0-cycles, will be given in Section 4.2. In the last section, we will give a few complements; for example we prove Proposition 5.1 which states the stability of $\mathrm{CH}(\cdot)_{\mathrm{flpshCh}}$ under the inclusion map of a smooth codimension 2 subvariety. We also establish in Theorem 5.4 a criterion for smoothability that is weaker than being in $\mathrm{CH}(\cdot)_{\mathrm{flpshCh}}$ (in the Whitney range).

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2 Whitney type statements

We first prove the following basic Whitney type result

Proposition 2.1. *Let $\phi : Y \to X$ be a flat morphism between smooth varieties. Let $n = \dim X$. Then for a smooth subvariety $Z \subset Y$ of dimension $d < \frac{n}{2}$ which is in general position and such that the restriction $\phi_{|Z}$ is proper, $\phi_{|Z} : Z \to \phi(Z)$ is an isomorphism, so the closed algebraic subset $\phi(Z) \subset X$ is smooth. Furthermore, if $n = 2d$, $\phi_{|Z}$ is an immersion and the image $\phi(Z)$ has finitely many singular points.*

Although this will be clear from the proof, let us first make precise what we mean by "in general position". This is a transversality condition with respect to ϕ and its infinitesimal properties. For the proposition above, it is enough that Z is the general fiber $\mathcal{Z}_b$ of a family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{f} & Y \\ p\downarrow & & \\ B & & \end{array} \tag{4}$$

where $\mathcal{Z}$ is smooth and p is smooth, which is very mobile at any point (x, y) of $Z \times Z \cong \mathcal{Z}_b \times Z_b$, $x \neq y$, (that is $f : \mathcal{Z} \to Y$ is a submersion at any point of Z, $(f, f) : \mathcal{Z} \times_B \mathcal{Z} \to Y \times Y$ is a submersion at any point (x, y) of $Z \times Z \setminus \Delta_Z$) and whose tangent space is mobile at any point of Z, that is, the morphism

$$F : \mathbb{P}(T_{\mathcal{Z}/B}) \to \mathbb{P}(T_Y)$$

given by the differential of the inclusions $f_b : \mathcal{Z}_b \to Y$, is submersive at any point of Z. The important fact for us is the following

Remark 2.2. Assuming $Y \subset \mathbb{P}^N$ is projective of dimension m, these general position properties will be satisfied by a general complete intersection of $m - d$ very ample divisors.

For the proof of Proposition 2.1, we will use the following consequence of the "general position" assumption.

Lemma 2.3. *Let $Z \subset Y$ be a smooth subvariety of dimension d which is in general position in the above sense. Then,*

(i) If $W \subset Y$ is a closed algebraic subvariety of codimension $> d$, Z does not intersect W.

(ii) If $W \subset Y \times Y$ is a subvariety of codimension $> 2d$, $Z \times Z$ does not intersect W away from the diagonal of Z.

(iii) If $W \subset \mathbb{P}(T_Y)$ is a subvariety of codimension $\geq 2d$, $\mathbb{P}(T_Z)$ does not intersect W.

Proof. (i) We use the notations of (4), with $Z = \mathcal{Z}_b$, $b \in B$ being a general point of B. As f is a submersion along $\mathcal{Z}_b$, there exists a Zariski neighborhood U of $\mathcal{Z}_b$ in $\mathcal{Z}$ such that any component of $f^{-1}(W) \cap U$ has codimension $> d$ in U. As $\dim B = \dim \mathcal{Z} - d$, it follows that

$$p_{|f^{-1}(W)\cap U} : f^{-1}(W) \cap U \to B$$

cannot be dominant, hence for a general $b \in B$, $\mathcal{Z}_b$ does not intersect $f^{-1}(W)$, that is, $Z = f(\mathcal{Z}_b)$ does not intersect W.

(ii) The argument is the same as above with f replaced by $(f, f) : \mathcal{Z} \times_B \mathcal{Z} \to Y \times Y$. The fibers of $(p, p) : \mathcal{Z} \times_B \mathcal{Z} \to B$ are now of dimension $2d$ and (f, f) is a submersion away from the diagonal of $\mathcal{Z}_b$, so if $W \subset Y \times Y$ has codimension $> 2d$, $(f, f)^{-1}(W)$ will have codimension $> 2d$ in $\mathcal{Z} \times_B \mathcal{Z} \setminus \Delta_{\mathcal{Z}}$, and will not dominate B, since $\dim B = \dim (\mathcal{Z} \times_B \mathcal{Z}) - 2d$.

(iii) The argument is the same as above except that we work now with $F : \mathbb{P}(T_{\mathcal{Z}/B}) \to \mathbb{P}(T_Y)$. We simply observe that the fibers of the natural map $\mathbb{P}(T_{\mathcal{Z}/B}) \to B$ are of dimension $2d - 1$. □

Proof of Proposition 2.1. Let $\Delta_Y \subset Y \times Y$ be the diagonal and let $Y' \subset Y \times Y \setminus \Delta_Y$ be the closed algebraic subset $Y \times_X Y \setminus \Delta_Y$. By flatness of ϕ, we have $\operatorname{codim}(Y' \subset Y \times Y) = n$. As $\dim Z \times Z = 2d < n$ and Z is in general position, $Z \times Z$ does not intersect Y' away from the diagonal by Lemma 2.3(ii), so $\phi_{|Z}$ is injective. When $n = 2d$, $Z \times Z$ intersects Y' away from the diagonal in at most finitely many points. It remains to prove the infinitesimal statement, for which we only assume that $2d \leq n$. Let $Y_k \subset Y$ be the locally closed subset of Y where the rank of ϕ is equal to k. Then $\dim \phi(Y_k) \leq k$, hence by flatness, $\operatorname{codim}(Y_k \subset Y) \geq n - k$, or equivalently $\dim Y_k \leq m + k$, where $m := \dim Y - n$. In particular, for $k < d$, we get $\operatorname{codim}(Y_k \subset Y) \geq n - k > n - d \geq d$, hence a Z in general position does not intersect Y_k by Lemma 2.3(i). In general, we observe the following: Along Y_k, we have the rank k morphism

$$\phi_k := (\phi_*)_{|Y_k} : T_{Y|Y_k} \to (\phi^* T_X)_{|Y_k}$$

with kernel a subbundle $\mathcal{K}_k \subset T_{Y|Y_k}$ of corank k. Let $W \subset \mathbb{P}(T_Y)$ be the set of pairs (y, u), $y \in Y$, $u \in \operatorname{Ker} \phi_{*,y}$. The stratification of Y by the Y_k's describes W as a union

$$W = \sqcup_k \mathbb{P}(\mathcal{K}_k).$$

As $\dim Y_k \leq k + m$ and $\operatorname{rk} \mathcal{K}_k = m + n - k$, we get

$$\dim \mathbb{P}(\mathcal{K}_k) \leq 2m + n - 1$$

for any k, and thus $\dim W \leq 2m + n - 1$. As $\dim \mathbb{P}(T_Y) = 2(m + n) - 1$, it follows that

$$\operatorname{codim}(W \subset \mathbb{P}(T_Y)) \geq n.$$

By Lemma 2.3(iii), Z being of dimension d and in general position with $n \geq 2d$, $\mathbb{P}(T_Z)$ does not intersect W. This means that $\phi_{|Z}$ is an immersion, which concludes the proof. □

We will combine Proposition 2.1 with the following easy result.

Lemma 2.4. *Let X be smooth of dimension n and let $z \in \mathrm{CH}_d(X)$ be a cycle. Assume that z belongs to the subring $\mathrm{CH}^*(X)_{\mathrm{Ch}}$ of $\mathrm{CH}^*(X)$ generated by Chern classes $c_i(E)$ for any coherent sheaf E on X. Then there exist a smooth projective variety Y and a smooth morphism $f : Y \to X$ such that $z = f_* z'$ in $\mathrm{CH}(X)$, where $z' \in \mathrm{CH}(Y)$ belongs to the subring generated by divisors on Y.*

Proof. First of all, using finite locally free resolutions and the Whitney formula, we know that z belongs to the subring of $\mathrm{CH}^*(X)$ generated by the Chern classes $c_i(E)$ for any locally free coherent sheaf E on X. Secondly, we can replace in this statement the Chern classes by the Segre classes, since the total Segre and Chern classes $s(E)$ and $c(E)$ satisfy the relation

$$s(E) = c(E)^{-1},\ c(E) = s(E)^{-1},$$

so any polynomial with integral coefficients in the Segre classes is a polynomial with integral coefficients in the Chern classes and vice-versa.

It thus suffices to prove that any monomial $s_{i_1}(E_1) \dots s_{i_k}(E_k) \in \mathrm{CH}(X)$, where the E_i's are locally free sheaves on X of rank r_i, satisfies the conclusion of Lemma 2.4. This statement follows from the definition of Segre classes (see [7]). Indeed, let $\pi_i : \mathbb{P}(E_i) \to X$ be the projectivization of E_i and let $H_i \in \mathrm{Pic}(\mathbb{P}(E_i))$ with first Chern class $c_1(H_i) \in \mathrm{CH}^1(\mathbb{P}(E_i))$ be the dual of its Hopf line bundle (so that $R^0\pi_{i*}H_i = E_i^*$). Then

$$s_j(E_i) = \pi_{i*}(c_1(H_i)^{j+r_i-1}) \text{ in } \mathrm{CH}(X). \tag{5}$$

It follows from (5) and the projection formula that

$$s_{i_1}(E_1) \dots s_{i_k}(E_k) = \pi_*(\mathrm{pr}_1^* c_1(H_1)^{i_1+r_1-1} \dots \mathrm{pr}_k^* c_1(H_k)^{i_k+r_k-1}) \text{ in } \mathrm{CH}(X), \tag{6}$$

where $\pi : \mathbb{P}(E_1) \times_X \dots \times_X \mathbb{P}(E_k) \to X$ is the fibred product of the $\pi_i : \mathbb{P}(E_i) \to X$ and pr_i is the projection from $\mathbb{P}(E_1) \times_X \dots \times_X \mathbb{P}(E_k)$ to its i-th factor. □

Corollary 2.5. *Let X be smooth of dimension n and let $z \in \mathrm{CH}_d(X)_{\mathrm{Ch}}$, with $2d < n$. Then z is smoothable, that is, z is rationally equivalent to a cycle $Z' = \sum_i n_i Z'_i$, where $Z'_i \subset X$ is smooth.*

Proof. Using Lemma 2.4, the result follows from Proposition 2.1 by Remark 2.2. □

Corollary 2.6. *(Hironaka [9], Kleiman [10]) If X is smooth and $2d < \dim X$, any cycle $z \in \mathrm{CH}_d(X)_{\mathbb{Q}}$ is rationally equivalent to a smooth cycle with $\mathbb{Q}$-coefficients. More precisely $(c-1)!z$ is smoothable, where $c := n - d$ is the codimension of z.*

Proof. Indeed, it suffices to prove the result when $z = [Z]$ is the class of a subvariety Z of X of dimension d. Let $\mathcal{O}_Z$ be the structural sheaf of Z, seen as a coherent sheaf on X. It follows from the Grothendieck-Riemann-Roch formula (see [7, Example 15.3.1]) that

$$c_{n-d}(\mathcal{O}_Z) = (-1)^{n-d-1}(n-d-1)![Z] \ \in\ \mathrm{CH}_d(X)_{\mathrm{Ch}} \subset \mathrm{CH}_d(X), \tag{7}$$

so Corollary 2.5 applies. □

Remark 2.7. In [9], which does not use Segre classes but the splitting principle to reduce Chern classes to products of divisors, the coefficient $(c-1)!$ appears multiplied by a constant, which is possibly 1.

Remark 2.8. Kleiman in [10] argues differently by studying singularities of Schubert varieties and proves a result which is of a different nature, as it also includes the case where $n = 2d$, which is not in the Whitney range.

Combining Lemma 2.4 and Proposition 2.1, we get the following criterion

Proposition 2.9. *(a) Let $\phi : Y \to X$ be a proper flat morphism between smooth varieties. Then for any cycle $z \in \mathrm{CH}_d(Y)_{\mathrm{Ch}}$ with $2d < n = \dim X$, the class $z' = \phi_* z$ is smoothable on X.*

(b) The same conclusion holds if we only assume that, away from a closed algebraic subset $Y_0 \subset Y$ of codimension $> d$, Y is smooth and ϕ is flat.

Proof. (a) By Lemma 2.4, the cycles z as above are of the form $\pi_* z'$ for a smooth proper morphism $\pi : P \to Y$ and for some cycle $z' \in \mathrm{CH}(P)$ which is a combination with integral coefficients of intersections of divisors on P. By Remark 2.2, Proposition 2.9 applies to z' and the flat morphism $\phi \circ \pi : P \to X$, proving the statement.

(b) We do the same construction as before and observe that, since $\pi : P \to Y$ is smooth and proper, the singular locus of P and the locus where $\phi \circ \pi$ is not flat are contained in $\pi^{-1}(P_0)$, which has codimension $> d$. A d-dimensional complete intersection of divisors in general position in P does not intersect P_0, hence is contained in $P \setminus P_0$, and we can apply Proposition 2.1 to $(\phi \circ \pi)_{|P \setminus P_0} : P \setminus P_0 \to X$. □

3 Flat pushforwards of Chern classes

3.1 Comments on Definition 1.3

We start with the following remarks on Definition 1.3.

Remark 3.1. By Lemma 2.4, in the definition of $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ and $\mathrm{CH}(X)_{\mathrm{smpshCh}}$, we can replace "intersections of divisors on P" by "elements of $\mathrm{CH}(P)_{\mathrm{Ch}}$". Indeed, if Y is smooth, $p : Y \to X$ is a proper flat morphism, and $z \in \mathrm{CH}_d(Y)_{\mathrm{Ch}}$, there exist by Lemma 2.4 a smooth variety P and a smooth morphism $p' : P \to Y$ such that

$$z = p'_*(w) \text{ in } \mathrm{CH}_d(Y),$$

where w belongs to the subring of $\mathrm{CH}(P)$ generated by divisors. The morphism $p \circ p' : P \to X$ is flat and projective, and $z = (p \circ p')_* w$.

Remark 3.2. *If $f : Y \to X$ is a flat (resp. smooth) morphism between smooth projective varieties, one has $f_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$, resp. $f_*(\mathrm{CH}(Y)_{\mathrm{smpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{smpshCh}}$.*

Let us now establish a few elementary facts.

Lemma 3.3. *(a) One has $\mathrm{CH}(X)_{\mathrm{Ch}} \subset \mathrm{CH}(X)_{\mathrm{smpshCh}} \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$.*

(b) The subgroup $\mathrm{CH}(X)_{\mathrm{smpshCh}}$ is a subring of $\mathrm{CH}(X)$.

(c) The subgroup $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ is a module over the ring $\mathrm{CH}(X)_{\mathrm{smpshCh}}$.

(d) If we work over an algebraically closed field of characteristic 0, (in particular over $\mathbb{C}$), we have $\mathrm{CH}(X)_{\mathrm{alg}} \subset \mathrm{CH}(X)_{\mathrm{Ch}}$, where $\mathrm{CH}(X)_{\mathrm{alg}}$ denotes the group of cycles algebraically equivalent to 0. The definition 1.3 thus makes sense for the groups of cycles modulo algebraic equivalence instead of the Chow groups.

For example, in the case of 0-cycles, the condition that z belongs to $\mathrm{CH}_0(X)_{\mathrm{flpshCh}}$ or $\mathrm{CH}_0(X)_{\mathrm{smpshCh}}$ depends only on the degree of z, assuming X is projective and connected and the field is algebraically closed of characteristic 0.

Proof of Lemma 3.3. (a) The second inclusion is obvious since smoothness implies flatness.

(b) and (c) Let $p_1 : P_1 \to X$, $p_2 : P_2 \to X$ be proper morphisms with P_1, P_2, X smooth and assume p_1 is smooth, p_2 is flat. Then $P_{12} := P_1 \times_X P_2$ is smooth. If Z_1, resp. Z_2 are intersections of divisors on P_1, resp. P_2, their pull-backs Z'_1, resp. Z'_2 to P_{12} via the projections

$$p'_1 : P_{12} \to P_2,\ p'_2 : P_{12} \to P_1$$

are also intersections of divisors, and the projection formula gives

$$p_{1*}Z_1 \cdot p_{2*}Z_2 = p_{12*}(Z_1' \cdot Z_2') \text{ in } \mathrm{CH}(X),$$

where $p_{12} : P_1 \times_X P_2 \to X$ is the natural morphism. This proves (b) and (c) since p_{12} is flat and it is smooth if p_2 is smooth.

(d) Indeed, it suffices by (a) to prove that $\mathrm{CH}^p(X)_{\mathrm{alg}} \subset \mathrm{CH}^p(X)_{\mathrm{Ch}}$. By formula (7), we have $(p-1)!\mathrm{CH}^p(X) \subset \mathrm{CH}^p(X)_{\mathrm{Ch}}$ for any smooth X and any closed algebraic subset $Z \subset X$ of codimension p. It follows that

$$\mathrm{CH}^p(X)_{\mathrm{alg}} \subset \mathrm{CH}^p(X)_{\mathrm{Ch}}$$

since the group $\mathrm{CH}^p(X)_{\mathrm{alg}}$ of cycles algebraically equivalent to zero is divisible, hence contained in $(p-1)!\mathrm{CH}^p(X)$, if we are over an algebraically closed field of characteristic 0. □

Another useful lemma is the following

Lemma 3.4. *Let $\phi : Y_1 \to Y_2$ be a morphism, with Y_1, Y_2 smooth. Then*

(i) One has

$$\phi^*(\mathrm{CH}(Y_2)_{\mathrm{smpshCh}}) \subset \mathrm{CH}(Y_1)_{\mathrm{smpshCh}}, \tag{8}$$

(ii) If ϕ is smooth, then

$$\phi^*(\mathrm{CH}(Y_2)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(Y_1)_{\mathrm{flpshCh}}. \tag{9}$$

Proof. Let $\psi : W \to Y_2$ be a flat (resp. smooth) projective morphism. Then

$$\psi_1 : W_1 := W \times_{Y_2} Y_1 \to Y_1$$

is flat (resp. smooth). Furthermore, if either ψ is smooth (Case (i)) or ϕ is smooth (Case (ii)), W_1 is smooth.

Let $\phi_1 : W_1 \to W$ be the first projection. If $\gamma \in \mathrm{CH}(W)_{\mathrm{Ch}}$, we have $\phi_1^*\gamma \in \mathrm{CH}(W_1)_{\mathrm{Ch}}$, and furthermore we have by [7, Proposition 1.7]

$$\psi_{1*}(\phi_1^*\gamma) = \phi^*(\psi_*\gamma) \text{ in } \mathrm{CH}(Y_1).$$

This proves (8) and (9). □

Remark 3.5. It is not clear if we can remove the assumption on ϕ in (ii). The problem is to guarantee the smoothness of the fibred product. In Proposition 4.2, we show how to go around this problem in some cases.

3.2 Some cases of Conjecture 1.5

We will give in this section the proof of Theorem 1.6, which implies in particular Conjecture 1.5 for cycles of dimension ≤ 7. We first note the following

Corollary 3.6. *Conjecture 1.5 is satisfied by smooth varieties of dimension ≤ 10.*

Proof. If $Z \subset X$ is a subvariety of codimension p of a smooth variety, one has the formula (7)

$$c_p(\mathcal{O}_Z) = (-1)^{p-1}(p-1)![Z] \text{ in } \mathrm{CH}^p(X),$$

hence for $p \leq 2$, the class $[Z] \in \mathrm{CH}^p(X)$ belongs to $\mathrm{CH}^p(X)_{\mathrm{Ch}}$. As cycles of dimension $d \leq 7$ belong to $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ by Theorem 1.6, we conclude that $\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{flpshCh}}$ when $\dim X \leq 10$ since this forces the cycles to have either dimension ≤ 7 or codimension ≤ 2. □

Theorem 1.6 is interesting when the considered cycles do not belong to $\mathrm{CH}(X)_{\mathrm{Ch}}$. Besides the case of 0-cycles on very general abelian varieties with high degree polarization (see [4]), some hypersurfaces in projective space provide such examples. For example, we have

Lemma 3.7. *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree 64. Then the class of a point $x \in X$ does not belong to $\mathrm{CH}_0(X)_{\mathrm{Ch}}$. More precisely, for any vector bundle E on X, the degree $\deg c_3(E)$ is divisible by 2.*

Proof. Let E be a vector bundle of rank r on X. By the Hirzebruch-Riemann-Roch formula, the holomorphic Euler-Poincaré characteristic of E is given by the formula

$$\chi(X,E) = \int_X \mathrm{Ch}(E)\mathrm{To}(X) = \alpha c_3(E) + \beta c_2(E)c_1(E) + \gamma c_2(E)c_1(X) + q(E), \tag{10}$$

where the constants α, β, γ, δ, ζ independent of E are rational and the quantity $q(E)$ is the part of the Riemann-Roch polynomial (in the Chern classes of E) which involves only $c_1(E)$ and the rank of E, and is an integer since it is equal to

$$\int_X (r-1)\mathrm{To}_3(X) + \mathrm{Ch}(\det E)\mathrm{To}(X) = (r-1)\chi(X,\mathcal{O}_X) + \chi(X,\det E).$$

The constants α and β are obtained by expressing $\mathrm{Ch}_3(E)$ as a polynomial in the Chern classes $c_i(E)$. One gets

$$\alpha = \frac{1}{2},\ \beta = -\frac{1}{2}. \tag{11}$$

Finally, the constant γ is obtained by expressing $\mathrm{Ch}_2(E)\mathrm{To}_1(X)$ using the Chern classes of E. One gets

$$\gamma = -\frac{1}{2}. \tag{12}$$

It is proved by Kollár (see [12]) that for X as above, any curve $C \subset X$ has degree divisible by 2. It follows that the numbers $\int_X c_2(E)c_1(E)$ and $\int_X c_2(E)c_1(X)$ are even. We thus deduce from (11) and (12) that $\chi(X,E) = \frac{1}{2}\int_X c_3(E) + C$, where C is an integer. Thus the degree of $c_3(E)$ has to be an integer. □

Remark 3.8. We see from the proof above that the obstruction to the existence of a vector bundle E on X (or more generally an element of $K_0(X)$) with $\deg c_3(E) = 1$ comes from the defect of the integral Hodge conjecture for degree 4 Hodge classes on X. Conversely, if the integral Hodge conjecture for degree 4 Hodge classes on X holds true, then the generator $\alpha \in H^2(X,\mathbb{Z})$ is algebraic, $\alpha = [Z]$, for some 1-cycle $Z \in \mathrm{CH}_1(X)$. As we have

$$\mathrm{CH}_1(X) = \mathrm{CH}^2(X) = \mathrm{CH}^2(X)_{\mathrm{Ch}},$$

the cycle Z belongs to $\mathrm{CH}^2(X)_{\mathrm{Ch}}$, so the cycle $H \cdot Z$ belongs to $\mathrm{CH}^3(X)_{\mathrm{Ch}}$. Hence there exists a degree 1 element in $\mathrm{CH}^3(X)_{\mathrm{Ch}}$ in this case.

Other examples of smooth projective varieties X for which $\mathrm{CH}(X) \neq \mathrm{CH}(X)_{\mathrm{Ch}}$ are given by generalized flag manifolds for certain affine algebraic groups with torsion index > 1 (see [6] and [17], [18] where this notion is discussed and computed for many groups). Merkurjev proved in [14] that for a simply connected semisimple algebraic group G, and for a closed subgroup H, the K_0-ring of G/H is generated by classes of homogeneous vector bundles on G/H that come from representations of H. If furthermore H is a Borel subgroup of G, then homogeneous vector bundles on G/H coming from representations of H are direct sums of line bundles. In the last case, it follows that the subgroup

$$\mathrm{CH}_0(G/H)_{\mathrm{Ch}} \subset \mathrm{CH}_0(G/H)$$

is also the subgroup generated by products of divisor classes. By definition of the torsion index of G, the index of the later subgroup is a multiple of the torsion index of G. The computations in [17], [18] thus give plenty of examples where $\mathrm{CH}_0(G/H)_{\mathrm{Ch}} \subset \mathrm{CH}_0(G/H)$ is a proper subgroup.

We now turn to the proof of Theorem 1.6. We will prove it in several steps, each one adding some arguments to the previous one. We first prove the following weaker statement, namely the inclusion

$$(d-1)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}}. \tag{13}$$

Proof of (13). Let X be smooth projective of dimension n and let $Z \subset X$ be a subvariety of dimension d. We choose a desingularization $\widetilde{Z} \to Z$ of Z and an embedding $\widetilde{Z} \subset X \times \mathbb{P}^m$ over X for some m. As the projection $p_X : X \times \mathbb{P}^m \to X$ to X is flat, it suffices to prove that

$$(d-1)!\widetilde{Z} \in \mathrm{CH}(X \times \mathbb{P}^m)_{\mathrm{flpshCh}}, \tag{14}$$

as it implies by Remark 3.2 that $(d-1)!Z \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$, which is the contents of (13). In other words, letting

$$Z' = \widetilde{Z},\ X' = X \times \mathbb{P}^m,$$

we reduced to the case of the class of a smooth subvariety $Z' \subset X'$, which we treat now. If $\dim X' \leq 2d$, this is finished by formula (7) since then $\mathrm{codim}\,(Z' \subset X') \leq d$. If not, let Y be a smooth general complete intersection of sufficiently ample hypersurfaces in X' containing Z', with Y of dimension $2d$. Such Y exists since Z' is smooth of dimension d and $\dim X' \geq 2d$. As $\mathrm{codim}\,(Z' \subset Y) = d$, we have $(d-1)![Z']_Y \in \mathrm{CH}_d(Y)_{\mathrm{Ch}}$ by formula (7), where $[Z']_Y$ is the class of Z' in Y. Let $j : Y \hookrightarrow X'$ be the inclusion of Y. By Corollary 3.10 stated below applied to the sequence of inclusions

$$Y_1 \cap Y_2 \cap \ldots \cap Y_{r+1} \hookrightarrow Y_1 \cap Y_2 \cap \ldots \cap Y_r$$

for $r \leq N-1$, we have

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X')_{\mathrm{flpshCh}},$$

hence $(d-1)![Z'] \in \mathrm{CH}_d(X')_{\mathrm{flpshCh}}$ since $[Z] = j_*[Z']_Y$. □

Corollary 3.10 is obtained as an immediate consequence of the following

Proposition 3.9. *Let X be a smooth variety and $j : Y \to X$ be a finite morphism, where Y is smooth projective with $\dim Y = \dim X - 1$. Then*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

Corollary 3.10. *If X is smooth projective of dimension n and $j : Y \to X$ is the inclusion of a smooth subvariety of codimension c, such that there exist inclusions*

$$Y = Y_c \subset Y_{c-1} \subset \ldots \subset Y_0 = X,$$

where Y_r is smooth of dimension $n-r$ for any $0 \leq r \leq n-c$, one has

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

Proof of Proposition 3.9. Let T be the smooth projective variety obtained by blowing-up the graph Γ_j of j in $Y \times X$. Let

$$\tau : T \to Y \times X$$

be the blow-up map and let pr_Y, pr_X be the two projections from $Y \times X$ to Y and X. We denote

$$p := \mathrm{pr}_Y \circ \tau : T \to Y,\ q := \mathrm{pr}_X \circ \tau : T \to X$$

the two natural morphisms.

Lemma 3.11. *The morphism p is smooth and the morphism q is flat.*

Proof. Indeed, the fiber of p over $y \in Y$ is the blow-up of X along y, which is smooth. The fiber of q over $x \in X$ is isomorphic to Y when $x \notin j(Y)$, hence it has dimension $n-1$, $n = \dim X$. We claim that all the fibers of q have dimension $\leq n-1$. To see this, we observe that

$$q^{-1}(x) = \tau^{-1}(Y \times \{x\})$$

is the set-theoretic union of several components, some being contained in the exceptional divisor E over Γ_j and mapping via τ to $(Y \times \{x\}) \cap \Gamma_j$, the other being birational to Y. The component which is birational to Y has dimension $\leq n-1$. The other components are also of dimension $\leq n-1$, since the morphism $\tau_{|E} : E \to \Gamma_j \cong Y$ has fibers of dimension $n-1$, and

$$(Y \times \{x\}) \cap \Gamma_j \cong j^{-1}(x) \subset Y \cong \Gamma_j$$

has dimension 0 because j is finite. This proves the claim. The fibers are thus equidimensional, hence q is flat since both T and X are smooth. □

For any class $w \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$, there exist by definition a (not necessarily connected but equidimensional) smooth projective variety W, a flat morphism $\phi : W \to Y$, and divisors $D_1, \ldots, D_{N-d} \in \mathrm{CH}^1(W)$, $N := \dim W$, such that

$$w = \phi_*(D_1 \ldots D_{N-d}) \text{ in } \mathrm{CH}_d(Y). \tag{15}$$

We now observe that

$$j_* w = \Gamma_{j*}(w) = \mathrm{pr}_{X*}(\mathrm{pr}_Y^* z' \cdot \Gamma_j) \tag{16}$$

in $\mathrm{CH}_d(X)$. Furthermore, we have as usual

$$\Gamma_j = \pm \tau_* E^n \text{ in } \mathrm{CH}^n(Y \times X),\ n = \dim X. \tag{17}$$

Let now

$$W_T := W_{\times Y} T$$

with first projection p_W to W, second projection p_T to T and morphism

$$\psi := q \circ p_T : W_T \to X.$$

Combining (15), (16), and (17), and applying the projection formula, we get

$$j_* w = \pm \psi_*(p_W^*(D_1 \ldots D_{N-d}) \cdot p_T^* E^n) \text{ in } \mathrm{CH}_d(X), \tag{18}$$

which proves that $j_* w$ belongs to $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$, since W_T is smooth by Lemma 3.11, and the morphism ψ is flat, being the composition of the two flat morphisms q and p_T. The proof of Proposition 3.9 is finished. □

For the proof of Theorem 1.6, we prove now the following stability results extending Proposition 3.9.

Proposition 3.12. *Let X be smooth projective and let $j : Y \hookrightarrow X$ be the inclusion of a smooth projective subvariety which is the zero-set of a transverse section σ of a vector bundle E on X. Then*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{19}$$

Proof. We prove the result by induction on the rank of E, the case of rank 1 being a particular case of Proposition 3.9. Let E be a rank r vector bundle on X and let $\pi : \mathbb{P}(E^*) = \mathrm{Proj}\,(\mathrm{Sym}^* E) \to X$ be the projectivization of E^*. Let $\pi^* E \to \mathcal{H}$ be the quotient line bundle on $\mathbb{P}(E^*)$. The section $\pi^*\sigma$ projects to section σ' of $\mathcal{H}$ and we have

Lemma 3.13. *(i) The* 0*-locus of* σ' *is a smooth hypersurface* X' *of* $\mathbb{P}(E^*)$*.*

(ii) Furthermore, the induced section σ'' *of* $F := \mathrm{Ker}\,(\pi^*E \to \mathcal{H})$ *on* X' *is transverse with* 0*-locus* $\pi^{-1}(Y) = \mathbb{P}(E^*_{|Y}) \subset X' \subset \mathbb{P}(E^*)$*.*

Proof. (i) The vanishing locus of σ' is a $\mathbb{P}^{r-2}$-bundle over the open subset $X \setminus Y$ of X where $\sigma \neq 0$, hence it is smooth over $X \setminus Y$. It obviously contains $\pi^{-1}(Y)$ and it remains to show that it is smooth there, which is easy.

(ii) The vanishing locus of σ'' on X' equals scheme-theoretically the vanishing locus of the section $\pi^*\sigma$ of π^*E on $\mathbb{P}(E^*)$, hence equals $\pi^{-1}(Y)$. It is thus smooth of codimension $r-1$ in X'. □

Denoting by $\pi_Y : \mathbb{P}(E^*_{|Y}) \to Y$ the restriction of π over Y, we know by Lemma 3.4 that $\pi_Y^* : \mathrm{CH}(Y) \to \mathrm{CH}(\mathbb{P}(E^*_{|Y}))$ maps $\mathrm{CH}(Y)_{\mathrm{flpshCh}}$ to $\mathrm{CH}(\mathbb{P}(E^*_{|Y}))_{\mathrm{flpshCh}}$. Denoting by

$$j' : \mathbb{P}(E^*_{|Y}) \hookrightarrow X',\ j'' : X' \hookrightarrow \mathbb{P}(E^*)$$

the inclusion maps, we get, first by the induction hypothesis on the rank and lemma 3.13, and secondly by Proposition 3.9, that

$$j'_*(\mathrm{CH}(\mathbb{P}(E^*_{|Y}))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X')_{\mathrm{flpshCh}},\ j''_*(\mathrm{CH}(X')_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

We conclude that the map $\gamma := j''_* \circ j'_* \circ \pi_Y^* : \mathrm{CH}(Y) \to \mathrm{CH}(\mathbb{P}(E^*))$ has the property that

$$\gamma(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

Recalling from [7, Proposition 1.7] that $\gamma = \pi^* \circ j_*$, we thus proved that

$$\pi^* \circ j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

Let $h = c_1(\mathcal{H}) \in \mathrm{CH}^1(\mathbb{P}(E^*))$. By Lemma 3.3, we have

$$h^{r-1}\mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}} \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}$$

and by Remark 3.2, $\pi_*(\mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)\mathrm{flpshCh}$. As $\pi_*(h^{r-1}\pi^*z) = z$ for any $z \in \mathrm{CH}(X)$, we conclude that for any $z \in \mathrm{CH}(Y)_{\mathrm{flpshCh}}$,

$$j_*z = \pi_*(h^{r-1}\pi^*(j_*z)) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

□

Another consequence of proposition 3.9 is the following

Proposition 3.14. *Let* X *be smooth projective and let* $Y \subset X$ *be a smooth projective subvariety which is the* 0*-set of a transverse section* σ *of a vector bundle* E *on* X*. Let* $\tau : \widetilde{X} \to X$ *be the blow-up of* X *along* Y*. Then*

$$\tau_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{20}$$

Proof. Let $\pi : \mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}^*E^*) \to X$ be the projectivization of E. Then the section σ gives a rational section $X \dashrightarrow \mathbb{P}(E)$ of π, whose image is isomorphic to $\widetilde{X}$. Furthermore, as shows a local computation, $\widetilde{X} \subset \mathbb{P}(E)$ is the zero-set of a transverse section $\overline{\sigma}$ of the quotient vector bundle $F := \pi^*E/\mathcal{S}$ on $\mathbb{P}(E)$, namely, $\overline{\sigma}$ is the projection of $\pi^*\sigma$ in $\pi^*E/\mathcal{S}$, where $\mathcal{S} \subset \pi^*E$ is the tautological subbundle. We have

$$\tau_* = \pi_* \circ j_* : \mathrm{CH}(\widetilde{X}) \to \mathrm{CH}(X), \tag{21}$$

where $j : \widetilde{X} \to \mathbb{P}(E)$ is the inclusion map. By Proposition 3.12, we have

$$j_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E))_{\mathrm{flpshCh}}$$

and $\pi_*(\mathrm{CH}(\mathbb{P}(E))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$ by Remark 3.2. Hence (21) implies (20). □

We now use these results to prove Corollary 1.11. More precisely, let us prove the following statement.

Proposition 3.15. *Let Y be smooth projective of dimension $2d-k$, and r a large integer. Denote $Y_\sharp :=:= Y \times \mathbb{P}^r$. Let $Z \subset Y$ be closed smooth of dimension d. Assume that for a general hypersurface $Y' \subset Y$ containing Z, there exist a smooth projective variety $\widetilde{Y}$ and a morphism $\tau : \widetilde{Y} \to Y_\sharp$ obtained by successive blow-ups $Y_{i+1} \to Y_i$, $Y_0 := Y_\sharp$ along smooth closed algebraic subsets $W_i \subset Y_i$ which are generalized complete intersections on Y_i, such that*

(1) the proper transform $\widetilde{Y'}$ of $Y' \times \mathrm{pt}$ in $\widetilde{Y}$ is smooth and a generalized complete intersection, and

(2) the proper transform Z' of Z is smooth, birational to Z.

Then

$$(d-k-2)![Z] \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}. \tag{22}$$

If the statement above is satisfied for any Z, Y as above with $\mathrm{codim}\,(Z \subset Y) \geq 3$, then

$$[Z] \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}. \tag{23}$$

Proof. Let $\tilde{j} : \widetilde{Y'} \hookrightarrow \widetilde{Y}$ be the inclusion. As $\widetilde{Y'}$ is a generalized complete intersection in $\widetilde{Y}$ by (1), Proposition 3.12 gives

$$\tilde{j}_*(\mathrm{CH}(\widetilde{Y'})_{\mathrm{flpshCh}}) \subset \mathrm{CH}_d(\widetilde{Y})_{\mathrm{flpshCh}}. \tag{24}$$

By iterated applications of Proposition 3.14,

$$\tau_*\mathrm{CH}_d(\widetilde{Y})_{\mathrm{flpshCh}} \subset \mathrm{CH}_d(Y_\sharp)_{\mathrm{flpshCh}}. \tag{25}$$

Finally, by (7),

$$(d-k-2)![Z'] \in \mathrm{CH}_d(\widetilde{Y'})_{\mathrm{flpshCh}} \tag{26}$$

since $\mathrm{codim}\,(Z' \subset \widetilde{Y'}) = d-k-1$. As we have

$$[Z \times \mathrm{pt}] = (\tau \circ \tilde{j})_*[Z'] \text{ in } \mathrm{CH}_d(Y), \tag{27}$$

(24), (25) and (26) imply that $(d-k-2)![Z \times \mathrm{pt}] \in \mathrm{CH}_d(Y_\sharp)_{\mathrm{flpshCh}}$, which implies (22) by applying the first projection $Y_\sharp \to Y$.

If for any pair (Z, Y) as above with $\dim Z = d$, $\dim Y \geq d+3$, and any general hypersurface $Y' \subset Y$ containing Z, there exists a $\widetilde{Y}$ satisfying the stated properties, the proof above provides inductively varieties $\widetilde{Y'_k}$ of dimension $2d-k$, $k \leq d-2$ containing a smooth subvariety Z_k of dimension d birational to Z, and morphisms

$$j_k : \widetilde{Y'_k} \to \widetilde{Y'_{k-1}}$$

such that

$$j_{k*}(\mathrm{CH}(\widetilde{Y'_k})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\widetilde{Y'_{k-1}})_{\mathrm{flpshCh}} \tag{28}$$

and $[Z_{k-1}] = j_{k*}[Z_k]$. Furthermore, $\widetilde{Y'_0} = Y$ and $Z_0 = Z$. For $k = d-2$, one has $\dim \widetilde{Y'_{d-2}} = d+2$, so $[Z_{d-2}] \in \mathrm{CH}(\widetilde{Y'_{d-2}})_{\mathrm{flpshCh}}$, hence $[Z] \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$ by (28). □

Let us show how Proposition 3.15 for $k = 0$ immediately gives the inclusion

$$(d-2)!\mathrm{CH}_d(X)_{\mathrm{flpshCh}} \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}} \tag{29}$$

for any smooth projective variety X.

Proof of (29). As before, if $Z \subset X$ is any subvariety, we replace it by a desingularization $\widetilde{Z} \subset X \times \mathbb{P}^N$, hence we can assume without loss of generality that Z is smooth. Z is then contained in a smooth subvariety $Y \subset X$ of dimension $2d$, which is a complete intersection of smooth hypersurfaces. By iterated applications of Proposition 3.9, it suffices to prove that the class z of Z in Y has the property that

$$(d-2)!z \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}. \tag{30}$$

Lemma 3.16. *A general sufficiently ample hypersurface $Y' \subset Y$ containing Z has isolated ordinary quadratic singularities.*

Proof. By Bertini, the singularities of Y' are on Z, and they correspond to the zeroes of the differential

$$d\sigma \in H^0(Z, N^*_Z(Y')) \tag{31}$$

along Z of the defining equation σ of Y'. The section (31) being generic, its zero locus is transverse, hence the singular locus of Y' is zero-dimensional. In fact, the transversality of the section $d\sigma$ also implies that the singularities are ordinary quadratic (see Lemma 3.17 below). □

By Lemma 3.16, the hypersurface Y' is desingularized by a single blow-up along the finite set $W \subset Y'$ of its singular points. We choose now a general 0-dimensional complete intersection $W' \subset Y$ containing W. We have $W' = W \cup W''$, where the set W'' is disjoint from Y'. It follows that the blow-up $\widetilde{Y}$ of Y along W' contains the blow-up $\widetilde{Y'}$ of Y' along W as a smooth hypersurface. It is clear that the proper transform Z' of Z under this blow-up is smooth birational to Z, so the assumptions of Proposition 3.15 are satisfied (with $k = 0$, $r = 0$) and (30) follows. □

We now prove the improved inclusion

$$(d-5)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}}. \tag{32}$$

Proof of (32). The proof mimics the argument above but the desingularization step is more involved. Let $Z \subset X$ be a subvariety of dimension d. By the previous proof, we can assume that Z is smooth and that there exists a smooth projective variety X' of dimension $2d-1$, mapping to X via a morphism $j' : X' \to X$ such that $j'_*(\mathrm{CH}(X')_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$ and containing a smooth subvariety $Z' \subset X'$ which is birational to Z via j. We are now going to reduce step by step the dimension of X' by taking hypersurfaces and desingularizing them, until we have constructed a variety X'' of dimension $2d-4$ containing a smooth subvariety Z'' of dimension d, together with a morphism $j'' : X'' \to X$ such that $j''(\mathrm{CH}(X'')_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$ and the restriction of j'' to Z'' is a birational morphism $Z'' \to Z$. We explain only the last step, the previous ones being similar and in fact easier. We thus assume that X is smooth of dimension $2d-3$ and $Z \subset X$ is smooth of dimension d. Let $Y \subset X$ be a general hypersurface containing Z.

Lemma 3.17. *Y has ordinary quadratic singularities along a 3-dimensional smooth subvariety $T \subset Z$ and is smooth away from Z.*

Proof. This follows from Bertini away from Z. Along Z, the differential df of the local equations defining Y provides a section

$$df \in H^0(Z, N^*_{Z/X}(Y)).$$

Taking Y sufficiently ample, the morphism df is arbitrary and the vector bundle $N^*_{Z/X}(Y)$ is generated by sections, hence the vanishing locus of df is a smooth subvariety $T \subset Z$ of codimension $d-3$, hence dimension 3. This implies that the singularities of Y are ordinary quadratic along T. This is proved by taking local holomorphic coordinates $z_1, \ldots, z_{2d-3}$ on

X, such that Z is defined by equations $z_j = 0$ for $j \geq d+1$. In these coordinates, the local holomorphic equation f defining Y takes the form

$$f = \sum_{j=d+1}^{2d-3} z_j g_j. \tag{33}$$

The locus T is the locus of points $z \in Z$ where the functions g_j all vanish. The locus T is smooth of codimension $d-3$ in Z if and only if the restriction to Z of the differentials dg_j are independent, or equivalently the differentials dg_j are independent modulo the dz_j, $j \geq d+1$. This is clearly equivalent by (33) to the fact that f has ordinary quadratic singularities along T. □

The variety Y can thus be desingularized by blowing-up the subvariety T in X. Unfortunately, T is not a generalized complete intersection in X, which we need in order to apply Proposition 3.15. We correct this as follows. First of all, we replace X by $X_\sharp := X \times \mathbb{P}^r$ for some $r \geq 3$, and Y by $Y \times \mathrm{pt}$. $Y \subset X_\sharp$ is a generalized complete intersection, and we are going to desingularize Y by successive blow-ups $\tau_i : X_i \to X_{i-1}$, $i \leq 4$, $X_0 = X_\sharp$ along smooth generalized complete intersections, keeping the property that the proper transforms $Y_i \subset X_i$ is a smooth generalized complete intersection. In other words, we are going to construct a resolution of singularities of $Y \times \mathrm{pt} \subset X_\sharp$ satisfying the conditions of Proposition 3.15, and (32) will follow from Proposition 3.15.

Let m be the dimension of $X_\sharp$ (which we can assume arbitrarily large). By choosing $m-3$ general hypersurfaces in $X_\sharp$ containing T, we construct a complete intersection subvariety

$$T_1 = T \cup T',$$

and the two components T, T' are smooth and meet transversally along a smooth surface $S \subset Y = Y \times \mathrm{pt} \subset X_\sharp$. This is where we use the fact that $\dim T \leq 3$, as it was observed by Hironaka in [9] that the residual subvariety T' obtained by general liaison starting from a smooth subvariety of dimension ≤ 3 is smooth. The fact that the two components intersect transversally along a smooth surface S is proved as Lemma 3.17. Furthermore, m being large, we can assume that T' intersects Y only along $S = T \cap T'$. We now choose $m-2$ general hypersurfaces in $X_\sharp$ containing S. Their intersection is a reducible surface

$$S_1 = S \cup S',$$

where the residual component S' is a smooth surface, the intersection $S \cap S'$ is as before a smooth curve $C \subset Y$, and S' intersects Y only along C. We next choose $m-1$ general hypersurfaces in $X_\sharp$ containing C. Their intersection is a reducible curve

$$C_1 = C \cup C',$$

where the second component C' is a smooth curve, the intersection $C \cap C'$ is a reduced 0-dimensional subscheme $W \subset Y$, and C' intersects Y only along W. We finally choose m general hypersurfaces in $X_\sharp$ containing W. Their intersection is a reduced 0-dimensional subscheme $W_1 \subset X$, which intersects Y along W.

We now blow-up $X_\sharp$ along W_1, getting X_1, then blow-up the proper transform $\widetilde{C}_1$ of C_1 in X_1 getting X_2, then blow-up the proper transform $\widetilde{S}_1$ of S_1 in X_2, getting X_3, and finally blow-up the proper transform $\widetilde{T}_1$ of T_1 in X_3. This gives us a variety X_4 birational to $X_\sharp$. We will denote by $\tau_i : X_i \to X_{i-1}$ (with $X_0 := X_\sharp$) the blow-up maps and by $\tau : X_4 \to X_\sharp$ their composition.

We observe that, after blowing-up W_1, the proper transforms of the curves C and C' are disjoint, so $\widetilde{C}_1 \subset X_1$ is in fact smooth. Similarly, after blowing-up $\widetilde{C}_1$, the proper transforms of the surfaces S and S' are disjoint, so $\widetilde{S}_1 \subset X_2$ is in fact smooth. Finally, after blowing-up $\widetilde{S}_1$, the proper transforms of T and T' are disjoint, hence $\widetilde{T}_1 \subset X_3$ is smooth. It follows that X_4 is smooth. As argued above, in order to complete the proof of (32), it now suffices to prove

Lemma 3.18. *(i) For any $1 \leq i \leq 4$, the blow-up maps $\tau_i : X_i \to X_{i-1}$ is the blow-up along a smooth center which is a generalized complete intersection in X_{i-1}.*

(ii) The proper transform $Y_4 \subset X_4$ of Y is smooth and is a generalized complete intersection in X_4.

Lemma 3.18 follows from the following general

Lemma 3.19. *Let N be a smooth variety and $M \subset N$ be a subvariety which has ordinary quadratic singularities along its singular locus $M_1 \subset M$ (in particular M_1 is a smooth subvariety of N). Let $T \subset N$ be a smooth subvariety, which is a disjoint union*

$$T = T_1 \sqcup T_2,$$

where $T_1 \subset M_1$ and T_2 is disjoint from M. Let $\widetilde{N}_T$ be the blow-up of N along T and $\widetilde{M} \subset \widetilde{N}_T$ be the proper transform of M. Then

(i) The singularities of $\widetilde{M}$ are ordinary quadratic along the proper transform of M_1.

(ii) If M is a generalized complete intersection in N, $\widetilde{M}$ is a generalized complete intersection in $\widetilde{N}_T$.

We apply indeed Lemma 3.19 to the proper transforms of the loci C_1, S_1, T_1 under the successive blow-ups. This guarantees us that at each step, these proper transforms are zero-sets of sections of adequate vector bundles on the intermediate blow-up varieties $X_1, \ldots, X_4$, which proves Lemma 3.18(i). We also apply Lemma 3.19 to Y itself and its successive proper transforms, using Lemma 3.17, which proves Lemma 3.18(ii).

Proof of Lemma 3.19. (i) Let $t := \dim T$, $m_1 := \dim M_1$, $m := \dim M$ and $n := \dim N$. By assumption, we can construct local coordinates $z_1, \ldots, z_n$ on N such that M_1 is defined by $z_i = 0$, $i \geq m_1 + 1$, T is defined by $z_i = 0$, $i \geq t + 1$, and M is defined by

$$z_i = 0,\ i \geq m + 2,\ f(z) := \sum_{i=m_1+1}^{m+1} z_i^2 = 0. \tag{34}$$

The blow-up $\widetilde{N}$ of N along T is defined inside

$$\mathbb{P}^{n-t-1} \times N$$

by the equations

$$Y_i z_j = Y_j z_i$$

for i, $j \geq t + 1$, the Y_i's being homogeneous coordinates on $\mathbb{P}^{n-t-1}$. On the open set where $Y_k \neq 0$, we have local coordinates

$$y_j, \text{ for } j \geq t + 1\, j \neq k,\ z_k$$

$$z_l, \text{ for } 1 \leq l \leq t$$

on $\widetilde{N}$ and the blow-up map is given by

$$z_j = z_k y_j,$$

the exceptional divisor being defined by $z_k = 0$. We examine the various cases $k \leq m_1$, $m_1 + 1 \leq k \leq m + 1$, $k > m + 1$.

If $k \leq m_1$, the local equations for $\widetilde{M}$ are

$$y_i = 0,\ i \geq m + 2,\ \sum_{i=m_1+1}^{m+1} y_i^2 = 0, \tag{35}$$

and $\widetilde{M_1}$ is defined by $y_i = 0$ for any $i \geq m_1 + 1$. It follows that $\widetilde{M}$ has ordinary quadratic singularities along $\widetilde{M_1}$.

If $k > m+1$, the local equations for $\widetilde{M}$ are

$$z_k = 0,\ y_i = 0,\ \text{for}\, i \geq m+2,\ i \neq k,\ \text{and} \sum_{i=m_1+1}^{m+1} y_i^2 = 0. \tag{36}$$

Furthermore $\widetilde{M_1}$ is defined by $y_i = 0$ for any $i \geq m_1+1$, $i \neq k$, and $z_k = 0$. It follows again that $\widetilde{M}$ has ordinary quadratic singularities along $\widetilde{M_1}$.

If $m_1 + 1 \leq k \leq m+1$, the local equations for $\widetilde{M}$ are

$$y_i = 0,\ i \geq m+2,\ 1 + \sum_{i=m_1+1, i\neq k}^{m+1} y_i^2 = 0, \tag{37}$$

and $\widetilde{M_1}$ is defined by $y_i = 0$ for any $i \geq m_1 + 1$. It follows that $\widetilde{M}$ is smooth in this open set.

(ii) Let E be the exceptional divisor of the blow-up map $\tau : \widetilde{N} \to N$, and E_1 be its component over T_1. We construct $\widetilde{\mathcal{F}}$ by modifying $\tau^*\mathcal{F}(-E)$ along E_1. As $T_1 \subset M$, the section s vanishes along T_1 and has a differential

$$ds : N_{T_1/N} \to \mathcal{E}_{|T_1}.$$

This differential has corank 1, as follows from the fact that M is singular with hypersurface singularities along T_1. We thus have a quotient line bundle $\mathcal{L}$ of $\mathcal{E}_{|T_1}$, and denoting $\tau_{E_1} : E_1 \to T_1$ the restriction of τ to E_1, we get a quotient map constructed as the composition

$$q : \tau^*\mathcal{E}(-E_1) \to \tau_{E_1}^*\mathcal{E}_{|T_1}(-E_1) \to \tau_{E_1}^*\mathcal{L}(-E_1).$$

We set

$$\widetilde{\mathcal{F}} := \operatorname{Ker} q.$$

This is a vector bundle of rank $n-m$ on $\widetilde{N}$. Furthermore, we observe that by construction, the section $\tau^*\sigma$ of $\tau^*\mathcal{F}$ provides in fact a section $\tilde{s}$ of $\widetilde{\mathcal{F}} \subset \tau^*\mathcal{F}$. It remains to check that the vanishing locus of $\tilde{s}$ is exactly $\widetilde{M}$. This easily follows from the local equations exhibited in the previous proof. □

□

Proof of Theorem 1.6. We have finally to explain how to improve the inclusion (32) to

$$(d-6)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}}. \tag{38}$$

We can assume that $d \geq 5$. Let $Z \subset X$ be a smooth subvariety of dimension d. From the previous proof, we know that there exist a smooth projective variety Y of dimension $2d-4$ and a morphism

$$j : Y \to X$$

such that $j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$, and a smooth subvariety $Z' \subset Z$ such that $j(Z') = Z$ and $j_{|Z'}$ induces a birational morphism between Z' and Z. We now choose a general hypersurface $Y' \subset Y$ containing Z'. By Lemma 3.17, Y' has ordinary quadratic singularities along a closed algebraic subset $Q \subset Z'$ which is smooth of dimension 4, namely Q is the zero-set of the section of $N^*_{Z/Y}(Y')$ defined as the differential of the local equation defining Y', which is well-defined along Z'. Assume that, after replacing Y by $Y_\sharp = Y \times \mathbb{P}^r$, we can construct a 4-dimensional complete intersection $Q_1 \subset Y$ which contains Q as an irreducible component, is the union $Q_1 = Q \cup Q'$, where Q' is smooth, the union has ordinary quadratic singularities along $T := Q' \cap Q$ and Q' meets Y' exactly along T. Then

the whole previous proof works, introducing complete intersections $T_1 = T \cup T'$, $S_1 = S \cap S'$, where $S := T \cap T'$ et caetera. In conclusion, we just have to explain how to reduce to this case, and we will do it by a blow-up of $Y_\sharp$ at finitely many points. The new problem that we meet here is that liaison of varieties of dimension ≥ 4 introduces points where the above description of Q_1 fails. Indeed, the general 4-dimensional complete intersection $Q_1 \subset Y_\sharp$ containing Q as an irreducible component is the zero-set of a section

$$\sigma \in H^0(Y_\sharp, \mathcal{E})$$

where $\mathcal{E}$ is a vector bundle of rank $m-4$, $m := \dim Y_\sharp$, on $Y_\sharp$, whose differential along Q provides a morphism

$$d\sigma : N_{Q/Y_\sharp} \to \mathcal{E}_{|Q}.$$

As $\dim Q = 4$, there are finitely points $y_i \in Q$, where the morphism $d\sigma$ has corank 2, and where the closed algebraic subset $Q_1 = Q \cup Q'$ does not have ordinary quadratic singularities along a smooth hypersurface $Q \cap Q'$ in Q.

We fix this point as follows.

Lemma 3.20. *Let $\widetilde{Y}$ be the blow-up of the finitely many points y_i in $Y_\sharp$ and at a number of other points which are not on $Y' = Y' \times \mathrm{pt}$. Then*

(i) The proper transform $\widetilde{Q}_1$ of Q_1 in $\widetilde{Y}$ has ordinary quadratic singularities along a 3-dimensional smooth subvariety T.

(ii) Furthermore, $\widetilde{Q}_1$ is the zero-set of a section of a rank $d-4$ vector bundle on $\widetilde{Y}$.

Proof. (i) At the points y_i, the embedding dimension of Q_1 is 6, so we can choose local coordinates $z_1, \ldots, z_{2d-4}$ on $Y_\sharp$ centered at y_i, such that Q_1 is contained in $M := \{z_j = 0,\ j \geq 7\}$, and Q is defined by the equations z_j, $j \geq 5$. The two equations for Q_1 in M have vanishing differential at y_i, and they vanish along Q. In an adequate choice of coordinates on M, they take the simplest form, namely

$$z_5 z_1 + z_6 z_2 = 0,\ z_5 z_3 + z_6 z_4 = 0. \tag{39}$$

Along the exceptional divisor of $\widetilde{Y}$ over y_i, we have homogeneous coordinates Z_i and two situations:

- In the Zariski open set where $Z_1 \neq 0$, we have coordinates z_1, $z'_2, \ldots, z'_6$ on the proper transform $\widetilde{M}$, the morphism τ being given by $\tau^* z_i = z_1 z'_i$, $i \geq 2$. Then the pull-backs of the equations (39) on $\widetilde{M}$ are

$$z_1^2(z'_5 + z'_6 z'_2),\ z_1^2(z'_5 z'_3 + z'_6 z'_4).$$

The proper transform of Q_1 is thus defined by the equations

$$z'_5 + z'_6 z'_2 = 0,\ z'_5 z'_3 + z'_6 z'_4 = 0.$$

It is clear that the closed analytic subset defined by these equations has two smooth irreducible components of dimension 4 meeting along a smooth hypersurface, namely $\{z'_5 = z'_6 = 0\}$ and $\{z'_5 + z'_6 z'_2 = 0,\ -z'_2 z'_3 + z'_4 = 0\}$.

- In the Zariski open set where $Z_5 \neq 0$, we have coordinates $z'_1, \ldots, z'_4$, z_5, z'_6 on the proper transform $\widetilde{M}$, the morphism τ being given by $\tau^* z_i = z_5 z'_i$, $i \neq 5$. Then the pull-backs of the equations (39) on $\widetilde{M}$ are

$$z_5^2(z'_1 + z'_6 z'_2),\ z_5^2(z'_3 + z'_6 z'_4).$$

The proper transform of Q_1 is thus defined by the equations

$$z'_1 + z'_6 z'_2 = 0,\ z'_3 + z'_6 z'_4 = 0.$$

It is clear that the closed analytic subset defined by these equations is smooth.

(ii) Recall that Q_1 is the zero-set of a section σ of a vector bundle $\mathcal{F}$ of rank $m-4$ on Y. As we saw in the computation above, at each point y_i, the differential $d\sigma : T_{Y_\sharp|Q_1} \to \mathcal{F}_{|Q_1}$ has rank only $m-6$. We thus have 2-dimensional quotients

$$q_i : \mathcal{F}_{|y_i} \to B_i.$$

Denoting E_{y_i} the exceptional divisor over y_i, these quotient maps provide a surjective morphism of sheaves on $\widetilde{Y}$

$$q : \tau^*\mathcal{F}(-\sum_i E_{y_i}) \to \oplus_i B_i \otimes \mathcal{O}_{E_{y_i}}(-E_{y_i}),$$

whose kernel is a vector bundle $\mathcal{F}'$ on $\widetilde{Y}$. It is clear from the explicit equations above that the section σ lifts to a section $\tilde{\sigma}$ of $\mathcal{F}'$, and that the proper transform $\widetilde{Q}_1$ is the zero-set of σ'. □

Let $\tau : \widetilde{Y} \to Y_\sharp$ be the blow-up of a finite reduced subset W of Y which contains all the points y_i above, is the zero set of a section of a vector bundle of rank m and is generic otherwise (in particular, the extra points are not in Y'). By Proposition 3.14, the morphism $\tau_* : \mathrm{CH}(\widetilde{Y}) \to \mathrm{CH}(Y_\sharp)$ maps $\mathrm{CH}(\widetilde{Y})_{\mathrm{flpshCh}}$ to $\mathrm{CH}(Y_\sharp)_{\mathrm{flpshCh}}$. The smooth projective variety $\widetilde{Y}$ contains the proper transforms $\widetilde{Z'}$ and $\widetilde{Y'}$ of Z' and Y' respectively. By Lemma 3.19(i), $\widetilde{Y'}$ has ordinary quadratic singularities along the proper transform $\widetilde{Q}$ of Q and by Lemma 3.19(ii), it is a generalized complete intersection in $\widetilde{Y}$. Finally, by Lemma 3.20 again, $\widetilde{Q}$ is an irreducible component of a 4-dimensional local complete intersection $\widetilde{Q_1}$, which is a generalized complete intersection on $\widetilde{Y}$ and has ordinary quadratic singularities along a 3-dimensional smooth subvariety $T \subset \widetilde{Q}$. By Propositions 3.14 and 3.12, the morphism $j \circ \mathrm{pr}_1 \circ \tau : \widetilde{Y} \to X$ maps the proper transform Z', which is smooth, birationally to Z, and satisfies

$$(j \circ \mathrm{pr}_1 \circ \tau)_*(\mathrm{CH}(\widetilde{Y})_{\mathrm{flpshCH}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCH}}.$$

Finally, as we already mentioned, the previous proof applies to desingularize the proper transform $\widetilde{Y'}$ by successive blow-ups of smooth generalized complete intersections starting from $\widetilde{Y}$, since $\widetilde{Q}$ is a component of a 4-dimensional generalized complete intersection in $\widetilde{Y}$ that has only ordinary quadratic singularities. We thus conclude by mimicking the previous proof that the proper transform $\widetilde{Z}$ of Z in $\widetilde{Y}$ satisfies

$$(d-6)![Z_1] \in \mathrm{CH}_d(\widetilde{Y})_{\mathrm{flpshCh}}.$$

By Propositions 3.14 applied to τ, it follows that $(d-6)![Z'] \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$, hence that

$$(d-6)![Z] = (d-6)!j_*[Z'] \in \mathrm{CH}_d(X)_{\mathrm{flpshCh}}.$$

□

4 The case of homogeneous varieties

4.1 Cycles on abelian varieties

We start with the proof of Theorem 1.13(3). The result in this case is stronger than Conjecture 1.5 since it states that $\mathrm{CH}_i(A) = \mathrm{CH}_i(A)_{\mathrm{smpshCh}}$ for any abelian variety A.

Proof of Theorem 1.13(3). Let $z \in \mathrm{CH}_d(A)$. We want to prove that

$$z \in \mathrm{CH}_d(A)_{\mathrm{smpshCh}}. \tag{40}$$

We can assume $z = [Z]$ for some subvariety $Z \subset A$ of dimension d. We denote by $\tau : \widetilde{Z} \to A$ a desingularization of Z. Consider the morphism

$$\phi : A \times \widetilde{Z} \to A$$

$$(x, \tilde{z}) \mapsto x + \tau(\tilde{z}).$$

Obviously ϕ is smooth, hence in particular flat. Furthermore, we have, denoting $0_A \in A$ the origin

$$z = \phi_*([0_A \times \widetilde{Z}]). \tag{41}$$

If $[0_A] \in \mathrm{CH}_0(A)$ belongs to the subring of $\mathrm{CH}^*(A)$ generated by Chern classes of coherent sheaves on A, so does $\mathrm{pr}_1^*([0_A]) = [0_A \times \widetilde{Z}] \in \mathrm{CH}(A \times \widetilde{Z})$, hence (41) implies (40) in this case. It is proved however in [4] that for a very general abelian variety A with sufficiently divisible polarization degree and high dimension, the class of a point does not belong to this subring of $\mathrm{CH}^*(A)$, so we cannot apply the argument directly to A. Nevertheless, Debarre also proves in *loc. cit.* that, if J is the Jacobian of a curve C of genus g, then for any point x of J, there exists a rank g vector bundle on J with a section whose zero locus is $\{x\}$ (with its reduced structure). In particular, the class $[x]$ belongs to the subring of $\mathrm{CH}^*(J)$ generated by Chern classes of coherent sheaves on J. Let now $j : C \hookrightarrow A$ be the inclusion of a smooth curve of genus g which is a complete intersection of ample hypersurfaces in A. Then by Lefschetz theorem on hyperplane sections, we have a surjective (hence smooth) morphism $\psi = j_* : J = JC \to A$ of abelian varieties, and $\psi_*([0_J]) = [0_A]$. Let

$$\phi_J : J \times \widetilde{Z} \to A$$

be the composite $\phi \circ (\psi, Id)$. Then ϕ_J is smooth and we have

$$z = \phi_{J*}([0_J \times \widetilde{Z}]). \tag{42}$$

As $[0_J]$ belongs to the subring of $\mathrm{CH}^*(J)$ generated by Chern classes of coherent sheaves on J, $[0_J \times \widetilde{Z}]$ belongs to the subring of $\mathrm{CH}^*(J \times \widetilde{Z})$ generated by Chern classes of coherent sheaves on $J \times \widetilde{Z}$, so (40) follows from formula (42). □

4.2 More general homogeneous varieties

We prove in this section Theorem 1.13(1), which is the following statement

Theorem 4.1. *(1) Let X be a projective variety which is homogeneous under a group G. Then*

$$\mathrm{CH}_i(X)_{\mathrm{flpshCh}} = \mathrm{CH}_i(X) \tag{43}$$

for all i.

(2) If furthermore there exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies $\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{smpshCh}}$, then

$$\mathrm{CH}_i(X)_{\mathrm{smpshCh}} = \mathrm{CH}_i(X) \tag{44}$$

for all i.

By (1), X satisfies Conjecture 1.5. It follows that cycles in $\mathrm{CH}_d(X)$ are smoothable for $2d < \dim X$ by Proposition 2.9.

Proof of Theorem 4.1. Let $z \in \mathrm{CH}_d(X)$ be the class of a subvariety $Z \subset X$ and let $\tau : \widetilde{Z} \to X$ be a desingularization of Z. We consider the morphism

$$f : G \times \widetilde{Z} \to X,$$

$$(g, \tilde{z}) \mapsto g \cdot \tau(\tilde{z}).$$

The morphism f is obviously smooth, since it is G-equivariant so its restriction to any $G \times \tilde{z}$ is smooth. It is however not proper, but it provides a G-equivariant rational map

$$F : \overline{G} \times \widetilde{Z} \dashrightarrow X, \tag{45}$$

where $\overline{G}$ is a smooth projective completion of G on which G acts (which exists by G-equivariant resolution of singularities [13]). The action of G on the left hand side of (45) is via its action on $\overline{G}$. By G-equivariant resolution of indeterminacies [16], there exist a smooth projective variety Y on which G acts, a G-equivariant birational morphism $\eta : Y \to \overline{G} \times \widetilde{Z}$, and a morphism

$$\widetilde{F} : Y \to X,$$

such that $F \circ \eta = \widetilde{F}$ as rational maps to X. The morphism $\widetilde{F}$ is proper since Y is projective, and it is again smooth because it is G-equivariant. A fortiori it is flat. Furthermore we have

$$z = \widetilde{F}_*(\eta^*([e \times \widetilde{Z}])) \text{ in } \mathrm{CH}_d(X), \tag{46}$$

where $e \in G \subset \overline{G}$ is the neutral element. We now recall from Theorem 1.6 that 0-cycles on a smooth projective variety T belong to $\mathrm{CH}_0(T)_{\mathrm{flpshCh}}$. For $T = \overline{G}$, this provides us with a smooth projective (nonnecessarily connected but that we can assume equidimensional) variety W and a flat proper morphism $\phi : W \to \overline{G}$ such that e can be written as

$$e = \phi_*(D_1 \dots D_N) \ \in \ \mathrm{CH}_0(\overline{G}), \ N = \dim W, \tag{47}$$

for some divisors $D_l \in \mathrm{CH}^1(W)$. Let $Y' := W \times_{\overline{G}} Y$, with first projection $q : Y' \to W$ and second projection $p : Y' \to Y$. Then p is flat and proper and denoting $\widetilde{F}' := \widetilde{F} \circ p : Y' \to X$, $\widetilde{F}'$ is also flat and proper. Moreover, by (46) and (47), z can be written as

$$z = \widetilde{F}'_*(q^* D_1 \dots q^* D_N) \in \ \mathrm{CH}_d(X). \tag{48}$$

If $\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{smpshCh}}$, then we can assume that the morphism $\phi : W \to \overline{G}$ is smooth, hence Y' is smooth and $p : Y' \to Y$ is smooth, so that the morphism $\widetilde{F}' : Y' \to X$ is smooth. In this case (48) shows that $z \in \mathrm{CH}_i(X)_{\mathrm{smpshCh}}$. This proves statement (2).

In general, in order to conclude from (48) that $z \in \mathrm{CH}_d(X)_{\mathrm{flpshCh}}$, that is, statement (1), we only need to know that we can arrange so that Y' is smooth. This is the contents of Proposition 4.2 below, which refines Theorem 1.6 for 0-cycles. The proof of the Theorem is finished. □

Proposition 4.2. *Let T be any smooth projective variety, and let $f : Y \to T$ be a morphism. Let $t \in T$ be any point. Then there exist a smooth projective variety W and a flat morphism $\phi : W \to T$ such that*

$$t = \phi_* t' \text{ in } \mathrm{CH}_0(T),$$

with $t' \in \mathrm{CH}_0(W)$ a combination of products of divisor classes on W, and

$$T' := W \times_T Y$$

is smooth.

Proof. We look at the proof of Theorem 1.6 for 0-cycles. It relies directly on Lemma 3.11. By Chow moving lemma, it suffices to prove the statement for a general point t of T. We choose a sequence of inclusion

$$\{t\} \subset T_1 \subset T_2 \subset \dots \subset T_{n-1} \subset T_n = T,$$

where each T_i is a general complete intersection of dimension i containing T_{i-1}. Lemma 3.11 tells us that the blow-up $\tau_i : M_i \to T_{i-1} \times T_i$ of the graph $\Gamma_i \subset T_{i-1} \times T_i$ of the inclusion $j_{i-1} : T_{i-1} \hookrightarrow T_i$ has the property that the morphism $p_i := \mathrm{pr}_1 \circ \tau_i : M_i \to T_{i-1}$ is smooth and the morphism $q_i := \mathrm{pr}_2 \circ \tau_i : M_i \to T_i$ is flat. Following the proof of Proposition 3.9, the variety W will be obtained by taking the successive fibred product of the M_i for $i \geq 2$, or equivalently

$$W := M_2 \times_{T_2} M_3 \times_{T_3} M_4 \times \dots \times_{T_n} M_n. \tag{49}$$

We denote by $p : W \to T_1$ the first projection and by $q : W \to T_n = T$ the last projection. Using the divisor t in T_1 and the exceptional divisors of each M_i, we conclude by combining formulas as in (18) that the variety W of (49) works. It remains to prove that the variety W constructed above has the property that $T' := W \times_T Y$ is smooth. Let $Y_{n-1} := f^{-1}(T_{n-1})$. Then by Bertini, Y_{n-1} is a smooth hypersurface in Y. This fact also says that the image $\operatorname{Im} f_{*,y}$ of the differential

$$f_{*,y} : T_{Y,y} \to T_{T,f(y)}$$

satisfies

$$T_{T_{n-1},y} + \operatorname{Im} f_{*,y} = T_{T,y}$$

at any point $y \in Y_{n-1}$. It follows that $W_n := Y \times_T M_n$ is smooth. Indeed the morphism $q_n : M_n \to T$ is smooth above $T \setminus T_{n-1}$ and, at any point m over $t \in T_{n-1}$, the image $\operatorname{Im} q_{n*,m}$ of its differential contains $T_{T_{n-1},t}$. As $\operatorname{Im} f_{*,y} + \operatorname{Im} q_{n*,m} = T_{T,t}$, $f(y) = t$, one finds that W_n is smooth. We now continue inductively with the morphism $W_n \to T_{n-1}$ instead of $Y \to T$ to conclude finally that T' is smooth. □

5 Further results

5.1 A further stability result for flat push-forwards of Chern classes

We establish in this section the following complement to Theorem 1.9.

Proposition 5.1. *Let X be smooth and projective and let $j_Z : Z \hookrightarrow X$ be the inclusion of a smooth codimension 2 subvariety of X. Then $j_{Z*}(\mathrm{CH}(Z)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$.*

Proof. Let $\tau : \widetilde{X} \to X$ be the blow-up of Z and $E_Z \subset \widetilde{X}$ be the exceptional divisor. We choose a very ample line bundle H on X and denote $H' := \tau^* H \in \operatorname{Pic} \widetilde{X}$. We have the following

Lemma 5.2. *Let $b \geq \dim Z$ be an integer. Then for $a >> 0$, the inclusion $\tilde{j} : Y \hookrightarrow \widetilde{X}$ of a general divisor in the linear system $|aH' - bE|$ on $\widetilde{X}$ has the property that the composite morphism $j := \tau \circ \tilde{j} : Y \to X$ is a finite morphism.*

Proof. It suffices to know that Y contains no fiber of $\tau_E : E \to Z$. Equivalently, it suffices to know that the section

$$\sigma_{|E} \in H^0(E, (aH' - bE)_{|E}) = H^0(Z, \tau_{E*}((aH' - bE)_{|E}) = H^0(Z, \mathrm{Sym}^b N^*_{Z/X}(aH)) \quad (50)$$

vanishes nowhere on Z, where $\tau_E := \tau_{|E} : E \to Z$. The vector bundle $\mathrm{Sym}^b N^*_{Z/X}(aH)$ has rank $b+1$ and is generated by sections for a large enough, hence a general section of $\mathrm{Sym}^b N^*_{Z/X}(aH)$ vanishes at no point of Z since $b \geq \dim Z$. Furthermore, the restriction morphism $H^0(\widetilde{X}, (aH' - bE)) \to H^0(E, (aH' - bE)_{|E})$ is surjective for $a >> 0$ by Serre vanishing. The lemma follows. □

By Lemma 3.4, $\tau_E^*(\mathrm{CH}(Z)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(E)_{\mathrm{flpshCh}}$, hence by Proposition 3.9,

$$j_{E*}(\tau_E^*(\mathrm{CH}(Z)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}, \quad (51)$$

where $j_E : E \to \widetilde{X}$ is the inclusion. We now prove the following

Lemma 5.3. *Let M be a smooth projective variety and let L be a very ample line bundle on M. Then if*

$$k : N \hookrightarrow M$$

is the inclusion of a very general member of the linear system $|L|$, we have

$$k^*(\mathrm{CH}(M)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(D)_{\mathrm{flpshCh}}(D).$$

Proof. Let $f : P \to M$ be a flat projective morphism, where P is smooth, and $D_i \in \mathrm{CH}^1(P)$, $i = 1, \dots, r$ be divisors. Then for any subvariety $N \subset M$, one gets by taking the Cartesian product a flat morphism $f_N : P_N := P \times_M N \to N$, and we have by [7, Proposition 2.3],

$$(f_*(D_1 \dots D_r))_{|N} = f_{N*}(D_{1|P_N} \dots D_{r|P_N}) \text{ in } \mathrm{CH}(N). \tag{52}$$

If now N is a general member of a linear system $|L|$, with L globally generated, P_N is smooth by Bertini, so (52) implies that $(f_*(D_1 \dots D_r))_{|N} \in \mathrm{CH}(N)_{\mathrm{flpshCh}}$ for general N. Here general may depend on $z := f_*(D_1 \dots D_r) \in \mathrm{CH}(M)_{\mathrm{flpshCh}}$ but by Lemma 3.3(d), one needs to check Lemma 5.3 only for countably many elements of $\mathrm{CH}(M)_{\mathrm{flpshCh}}$. A very general N will satisfy the property above for countably many choices of f, D_i generating $\mathrm{CH}(M)_{\mathrm{flpshCh}}$ modulo algebraic equivalence, so the proof is finished. □

Let b, a and $\tilde{j} : Y \hookrightarrow \widetilde{X}$ be very general as in Lemmas 5.2 and 5.3. Then by (51) and Lemma 5.3, for any $z \in \mathrm{CH}(Z)_{\mathrm{flpshCh}}$, we have

$$(j_{E*}(\tau_E^* z))_{|Y} \in \mathrm{CH}(Y)_{\mathrm{flpshCh}}. \tag{53}$$

By Lemma 5.2, $j : \tau \circ \tilde{j} : Y \to X$ is finite, hence by Proposition 3.9, (53) implies

$$j_*((j_{E*}(\tau_E^* z))_{|Y}) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{54}$$

We have

$$j_*((j_{E*}(\tau_E^* z))_{|Y}) = b j_{Z*}(z) \text{ in } \mathrm{CH}(X),$$

hence we conclude that $b j_{Z*}(z) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$. It follows as well that $(b+1) j_{Z*}(z) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$, hence $j_{Z*}(z) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$. □

5.2 A weaker criterion for smoothability

The general principle underlying the proof of Theorem 1.13 is that movable cycles (in a precise sense) are flat pushforwards of Chern classes because 0-cycles are, hence they are smoothable in the Whitney range. The same idea leads to the following criterion for smoothability.

Theorem 5.4. *Let $p : \mathcal{Z} \to X$ be a projective morphism, where X is smooth projective. Let $q : \mathcal{Z} \to B$ be a surjective morphism of relative dimension d, where B is smooth projective. Assume that the locus of points $z \in \mathcal{Z}$ where either $\mathcal{Z}$ is not smooth, or p is not flat has codimension $> d$ in $\mathcal{Z}$. Then for any point $b \in B$, the cycle is smoothable if $2d < \dim X$. Moreover, if $\mathcal{Z}$ is smooth and p is flat, $p_* \mathcal{Z}_b$ belongs to $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$.*

Proof. Assume first that $\mathcal{Z}$ is smooth and $p : \mathcal{Z} \to X$ is flat. Let $q : \mathcal{Z} \to B$ be a surjective morphism of relative dimension d, where B is smooth projective. By Proposition 4.2 applied to the morphism $q : \mathcal{Z} \to B$ (in place of $Y \to T$), for any $b \in B$, there exist a smooth projective variety W and a flat morphism $\phi : W \to B$ such that

$$b = \phi_* b' \text{ in } \mathrm{CH}_0(B),$$

with $b' \in \mathrm{CH}_0(W)$ a combination of products of divisor classes on W, and $\mathcal{Z}_W := W \times_B \mathcal{Z}$ is smooth. We now consider the morphisms

$$p_W : \mathcal{Z}_W \to X, \ q_W : \mathcal{Z}_W \to W.$$

The morphism p_W is flat, as the composition of the flat morphisms $\mathcal{Z}_W \to \mathcal{Z}$ and $\mathcal{Z} \to X$. Furthermore we have

$$p_*(\mathcal{Z}_b) = p_{W*}(q_W^*(b')) \text{ in } \mathrm{CH}_d(X). \tag{55}$$

Hence $p_*(\mathcal{Z}_b) \in \mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ since $b' \in \mathrm{CH}_0(W)_{\mathrm{Ch}}$, so $q_W^*(b') \in \mathrm{CH}_d(\mathcal{Z}_W)_{\mathrm{Ch}}$. This proves the second statement of the theorem. If, in the situation above, we only assume B smooth and p, q projective, formula (55) is still true. If furthermore $\mathcal{Z}$ is smooth in codimension d, so is $\mathcal{Z}_W$, by Proposition 4.2 applied to the smooth locus of $\mathcal{Z}$ and by flatness of ϕ. If p is flat in codimension d, so is $p_W : \mathcal{Z}_W \to X$, again by flatness of ϕ. We are thus in this case in the situation of Proposition 2.9(b), which tells us that $p_*(\mathcal{Z}_b)$ is smoothable if $2d < \dim X$. □

5.3 A generalization of the diagonal property

In the paper [15], the following "diagonal property" is considered, for any smooth variety X of dimension n:

(*) *There exist a rank n vector bundle E on $X \times X$ and a section σ of E whose (schematic) zero set is the diagonal $\Delta_X \subset X \times X$.*

Property (*) implies in particular that $\Delta_X = c_n(E)$ in $\mathrm{CH}(X \times X)$ for some vector bundle E on $X \times X$, so that $\Delta_X \in \mathrm{CH}(X \times X)_{\mathrm{Ch}}$. This last property is not satisfied by general abelian varieties with sufficiently divisible polarization degree (see [4]).

Let us instead consider the following weaker property:

(**) *The class $\Delta_X \in \mathrm{CH}(X \times X)$ of the diagonal $\Delta_X \subset X \times X$ belongs to $\mathrm{CH}(X \times X)_{\mathrm{smpshCh}}$.*

By Proposition 2.4, this property is satisfied by X if $\Delta_X \in \mathrm{CH}(X \times X)_{\mathrm{Ch}}$. Condition (**) is satisfied by abelian varieties by Theorem 1.13 applied to $A \times A$.

Proposition 5.5. *Assume X satisfies property (**). Then $\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{smpshCh}}$. That is, any cycle $z \in \mathrm{CH}(X)$ is of the form $f_*\beta$, for some smooth proper morphism $f : Y' \to X$ and class β in the subring of $\mathrm{CH}(Y')$ generated by Chern classes of coherent sheaves on Y'.*

In particular, any cycle $z \in \mathrm{CH}_d(X)$ is smoothable if $2d < n = \dim X$.

Proof. The second statement follows from the first by Proposition 2.9. To prove the first statement, it suffices to consider the case of the class $[Z]$ of a subvariety $Z \subset X$. Let $\tau : \widetilde{Z} \to X$ be a desingularization of Z. Then we have the following relation in $\mathrm{CH}(X)$

$$[Z] = \mathrm{pr}_{2*}((\tau, Id_X)^*\Delta_X), \tag{56}$$

where $\mathrm{pr}_2 : \widetilde{Z} \to X \to X$ is the second projection. Our assumption is that $\Delta_X = \pi_*\alpha$, where Y is smooth, $\pi : Y \to X \times X$ is a smooth proper morphism, and $\alpha \in \mathrm{CH}(Y)$ belongs to the subring generated by Chern classes of coherent sheaves on Y. We thus have

$$(\tau, Id_X)^*\Delta_X = \pi'_*\alpha' \text{ in } \mathrm{CH}(\widetilde{Z} \times X), \tag{57}$$

where

$$Y' = (\widetilde{Z} \times X) \times_{X \times X} Y, \tag{58}$$

$\pi' : Y' \to \widetilde{Z} \times X$ is the first projection and α' is the pull-back of α to Y' via the second projection $Y' \to Y$. As the morphism π is smooth, the morphism π' is also smooth (hence in particular Y' is smooth). Furthermore, α' belongs to the subring of $\mathrm{CH}(Y')$ generated by Chern classes of coherent sheaves on Y', hence the result follows from (56), (57). □

Remark 5.6. We could weaken property (**) by asking that Δ_X belongs to $\mathrm{CH}(X \times X)_{\mathrm{flpshCh}}$. However, this does not imply as above that $\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{flpshCh}}$. The reason is that, in the proof above, if we only assume that π is flat, the variety Y' of (58) is not necessarily smooth.

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