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Shimura varieties (version of November 7, 2023)

Sophie Morel

ABSTRACT. Lecture 1 gives an introduction to Shimura varieties over the complex numbers (defined here as a special type of locally symmetric spaces) and to the general theory of canonical models; it also discusses in more detail the example of the Siegel modular varieties. Lecture 2 presents some families of Shimura varieties (PEL type, Hodge type, abelian type) and the results that are known about their canonical and integral models. Finally, lecture 3 discusses the cohomology of Shimura varieties, concentrating mostly on the compact non-endoscopic case.

1. Lecture 1: locally symmetric spaces and Shimura varieties

1.1. Locally symmetric spaces. Let G be a semi-simple algebraic group over \mathbb{Q} , for example SL_n , Sp_{2n} , or a special orthogonal or special unitary group. Locally symmetric spaces for $G(\mathbb{R})$ are “nice enough” spaces whose cohomology is related to automorphic representations of G . A good reference for locally symmetric spaces is the introductory paper [51] by Ji.

To simplify the presentation, we will assume here that $G(\mathbb{R})$ is connected. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and let $X = G(\mathbb{R})/K_\infty$. If Γ is a discrete subgroup of $G(\mathbb{R})$ such that $\Gamma \backslash G(\mathbb{R})$ (or equivalently $\Gamma \backslash X$) is compact and that Γ acts properly and freely on X ,¹ then there is a classical connection between the cohomology of $\Gamma \backslash X$ and automorphic representations of $G(\mathbb{R})$, called *Matsushima’s formula* (see Matsushima’s paper [81]). We will state a modern reformulation in Lecture 3 (see Theorem 3.1), but roughly it relates the Betti numbers of $\Gamma \backslash X$ and the multiplicities of representations of $G(\mathbb{R})$ in $L^2(\Gamma \backslash G(\mathbb{R}))$.

In fact, Matsushima’s paper deals with semi-simple real Lie groups. Here, we have an algebraic group defined over \mathbb{Q} , so we have a particularly nice way to produce discrete subgroups of $G(\mathbb{R})$. Remember that a subgroup Γ of $G(\mathbb{Q})$ is called an *arithmetic subgroup* if there exists a closed embedding of algebraic groups $G \subset \mathrm{GL}_N$ such that, setting $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$, we have that $\Gamma \cap G(\mathbb{Z})$ is

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¹This holds for example if Γ is torsion free.

of finite index in Γ and in $G(\mathbb{Z})$.² If Γ is small enough, then it acts properly and freely on X ([51, Proposition 5.5]), so the quotient $\Gamma \backslash X$ is a real analytic manifold. Also, the quotient $\Gamma \backslash G(\mathbb{R})$ is compact if and only if G is anisotropic (over \mathbb{Q}), which means that G has no nontrivial parabolic subgroup defined over \mathbb{Q} ([51, Theorem 5.10]). If $\Gamma \backslash X$ is not compact but $G(\mathbb{R})$ has a discrete series, then there is an extension of Matsushima's formula, due to Borel and Casselman in [19], that involves L^2 cohomology of $\Gamma \backslash X$; see 3.5.3.

We actually would like to see automorphic representations of $G(\mathbb{A})$ (not just $G(\mathbb{R})$) in the cohomology of our spaces, so we will use adelic versions of $\Gamma \backslash X$. Let K be an open compact subgroup of $G(\mathbb{A}_f)$, where \mathbb{A}_f is the ring of finite adèles; for example, if we have chosen an embedding $G \subset \mathrm{GL}_N$, then we could take $K = G(\mathbb{A}_f) \cap \mathrm{Ker}(\mathrm{GL}_N(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GL}_N(\mathbb{Z}/n\mathbb{Z}))$, for some positive integer n (these are called *principal congruence subgroups*). Let

$$M_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K,$$

where the group K acts by right translations on the factor $G(\mathbb{A}_f)$, and the group $G(\mathbb{Q})$ acts by left translations on both factors simultaneously. Choose a system of representatives $(x_i)_{i \in I}$ of the finite³ quotient $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K$, and set $\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}$ for every $i \in I$. Then the Γ_i are arithmetic subgroups of $G(\mathbb{Q})$, and we have

$$M_K = \coprod_{i \in I} \Gamma_i \backslash X,$$

so M_K is a real analytic manifold if K is small enough. But now we have an action of $G(\mathbb{A}_f)$ on the projective system $(M_K)_{K \subset G(\mathbb{A}_f)}$, so we get an action on $\varinjlim_K \mathrm{H}^*(M_K)$, where H^* is any "reasonable" cohomology theory, for example Betti cohomology. If G is anisotropic over \mathbb{Q} , then Matsushima's result can be reformulated to give a description of this action in terms of irreducible representations of $G(\mathbb{A})$ appearing in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$; this is the version we give in Theorem 3.1. There is also a version of the Borel-Casselman generalization (see 3.5.3).

There is another way to think about the action of $G(\mathbb{A}_f)$ on $(M_K)_{K \subset G(\mathbb{A}_f)}$, which does not involve a limit on K . Fix a Haar measure on $G(\mathbb{A}_f)$ such that open compact subgroups of $G(\mathbb{A}_f)$ have rational volume (this is possible because these groups are all commensurable); then every open subset of $G(\mathbb{A}_f)$ has rational volume, because it is a finite union of translates of open compact subgroups. The *Hecke algebra* of G is the space \mathcal{H}_G of locally constant functions with compact support from $G(\mathbb{A}_f)$ to \mathbb{Q} ; if $f, g \in \mathcal{H}_G$, then the convolution product $f * g$ still has rational values by the choice of Haar measure, so convolution defines a multiplication on \mathcal{H}_G . For every open compact subgroup K of $G(\mathbb{A}_f)$, the *Hecke algebra at level K* is the subalgebra $\mathcal{H}_{G,K}$ of bi- K -invariant functions in \mathcal{H}_G ; we have $\mathcal{H}_G = \bigcup_K \mathcal{H}_{G,K}$.

Fix K small enough. Then $\mathrm{H}^*(M_K)$ is the set of K -invariant vectors in $\varinjlim_{K' \subset G(\mathbb{A}_f)} \mathrm{H}^*(M_{K'})$, so it has an action of $\mathcal{H}_{G,K}$.⁴ We can describe this action using Hecke correspondences: let $g \in G(\mathbb{A}_f)$ and let K' be an open compact

²We can check that this definition does not depend on the embedding $G \subset \mathrm{GL}_N$, see [51, Proposition 4.2].

³See Lemmas 5.11 and 5.12 of [87].

⁴In fact, we can recover the action of $G(\mathbb{A}_f)$ on $\varinjlim_{K \subset G(\mathbb{A}_f)} \mathrm{H}^*(M_K)$ from the action of $\mathcal{H}_{G,K}$ on M_K for every K small enough.

subgroup of $G(\mathbb{A}_f)$ such $K' \subset K \cap gKg^{-1}$. Then we have a *Hecke correspondence* $(T_1, T_g) : M_{K'} \rightarrow M_K^2$ sending the class of (x, h) in $M_{K'}$ to that of $((x, h), (x, hg))$ in $M_K \times M_K$, and T_1, T_g are both finite covering maps if K is small enough. Then the action of $\frac{1}{\text{vol}(K')} \mathbb{1}_{KgK}$ on $H^*(M_K)$ is given by pulling back cohomology classes along T_1 , then pushing them forward along T_g (see Section 16 of [64]).

We can also ask whether there is more structure on the spaces $\Gamma \backslash X$ (or M_K). For example, suppose that $G = \text{SL}_2$ and $K_\infty = \text{SO}(2)$. Then, for Γ an arithmetic subgroup of $\text{SL}_2(\mathbb{Z})$, the space $\Gamma \backslash X$ is a modular curve, so it is the set of complex points of an algebraic variety defined over a number field F , and we can use the commuting actions of Hecke correspondences and of the absolute Galois group of F on the étale cohomology of this variety to construct some instance of the global Langlands correspondence for GL_2 . See for example Deligne's Bourbaki seminar [34].

In order to generalize this picture, as outlined in Langlands's papers [69] and [70], we first need to know when the spaces $\Gamma \backslash X$ or M_K are the set of \mathbb{C} -points of an algebraic variety, and whether this algebraic variety is defined over a number field. As we will see in the section on canonical models (1.4), another advantage of M_K over $\Gamma \backslash X$ is that, when the answer to both questions above is "yes", then the M_K for K varying tend to all be defined over the same field, while this is not the case for the $\Gamma \backslash X$.

REMARK 1.1. The first step is to check whether $\Gamma \backslash X$ has the structure of a complex manifold, and there are obvious obstructions to that. For example, if $G = \text{SL}_3$ and $K_\infty = \text{SO}(3)$, then $\Gamma \backslash X$ is 5-dimensional as a real manifold, so it cannot have the structure of a complex manifold.

Choose a $G(\mathbb{R})$ -invariant Riemannian metric on $X = G(\mathbb{R})/K_\infty$ (such a metric is unique up to rescaling on each irreducible factor). Then X is a *symmetric space*, that is, a Riemannian manifold such that:

- (a) The group of isometries of X acts transitively on X ;
- (b) For every $p \in X$, there exists a symmetry s_p of X (i.e. an involutive isometry) such that p is an isolated fixed point of s_p .

Moreover, the symmetric space X is of *noncompact type*, that is, it has negative curvature. For Γ a small enough arithmetic subgroup of $G(\mathbb{Q})$, the Riemannian manifold $\Gamma \backslash X$ is a *locally symmetric space*; in particular, it does not satisfy condition (a) anymore, and it satisfies a variant of condition (b) where we only ask for the symmetry to be defined in a neighborhood of the point. See Ji's notes [51] for a review of locally symmetric spaces.

We say that X is a *Hermitian symmetric domain* if it admits a $G(\mathbb{R})$ -invariant Hermitian metric. See Section 1 of Milne's notes [87] for a review of Hermitian symmetric domains.

EXAMPLE 1.2. Let d be a positive integer. The *Siegel upper half space* \mathfrak{h}_d^+ is the set of symmetric $d \times d$ complex matrices $Y \in M_d(\mathbb{C})$ such $\text{Im}(Y)$ is positive definite; if $d = 1$, then this is just the usual upper half plane. We claim that \mathfrak{h}_d^+ is a Hermitian symmetric domain. For the proofs of the basic properties of \mathfrak{h}_d^+ that we state below, a good reference is Siegel's paper [112].

We first need to see \mathfrak{h}_d^+ as a symmetric space. Let Sp_{2d} be the symplectic group of the symplectic form with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$, where $I_d \in \text{GL}_d(\mathbb{Z})$ is the identity

matrix. For every commutative ring R , we have

$$\mathrm{Sp}_{2d}(R) = \left\{ g \in \mathrm{GL}_{2d}(R) \mid {}^t g \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \right\}.$$

Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$. We make $\mathrm{Sp}_{2d}(\mathbb{R})$ act on \mathfrak{h}_d^+ by the following formula:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Y = (AY + B)(CY + D)^{-1},$$

where A, B, C, D are $d \times d$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2d}(\mathbb{R})$ (see page 9 of [112]). Then this action is transitive (see page 10 of [112]). Let K_∞ be the stabilizer in $\mathrm{Sp}_{2d}(\mathbb{R})$ of $iI_d \in \mathfrak{h}_d^+$. Then $K_\infty = O(2d) \cap \mathrm{Sp}_{2d}(\mathbb{R})$ (this is easy to check directly), so it is a maximal compact subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$,⁵ and we have $\mathfrak{h}_d^+ \simeq \mathrm{Sp}_{2d}(\mathbb{R})/K_\infty$ as real analytic manifolds.

Also, the space \mathfrak{h}_d^+ is an open subset of the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, so it has an obvious structure of complex manifold. It remains to construct a $\mathrm{Sp}_{2d}(\mathbb{R})$ -invariant Hermitian metric on \mathfrak{h}_d^+ . Let \mathcal{D}_d be the set of symmetric matrices $A \in M_d(\mathbb{C})$ such that $I_d - A^*A$ is positive definite, this is a bounded domain in the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, hence is equipped with a canonical Hermitian metric called the *Bergman metric*, which has negative curvature (see for example [87, Theorem 1.3]); in particular, this metric is invariant by all holomorphic automorphisms of \mathcal{D}_d . Now note that we have an isomorphism $\mathfrak{h}_d^+ \xrightarrow{\sim} \mathcal{D}_d$ sending $X \in \mathfrak{h}_d^+$ to $(iI_d - X)(iI_d + X)^{-1}$ (whose inverse sends $A \in \mathcal{D}_d$ to $i(I_d - A)(I_d + A)^{-1}$), see pages 8-9 of [112]. We can give a formula for the resulting Hermitian metric on \mathfrak{h}_d^+ : up to a positive scalar, it is given by

$$ds^2 = \mathrm{Tr}(\mathrm{Im}(Y)^{-2} dY \mathrm{Im}(Y)^{-1} d\bar{Y})$$

(see formula (28) on page 17 of [112]).

The isomorphism $\mathfrak{h}_d^+ \xrightarrow{\sim} \mathcal{D}_d$ is called a *bounded realization* of \mathfrak{h}_d^+ .

We now state the classification of Hermitian symmetric domains in terms of real algebraic groups. Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

THEOREM 1.3. [87, Theorem 1.21] *Suppose that $G(\mathbb{R})$ is adjoint and connected. The locally symmetric space X is a Hermitian symmetric domain if and only if there exists a morphism of real Lie groups $u : U(1) \rightarrow G(\mathbb{R})$ such that:*

- (a) *The only characters of $U(1)$ that appear in its representation on $\mathrm{Lie}(G(\mathbb{R}))$ via $\mathrm{Ad} \circ u$ are 1, z and z^{-1} ;*
- (b) *Conjugation by $u(-1)$ is a Cartan involution of $G(\mathbb{R})$, which means that the real Lie subgroup $\{g \in G(\mathbb{C}) \mid g = u(-1)\bar{g}u(-1)^{-1}\}$ of $G(\mathbb{C})$ is compact;*
- (c) *The image of $u(-1)$ in every simple factor of $G(\mathbb{R})$ is nontrivial.*

Moreover, we can choose u such that K_∞ is the centralizer of u in $G(\mathbb{R})$, which means that X is isomorphic to the set of conjugates of u by elements of $G(\mathbb{R})$.

⁵In fact, we have an isomorphism $U(d) \xrightarrow{\sim} K_\infty$ sending $X + iY \in U(d)$ (with $X, Y \in \mathrm{GL}_d(\mathbb{R})$) to $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$.

We explain the construction of the morphism u . Suppose that X is a Hermitian symmetric domain, and let $p \in X$. For every $z \in \mathbb{C}$ with $|z| = 1$, multiplication by z on $T_p X$ preserves the Hermitian metric and sectional curvatures, so there exists a unique isometry $u_p(z)$ of X fixing p and such that $T_p u_p(z)$ is multiplication by z . The uniqueness implies that $u_p(z)u_p(z') = u_p(zz')$ if $|z| = |z'| = 1$, so we get a morphism of groups from $U(1)$ to the group of isometries of X , which is equal to $G(\mathbb{R})_{\text{ad}}^0 = G(\mathbb{R})$.

REMARK 1.4. If we don't assume that $G(\mathbb{R})$ is adjoint, then u goes from $U(1)$ to $G(\mathbb{R})_{\text{ad}}$, as we see in the sketch above, and it will not necessarily lift to a morphism from $U(1)$ to $G(\mathbb{R})$.

EXAMPLE 1.5. (1) If $G = \text{Sp}_{2d}$, let $h : \mathbb{C}^\times \rightarrow G(\mathbb{R})$ be defined by

$$h(a + ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

Then we can take $u : U(1) \rightarrow \text{PSp}_{2d}(\mathbb{R})$ given by $u(z) = h(\sqrt{z})$. Note that u does not lift to a morphism from $U(1)$ into $G(\mathbb{R})$.

(2) If $G = \text{PGL}_n$ with $n \geq 3$, then the centralizer of a character $u : U(1) \rightarrow G(\mathbb{R})$ cannot be a maximal compact subgroup of $G(\mathbb{R})$ (exercise), so the locally symmetric space of maximal compact subgroups of $G(\mathbb{R})$ is not Hermitian.

Theorem 1.3 puts some pretty strong restrictions on the root systems of the simple factors of $G(\mathbb{R})$, see Theorem 1.25 of [87] and the table following it. In particular, the type A simple factors of $G(\mathbb{R})$ must be of the form $\text{PSU}(p, q)$, and $G(\mathbb{R})$ can have no simple factor of type E_8, F_4 or G_2 .

The natural next step would be to wonder for which Hermitian symmetric domains X the quotients $\Gamma \backslash X$ are algebraic varieties, but in fact it turns out that the answer is “for all of them”, as was proved by Baily and Borel. Borel later proved that this structure of algebraic variety is unique.

THEOREM 1.6 (Baily-Borel and Borel, [12] and [17]). *Suppose that $X = G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain. Then, for any torsion free arithmetic subgroup Γ of $G(\mathbb{Q})$, the quotient $\Gamma \backslash X$ has a unique structure of algebraic variety over \mathbb{C} compatible with its structure of complex analytic manifold.*

The very rough idea is that the sheaf of automorphic forms on $\Gamma \backslash X$ of sufficiently high weight will define an embedding of $\Gamma \backslash X$ into a projective space. Borel's uniqueness theorem actually says that, if V is a quasi-projective complex algebraic variety, then any holomorphic map $V \rightarrow X$ is regular; see the discussion below Theorem 3.14 of [87] for more details on its proof.

Remember that we did not just want the locally symmetric spaces $\Gamma \backslash X$ to be algebraic varieties, we also wanted them to be defined over a number field, and we would ideally like the number field in question to only depend on G and K_∞ . For this, it will actually be easier to work with reductive groups instead of semi-simple groups. As a motivation for this, and for the definition of Shimura varieties, we now spend some more time on the case of the symplectic group.

1.2. The Siegel modular variety. :

See the end of this subsection (1.2.6 on page 13) for some background on abelian schemes.

1.2.1. *Symplectic groups.* We use the notation of Example 1.2. It is well-known that \mathfrak{h}_1^+ parametrizes elliptic curves over \mathbb{C} : an element $\tau \in \mathfrak{h}_1^+$ is sent to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and $E_\tau \simeq E_{\tau'}$ if and only if $\tau, \tau' \in \mathfrak{h}_1^+$ in the same orbit for the action of $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z})$; so we can recover τ from E_τ and the data of a symplectic isomorphism $\mathrm{H}_1(E_\tau, \mathbb{Z}) \simeq \mathbb{Z}^2$ where \mathbb{Z}^2 is equipped with the standard symplectic form. We want to give a similar picture for higher-dimensional abelian varieties; in fact, the analogy works best if we consider abelian varieties with a principal polarization (Definition 1.30).

We first introduce some notation and recall the definition of the (general) symplectic group as a group scheme over \mathbb{Z} . If R is a commutative ring, we denote by ψ_R the perfect symplectic pairing on R^{2d} with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$. So we have

$$\psi_R((x_1, \dots, x_d, y_1, \dots, y_d), (x'_1, \dots, x'_d, y'_1, \dots, y'_d)) = \sum_{i=1}^d x_i y'_i - \sum_{i=1}^d x'_i y_i.$$

The *general symplectic group* GSp_{2d} is the reductive group scheme over \mathbb{Z} whose points over a commutative ring R are given by:

$$\mathrm{GSp}_{2d}(R) = \{g \in \mathrm{GL}_{2d}(R) \mid \exists c(g) \in R^\times, \psi_R(g \cdot, g \cdot) = c(g)\psi_R\}.$$

The scalar $c(g)$ is called the *multiplier* of $g \in \mathrm{GSp}_{2d}(R)$. Sending g to $c(g)$ defines a morphism of group schemes $c : \mathrm{GSp}_{2d} \rightarrow \mathrm{GL}_1$, whose kernel Sp_{2d} is called the *symplectic group*.

EXAMPLE 1.7. We have $\mathrm{GSp}_2 = \mathrm{GL}_2$, $c = \det$ and $\mathrm{Sp}_2 = \mathrm{SL}_2$.

1.2.2. *Complex abelian varieties.* Let A be complex abelian variety of dimension d ; we identify A and its set of complex points. Then A is a connected complex Lie group of dimension d , so we have $A \simeq \mathrm{Lie}(A)/\Lambda$, with $\mathrm{Lie}(A) \simeq \mathbb{C}^d$ the universal cover of A and $\Lambda = \pi_1(A) = \mathrm{H}_1(A, \mathbb{Z}) \simeq \mathbb{Z}^{2d}$ a lattice in the underlying \mathbb{R} -vector space. Let A^\vee be the dual abelian variety, i.e. the space of degree 0 line bundles on A (Definition 1.28). We can identify $\mathrm{Lie}(A^\vee)$ with the space of semi-linear forms on $\mathrm{Lie}(A)$ and $\mathrm{H}_1(A^\vee, \mathbb{Z})$ with the subspace Λ^\vee of forms whose imaginary part takes integer values on Λ (see [92], §9). For every positive integer n , we have $A[n] = \frac{1}{n}\Lambda/\Lambda$ and $A^\vee[n] = \frac{1}{n}\Lambda^\vee/\Lambda^\vee$, and the canonical pairing $A[n] \times A^\vee[n] \rightarrow \mu_n(\mathbb{C})$ is given by

$$(v, u) \mapsto e^{-2i\pi n \mathrm{Im}(u(v))}$$

(see [92], §24). We then have a bijection between the set of polarizations on A (Definition 1.30) and the set of positive definite Hermitian forms⁶ H on $\mathrm{Lie}(A)$ such that the symplectic form $\mathrm{Im}(H)$ takes integer values on Λ ; given such a form H , the corresponding isogeny λ_H from A to A^\vee is given on \mathbb{C} -points by:

$$\begin{cases} \lambda_H : \mathrm{Lie}(A)/\Lambda & \rightarrow & \mathrm{Lie}(A^\vee)/\Lambda^\vee \\ & w & \mapsto & (v \mapsto H(v, w)) \end{cases}$$

It follows that the Weil pairing (see Remark 1.31(2)) corresponding to λ_H is the map

$$(1.1) \quad \begin{cases} A[n] \times A[n] & \rightarrow & \mu_n(\mathbb{C}) \\ & (v, w) & \mapsto & e^{-2i\pi n \mathrm{Im}(H(v, w))}. \end{cases}$$

⁶We take Hermitian forms to be semi-linear in the first variable and linear in the second variable.

As $v, w \in \frac{1}{n}\Lambda$, we have $\text{Im}(H(v, w)) \in \frac{1}{n^2}\mathbb{Z}$, so the formula does define a n th root of 1 in \mathbb{C} .

In particular, the polarization λ_H is principal if and only if Λ is self-dual with respect to the symplectic form $\text{Im}(H)$, that is,

$$\Lambda = \{w \in \text{Lie}(A) \mid \forall v \in \Lambda, \text{Im}(H(v, w)) \in \mathbb{Z}\}.$$

In that case, the symplectic \mathbb{Z} -module $(\Lambda, \text{Im}(H))$ is isomorphic to \mathbb{Z}^{2d} with the form $\psi_{\mathbb{Z}}$.

Let $\widetilde{\mathcal{M}}_d$ be the set of isomorphism classes of triples $(A, \lambda, \eta_{\mathbb{Z}})$, where A is a d -dimensional complex abelian variety, λ is a principal polarization on A and $\eta_{\mathbb{Z}}$ is an isomorphism of symplectic spaces from $H_1(A, \mathbb{Z})$ to $(\mathbb{Z}^{2d}, \psi_{\mathbb{Z}})$. We have an action of $\text{Sp}_{2d}(\mathbb{Z})$ on $\widetilde{\mathcal{M}}_d$: if $c = (A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$ and $x \in \text{Sp}_{2g}(\mathbb{Z})$, set $x \cdot c = (A, \lambda, x \circ \eta_{\mathbb{Z}})$.

Let $(A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$. Then $\Lambda = H_1(A, \mathbb{Z})$ is a lattice in the real vector space $\text{Lie}(A)$, we have $A = \text{Lie}(A)/\Lambda$ and, if H_{λ} is the Hermitian form associated to the polarization λ , then the symplectic form $\text{Im}(H_{\lambda})|_{\Lambda}$ is sent to the form $\psi_{\mathbb{Z}}$ on \mathbb{Z}^{2d} by the isomorphism $\eta_{\mathbb{Z}} : \Lambda \xrightarrow{\sim} \mathbb{Z}^{2d}$. So, if we see \mathbb{R}^{2d} as a complex vector space via the isomorphisms (of real vector spaces) $\text{Lie}(A) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d}$, then the Hermitian form on \mathbb{R}^{2d} corresponding to H_{λ} is $(v, w) \mapsto \psi_{\mathbb{R}}(iv, w) + i\psi_{\mathbb{R}}(v, w)$.

This shows $\eta_{\mathbb{Z}}$ determines all the data of the isomorphism class of $(A, \lambda, \eta_{\mathbb{Z}})$, except for the structure of complex vector space on $\mathbb{R}^{2d} \simeq \text{Lie}(A)$. This structure of complex vector space is equivalent to the data of an \mathbb{R} -linear endomorphism J of \mathbb{R}^{2d} such that $J^2 = -1$ (the endomorphism J corresponds to multiplication by i). We also need the \mathbb{R} -bilinear map $\mathbb{R}^{2d} \times \mathbb{R}^{2d} \rightarrow \mathbb{C}$ defined by $(v, w) \mapsto \psi_{\mathbb{R}}(J(v), w) + i\psi_{\mathbb{R}}(v, w)$ to be a positive definite Hermitian form on \mathbb{R}^{2d} , which is equivalent to the following conditions:

- (a) $\psi_{\mathbb{R}}(J(v), J(w)) = \psi_{\mathbb{R}}(v, w)$ for all $v, w \in \mathbb{R}^{2d}$;
- (b) the \mathbb{R} -bilinear form $(v, w) \mapsto \psi_{\mathbb{R}}(J(v), w)$ on \mathbb{R}^{2d} (which is symmetric by (a)) is positive definite.

Conversely, if we have a complex structure J on \mathbb{R}^{2d} satisfying (a) and (b), then we get a positive definite Hermitian form H on \mathbb{R}^{2d} whose imaginary part takes integer values on the lattice \mathbb{Z}^{2d} , so the complex torus $A = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ has a polarization λ induced by H , hence is an abelian variety (for example by the Kodaira embedding theorem), and we get an element $(A, \lambda, \mathbf{id}_{\mathbb{Z}^{2d}})$ of $\widetilde{\mathcal{M}}_d$.

To sum up, we have defined a bijection from $\widetilde{\mathcal{M}}_d$ to the set X' of endomorphisms J of \mathbb{R}^{2d} such that $J^2 = -1$ and that J satisfies conditions (a) and (b).

Now observe that, if W is a \mathbb{R} -vector space, then the data of an endomorphism J of W such that $J^2 = -1$ (i.e. of the structure of a \mathbb{C} -vector space on W) is equivalent to the data of a \mathbb{C} -linear endomorphism $J_{\mathbb{C}}$ of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $W \otimes_{\mathbb{R}} \mathbb{C} = \text{Ker}(J_{\mathbb{C}} - i \cdot \mathbf{id}_W) \oplus \text{Ker}(J_{\mathbb{C}} + i \cdot \mathbf{id}_W)$ and $\text{Ker}(J_{\mathbb{C}} + i \cdot \mathbf{id}_W) = \overline{\text{Ker}(J_{\mathbb{C}} - i \cdot \mathbf{id}_W)}$, where $v \mapsto \bar{v}$ is the involution of $W \otimes_{\mathbb{R}} \mathbb{C}$ induced by complex conjugation on \mathbb{C} . This is equivalent to giving a \mathbb{C} -vector subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $W \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \bar{E}$, i.e. a d -dimensional complex subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $E \cap \bar{E} = \{0\}$ (take $E = \text{Ker}(J_{\mathbb{C}} - i \cdot \mathbf{id}_W)$).⁷

⁷In fancier terms, we are saying that putting a structure of complex vector space on W is the same as putting a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on it (or of type $\{(1, 0), (0, 1)\}$, depending on your normalization). When $W = \mathbb{R}^{2d}$ and the complex structure comes from an

We apply this to $W = \mathbb{R}^{2d}$. Let J be a complex structure on \mathbb{R}^{2d} , and let E be the corresponding \mathbb{C} -vector subspace of $W \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2d}$. Then condition (a) on J is equivalent to the fact that:

$$(a') \quad \psi_{\mathbb{C}}(v, w) = 0 \text{ for all } v, w \in E$$

(i.e. to the fact that E is a Lagrangian subspace⁸ of \mathbb{C}^{2d}), and condition (b) on J is equivalent to the fact that

$$(b') \quad -i\psi_{\mathbb{C}}(v, \bar{v}) \in \mathbb{R}_{>0} \text{ for all } v \in E \setminus \{0\}.$$

Note that these two conditions on a \mathbb{C} -vector subspace E of \mathbb{C}^{2d} imply that $E \cap \bar{E} = \{0\}$. So we get a bijection from X' to the set of Lagrangian subspaces E of \mathbb{C}^{2d} satisfying (b').

If we represent Lagrangian subspaces of \mathbb{C}^{2d} by their bases, seen as complex matrices of size $2d \times d$, then the action of $\mathrm{Sp}_{2d}(\mathbb{R})$ is just left multiplication. For example, if we see \mathbb{R}^{2d} as a complex vector space via the isomorphism $\mathbb{R}^{2d} \xrightarrow{\sim} \mathbb{C}^d$ sending $(x_1, \dots, x_d, y_1, \dots, y_d)$ to $(x_1 + iy_1, \dots, x_d + iy_d)$, then the corresponding endomorphism of \mathbb{R}^{2d} is $\begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix} \in X'$, and the corresponding subspace of \mathbb{C}^{2d} is the one with basis $\begin{pmatrix} iI_d \\ I_d \end{pmatrix}$.

More generally, if $Y \in \mathfrak{h}_d^+$, the subspace of \mathbb{C}^{2d} with basis $\begin{pmatrix} Y \\ I_d \end{pmatrix}$ is a Lagrangian subspace satisfying condition (b'), and every such Lagrangian subspace is of that form. So we get bijections $\widetilde{\mathcal{M}}_d \simeq X' \simeq \mathfrak{h}_d^+$, and we can check that the second bijection is $\mathrm{Sp}_{2d}(\mathbb{R})$ -equivariant. Unraveling the definitions, we see that $Y \in \mathfrak{h}_d^+$ corresponds to the element $(A_Y, \lambda_Y, \eta_{\mathbb{Z}, Y})$ of $\widetilde{\mathcal{M}}_d$ such that $A_Y = \mathbb{C}^d / (\mathbb{Z}^d + Y\mathbb{Z}^d)$, λ_Y is the principal polarization given by the Hermitian form with matrix $\mathrm{Im}(Y)^{-1}$ on \mathbb{C}^d , and $\eta_{\mathbb{Z}, Y} : \mathbb{Z}^d + Y\mathbb{Z}^d \xrightarrow{\sim} \mathbb{Z}^{2d}$ is the isomorphism sending $a \in \mathbb{Z}^d$ to $(a, 0) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$ and $Ya \in Y\mathbb{Z}^d$ to $(0, a) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$.

Now we want an interpretation of the quotients $\Gamma \backslash \mathfrak{h}_d^+$, for Γ an arithmetic subgroup of $\mathrm{Sp}_{2d}(\mathbb{Q})$. We will do this for the groups $\Gamma(n) = \mathrm{Ker}(\mathrm{Sp}_{2d}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$, where n is a positive integer ($\Gamma(n)$ is called the *principal congruence subgroup* at level n). An arithmetic subgroup of $\mathrm{Sp}_{2d}(\mathbb{Q})$ that contains some $\Gamma(n)$ is called a *congruence subgroup*. For $d = 1$, there exist arithmetic subgroups that are not congruence subgroups (see the introduction of Raghunathan's paper [101] for a counterexample, attributed to Fricke-Klein); for $d \geq 2$, it is known that every arithmetic subgroup is a congruence subgroup (this was proved independently by Bass-Lazard-Serre in [13] and Mennicke in [82]).

We will need the notion of a level structure; we give the general definition here.

DEFINITION 1.8. Let S be a scheme, (A, λ) be a principally polarized abelian scheme of relative dimension d over S , and n be a positive integer. A *level n structure* on (A, λ) is a couple (η, φ) , where $\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2d}$ and $\varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ are isomorphisms of group schemes such that $\varphi \circ \psi_{\mathbb{Z}/n\mathbb{Z}} \circ (\eta \times \eta)$ is the Weil pairing associated to λ on $A[n]$.

element $(A, \lambda, \eta_{\mathbb{Z}})$ of $\widetilde{\mathcal{M}}_d$, then this Hodge structure is the one coming from the Hodge structure on $H_1(A, \mathbb{R})$ via the isomorphism $H_1(A, \mathbb{R}) \xrightarrow{\eta_{\mathbb{Z}} \otimes \mathbb{R}} \mathbb{Z}^{2d} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d}$.

⁸A maximal isotropic subspace.

REMARK 1.9. A level n structure on (A, λ) can only exist if n is invertible on S and $\mu_{n,S}$ is a constant group scheme.

Note that isomorphisms $\varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ correspond to sections $\zeta \in \mu_n(S)$ generating $\mu_{n,S}$ (i.e. to primitive n th roots of 1 over S), by sending φ to $\zeta = \varphi(1)$. So we will also see level structures as couples (η, ζ) , with $\zeta \in \mu_n(S)$ primitive.

Let $\zeta_n = e^{-2i\pi/n} \in \mu_n(\mathbb{C})$. If $Y \in \mathfrak{h}_d^+$, then $\frac{1}{n}\eta_{\mathbb{Z},Y}$ defines an isomorphism of groups $\eta_Y : A_Y[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$, and it follows from formula (1.1) that (η, ζ_n) is a level n structure on (A_Y, λ_Y) .

Using the fact that $\mathrm{Sp}_{2d}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ is surjective for every $n \in \mathbb{N}$, which follows from strong approximation for Sp_{2d} ,⁹ we finally get:

PROPOSITION 1.10. *Let n be a positive integer. The map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces a bijection from $\Gamma(n) \backslash \mathfrak{h}_d^+$ to the set of isomorphism classes of triples (A, λ, η) , where (A, λ) is a principally polarized complex abelian variety of dimension d and $\eta : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$ is an isomorphism of groups such that (η, ζ_n) is a level n structure on (A, λ) .*

So we have reinterpreted $\Gamma(n) \backslash \mathfrak{h}_d^+$ as the set of \mathbb{C} -points of a certain moduli problem. Considering this moduli problem over more general bases, we have an ‘‘obvious’’ way to make $\Gamma(n) \backslash \mathfrak{h}_d^+$ into an algebraic variety, which we now review.

1.2.3. *The connected Siegel modular variety.* Let $\mathcal{O}_n = \mathbb{Z}[1/n][T]/(T^n - 1)$. If S is a scheme over \mathcal{O}_n , we denote by $\varphi_0 : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ the isomorphism sending 1 to the class of T .

DEFINITION 1.11. Let $\mathcal{M}'_{d,n}$ be the contravariant functor from the category of \mathcal{O}_n -schemes to the category of sets sending S to the set of isomorphism classes of triples (A, λ, η) , where (A, λ) is a principally polarized abelian scheme of relative dimension d over S and $\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2d}$ is an isomorphism of group schemes such that (η, φ_0) is a level n structure on (A, λ) .

An isomorphism from (A, λ, η) to (A', λ', η') is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$, and, for $f : T \rightarrow S$ a morphism of \mathcal{O}_n -schemes, the map $\mathcal{M}'_{d,n}(S) \rightarrow \mathcal{M}'_{d,n}(T)$ is base change along f .

THEOREM 1.12 (Mumford, cf. [41]). *Suppose that $n \geq 3$. Then the functor $\mathcal{M}'_{d,n}$ is representable by a smooth quasi-projective \mathcal{O}_n -scheme purely of dimension $d(d+1)/2$ and with connected geometric fibers, which we still denote by $\mathcal{M}'_{d,n}$ and call the connected Siegel modular variety of level n .*

REMARK 1.13. If $n \in \{1, 2\}$, then triples (A, λ, η) as in Definition 1.11 may have automorphisms, so we should see $\mathcal{M}'_{d,n}$ as a stack. It will then be representable by a smooth Deligne-Mumford stack over \mathcal{O}_n that is a finite étale quotient of the scheme $\mathcal{M}'_{d,3n}$.

We can now strengthen Proposition 1.10 to the following result.

PROPOSITION 1.14. *Let $n \geq 3$ be an integer. Then the map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces an isomorphism of complex algebraic varieties from $\Gamma(n) \backslash \mathfrak{h}_d^+$ to $\mathcal{M}'_{d,n}(\mathbb{C})$.*

Let α be the map of the proposition. To show that α is regular, we use the fact that the triples (A_Y, λ_Y, η_Y) are obtained as the fibers of an abelian scheme

⁹See the papers [99] and [100] of Platonov.

$\mathcal{A} \rightarrow \Gamma(n) \backslash \mathfrak{h}_d^+$ given together with a polarization λ and a level n structure η . The abelian scheme \mathcal{A} is a locally symmetric space for the non-reductive group $\mathbb{Q}^{2d} \rtimes \mathrm{Sp}_{2d}$, where Sp_{2d} acts on \mathbb{Q}^{2d} via its standard representation, and is an example of a *mixed Shimura variety* (see for example Pink's thesis [96] for the theory of mixed Shimura varieties and their compactifications), and λ and η can also be given by explicit formulas interpolating those for λ_Y and η_Y . Once we have this fact, the map α corresponds to the $\Gamma(n) \backslash \mathfrak{h}_d^+$ -point of $\mathcal{M}'_{d,n}$ defined by $(\mathcal{A}, \lambda, \eta)$, hence it is a morphism of algebraic varieties.

Let us explain why α^{-1} is holomorphic (for more details, see the proof of Proposition 1.1 in Milne's [88]); by Borel's uniqueness theorem (see Theorem 1.6), this will imply that it is an isomorphism of algebraic varieties. We don't know anything explicit about the algebraic variety structure on $\mathcal{M} := \mathcal{M}'_{d,n}(\mathbb{C})$, but we do know that we have a universal abelian scheme $f : \mathcal{A} \rightarrow \mathcal{M}$ of relative dimension d , with a principal polarization λ and a level n structure η . By definition, the map α^{-1} sends $x \in \mathcal{M}$ to the element of $\Gamma(n) \backslash \mathfrak{h}_d^+$ corresponding to the triple $(\mathcal{A}_x, \lambda_x, \eta_x)$, so we need to understand how to construct this. The exponential map gives a surjective morphism $\exp : \mathrm{Lie}(\mathcal{A}/\mathcal{M}) \rightarrow \mathcal{A}$ of abelian groups schemes over \mathcal{M} (where the source is the relative Lie algebra of $\mathcal{A} \rightarrow \mathcal{M}$), whose kernel is the relative homology of $\mathcal{A} \rightarrow \mathcal{M}$ with coefficients in \mathbb{Z} . So we get an exact sequence of abelian sheaves over \mathcal{M} :

$$0 \rightarrow \mathcal{L} := (R^1 f_* \mathbb{Z})^\vee \rightarrow \mathrm{Lie}(\mathcal{A}/\mathcal{M}) \xrightarrow{\exp} \mathcal{A} \rightarrow 0$$

Let $x \in \mathcal{M}$. The level structure η_x defines an isomorphism $(\mathbb{Z}/n\mathbb{Z})^{2d} \xrightarrow{\sim} \mathcal{A}_x[n]$, that is, a basis of $\mathcal{A}_x[n]$; we can see the entries e_1, \dots, e_{2d} of this basis as elements of $\frac{1}{n}\mathcal{L}_x/\mathcal{L}_x \subset \mathrm{Lie}(\mathcal{A}_x/\mathcal{M}_x)/\mathcal{L}_x \xrightarrow{\sim} \mathcal{A}_x$, and lift them to a basis $\tilde{e}_1, \dots, \tilde{e}_{2d}$ of $\frac{1}{n}\mathcal{L}_x$, which gives an isomorphism $\mathbb{Z}^{2d} \xrightarrow{\sim} \mathcal{L}_x$, hence an isomorphism $\mathbb{R}^{2d} \xrightarrow{\sim} \mathrm{Lie}(\mathcal{A}_x/\mathcal{M}_x)$. Now we go back to the discussion in Subsection 1.2.2 and we see that the element Y_x of \mathfrak{h}_d^+ whose class is equal to $\alpha^{-1}(x)$ is basically a basis of the Lagrangian subspace corresponding to the complex structure on \mathbb{R}^{2d} coming from the isomorphism $\mathbb{R}^{2d} \xrightarrow{\sim} \mathrm{Lie}(\mathcal{A}_x/\mathcal{M}_x)$ that we just defined. So we need to show that we can find a neighborhood U of x such that $\tilde{e}_1, \dots, \tilde{e}_{2d}$ extend to holomorphic sections of $\frac{1}{n}\mathcal{L}|_U \rightarrow U$ that define a level n structure on \mathcal{A}_y at each point y of U . The extensions exist because the morphism $\frac{1}{n}\mathcal{L} \rightarrow \mathcal{M}$ is étale, and the condition on the extensions is open because it only involves finite group schemes over \mathcal{M} .

In particular, we showed that $\Gamma(n) \backslash \mathfrak{h}_d^+$ is the set of complex points of an algebraic variety defined over the number field $\mathbb{Q}(\zeta_n)$. Unfortunately, this number field depends on the level n . The issue is that we need a fixed primitive n th root of 1 in order to define the moduli problem $\mathcal{M}'_{d,n}$, so we need to be over a basis where such a primitive n th root exists. To fix this problem, we will allow the primitive n th root of 1 to vary.

1.2.4. The Siegel modular variety.

DEFINITION 1.15. Let n be a positive integer. The *Siegel modular variety* $\mathcal{M}_{d,n}$ is the contravariant functor from the category of $\mathbb{Z}[1/n]$ -schemes to the category of sets sending a scheme S to the set of isomorphism classes of triples $(A, \lambda, \eta, \varphi)$, where (A, λ) is a principally polarized abelian scheme of relative dimension d over S and (η, φ) is a level n structure on (A, λ) .

An isomorphism from $(A, \lambda, \eta, \varphi)$ to $(A', \lambda', \eta', \varphi')$ is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$.

The group $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ acts on $\mathcal{M}_{d,n}$: if $g \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ and $(A, \lambda, \eta, \varphi) \in \mathcal{M}_{d,n}(S)$, then

$$g \cdot (A, \lambda, \eta, \varphi) = (A, \lambda, g \circ \eta, c(g)^{-1} \varphi).$$

The kernel of this action is the group $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$. If n divides m , then we have a morphism $\mathcal{M}_{d,m} \rightarrow \mathcal{M}_{d,n}$ that forgets part of the level m structure; this morphism is representable finite étale, and in fact it is a torsor under the finite group $K(n)/K(m)$.

We have the following variant of Theorem 1.12.

THEOREM 1.16 (Mumford, cf. [41]). *Suppose that $n \geq 3$. Then the functor $\mathcal{M}_{d,n}$ is representable by a smooth quasi-projective $\mathbb{Z}[1/n]$ -scheme purely of dimension $d(d+1)/2$, which we still denote by $\mathcal{M}_{d,n}$ and call the Siegel modular variety of level n .*

REMARK 1.17. The scheme $\mathcal{M}_{d,n}$ is the Shimura variety for GSp_{2d} with level $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$, or rather its integral model. If K is an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ that is small enough,¹⁰ then we can also define the Shimura variety $\mathcal{M}_{d,K}$ with level K , and even its integral model: choose n such that $K(n) \subset K$. Then $K(n)$ is a normal subgroup of K , so the group $K/K(n) \subset \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{M}_{d,n}$, and we set $\mathcal{M}_{d,K} = \mathcal{M}_{d,n}/(K/K(n))$. It is easy to check that $\mathcal{M}_{d,K,\mathbb{Q}}$ does not depend on the choice of n ; however, with our definition, the localization of \mathbb{Z} over which $\mathcal{M}_{d,K}$ is defined depends on the choice of n , so this is not ideal.

In fact, for K an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, we have a direct definition of a level K structure on a principally polarized abelian scheme (see Section 5 of Kottwitz's paper [64]). For K small enough, the scheme $\mathcal{M}_{d,K,\mathbb{Q}}$ is the moduli space of principally polarized abelian schemes with level K structure. In general, this moduli space is representable by a Deligne-Mumford stack. We can also define the moduli space over a localization of \mathbb{Z} , but the primes that we invert depend on K ; see the discussion in Subsubsection 2.1.4.

Let us explain the relationship between $\mathcal{M}_{d,n}$ and $\mathcal{M}'_{d,n}$. We define a map $s : (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ by $s(\alpha) = \begin{pmatrix} 0 & \alpha I_d \\ I_d & 0 \end{pmatrix}$; note that s is a section of the multiplier $c : \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}) \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$, and that it is not a morphism of groups.

PROPOSITION 1.18. *The morphism $\mathcal{M}'_{d,n} \times (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathcal{M}_{d,n,\mathcal{O}_n}$ sending $((A, \lambda, \eta), \alpha)$ to $(A, \lambda, s(\alpha) \circ \eta, \varphi_0 \circ \alpha)$ (where we see α as an automorphism of $\underline{\mathbb{Z}/n\mathbb{Z}}_S$ for any scheme S) is an isomorphism.*

As a corollary, we get a description of the complex points of $\mathcal{M}_{d,n}$. Let $\mathfrak{h}_d = \mathfrak{h}_d^+ \cup (-\mathfrak{h}_d^+)$ be the set of symmetric matrices $Y \in M_d(\mathbb{C})$ such that $\mathrm{Im}(Y)$ is positive definite or negative definite. The action of $\mathrm{Sp}_{2d}(\mathbb{R})$ on \mathfrak{h}_d extends to a transitive action of $\mathrm{GSp}_{2d}(\mathbb{R})$, given by the same formula. The stabilizer of $iI_d \in \mathfrak{h}_d$ in $\mathrm{GSp}_{2d}(\mathbb{R})$ is $\mathbb{R}_{>0}K_\infty$, where K_∞ is as before the stabilizer of iI_d in $\mathrm{Sp}_{2d}(\mathbb{R})$, so $\mathfrak{h}_d \simeq \mathrm{GSp}_{2d}(\mathbb{R})/\mathbb{R}_{>0}K_\infty$ as real analytic manifolds.

¹⁰For example, $K \subset K(N)$ with $N \geq 3$.

COROLLARY 1.19. *We have an isomorphism of complex algebraic varieties*

$$\mathcal{M}_{d,n}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n))$$

extending the isomorphism of Proposition 1.14, where $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$ and $\mathrm{GSp}_{2g}(\mathbb{Q})$ acts diagonally on $\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)$.

This follows from the fact that

$$\mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n)) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})^+ \backslash (\mathfrak{h}_d^+ \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n)),$$

where $\mathrm{GSp}_{2d}(\mathbb{Q})^+ = \{g \in \mathrm{GSp}_{2d}(\mathbb{Q}) \mid c(g) > 0\}$, and from strong approximation for Sp_{2d} ,¹¹ which implies that c induces a bijection

$$\mathrm{GSp}_{2d}(\mathbb{Q})^+ \backslash \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n) \xrightarrow{\sim} \mathbb{Q}_{>0} \backslash \mathbb{A}_f^\times / c(K(n)) \simeq \widehat{\mathbb{Z}}^\times / (1 + n\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times.$$

For every $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we choose $x_i \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$ lifting i and we set

$$\Gamma(n)_i = \mathrm{GSp}_{2d}(\mathbb{Q})^+ \cap x_i K(n) x_i^{-1} = \mathrm{Sp}_{2d}(\mathbb{Q}) \cap x_i K(n) x_i^{-1}.$$

Then the $\Gamma(n)_i$ are arithmetic subgroups of $\mathrm{Sp}_{2d}(\mathbb{Q})$, and we have

$$\mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n)) \simeq \coprod_{i \in (\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)_i \backslash \mathfrak{h}_d^+$$

as complex manifolds.

In fact, for $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we can take $x_i = \begin{pmatrix} 0 & a_i I_d \\ I_d & 0 \end{pmatrix}$ with $a_i \in \widehat{\mathbb{Z}}^\times$ lifting i .

In particular, we have $x_i \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$; as $K(n)$ is a normal subgroup of $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$, we get $x_i K(n) x_i^{-1} = K(n)$, hence $\Gamma(n)_i = \Gamma(n)$, and finally

$$\mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K(n)) \simeq \coprod_{i \in (\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n) \backslash \mathfrak{h}_d^+.$$

REMARK 1.20. If K is a small enough open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, then we get an isomorphism of complex manifolds:

$$\mathcal{M}_{d,K}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f) / K).$$

1.2.5. *Hecke correspondences.* We can also descend the Hecke correspondences of Subsection 1.1 to morphisms of schemes.

We proceed as in Section 3 of Laumon's paper [74]. Let $g \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$, and let K, K' be small enough open compact subgroups of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$. We want to define finite étale morphisms $T_1, T_g : \mathcal{M}_{d,K'} \rightarrow \mathcal{M}_{d,K}$, and the Hecke correspondence associated to (g, K, K') is the couple (T_1, T_g) .

Choose $n \geq 3$ such that $K(n) \subset K'$; then $\mathcal{M}_{d,K'} = \mathcal{M}_{d,n}/(K'/K(n))$ and $\mathcal{M}_{d,K} = \mathcal{M}_{d,n}/(K/K(n))$. The morphism T_1 should just forget part of the level structure: as $K'/K(n) \subset K/K(n)$, we have an obvious choice for $T_1 : \mathcal{M}_{d,K'} \rightarrow \mathcal{M}_{d,K}$: we take the morphism induced by the identity of $\mathcal{M}_{d,n}$.

To define T_g , we first consider the following special case: if $g \in M_{2d}(\widehat{\mathbb{Z}}) \cap \mathrm{GSp}_{2d}(\mathbb{A}_f)$, let $x = (A, \lambda, \eta, \varphi) \in \mathcal{M}_{d,n}(S)$. Let u be the endomorphism of $(\mathbb{Z}/n\mathbb{Z})^{2d}$ with matrix g . Then T_g sends the class of x in $\mathcal{M}_{d,K'}(S)$ to the class of $(A', \lambda', \eta', \varphi) \in \mathcal{M}_{d,n}(S)$, where A' is the quotient of A by the finite flat subgroup scheme $\eta^{-1}(\underline{\mathrm{Ker}} u)$ of $A[n]$ and λ', η' are the morphisms deduced from $\lambda, g \circ \eta$.

¹¹See the papers [99] and [100] of Platonov.

Note that, if $g = a \cdot I_{2d}$ with $a \in \widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times$ and $K' = K$, then the morphism $T_g : \mathcal{M}_{d,K} \rightarrow \mathcal{M}_{d,K}$ is an isomorphism.

Finally, for a general $g \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$, we write $g = g_0^{-1}g_1$ with $g_0 \in (\widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times)I_{2d}$ and $g_1 \in M_{2d}(\widehat{\mathbb{Z}}) \cap \mathrm{GSp}_{2d}(\mathbb{A}_f)$, and we set $T_g = T_{g_0}^{-1} \circ T_{g_1}$.

REMARK 1.21. If we use instead the general definition of a level K structure from Section 5 of [64], then it becomes much easier to define the Hecke correspondences on the moduli problems; see Section 6 of loc. cit.

REMARK 1.22. We have two ways to think of $\mathcal{M}_{d,n}(\mathbb{C})$: as an adelic double quotient or as finite disjoint union of spaces $\Gamma(n) \backslash \mathfrak{h}_d^+$, which are locally symmetric spaces associated to the semi-simple group Sp_{2d} . The first description is more convenient to see the action of adelic Hecke operators, and the second description is a bit more concrete and has simpler combinatorics. Note also that the complex manifold $\Gamma(n) \backslash \mathfrak{h}_d^+$ is isomorphic to the set of \mathbb{C} -points of the algebraic variety $\mathcal{M}'_{d,n}$, but this algebraic variety is defined over the field $\mathbb{Q}[T]/(T^n - 1)$, which depends on n . On the other hand, the adelic double quotient $\mathrm{GSp}_{2d}(\mathbb{Q}) \backslash (\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)$ is isomorphic to the set of \mathbb{C} -points of the algebraic variety $\mathcal{M}_{d,n}$, which is defined over \mathbb{Q} (and even $\mathbb{Z}[1/n]$). So if we want to consider Shimura varieties as a projective system of algebraic varieties over a number field, then it makes sense to use the adelic double quotients, because they are all defined over the same field.

1.2.6. *Background on abelian schemes.* Let S be a scheme. We denote by Sch/S the category of S -schemes.

DEFINITION 1.23. An *abelian scheme* over S is an S -group scheme $A \rightarrow S$ which is smooth and proper with geometrically connected fibers. If S is the spectrum of a field k , an abelian scheme over S is also called an *abelian variety* over k .

A *morphism of abelian schemes* over S is a morphism of S -group schemes between abelian schemes over S .

PROPOSITION 1.24. *Let A be an abelian scheme over S . Then the S -group scheme $A \rightarrow S$ is commutative.*

DEFINITION 1.25. Let A be an abelian scheme over S , and let $e : S \rightarrow A$ be its zero section. We consider the following two functors from $(\mathrm{Sch}/S)^{\mathrm{op}}$ to the category of sets:

- (a) The functor $\mathbf{Pic}_{A/S,e}$ sending an S -scheme $T \rightarrow S$ to the set of isomorphism classes of couples (\mathcal{L}, φ) , where \mathcal{L} is an invertible sheaf on $A \times_S T$ and $\varphi : \mathcal{O}_T \xrightarrow{\sim} e_T^* \mathcal{L}$ is an isomorphism, with $e_T = e \times_S T : T \rightarrow A \times_S T$. An isomorphism from (\mathcal{L}, φ) to (\mathcal{L}', φ') is an isomorphism of \mathcal{O}_T -modules $\alpha : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $(e_T^* \alpha) \circ \varphi = \varphi'$;
- (b) The subfunctor $\mathbf{Pic}_{A/S,e}^0$ of $\mathbf{Pic}_{A/S,e}$ sending an S -scheme $T \rightarrow S$ to the set of isomorphism classes of couples (\mathcal{L}, φ) as in (a) such that, for every point t of T , every smooth projective curve over the residue field $\kappa(t)$ of t and every morphism of $\kappa(t)$ -schemes $f : C \rightarrow A \times_S t$, the line bundle $f^*(\mathcal{L}|_{A \times_S t})$ is of degree 0 on C .

REMARK 1.26. (1) The functor $\mathbf{Pic}_{A/S,e}$ can be made into a functor into the category of abelian groups: if T is an S -scheme and (\mathcal{L}, φ) , (\mathcal{L}', φ') represent elements of $\mathbf{Pic}_{A/S,e}(T)$, their product is represented

by $(\mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{L}', \varphi \otimes \varphi')$, where $\varphi \otimes \varphi'$ is the isomorphism

$$\mathcal{O}_T = \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{O}_T \xrightarrow[\varphi \otimes \varphi']{\sim} (e_T^* \mathcal{L}) \otimes_{\mathcal{O}_T} (e_T^* \mathcal{L}') = e_T^*(\mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{L}').$$

Moreover, for every S -scheme T , the set $\mathbf{Pic}_{A/S,e}^0(T)$ is a subgroup of $\mathbf{Pic}_{A/S,e}(T)$.

- (2) If $X \rightarrow S$ is a scheme over S , then the relative Picard functor $\mathbf{Pic}_{X/S}$ on Sch/S is the fppf sheafification of the functor $T \mapsto \mathbf{Pic}(X \times_S T)$, where, for Y a scheme, we denote by $\mathbf{Pic}(Y)$ the set of isomorphism classes of line bundles on Y (that is an abelian group for the tensor product); see [114, Situation 0D25]. We can also define a subfunctor $\mathbf{Pic}_{X/S}^0$ of $\mathbf{Pic}_{X/S}$ as in Definition 1.25. By [114, Lemma 0D28], if A is an abelian scheme over S , then there is an isomorphism of functors in abelian groups $\mathbf{Pic}_{A/S} \simeq \mathbf{Pic}_{A/S,e}$, inducing an isomorphism $\mathbf{Pic}_{A/S}^0 \simeq \mathbf{Pic}_{A/S,e}^0$.
- (3) We can upgrade $A \mapsto \mathbf{Pic}_{A/S,e}$ and $A \mapsto \mathbf{Pic}_{A/S,e}^0$ to contravariant functors in A : if $f : A \rightarrow B$ is a morphism of abelian schemes over S , then it induces a natural transformation $f^* : \mathbf{Pic}_{B/S,e} \rightarrow \mathbf{Pic}_{A/S,e}$ sending (\mathcal{L}, φ) to $(f^*(\mathcal{L}), f^*(\varphi))$, and f^* sends $\mathbf{Pic}_{B/S,e}^0$ to $\mathbf{Pic}_{A/S,e}^0$.

THEOREM 1.27. *Let A be an abelian scheme over S . Then $\mathbf{Pic}_{A/S,e}^0$ is representable by an abelian scheme over S .*

PROOF. We know that $\mathbf{Pic}_{A/S,e}^0$ is representable by an algebraic space over S by a result of M. Artin (see [10] or [114, Proposition 0D2C]). We can check on the moduli problem that this algebraic space is proper and smooth, and its fibers over points of S are abelian varieties by the classical theory of the dual abelian variety (see sections II.8 and III.13 of Mumford's book [92]). It remains to prove that the algebraic space representing $\mathbf{Pic}_{A/S,e}^0$ is a scheme; this is due to Raynaud, and a proof is given in Theorem 1.9 of [41]. \square

DEFINITION 1.28. Let A be an abelian scheme over S . The abelian scheme over S representing $\mathbf{Pic}_{A/S,e}^0$ is called the *dual abelian scheme* of A and denoted by A^\vee . In particular, we get a couple $(\mathcal{P}_A, \varphi_A)$ representing the element of $\mathbf{Pic}_{A/S,e}^0(A^\vee)$ corresponding to \mathbf{id}_{A^\vee} , with \mathcal{P}_A a line bundle on $A \times_S A^\vee$, called the *Poincaré line bundle*.

If $f : A \rightarrow B$ is a morphism of abelian schemes over S , we denote by $f^\vee : B^\vee \rightarrow A^\vee$ the morphism corresponding to the natural transformation $f^* : \mathbf{Pic}_{B/S,e}^0 \rightarrow \mathbf{Pic}_{A/S,e}^0$ of Remark 1.26.

REMARK 1.29. Let $e : S \rightarrow A^\vee$ be the unit section. Then the pullback of \mathcal{P}_A by $A \times_S e : A = A \times_S S \rightarrow A \times_S A^\vee$ is the line bundle on A corresponding to the element e of $A^\vee(S) = \mathbf{Pic}_{A/S,e}^0(S)$; in other words, it is isomorphic to the trivial line bundle \mathcal{O}_A . So \mathcal{P}_A defines an element of $\mathbf{Pic}_{A^\vee/S,e}^0(A)$, that is, a morphism of S -schemes $A \rightarrow A^{\vee\vee}$, called the *biduality morphism*. The *biduality theorem* says that the biduality morphism is an isomorphism. For S the spectrum of a field, this is proved in Section III.13 of [92], and the general case reduces to this by looking at the fibers over points of S .

Let A be an abelian scheme over S and let \mathcal{L} be a line bundle on A . We denote by $\mu, p_1, p_2, \varepsilon : A \times_S A \rightarrow A$ the addition morphism, the first projection, the second projection and the zero morphism respectively. Then the line bundle $(\mu^*\mathcal{L}) \otimes (p_1^*\mathcal{L}^{\otimes -1}) \otimes (p_2^*\mathcal{L}^{\otimes -1}) \otimes (\varepsilon^*\mathcal{L})$ on $A \times_S A$ is trivial when restricted to $S \times_S A$ via the zero section of A , hence it defines an element of $\mathbf{Pic}_{A/S, e}^0(A)$, corresponding to a morphism of S -schemes $\lambda(\mathcal{L}) : A \rightarrow A^\vee$, which is a morphism of abelian varieties by the theorem of the cube (see for example Section III.10 of [92]).

DEFINITION 1.30. Let A be an abelian scheme over S . A *polarization* on A is an isogeny $\lambda : A \rightarrow A^\vee$ (i.e. a finite faithfully flat morphism of abelian schemes) such that, for every algebraically closed field k and every morphism $\mathrm{Spec} k \rightarrow S$, the morphism $\lambda \times_S \mathrm{Spec} k : A \times_S \mathrm{Spec} k \rightarrow A^\vee \times_S \mathrm{Spec} k = (A \times_S \mathrm{Spec} k)^\vee$ is of the form $\lambda(\mathcal{L})$, for \mathcal{L} an ample line bundle on $A \times_S \mathrm{Spec} k$. We say that a polarization is *principal* if it is an isomorphism.

A *principally polarized abelian scheme* over S is a pair (A, λ) , where A is an abelian scheme over S and λ is a principal polarization on A .

REMARK 1.31. Let λ be a polarization on A , and let n be a positive integer. Then, composing $\lambda : A[n] \rightarrow A^\vee[n]$ with the canonical Cartier pairing $A[n] \times A^\vee[n] \rightarrow \mu_{n, S}$,¹² we get a pairing $A[n] \times A[n] \rightarrow \mu_{n, S}$, called the *Weil pairing* associated to λ . If λ is principal, this is a perfect pairing.

1.3. Shimura varieties over \mathbb{C} . Remember the upshot of Subsection 1.2: if we want algebraic varieties that are all defined over the same number field, and Hecke correspondences that are also defined on this number field, it is better to work with adelic double quotients for a reductive group such as GSp_{2d} rather than with locally symmetric spaces for a semi-simple group such as Sp_{2d} . This (and Theorem 1.3) motivates the definition of Shimura data, due to Deligne in [35].

1.3.1. The Serre torus. Let \mathbb{S} be \mathbb{C}^\times seen as an algebraic group over \mathbb{R} ; this is called the *Serre torus*. In other words, the group \mathbb{S} is the Weil restriction of scalars from \mathbb{C} to \mathbb{R} of GL_1 , so that $\mathbb{S}(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^\times$ for every \mathbb{R} -algebra R . We denote by w the injective morphism $\mathrm{GL}_{1, \mathbb{R}} \rightarrow \mathbb{S}$ corresponding to the inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$.

We have $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$, where the isomorphism sends $a \otimes 1 + b \otimes i$ to $(a + ib, a - ib)$. So the abelian group $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathrm{GL}_{1, \mathbb{C}})$ of characters of \mathbb{S} is free of rank 2 and generated by the characters z and \bar{z} corresponding to the two projections of $\mathbb{C}^\times \times \mathbb{C}^\times$ on \mathbb{C}^\times . We denote by $r : \mathrm{GL}_{1, \mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$ the injective morphism corresponding to the injection of the first factor in $\mathbb{C}^\times \times \mathbb{C}^\times$.

If V is a real vector space and $\rho : \mathbb{S} \rightarrow \mathrm{GL}(V)$ is a morphism of algebraic groups (i.e. a representation of \mathbb{S} on V), then we have $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p, q \in \mathbb{Z}} V^{p, q}$, where $V^{p, q}$ is the subspace of $V_{\mathbb{C}}$ on which $\mathbb{S}_{\mathbb{C}}$ acts by the character $z^{-p}\bar{z}^{-q}$; moreover, as ρ is defined over \mathbb{R} , we have $\overline{V^{p, q}} = V^{q, p}$ for all $p, q \in \mathbb{Z}$. Let $m \in \mathbb{Z}$. We say that ρ is of *weight* m if $\rho \circ w : \mathrm{GL}_1 \rightarrow \mathrm{GL}(V)$ is equal to $x \mapsto x^{-m} \mathrm{id}_V$.

REMARK 1.32. If $\rho : \mathbb{S} \rightarrow \mathrm{GL}(V)$ is of weight m , we have $V^{p, q} = 0$ unless $p + q = m$, so the decomposition $V_{\mathbb{C}} = \bigoplus_{p, q} V^{p, q}$ is a pure Hodge structure of weight m on V . In fact, representations of weight m of \mathbb{S} on V are in bijection with pure Hodge structures of weight m on V .

¹²See page 183 of Mumford's book [92] for the case of abelian varieties over a field. For the case of general abelian schemes, see the beginning of Section 1 of Oda's paper [95], applied to the isogeny given by multiplication by n .

1.3.2. Shimura data.

DEFINITION 1.33. A *Shimura datum* is a couple (G, h) , where G is a connected reductive algebraic group over \mathbb{Q} and $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ is a morphism of real algebraic groups such that:

- (a) The image of $h \circ w : \mathrm{GL}_{1, \mathbb{R}} \rightarrow G_{\mathbb{R}}$ is central;
- (b) If $\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{R}})$ and $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p, q \in \mathbb{Z}} \mathfrak{g}^{p, q}$ is the decomposition induced by the representation $\mathrm{Ad} \circ h : \mathbb{S} \rightarrow \mathrm{GL}(\mathfrak{g})$, then we have $\mathfrak{g}^{p, q} = 0$ unless $(p, q) \in \{(-1, 1), (0, 0), (1, -1)\}$;
- (c) Conjugation by $h(i)$ induces a Cartan involution of $G_{\mathrm{der}}(\mathbb{R})$ (see Theorem 1.3);
- (d) G_{ad} has no normal subgroup (defined over \mathbb{Q}) whose group of \mathbb{R} -points is compact.¹³

Note that condition (b) implies condition (a), because it implies that the image of $h \circ w$ centralizes $\mathrm{Lie}(G_{\mathbb{R}})$.

Let (G, h) be a Shimura datum. We denote by K_{∞} the centralizer of h in $G(\mathbb{R})$ and by X the set of $G(\mathbb{R})$ -conjugates of h . Then K_{∞} contains the center of $G(\mathbb{R})$, and $K_{\infty} \cap G_{\mathrm{der}}(\mathbb{R})^0$ is equal to the centralizer of $h(i)$ in $G_{\mathrm{der}}(\mathbb{R})^0$, hence is a maximal compact subgroup of $G_{\mathrm{der}}(\mathbb{R})^0$ by condition (c). We have $X \simeq G(\mathbb{R})/K_{\infty}$, and Theorem 1.3 implies that there is a $G(\mathbb{R})$ -invariant complex structure on X such that the connected components of X are Hermitian symmetric domains.

REMARK 1.34. There are many variants in the definition of a Shimura datum. The definition we gave here is equivalent to Definition 5.5 of Milne's notes [87], but it is possible to impose additional conditions to make some statements simpler; see for example the discussion of additional axioms on page 63 of [87]. On the other hand, some authors use a slightly more general definition, where X is allowed to be finite over the set of morphisms $\mathbb{S} \rightarrow G_{\mathbb{R}}$. This is for example the case of Pink in [96] (see Definition 2.1) and [97] (see (3.1)); in the first of these references, Pink also generalizes the definition to include groups that are not necessarily reductive.

EXAMPLE 1.35. Take $G = \mathrm{GSp}_{2d}$. Up to conjugation, there exists a unique morphism $h : \mathbb{S} \rightarrow \mathrm{GSp}_{2d}$ satisfying conditions (a)-(c) of Definition 1.33 and such that $(h \circ w)(x) = xI_{2d}$ for every $x \in \mathbb{R}^{\times}$. An element of that class is given by

$$h(a + ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

For this h , we have $K_{\infty} = \mathrm{GSp}_{2d}(\mathbb{R}) \cap \mathrm{GO}(2d)$, and we can check that the map $\mathrm{GU}(d) \rightarrow K_{\infty}$ sending $X + iY \in \mathrm{GU}(d)$ (with $X, Y \in M_d(\mathbb{R})$) to $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ is an isomorphism of Lie groups. So $K_{\infty} = \mathbb{R}_{>0}K'_{\infty}$, where $K'_{\infty} = \mathrm{Sp}_{2d}(\mathbb{R}) \cap \mathrm{O}(d)$ is the maximal compact subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$ that was called K_{∞} in Subsections 1.1 and 1.2. This implies that $X \simeq \mathfrak{h}_d$.

The couple (GSp_{2d}, h) is called a *Siegel Shimura datum*.

Let K be an open compact subgroup of $G(\mathbb{A}_f)$. We set

$$M_K(G, h)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K,$$

¹³Note that $G_{\mathrm{ad}}(\mathbb{R})$ could still have compact normal algebraic subgroups, as long as they are not defined over \mathbb{Q} .

where the group K acts by right translations on the factor $G(\mathbb{A}_f)$, and the group $G(\mathbb{Q})$ acts by left translations on both factors simultaneously. This is the *Shimura variety at level K* associated to the Shimura datum (G, h) .

As in Subsection 1.1, if $(x_i)_{i \in I}$ is a system of representatives of the finite quotient $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K$, and if $\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}$ for every $i \in I$, then the Γ_i are arithmetic subgroups of $G(\mathbb{Q})$, and we have

$$M_K(G, h)(\mathbb{C}) = \coprod_{i \in I} \Gamma_i \backslash X.$$

Hence it follows from Theorem 1.6 that $M_K(G, h)(\mathbb{C})$ is the set of complex points of a quasi-projective algebraic variety over \mathbb{C} , smooth if K is small enough.

Again as in Subsection 1.1, we have Hecke correspondences between the varieties $M_K(G, h)(\mathbb{C})$, which are finite maps, hence morphisms of algebraic varieties. This defines an action of $G(\mathbb{A}_f)$ on the projective system $(M_K(G, h)(\mathbb{C}))_{K \subset G(\mathbb{A}_f)}$, or on its limit

$$M(G, h)(\mathbb{C}) = \varprojlim_K M_K(G, h)(\mathbb{C}).$$

Under the additional assumption that the \mathbb{R} -split and \mathbb{Q} -split maximal subtori of the center of G are equal (which is sometimes required in the definition of a Shimura datum), the limit is equal to $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$ (see Proposition 4.19 of [87]). Moreover, for every open compact subgroup K of $G(\mathbb{A}_f)$, we have $M_K(G, h)(\mathbb{C}) = M(G, h)(\mathbb{C}) / K$.

1.3.3. Morphisms of Shimura varieties. Let $(G_1, h_1), (G_2, h_2)$ be Shimura data, and let $u : G_1 \rightarrow G_2$ be a morphism of algebraic groups such that $u \circ h_1$ and h_2 are conjugated under $G_2(\mathbb{R})$; we say that u is a *morphism of Shimura data*. Then u induces a morphism of complex manifolds $X_1 \rightarrow X_2$. Hence, for all $K_1 \subset G_1(\mathbb{A}_f)$, $K_2 \subset G_2(\mathbb{A}_f)$ open compact subgroups such that $u(K_1) \subset K_2$, we get a morphism of quasi-projective varieties $u(K_1, K_2) : M_{K_1}(G_1, h_1)(\mathbb{C}) \rightarrow M_{K_2}(G_2, h_2)(\mathbb{C})$. We can also think of the collection of all $u(K_1, K_2)$ as a morphism of \mathbb{C} -schemes $u : M(G_1, h_1)(\mathbb{C}) \rightarrow M(G_2, h_2)(\mathbb{C})$.

PROPOSITION 1.36 (Deligne, see Proposition 1.15 of [35]). *If G_1 is an algebraic subgroup of G_2 and u is the inclusion, then, for every open compact subgroup K_1 of $G_1(\mathbb{A}_f)$, there exists an open compact subgroup $K_2 \supset K_1$ of $G_2(\mathbb{A}_f)$ such that $u(K_1, K_2) : M_{K_1}(G_1, h_1)(\mathbb{C}) \rightarrow M_{K_2}(G_2, h_2)(\mathbb{C})$ is a closed immersion.*

1.3.4. Connected components. For (G, h) equal to the Shimura datum of Example 1.35 and $K = \text{Ker}(G\text{Sp}_{2d}(\widehat{\mathbb{Z}}) \rightarrow G\text{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$, we have seen that the multiplier $c : G\text{Sp}_{2d} \rightarrow \text{GL}_1$ induces a bijection

$$\pi_0(M_K(G, h)(\mathbb{C})) \xrightarrow{\sim} \mathbb{Q}_{>0} \backslash \mathbb{A}_f^\times / c(K) = \mathbb{Q}^\times \backslash \mathbb{A}^\times / c(K_\infty K).$$

In fact, it follows from real approximation and the Hasse principle that this works for many Shimura varieties:

THEOREM 1.37 (Deligne, see [35] 2.7). *Let $\nu : G \rightarrow T := G/G_{\text{der}}$ be the quotient morphism, and suppose that G_{der} is simply connected. Then, for every open compact subgroup K of $G(\mathbb{A}_f)$, the map ν induces a bijection*

$$\pi_0(M_K(G, h)(\mathbb{C})) \xrightarrow{\sim} T(\mathbb{Q}) \backslash T(\mathbb{A}) / \nu(K_\infty \times K).$$

1.4. Canonical models. In the situation of Example 1.35, we have seen that the algebraic varieties $M_K(G, h)(\mathbb{C})$ and all the Hecke correspondences are defined over \mathbb{Q} . We would like to generalize this kind of result to other Shimura varieties.

1.4.1. *Model of a Shimura variety.* First we need to say what we mean by a model.

DEFINITION 1.38. Let (G, h) be a Shimura datum, and F be a subfield of \mathbb{C} . A *model* of the projective system $(M_K(G, h)(\mathbb{C}))_K$ over F is the data:

- for every open compact subgroup K of $G(\mathbb{A}_f)$, of a quasi-projective variety M_K over F and an isomorphism $\iota_K : M_K \otimes_F \mathbb{C} \xrightarrow{\sim} M_K(G, h)(\mathbb{C})$;
- for every $g \in G(\mathbb{A}_f)$ and all open compact subgroups K, K' of $G(\mathbb{A}_f)$ such that $gK'g^{-1} \subset K$, of a morphism of F -varieties $T_{g, K, K'} : M_{K'} \rightarrow M_K$,

such that:

- (i) For all g, K, K' as above, the morphism $\iota_K \circ T_{g, K, K'} \circ \iota_{K'}^{-1} : M_{K'}(G, h)(\mathbb{C}) \rightarrow M_K(G, h)(\mathbb{C})$ sends the class of (x, h) in $M_{K'}(G, h)(\mathbb{C})$ to the class of (x, hg) in $M_K(G, h)(\mathbb{C})$;
- (ii) If K is an open compact subgroup of $G(\mathbb{A}_f)$ and $g \in K$, then $T_{g, K, K} = \text{id}_{M_K}$;
- (iii) If K, K', K'' are open compact subgroups of $G(\mathbb{A}_f)$ and $g, h \in G(\mathbb{A}_f)$ are such that $gK'g^{-1} \subset K$ and $hK''h^{-1} \subset K'$, then $T_{g, K, K'} \circ T_{h, K', K''} = T_{gh, K, K''}$;
- (iv) If K, K' are open compact subgroups of $G(\mathbb{A}_f)$ such that K' is a normal subgroup of K , then the morphisms $T_{g, K', K'}$ for $g \in K$ define an action of K/K' on $M_{K'}$ (this follows from (ii) and (iii)), and $T_{1, K', K} : M_{K'} \rightarrow M_K$ induces an isomorphism $M_{K'}/(K/K') \rightarrow M_K$.

REMARK 1.39. We phrased Definition 1.38 in this way to stay close to the definition of a canonical model given by Deligne in Définition 3.1 of [35]. In Deligne's words, a family of E -schemes (M_K) with a family of morphisms $(T_{g, K, K'})$ satisfying conditions (ii), (iii) and (iv) is a *scheme over E with a continuous action of $G(\mathbb{A}_f)$* , and the family of isomorphisms (ι_K) satisfying condition (i) is an *isomorphism of $(M_K) \otimes_F \mathbb{C}$ with $M(G, h)(\mathbb{C})$ compatible with the $G(\mathbb{A}_f)$ -actions*. However, we did not phrase the conditions in the most optimal way possible; for example, condition (i) implies conditions (ii) and (iii).

If we have a model $(M_K)_K$ of $(M_K(G, h)(\mathbb{C}))_K$ over F , we write $M = \varprojlim_K M_K$ (where the transition morphisms are given by the $T_{1, K', K}$). This is an \bar{F} -scheme with an action of $G(\mathbb{A}_f)$, and we have a $G(\mathbb{A}_f)$ -equivariant isomorphism $M \otimes_F \mathbb{C} \xrightarrow{\sim} M(G, h)(\mathbb{C})$.

In particular we get an action of $\text{Gal}(\bar{F}/F)$ on $\pi_0(M(G, h)(\mathbb{C})) \xrightarrow{\sim} \pi_0(M \otimes_F \bar{F})$, which must commute with the action of $G(\mathbb{A}_f)$. Under the hypothesis of Theorem 1.37, we have $\pi_0(M \otimes_F \bar{F}) \xrightarrow{\sim} \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$ with $T = G/G_{\text{der}}$, and $G(\mathbb{A}_f)$ acts transitively on this set of connected components (Proposition 2.2 of [35]). So every element of $\text{Gal}(\bar{F}/F)$ acts by translation by an element of $\pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$, and the action of $\text{Gal}(\bar{F}/F)$ comes from a morphism of groups $\text{Gal}(\bar{F}/F) \rightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$, that necessarily factors through the maximal abelian quotient $\text{Gal}(\bar{F}/F)^{\text{ab}}$.

Suppose that F is a number field. Then global class field theory¹⁴ gives an isomorphism

$$\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \xrightarrow{\sim} \pi_0(F^\times \backslash \mathbb{A}_F^\times)$$

where \mathbb{A}_F is the ring of adèles of F , so the action of $\mathrm{Gal}(\overline{F}/F)$ on $\pi_0(M \otimes_F \overline{F})$ comes from a morphism of groups

$$\lambda_M : \pi_0(F^\times \backslash \mathbb{A}_F^\times) \rightarrow \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})) / \pi_0(K_\infty),$$

called the *reciprocity law* of the model.

1.4.2. *The case of tori.* We consider the case where $G = T$ is a torus. Let $h : \mathbb{S} \rightarrow T_{\mathbb{R}}$ be any morphism of real algebraic groups. Then h trivially satisfies the conditions of Definition 1.33, so we get a Shimura datum (T, h) , and $X = T(\mathbb{R}) / \mathrm{Cent}_{T(\mathbb{R})}(h)$ is a singleton. For every open compact subgroup K of $T(\mathbb{A}_f)$,

$$M_K(T, h)(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / K$$

is a finite group. Let $\overline{T(\mathbb{Q})}$ be the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$; then

$$M(T, h)(\mathbb{C}) = \overline{T(\mathbb{Q})} \backslash T(\mathbb{A}_f),$$

which is a profinite group isomorphic to $\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}_f))$ by the obvious map. Giving a model of the Shimura variety of (T, h) over a subfield F of \mathbb{C} is the same as giving an action of $\mathrm{Gal}(\overline{F}/F)$ over $\pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}_f))$ (commuting with the action of $T(\mathbb{A}_f)$ by translations), i.e. a morphism of groups $\mathrm{Gal}(\overline{F}/F) \rightarrow \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}_f))$. If F is a number field, this is equivalent to giving a morphism of groups

$$\lambda_M : \pi_0(F^\times \backslash \mathbb{A}_F^\times) \rightarrow \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})).$$

It is natural to construct such a morphism from a morphism of algebraic groups $F^\times \rightarrow T$, where F^\times is seen as an algebraic group over \mathbb{Q} (so that, for example, we have $F^\times(\mathbb{A}) = (\mathbb{A} \otimes_{\mathbb{Q}} F)^\times = \mathbb{A}_F^\times$). We already have a morphism $h : \mathbb{S} \rightarrow T_{\mathbb{R}}$, which gives a morphism of complex algebraic groups $h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \rightarrow T_{\mathbb{C}}$. Remember that $\mathbb{S}_{\mathbb{C}} \simeq \mathrm{GL}_{1, \mathbb{C}} \times \mathrm{GL}_{1, \mathbb{C}}$, and that we denoted by $r : \mathrm{GL}_{1, \mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$ the embedding of the first factor (see page 15). We get a morphism $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1, \mathbb{C}} \rightarrow T_{\mathbb{C}}$. As T is an algebraic group over \mathbb{Q} , this morphism is defined over a finite extension of \mathbb{Q} in \mathbb{C} , and we call this extension F . We get a morphism of F -algebraic groups $\mathrm{GL}_{1, F} \rightarrow T_F$, hence a morphism of \mathbb{Q} -algebraic groups $F^\times \rightarrow \mathrm{Res}_{F/\mathbb{Q}} T_F$, where $\mathrm{Res}_{F/\mathbb{Q}} T_F$ is the algebraic group that sends a \mathbb{Q} -algebra R to $T(R \otimes_{\mathbb{Q}} F)$. Composing this with the norm $N_{F/\mathbb{Q}} : \mathrm{Res}_{F/\mathbb{Q}} T_F \rightarrow T$, we finally get a morphism $r(h) : F^\times \rightarrow T$, called the *reciprocity morphism for (T, h)* . We take λ_M to be induced by $r(h)$.¹⁵

So if (G, h) is a Shimura datum with G a torus, we get a canonically defined model of the associated Shimura variety over the field of definition of $h_{\mathbb{C}} \circ r$.

1.4.3. *The reflex field.* Let (G, h) be a Shimura datum satisfying the hypothesis of Theorem 1.37, and let $\nu : G \rightarrow T := G/G_{\mathrm{der}}$ be the quotient morphism. For every open compact subgroup K of $G(\mathbb{A}_f)$, we have a bijection, induced by ν :

$$\pi_0(M_K(G, h)(\mathbb{C})) \xrightarrow{\sim} T(\mathbb{Q}) \backslash T(\mathbb{A}) / \nu(K_\infty \times K).$$

Suppose that we have a model $(M_K)_K$ of the Shimura variety of (G, h) over a number field $F \subset \mathbb{C}$. We would expect the Shimura variety of $(T, \nu \circ h)$ to also have a model over F , and the isomorphism above to be $\mathrm{Gal}(\overline{F}/F)$ -equivariant, where

¹⁴Normalized so that local uniformizers correspond to geometric Frobenius elements.

¹⁵This is the opposite of Deligne's convention in [35] (3.9.1); I have chosen to follow Pink's convention in [96] Chapter 11.

the action on the right hand side is given by the morphism $r(\nu \circ h) : F^\times \rightarrow T$ constructed in 1.4.2.

Remember that $r : \mathrm{GL}_{1,\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}} \simeq \mathrm{GL}_{1,\mathbb{C}} \times \mathrm{GL}_{1,\mathbb{C}}$ is the embedding of the first factor (see page 15). By the previous paragraph, we would expect F to contain the field of definition of $\nu_{\mathbb{C}} \circ h_{\mathbb{C}} \circ r$. In fact it would make sense to take F to be the field of definition of $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1,\mathbb{C}} \rightarrow G_{\mathbb{C}}$, except that h is only significant up to conjugation. This motivates the following definition.

DEFINITION 1.40. Let (G, h) be a Shimura datum. The *reflex field* $F(G, h)$ of (G, h) is the field of definition of the conjugacy class of $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1,\mathbb{C}} \rightarrow G_{\mathbb{C}}$.

Let $F = F(G, h)$. Then F is a finite extension of \mathbb{Q} in \mathbb{C} , and, for every morphism ρ of G into a commutative algebraic group, the morphism $\rho_{\mathbb{C}} \circ h_{\mathbb{C}} \circ r$ is defined over F . Note that $h_{\mathbb{C}} \circ r$ itself is not necessarily defined over F .

EXAMPLE 1.41. Let $E = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic extension of \mathbb{Q} , let $p \geq q \geq 1$ be integers, and set $n = p + q$. Let $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$. For every commutative ring R , we denote by $x \mapsto \bar{x}$ the involution of $R \otimes_{\mathbb{Z}} \mathcal{O}_E$ induced by the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q})$, and, for every $Y \in M_n(R \otimes_{\mathbb{Q}} E)$, we write $Y^* = {}^t \bar{Y}$.

The general unitary group $\mathrm{GU}(p, q)$ is the \mathbb{Z} -group scheme defined by

$$\mathrm{GU}(p, q)(R) = \{g \in \mathrm{GL}_n(R \otimes_{\mathbb{Z}} \mathcal{O}_E) \mid \exists c(g) \in R^\times, g^* J g = c(g) J\}$$

for every commutative ring R . Then $\mathrm{GU}(p, q)_{\mathbb{Q}}$ is a connected reductive algebraic group, and we have a morphism of group schemes $c : \mathrm{GU}(p, q) \rightarrow \mathrm{GL}_1$, whose kernel is the unitary group $\mathrm{U}(p, q)$.

Let $h : \mathbb{S} \rightarrow \mathrm{GU}(p, q)_{\mathbb{R}}$ be the morphism defined by

$$h(z) = \begin{pmatrix} zI_p & 0 \\ 0 & \bar{z}I_q \end{pmatrix} \in \mathrm{GU}(p, q)(\mathbb{R}).$$

Then (G, h) is a Shimura datum, and K_∞ is the set of matrices $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ such that $g_1 \in \mathrm{GL}_p(\mathbb{C})$, $g_2 \in \mathrm{GL}_q(\mathbb{C})$ and there exists $c \in \mathbb{R}$ with $g_1^* g_1 = cI_p$ and $g_2^* g_2 = cI_q$.

We use the isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ sending $x \otimes 1 + y \otimes \sqrt{-d}$ to $(x + \sqrt{-d}y, x - \sqrt{-d}y)$ to identify $\mathrm{GU}(p, q)(\mathbb{C})$ to a subgroup of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$; note that the involution $g \mapsto \bar{g}$ of $\mathrm{GU}(p, q)(\mathbb{C})$ corresponds to switching the two factors.

With this convention, we have $h_{\mathbb{C}} \circ r(z) = \left(\begin{pmatrix} zI_p & 0 \\ 0 & I_q \end{pmatrix}, \begin{pmatrix} I_p & 0 \\ 0 & zI_q \end{pmatrix} \right)$ for every $z \in \mathbb{C}^\times$.

It is easy to see that $h_{\mathbb{C}} \circ r$ is defined over E but not over \mathbb{Q} . On the other hand, the reflex field of (G, h) is E if $p > q$ and \mathbb{Q} if $p = q$.

EXAMPLE 1.42. Let (G, h) be the Shimura datum of Example 1.35 (so that $G = \mathrm{GSp}_{2d}$). For every $z \in \mathbb{C}^\times$, we have

$$h_{\mathbb{C}} \circ r(z) = \begin{pmatrix} \frac{1}{2}(z+1)I_d & -\frac{1}{2i}(z-1)I_d \\ \frac{1}{2i}(z-1)I_d & \frac{1}{2}(z+1)I_d \end{pmatrix} = P \begin{pmatrix} zI_d & 0 \\ 0 & I_d \end{pmatrix} P^{-1},$$

where

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}I_d & \frac{i}{\sqrt{2}}I_d \\ \frac{i}{\sqrt{2}}I_d & \frac{1}{\sqrt{2}}I_d \end{pmatrix}.$$

So the reflex field of (G, h) is \mathbb{Q} .

1.4.4. *Canonical models.* We are now ready to define canonical models.

DEFINITION 1.43. Let (G, h) be a Shimura datum and let $F = F(G, h)$. A *canonical model* of $M(G, h)(\mathbb{C})$ is a model $(M_K)_K$ over F such that, for every torus $u : H \subset G$ and every $h' : \mathbb{S} \rightarrow H_{\mathbb{R}}$ such $u \circ h'$ and h are $G(\mathbb{R})$ -conjugated (i.e. such that u induces a morphism of Shimura data from (H, h') to (G, h)), the morphism

$$u : M(H, h')(\mathbb{C}) \rightarrow M(G, h)(\mathbb{C})$$

is defined over the compositum $F \cdot F(H, h') \subset \mathbb{C}$, where we use as model of $M(H, h')(\mathbb{C})$ over $F(H, h')$ the one defined in 1.4.2.

- EXAMPLES 1.44. (1) If G is a torus, then the model of 1.4.2 is a canonical model of $M(G, h)(\mathbb{C})$.
 (2) If (G, h) is the Shimura datum of Example 1.35 (so that $G = \mathrm{GSp}_{2d}$), then the schemes $(\mathcal{M}_{d, K, \mathbb{Q}})_{K \subset \mathrm{GSp}_{2d}(\mathbb{A}_f)}$ of 1.2.4 form a canonical model of $M(G, h)(\mathbb{C})$. This is not obvious but follows from the main theorem of complex multiplication; see Section 4 of [35].

At the time of Deligne's paper [35], it was not known whether all Shimura varieties have canonical models (spoiler: this is now known to be true, see Theorem 2.30), but it was possible to prove their uniqueness. If $u : (H, h') \rightarrow (G, h)$ is a morphism of Shimura data as in Definition 1.43 (so that H is a subtorus of G), the points of the image of $M(H, h')(\mathbb{C})$ in $M(G, h)(\mathbb{C})$ are called *special points*. The fact that canonical models are unique if they exist relies on the following two points:

- (i) The set of all $G(\mathbb{A}_f)$ -translates of special points are dense in $M(G, h)(\mathbb{C})$ (see Proposition 5.2 of [35]; in fact, we just need to take $G(\mathbb{A}_f)$ -translates of one special point);
- (ii) For every finite extension $F' \subset \mathbb{C}$ of $F(G, h)$, there exists $u : (H, h') \rightarrow (G, h)$ as above such that $F(H, h')$ and F' are linearly disjoint over $F(G, h)$ (Théorème 5.1 of [35]).

From this, we can deduce:

THEOREM 1.45 (Corollaire 5.4 of [35]). *Let $u : (G_1, h_1) \rightarrow (G_2, h_2)$ be a morphism of Shimura data, and suppose that the Shimura varieties of (G_1, h_1) and (G_2, h_2) have canonical models. Then the morphism $M(G_1, h_1)(\mathbb{C}) \rightarrow M(G_2, h_2)(\mathbb{C})$ of Shimura varieties corresponding to u is defined over any common extension F of $F(G_1, h_1)$ and $F(G_2, h_2)$ in \mathbb{C} .*

COROLLARY 1.46. *Let (G, h) be a Shimura datum. Then a canonical model of $M(G, h)(\mathbb{C})$ is unique up to unique isomorphism if it exists.*

COROLLARY 1.47. *Let (G, h) be a Shimura datum satisfying the hypothesis of Theorem 1.37, let $\nu : G \rightarrow T := G/G_{\mathrm{der}}$ be the quotient morphism, and let $F = F(G, h)$. Then the action of $\mathrm{Gal}(\overline{F}/F)$ on $\pi_0(M(G, h)(\mathbb{C})) \simeq \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A})) / \pi_0(K_{\infty})$ is given by the reciprocity morphism for $(T, \nu \circ h)$.*

PROOF. Apply Theorem 1.45 to $\nu : (G, h) \rightarrow (T, \nu \circ h)$. □

Using the same techniques as for Theorem 1.45, we also get the following very useful result.

PROPOSITION 1.48 (Corollaire 5.7 of [35]). *Let $u : (G_1, h_1) \rightarrow (G_2, h_2)$ be a morphism of Shimura data such that the underlying morphism of algebraic groups is a closed immersion and that $F(G_1, h_1) \subset F(G_2, h_2)$. If $M(G_2, h_2)(\mathbb{C})$ has a canonical model, then so does $M(G_1, h_1)(\mathbb{C})$.*

2. Lecture 2: arithmetic Shimura varieties

In lecture 1, we have defined Shimura varieties over \mathbb{C} and introduced the notion of canonical model of a Shimura variety. We also discussed in some detail the example of the Siegel modular varieties, which have canonical models coming from their modular interpretation. In this lecture, we first want to present different types of Shimura varieties, with each type being contained in the next one:

- **The Siegel modular variety:** it is a moduli space of principally polarized abelian schemes with some level structure;
- **PEL type Shimura varieties:** they have an interpretation as moduli spaces of polarized abelian schemes with multiplication by the ring of integers \mathcal{O} of some number field and some level structure (here “P” means “polarization”, “E” means “endomorphisms” in reference to the action of \mathcal{O} and “L” means “level structure”);
- **Hodge type Shimura varieties:** they come from Shimura data (G, h) that have an injective morphism into a Siegel Shimura datum (Example 1.35);
- **Abelian type Shimura varieties:** their Shimura datum is “isogenous” to a Hodge type Shimura datum (in a way to be made precise later);
- **General Shimura varieties:** all Shimura varieties.

The further we go down in the list, the less is known about the geometry of the Shimura variety (and the higher the price for what we know), because many of the techniques we have rely on the interpretation of the Shimura varieties as moduli spaces of abelian schemes, and this is only really available for PEL type Shimura varieties (there is a modular interpretation for Hodge type Shimura varieties, but it is harder to use).

For example, PEL type Shimura varieties naturally come with an *integral model* defined over a localization of the ring of integers of their reflex field, but it took a lot of effort to construct integral models for Hodge type and abelian type Shimura varieties, and to formulate their properties (see 2.1.4 and Theorems 2.22 and 2.29); as far as we know, nothing is known for general Shimura varieties.

We are not claiming that the classification above is the only measure of the complexity of a Shimura variety, whatever that means. For example, as we will discuss in lecture 3, if one wants to study the cohomology of Shimura varieties and their zeta functions, then the simplest case is not the case of Siegel modular varieties, but rather of compact PEL type Shimura varieties whose group has no endoscopy and a simply connected derived subgroup; we will introduce some examples of these, known as *Kottwitz’s simple Shimura varieties*, in 2.4.

2.1. PEL type Shimura varieties. These Shimura varieties were introduced by Kottwitz in [64], but we will follow the presentation of Lan for the moduli

problems (cf. [67] 1.4.1), which is closer to our definition of the Siegel moduli problem. The equivalence between the two definitions is proved in [67] 1.4.3.

2.1.1. PEL data.

DEFINITION 2.1 (see [68] 5.1 or Definition 1.2.13 of [67]). An (*integral*) *PEL datum* is a quintuple $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$, where:

- (1) \mathcal{O} is an order in a finite-dimensional semisimple \mathbb{Q} -algebra B (that is, \mathcal{O} is a subring of B that is a free \mathbb{Z} -module and spans the \mathbb{Q} -vector space B);
- (2) $*$ is a *positive involution* of \mathcal{O} , i.e. an anti-automorphism of rings of order 2 such that, for every $x \in \mathcal{O} \setminus \{0\}$, we have $\mathrm{Tr}_{(B \otimes_{\mathbb{Q}} \mathbb{R})/\mathbb{R}}(xx^*) > 0$;
- (3) Λ is an \mathcal{O} -module that is finitely generated and free as a \mathbb{Z} -module;
- (4) $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ is a \mathbb{Z} -bilinear alternating map such that, for all $x, y \in \Lambda$ and $b \in \mathcal{O}$, we have

$$\langle bx, y \rangle = \langle x, b^*y \rangle;$$

- (5) $h : \mathbb{C} \rightarrow \mathrm{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is an \mathbb{R} -algebra morphism such that:
 - (a) For $z \in \mathbb{C}$ and $x, y \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, we have

$$\langle h(z)(x), y \rangle = \langle x, h(\bar{z})(y) \rangle;$$

- (b) The \mathbb{R} -bilinear pairing $\langle \cdot, h(i)(\cdot) \rangle$, which is symmetric by (a), is also positive definite.

Let $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$ be a PEL datum. We define a group scheme G over \mathbb{Z} by

$$G(R) = \{g \in \mathrm{End}_{\mathcal{O} \otimes_{\mathbb{Z}} R}(\Lambda \otimes_{\mathbb{Z}} R) \mid \exists c(g) \in R^\times, \langle g(\cdot), g(\cdot) \rangle = c\langle \cdot, \cdot \rangle\}$$

for every commutative ring R . We also get a morphism of group schemes $c : G \rightarrow \mathrm{GL}_1$. The morphism of \mathbb{R} -algebras $h : \mathbb{C} \rightarrow \mathrm{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ induces a morphism of \mathbb{R} -algebraic groups $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$.

PROPOSITION 2.2. *The couple $(G_{\mathbb{Q}}^0, h)$ satisfies conditions (a)-(c) in the definition of a Shimura datum (Definition 1.33).*

Let us explain why this proposition is true. Condition (a) of Definition 1.33 follows from the fact that $h : \mathbb{C}^\times \rightarrow \mathrm{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is a morphism of \mathbb{R} -algebras, and condition (c) of loc. cit. follows from condition (5)(b) of Definition 2.1. We prove condition (b), on the decomposition of $\mathrm{Lie}(G_{\mathbb{C}})$ into eigenspaces for the action of $\mathrm{Ad} \circ h_{\mathbb{C}}$. Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, a finite-dimensional \mathbb{R} -vector space. Then $h : \mathbb{C} \rightarrow \mathrm{End}_{\mathbb{R}}(V)$ is a morphism of \mathbb{R} -algebras, hence defines a structure of \mathbb{C} -vector space on V . In particular, we have $V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus V_2$, where \mathbb{C} acts on V_1 (resp. V_2) via multiplication by $z \mapsto z$ (resp. $z \mapsto \bar{z}$). Condition (b) follows from this, because $\mathrm{End}(V) \simeq V^* \otimes_{\mathbb{R}} V$ and $\mathfrak{g} := \mathrm{Lie}(G_{\mathbb{R}}) \subset \mathrm{End}_{\mathbb{R}}(V)$.

We keep the same notation. Then $A := V/\Lambda$ is a complex torus. We would like this torus to be an abelian variety, so we need a polarization on it, that is, a positive definite Hermitian form H on V such that $\mathrm{Im}(H)$ takes integer values on Λ . But we already have an alternating form on Λ , so we know from 1.2.2 how to proceed: define H by

$$H(v, w) = \langle h(i)(v), w \rangle + i\langle v, w \rangle.$$

The fact that $\mathrm{Im}(H)$ takes integral values on Λ is clear, the pairing H is Hermitian by condition (5)(a) and positive definite by condition (5)(b). So the torus A is an

abelian variety, with dual abelian variety $A^\vee = V/\Lambda^\vee$, where $\Lambda^\vee = \{v \in V \mid \forall w \in \Lambda, \langle v, w \rangle \in \mathbb{Z}\}$ is the dual lattice of Λ . The polarization $\lambda : A \rightarrow A^\vee$ defined by H is then just the map induced by $\Lambda \subset \Lambda^\vee$. We have a morphism of algebraic groups $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \subset \mathrm{GL}(V)$, and the decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ induced by this morphism (see 1.3.1) is the same as the Hodge structure coming from the isomorphism $V \simeq H_1(A, \mathbb{R})$. Note also that the action of \mathcal{O} on L defines a morphism of rings $\iota : \mathcal{O} \rightarrow \mathrm{End}_{\mathbb{C}}(A)$ satisfying the *Rosati condition*, which means that, for every $b \in \mathcal{O}$, we have

$$\lambda \circ i(b^*) = i(b)^\vee \circ \lambda.$$

REMARK 2.3 ([64], sections 5 and 7). Let $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$ be a PEL datum, and let G be the associated group scheme. If the \mathbb{Q} -algebra $B = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is simple, then all the simple factors of $G_{\mathrm{der}}(\mathbb{C})$ are of the same type, which is A , C or D . If we are in type A or C , then $G_{\mathbb{Q}}$ is connected and reductive. In type D , the group $G_{\mathbb{Q}}$ is reductive and has $2^{[F_0:\mathbb{Q}]}$ connected components, where F_0 is the field of fixed points of $*$ in the center of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$, so it is never connected.

We now give some examples covering all the types.

EXAMPLE 2.4. Take $\mathcal{O} = \mathbb{Z}$, $* = \mathrm{id}_{\mathcal{O}}$, $\Lambda = \mathbb{Z}^{2d}$, $\langle \cdot, \cdot \rangle$ the perfect symplectic pairing with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ in the canonical basis of \mathbb{Z}^{2d} , and $h : \mathbb{C} \rightarrow M_{2d}(\mathbb{R})$ defined by

$$h(a + ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

Then we get a PEL datum, and the couple $(G_{\mathbb{Q}}, h)$ is the Siegel Shimura datum of Example 1.35.

EXAMPLE 2.5. Let $B = E \subset \mathbb{C}$ be an imaginary quadratic extension of \mathbb{Q} , \mathcal{O} be an order in E (for example the ring of integers \mathcal{O}_E), $*$ be the restriction to \mathcal{O} of complex conjugation,¹⁶ $\Lambda = \mathcal{O}^{p+q}$ with $p \geq q \geq 0$. Choose $\varepsilon \in \mathcal{O}$ such that $-i\varepsilon \in \mathbb{R}_{>0}$, let H be the Hermitian pairing on Λ with matrix $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and $\langle \cdot, \cdot \rangle$ be the alternating pairing $\mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(\varepsilon H)$ on Λ . Finally, define $h : \mathbb{C} \rightarrow \mathrm{End}_{\mathbb{C}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) = M_{p+q}(\mathbb{C})$ by $h(z) = \begin{pmatrix} zI_p & 0 \\ 0 & \bar{z}I_q \end{pmatrix}$. Then we get a PEL datum, and the couple $(G_{\mathbb{Q}}, h)$ is the Shimura datum of Example 1.41.

If $q = 0$, then (G, h) does not satisfy condition (d) of Definition 1.33.

EXAMPLE 2.6. At the other extreme of type A, we have the PEL data considered by Kottwitz in [63] (see the beginning of Section 1 of that paper). Let F_0 be a totally real number field, F be a totally imaginary quadratic extension of F_0 and B be a division algebra with center F ; we suppose given a positive involution $*$ on B whose restriction to F is the nontrivial element of $\mathrm{Gal}(F/F_0)$.

We take $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = B$ with its action by left translations. The alternating form $\langle \cdot, \cdot \rangle$ on $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by $\langle x, y \rangle = \mathrm{Tr}_{B/\mathbb{Q}}(xby^*)$, where b is an element of B^\times such that $b^* = -b$. We have $\mathrm{End}_B(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}) = B^{\mathrm{op}}$, so $G_{\mathbb{Q}} = \{x \in (B^{\mathrm{op}})^\times \mid xx^\ddagger \in \mathbb{Q}^\times\}$,

¹⁶If $(1, a)$ is a \mathbb{Z} -basis of \mathcal{O}_E , then every order \mathcal{O} of E is contained in \mathcal{O}_E and of the form $\mathbb{Z} \oplus fa\mathbb{Z}$, where $f = [\mathcal{O}_E : \mathcal{O}]$ is the *conductor* of \mathcal{O} . In particular, the order \mathcal{O} is stable by the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q})$, which is the restriction of complex conjugation to E .

where $(\cdot)^\ddagger$ is the involution of B^{op} define by $x^\ddagger = bx^*b^{-1}$ (which is not positive in general).

Warning: What Kottwitz calls D and $*$ in [63] is what we call B^{op} and \ddagger here, so in particular the involution $*$ of [63] is not assumed to be positive (and the PEL datum is not defined there). We changed the notation to be consistent with the rest of the text and with Kottwitz's other paper [64].

We have $F_0 \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r$, $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^r$ and $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_n(\mathbb{C})^r$, where $n = \sqrt{\dim_{\mathbb{Q}} B}$; the involution $*$ is conjugate on each factor to the positive involution $A \mapsto {}^t\bar{A}$. So, if $G_0 = \{x \in (B^{\text{op}})^{\times} | xx^\ddagger = 1\}$, then $G_{0,\mathbb{R}}$ is isomorphic to a product $\prod_{i=1}^r U(p_i, q_i)$ of unitary groups, and $G_{\mathbb{R}}$ is isomorphic to the corresponding general group (i.e. the subgroup of $\prod_{i=1}^r \text{GU}(p_i, q_i)$ where all the multipliers are equal). We define $h : \mathbb{C}^{\times} \rightarrow G_{\mathbb{R}}$ by sending $z \in \mathbb{C}^{\times}$ to element of $G(\mathbb{R})$ corresponding to the family of matrices $\left(\begin{pmatrix} zI_{p_i} & 0 \\ 0 & \bar{z}I_{q_i} \end{pmatrix} \right)_{1 \leq i \leq r}$. This extend to a morphism of \mathbb{R} -algebras $h : \mathbb{C} \rightarrow B^{\text{op}} \otimes_{\mathbb{Q}} \mathbb{R}$ sending 0 to 0; as \mathbb{C} is commutative, we can also see h as a morphism of \mathbb{R} -algebras $\mathbb{C} \rightarrow B \otimes_{\mathbb{Q}} \mathbb{R}$.

We have defined what is known as a rational PEL datum. To get an integral PEL datum, we must suppose that there exists an order \mathcal{O} in B such that $\mathcal{O}^* = \mathcal{O}$ and $\langle \cdot, \cdot \rangle$ takes integer values on \mathcal{O} .

EXAMPLE 2.7. Let B be a quaternion algebra over \mathbb{Q} such that $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H}$, and let \mathcal{O} be an order in B that is stable by the involution of \mathbb{H} defined by $(x + iy + jz + kt)^* = x - iy - jz - kt$. Let $\Lambda = \mathcal{O}^{2n}$, let $\langle \cdot, \cdot \rangle$ be $\text{Tr}_{\mathcal{O}/\mathbb{Z}} \circ H$, where H is the skew-Hermitian pairing on Λ with matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Define $h : \mathbb{C} \rightarrow \text{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) = M_n(\mathbb{H})$ by $h(a + ib) = \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$.

The group $(G^0)_{\mathbb{R}, \text{der}}$ is often denoted by SO_{2n}^* ; it is a quasi-split outer form of the split orthogonal group SO_{2n} , hence is of type D_n . But note that the algebraic group $G_{\mathbb{Q}}$ is not connected.

DEFINITION 2.8. We say that a Shimura datum is of *PEL type* if it is of the form $(G_{\mathbb{Q}}^0, h)$, where (G, h) comes from a PEL datum. The corresponding Shimura varieties are called *PEL type Shimura varieties*.

2.1.2. PEL moduli problems. Just as in the case of the Siegel modular variety, PEL type Shimura varieties are the solution of a moduli problem,¹⁷ known as a *PEL moduli problem*. We now discuss these.

Let $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$ be a PEL datum, and let (G, h) be defined as before. If $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, then we saw that the morphism $h : \mathbb{C} \rightarrow \text{End}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}}(V)$ defines a decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$. The *reflex field* F of the PEL datum is the field of definition of the isomorphism class of $V^{-1,0}$ as an $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}$ -module, that is, the subfield of \mathbb{C} generated by the elements $\text{Tr}(b, V^{1,0})$, for $b \in \mathcal{O}$. It is also equal to the field of the definition of the conjugacy class of $h_{\mathbb{C}} \circ r : \text{GL}_{1,\mathbb{C}} \rightarrow G_{\mathbb{C}}$, that is, the reflex field of $(G_{\mathbb{Q}}^0, h)$ when this couple is a Shimura datum.

DEFINITION 2.9. We say that a prime number p is *good* for the PEL datum if:

- p is unramified in \mathcal{O} (i.e. it does not divide the discriminant of \mathcal{O}/\mathbb{Z});

¹⁷Well, almost. See Proposition 2.14 below for a more precise statement.

- p does not divide $[\Lambda^\vee : \Lambda]$, where Λ^\vee is as before the lattice $\{v \in V \mid \forall w \in \Lambda, \langle v, w \rangle \in \mathbb{Z}\}$;
- $p \neq 2$ if the PEL datum has a factor of type D (i.e. if $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}$ has a simple factor isomorphic to an algebra $M_n(\mathbb{H})$ with its canonical positive involution).

If p is not good we say that it is bad. Note that, if p is good, then F and $G_{\mathbb{Q}}$ are unramified at p ; in fact, the group $G(\mathbb{Z}_p)$ is then a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$.

Let T be a set of good primes (finite or infinite), and let $\mathcal{O}_{F,T}$ be the localization $\mathcal{O}_F[\frac{1}{p}, p \notin T]$.

DEFINITION 2.10. Let n be a positive integer that is prime to all the elements of T . Then the PEL moduli problem at level n defined by the fixed PEL datum is the contravariant functor \mathcal{M}_n from the category of $\mathcal{O}_{F,T}$ -schemes to the category of sets sending an $\mathcal{O}_{F,T}$ -scheme S to the set of isomorphism classes of quadruples $(A, \lambda, \iota, (\eta, \varphi))$, where:

- A is an abelian scheme over S ;
- $\lambda : A \rightarrow A^\vee$ is a polarization whose degree is prime to all the elements of T ;
- $\iota : \mathcal{O} \rightarrow \text{End}_S(A)$ is a morphism of rings satisfying the Rosati condition: for every $b \in \mathcal{O}$, we have $\lambda \circ \iota(b^*) = \iota(b)^\vee \circ \lambda$;
- (η, φ) is a level n structure on A , i.e., $\eta : A[n] \xrightarrow{\sim} (\Lambda/n\Lambda)_S$ is an \mathcal{O} -equivariant isomorphism of group schemes and $\varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ is an isomorphism of group schemes such that $\varphi \circ \langle \cdot, \cdot \rangle \circ \eta$ is the Weil pairing defined by λ on $A[n]_S$, and moreover (η, φ) are liftable to level m structures for every prime-to- T multiple m of n , in the sense of [67] Definition 1.3.6.2.

We furthermore require that this quadruple satisfy the following *determinant condition* (see Definition 1.3.4.1 of [67]): let $\alpha_1, \dots, \alpha_t$ be a basis of the \mathbb{Z} -module \mathcal{O} , and let X_1, \dots, X_t be indeterminates. Then $\det(\iota(\alpha_1)X_1 + \dots + \iota(\alpha_t)X_t, \text{Lie}(A))$ is a polynomial in $\mathcal{O}_S[X_1, \dots, X_t]$, and the condition says that this polynomial is equal to the image by the map $\mathcal{O}_{F,T} \rightarrow \mathcal{O}_S$ of the polynomial $\det(\alpha_1 X_1 + \dots + \alpha_t X_t, V^{1,0}) \in \mathcal{O}_{F,T}[X_1, \dots, X_t]$.¹⁸

- REMARK 2.11. (1) We did not specify the relative dimension of the abelian scheme A in the moduli problem, because it is necessarily equal to $\dim_{\mathbb{C}}(V^{1,0})$ by the determinant condition.
- (2) For $n \in \{1, 2\}$, the objects of the moduli problem \mathcal{M}_n can have nontrivial automorphisms, so it would make more sense to see \mathcal{M}_n as a functor with values in groupoids, i.e. as a stack. See Remark 1.13.
- (3) As in Remark 1.17, it is also possible to define the moduli problem \mathcal{M}_K for more general levels K , i.e. for open compact subgroups K of $G(\mathbb{A}_f)$, though there is a condition on K corresponding to the condition that n be prime to T . Let \mathbb{A}_f^T be the ring of prime-to- T adèles of \mathbb{Q} , i.e. the restricted product of the \mathbb{Q}_p for $p \notin T$, and \mathbb{A}_T be the restricted product

¹⁸It is not totally obvious that the second polynomial, which is a priori in $F[X_1, \dots, X_t]$, has its coefficients in $\mathcal{O}_{F,T}$. But it is also not too hard to check. See for example pages 389-390 of [64].

of the \mathbb{Q}_p for $p \in T$. We have $\mathbb{A}_f = \mathbb{A}_T \times \mathbb{A}_f^T$, and the condition on K is that $K = K_T K^T$, where $K^T \subset G(\mathbb{A}_f^T)$ and $K_T = \prod_{p \in T} K_p$ with $K_p = G(\mathbb{Z}_p)$ for every $p \in T$.

The following theorem is a consequence of Mumford's representability result (Theorem 1.16), as explained in Section 5 of Kottwitz's paper [64]. See also Corollaries 1.4.1.12 and 7.2.3.10 of Lan's book [67] for more details.

THEOREM 2.12. *If $n \geq 3$, then the functor \mathcal{M}_n is representable by a smooth quasi-projective scheme over $\mathcal{O}_{F,T}$.*

In fact, by Theorem 1.4.1.11 and Corollary 7.2.3.10 of [67], the functor \mathcal{M}_K is representable by a smooth quasi-projective scheme for K small enough.

REMARK 2.13. We also have an action of the Hecke operators defined by elements $g \in G(\mathbb{A}_f^T)$ (i.e. the Hecke operators that are trivial at primes of T) on the tower (\mathcal{M}_K) ; as for the Siegel moduli problem, the element g acts on the level structure. See Remark 1.4.3.11 of [67] (and the comparison result of Proposition 1.4.3.4 of loc. cit.).

2.1.3. PEL moduli problems and canonical models. Consider the PEL datum of Example 2.4. The associated couple $(G_{\mathbb{Q}}, h)$ is the Siegel Shimura datum of Example 1.35, and we have seen in Example 1.44(2) that the corresponding moduli problem, with $T = \emptyset$, defines a canonical model of the Shimura variety of $(G_{\mathbb{Q}}, h)$.

We would like something like this to be true for general PEL data, but one obstruction is that the moduli problem only depends on the completions of the symplectic \mathcal{O} -module $(\Lambda, \langle \cdot, \cdot \rangle)$ at the places of \mathbb{Q} ; this is not obvious in Definition 2.10, but it becomes so if we use the moduli problem of Definition 1.4.2.1 of [67] or of Section 5 of [64] (the equivalence of the two moduli problems is proved in [67] 1.4.3). Suppose that we have a finitely generated \mathcal{O} -module Λ' that is free as a \mathbb{Z} -module, with a symplectic form $\langle \cdot, \cdot \rangle'$ satisfying condition (4) of Definition 2.1 and such that, for every place v of \mathbb{Q} , $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}_v$ and $\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}_v$ are isomorphic as symplectic $B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ -modules. Then we can see h as a morphism $\mathbb{C} \rightarrow \text{End}_{B \otimes_{\mathbb{Q}} \mathbb{R}}(\Lambda' \otimes_{\mathbb{Z}} \mathbb{R})$, and it will satisfy condition (5) of Definition 2.1, so we get a PEL datum $(\mathcal{O}, *, \Lambda', \langle \cdot, \cdot \rangle', h)$ that defines the same moduli problem as the original PEL datum. In particular, we will get a Shimura datum $(G'_{\mathbb{Q}}, h)$, and we should also see the canonical model for the Shimura variety of $(G'_{\mathbb{Q}}, h)$ in the PEL moduli problem. A priori, there is no reason for G and G' to be isomorphic, and they won't be in general; we only know that $G_{\mathbb{Q}_v} \simeq G'_{\mathbb{Q}_v}$ for every place v of \mathbb{Q} .

In fact, we have the following result.

PROPOSITION 2.14. *Suppose that the semisimple \mathbb{Q} -algebra $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ has no simple factor of type D (see Definition 2.9). Then:*

- (i) *(Remark 1.4.4.4 of [67].) Let $T \subset T'$ be two sets of good prime numbers, let $K^{T'}$ be an open compact subgroup of $G(\mathbb{A}_f^{T'})$, and let $K = K^{T'} \prod_{p \in T'} G(\mathbb{Z}_p)$. Let \mathcal{M}_K (resp. \mathcal{M}'_K) be the moduli problem over $\text{Spec}(\mathcal{O}_{F,T})$ (resp. $\text{Spec}(\mathcal{O}_{F,T'})$) from Definition 2.10, where we take the set of good primes to be T (resp. T'). Then the forgetful functor $\mathcal{M}_K \rightarrow \mathcal{M}'_K \times_{\text{Spec}(\mathcal{O}_{F,T'})} \text{Spec}(\mathcal{O}_{F,T})$ is an isomorphism.*
- (ii) *(Sections 7-8 of [64].) Let $\text{Ker}^1(\mathbb{Q}, G)$ be the kernel of the diagonal map $\text{H}^1(\mathbb{Q}, G) \rightarrow \prod_v \text{H}^1(\mathbb{Q}_v, G)$, where we take the product over all places*

v of \mathbb{Q} ; this is a finite set. Suppose that $(G_{\mathbb{Q}}, h)$ is a Shimura datum,¹⁹ and use $T = \emptyset$ to define the moduli problem of Definition 2.10, so that it is a moduli problem over $\text{Spec}(F)$.²⁰ Then the projective system $(\mathcal{M}_K)_{K \subset G(\mathbb{A}_f)}$ is a disjoint union indexed by $i \in \text{Ker}^1(\mathbb{Q}, G)$ of projective systems $(\mathcal{M}_K^{(i)})_{K \subset G(\mathbb{A}_f)}$, and each $(\mathcal{M}_K^{(i)})_{K \subset G(\mathbb{A}_f)}$ is a canonical model of the Shimura variety of $(G_{\mathbb{Q}}, h)$.

- REMARK 2.15. (1) For every $i \in \text{Ker}^1(\mathbb{Q}, G)$, let G_i be the corresponding inner form of $G_{\mathbb{Q}}$;²¹ we have $G_{i,v} \simeq G_v$ for every place v of \mathbb{Q} , and we denote by h_i the morphism $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \simeq G_{i,\mathbb{R}}$. If $(G_{\mathbb{Q}}, h)$ is a Shimura datum, then all (G_i, h_i) are, and the projective system $(\mathcal{M}_K^{(i)})_{K \subset G(\mathbb{A}_f)}$ in (ii) of the proposition is actually a canonical model of the Shimura variety of (G_i, h_i) (note that $G_i(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$). But, as noted by Kottwitz at the end of Section 8 of [64], under our hypothesis that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ has no factor of type D , all the groups G_i are isomorphic to $G_{\mathbb{Q}}$ and the h_i correspond to h . (In fact Kottwitz calculates $\text{Ker}^1(\mathbb{Q}, G)$ in Section 7 of [64], and finds that in our situation it is either trivial or isomorphic to $\text{Ker}^1(\mathbb{Q}, Z(G))$ by the canonical map, where $Z(G)$ is the center of G .)
- (2) As mentioned in (1), there is some information about $\text{Ker}^1(\mathbb{Q}, G)$ in Section 7 of [64], and ways to calculate it. For example, if $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is simple of type C , we have $\text{Ker}^1(\mathbb{Q}, G) = \{1\}$, which explains why the moduli problem of Definition 1.15 gives a canonical model of the Siegel Shimura variety and not of a finite disjoint union of copies of it. If $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is simple of type A , then $\text{Ker}^1(\mathbb{Q}, G)$ is automatically trivial in “half” of the cases, and it is always isomorphic to $\text{Ker}^1(\mathbb{Q}, Z(G))$; for example, it is trivial for the Shimura datum of Example 1.41.
- (3) Kottwitz does not say much about the case where $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ has simple factors of type D . The situation is complicated for many reasons: the group $G_{\mathbb{Q}}$ is not connected, $\text{Ker}(\mathbb{Q}, G)$ is not trivial, and point (i) of Proposition 2.14 is not true in general, so we must also be careful about the choice of T . In any case, it is still true that the Shimura varieties $M_K(G_{\mathbb{Q}}^0, h)(\mathbb{C})$ are open and closed subschemes of $\mathcal{M}_{K,\mathbb{C}}$ (see [66] 2.5), and we might even get canonical models out of this, but we will not pursue this here because there are other ways to construct canonical models for these Shimura data (see 2.2.1).
- (4) If we want $(G_{\mathbb{Q}}, h)$ to be a PEL Shimura datum, then this puts pretty strict conditions on the center of $G_{\mathbb{Q}}$. For example, if F is a nontrivial totally real extension of \mathbb{Q} , then the group $\text{Res}_{F/\mathbb{Q}} \text{GL}_2$ (defined by $\text{Res}_{F/\mathbb{Q}} \text{GL}_2(R) = \text{GL}_2(R \otimes_{\mathbb{Q}} F)$) is part of a Shimura datum, but this Shimura datum cannot be PEL; more generally, we have the issue with the group $\text{Res}_{F/\mathbb{Q}} \text{GSp}_{2d}$. This is somewhat annoying, as sometimes we really do want to consider the Shimura varieties for these precise groups (see for example Nekovar and Scholl’s [93]). Fortunately, these Shimura

¹⁹As we excluded case D , the group $G_{\mathbb{Q}}$ is connected. So, by Proposition 2.2, the only obstacle to $(G_{\mathbb{Q}}, h)$ being a Shimura datum is condition (d) of Definition 1.33.

²⁰By (i), this moduli problem is the generic fiber of the moduli problems defined by nonempty sets T .

²¹See 2.4.1 for the definition of inner forms.

data are of abelian type (see 2.2.2), so their Shimura varieties are still understood reasonably well.

2.1.4. *Canonical integral models.* PEL moduli problems don't just give canonical models of Shimura varieties, they also give models over various localizations of the ring of integers of the reflex field, at least when $\text{Ker}^1(\mathbb{Q}, G)$ is trivial. Here are some things that we can learn from this example:

- (1) The ring of integers over which we can expect to have a “good” integral model depends on the level K . More precisely, to have a good integral model defined over $\mathcal{O}_{F,(p)}$, we need K to be of the form $K^p K_p$, where $K^p \subset G(\mathbb{A}_f^p)$ and K_p is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. Likewise, the Hecke correspondences that will extend to finite étale morphisms between integral models over $\mathcal{O}_{F,(p)}$ are the ones that are trivial at p , i.e. defined by elements of $G(\mathbb{A}_f^p)$.
- (2) We need a notion of what a “good” integral model is. If the Shimura varieties for (G, h) are compact, then we can just ask for the integral model to be projective smooth over the localization of \mathcal{O}_F that we are using. But this does not suffice in the noncompact case.

REMARK 2.16. In these notes, we will only talk about integral models at places of good reduction, so we want these models to be smooth over the base ring and this is why we impose the conditions of (1). There is of course a (much more difficult and very interesting) theory for places of bad reduction.

To solve problem (2), Milne suggested only looking at models with a certain extension property. Let (G, h) be a Shimura datum, let $F = F(G, h)$, let p be a prime number at which G is unramified, and let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial maximal compact subgroup.²² For every level K , write $M_K = M_K(G, h)$. We want to define the notion of a canonical integral model \mathcal{M}_{K_p} over $\mathcal{O}_{F,(p)}$ of the projective system $(M_{K_p K^p})_{K^p \subset G(\mathbb{A}_f^p)}$, or of its limit M_{K_p} . The idea, first suggested by Milne in [85] (see also Moonen's paper [90]), is to require that, for every S in a class of “admissible test schemes” over $\mathcal{O}_{F,(p)}$, any morphism $S \otimes_{\mathcal{O}_{F,(p)}} F \rightarrow M_{K_p}$ should extend to a morphism $S \rightarrow \mathcal{M}_{K_p}$. The problem is to decide what class of admissible test schemes one should use. We will follow Kisin's presentation in [54].

DEFINITION 2.17 (See [54] 2.3.7.). A *canonical integral model* of the projective system $(M_{K_p K^p})_{K^p \subset G(\mathbb{A}_f^p)}$ (or of its limit M_{K_p}) over $\mathcal{O}_{F,(p)}$ is a projective system $(\mathcal{M}_{K_p K^p})_{K^p \subset G(\mathbb{A}_f^p)}$ of smooth $\mathcal{O}_{F,(p)}$ -schemes with finite étale transition maps, given with finite étale morphisms $T_{g, K^p, K'^p} : \mathcal{M}_{K_p K'^p} \rightarrow \mathcal{M}_{K_p K^p}$ for all $g \in G(\mathbb{A}_f^p)$ and K^p, K'^p open compact subgroups of $G(\mathbb{A}_f^p)$ such that $K'^p \subset K^p \cap gK^p g^{-1}$, and with an isomorphism of projective systems $\iota : (\mathcal{M}_{K_p K^p}) \otimes_{\mathcal{O}_{F,(p)}} F \xrightarrow{\sim} (M_{K_p K^p})$, such that:

- (a) The morphisms T_{g, K^p, K'^p} satisfy the analogues of conditions (ii), (iii) and (iv) of Definition 1.38, and they correspond to the morphisms $T_{g, K_p K^p, K_p K'^p}$ between canonical models by the isomorphism ι (in other words, ι is $G(\mathbb{A}_f^p)$ -equivariant);

²²The condition on p means that G extends to a reductive group scheme \mathcal{G} over \mathbb{Z}_p , and then we can take $K_p = \mathcal{G}(\mathbb{Z}_p)$.

- (b) The scheme $\mathcal{M}_{K_p} := \varprojlim_{K^p} \mathcal{M}_{K_p K^p}$ satisfies the following extension property: if S is a regular formally smooth $\mathcal{O}_{F,(p)}$ -scheme, then any morphism $S \otimes_{\mathcal{O}_{F,(p)}} F \rightarrow M_{K_p}$ extends to a morphism $S \rightarrow \mathcal{M}_{K_p}$.

As in the definition of canonical models, the properties listed in this definition are redundant, see Remark 1.39.

In particular, by applying the extension property with $S = \mathcal{M}_{K_p}$, we see that integral canonical models are unique up to unique isomorphism. Now the problem is existence. For PEL type Shimura varieties of type A or C satisfying the condition that $\text{Ker}^1(\mathbb{Q}, G)$ is trivial, the PEL moduli problem will give a canonical integral model, though that is not trivial: the main ingredient is the Néron-Ogg-Shafarevich criterion on the good reduction of abelian varieties. For more details, see Theorem 2.10 of [85], and also Corollary 3.8 and Example 3.12(ii) of [90].

But what about other Shimura varieties ?

2.2. Hodge type and abelian type Shimura varieties.

2.2.1. *Hodge type Shimura varieties.* For every $d \geq 1$, we denote by (GSp_{2d}, h_d) the Siegel Shimura datum of Example 1.35.

The following condition was introduced in Deligne's paper [37] (Section 2.3) and named in Milne's paper [84] (at the end of Section 3).

DEFINITION 2.18. A Shimura datum (G, h) is of *Hodge type* if there exists an integer $d \geq 1$ and a morphism of Shimura data $u : (G, h) \rightarrow (\text{GSp}_{2d}, h_d)$ such that the underlying morphism of algebraic groups $G \rightarrow \text{GSp}_{2d}$ is injective. In this case, we also say that the corresponding Shimura varieties are of Hodge type.

EXAMPLE 2.19. (1) Every Shimura datum of PEL type is of Hodge type, pretty much by definition (or by Proposition 2.3.2 of [37]): if $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$ is a PEL datum and G is the corresponding group scheme, then G embeds into the group scheme H defined by

$$H(R) = \{g \in \text{End}_R(\Lambda \otimes_{\mathbb{Z}} R) \mid \exists c(g) \in R^\times, \langle g(\cdot), g(\cdot) \rangle = c(g)\langle \cdot, \cdot \rangle\}$$

for every commutative ring R . As the alternating pairing $\langle \cdot, \cdot \rangle$ is nondegenerate by condition (5)(b) of Definition 2.1, and as all nondegenerate alternating pairings on $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ are equivalent, we have $H_{\mathbb{Q}} \simeq \text{GSp}_{2d, \mathbb{Q}}$ for $2d = \dim_{\mathbb{Q}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$. Let h' be the composition of $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ and of the embedding $G_{\mathbb{R}} \rightarrow H_{\mathbb{R}} = \text{GSp}_{2d, \mathbb{R}}$. We have seen in the discussion after Proposition 2.2 that $h : \mathbb{C} \rightarrow \text{End}_{\mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ induces a Hodge structure of type $\{(-1, 0), (0, -1)\}$ on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$,²³ so h' satisfies condition (b) of Definition 1.33. It also satisfies condition (c) of loc. cit. because the \mathbb{R} -bilinear pairing $\langle \cdot, h(i)(\cdot) \rangle$ on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is symmetric definite positive. Finally, for every $a \in \mathbb{R}^\times$, the element $h(a)$ of $\text{End}_{\mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is $a \cdot \text{id}$ (because h is a morphism of \mathbb{R} -algebras), so $h'(a) = aI_{2d} \in \text{GSp}_{2d}(\mathbb{R})$. This implies that h' and h_d are conjugated by $\text{GSp}_{2d}(\mathbb{R})$.

- (2) The list of groups G that have Shimura data of Hodge type is given (at least in theory) in Section 2.3 of [37]. For example, the group G can be of type B , while that is not possible for PEL type Shimura data.

²³Which means that, in the decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ induced by h , we have $V^{p,q} = 0$ unless $(p, q) \in \{(-1, 0), (0, -1)\}$.

The following result is due to Deligne; it follows from Corollaire 5.7 of [35], which was already cited as Proposition 1.48.

PROPOSITION 2.20. *Every Shimura variety of Hodge type admits a canonical model.*

REMARK 2.21. There is a general philosophy that Shimura varieties should be moduli spaces of motives (the conditions that we put on a Shimura datum (G, h) are basically there to force the $G(\mathbb{R})$ -conjugacy class of h to be a parameter space for Hodge structures); see Section 3 of [84] for more precise hopes.

For (G, h) of Hodge type, we are a bit closer to that hope: Milne has proved that $M_K(G, h)(\mathbb{C})$ is a moduli space of abelian varieties with Hodge cycles of a certain type and level structure (see Theorem 3.11 of [84]). As Hodge cycles on complex abelian varieties are absolute by a theorem of Deligne (see Theorem 2.11 of Chapter I of [38]), we can also see the action of $\text{Aut}(\mathbb{C}/F)$ (where $F = F(G, h)$) on this modular interpretation. So far, this has not allowed people to give a moduli interpretation of the integral models of the Shimura variety of (G, h) , but it does help with the construction of integral models, that we now discuss.

Let (G, h) be a Shimura datum of Hodge type, and let $u : G \rightarrow \text{GSp}_{2d}$ be an injective morphism inducing a morphism of Shimura data $(G, h) \rightarrow (\text{GSp}_{2d}, h_d)$. Let $F = F(G, h)$ be the reflex field of (G, h) . We fix a prime number p such that $G_{\mathbb{Q}_p}$ extends to a reductive group scheme \mathcal{G} over \mathbb{Z}_p . To simplify the presentation, we will assume that the embedding $G_{\mathbb{Q}_p} \rightarrow \text{GSp}_{2d, \mathbb{Q}_p}$ extends to an embedding $\mathcal{G} \rightarrow \text{GSp}_{2d, \mathbb{Z}_p}$, though that is not necessary. We set $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $K'_p = \text{GSp}_{2d}(\mathbb{Z}_p)$.

For each sufficiently small open compact subgroup K^p of $G(\mathbb{A}_f^p)$, we fix an open compact subgroup K'^p of $\text{GSp}_{2d}(\mathbb{A}_f^p)$ such that, setting $K = K_p K^p$ and $K' = K'_p K'^p$, the morphism u defines a closed immersion $M_K(G, h) \rightarrow M_{K'}(\text{GSp}_{2d}, h_d)_F$ (this is possible by Proposition 1.15 of [35]).

Let $\mathcal{M}_{K'}$ be the model of $M_{K'}(\text{GSp}_{2d}, h_d)$ over $\mathcal{O}_{F, (p)}$ given by base change from its canonical integral model over $\mathbb{Z}_{(p)}$. We denote by \mathcal{M}_K the normalization of the closure of the image of $M_K(G, h)$ in $\mathcal{M}_{K'} \supset M_{K'}(\text{GSp}_{2d}, h_d)_F$. Kisin proved the following result.

THEOREM 2.22 (Theorem 2.3.8 of [54]). *Suppose that $p > 2$. Then $\varprojlim_{K^p} \mathcal{M}_{K_p K^p}$ is a canonical integral model of the Shimura variety of (G, h) .*

In particular, the schemes \mathcal{M}_K do not depend on the choice of K'^p or on the embedding $G \rightarrow \text{GSp}_{2d}$.

REMARK 2.23. In fact, Theorem 2.3.8 of [54] is more general, and gives a construction of a canonical integral model without the assumption on the embedding $G_{\mathbb{Q}_p} \rightarrow \text{GSp}_{2d, \mathbb{Q}_p}$. It even allows the case $p = 2$ under some conditions.

REMARK 2.24. We have given the original construction of Kisin, but Yujie Xu recently proved in [123] that the normalization step is unnecessary, so \mathcal{M}_K is just a closed subscheme of $\mathcal{M}_{K'}$.

2.2.2. Abelian type Shimura varieties.

DEFINITION 2.25. Let (G, h) be a Shimura datum. We say that (G, h) is of *abelian type* if there exists a Shimura datum of Hodge type (G_1, h_1) and a central isogeny $G_{1, \text{der}} \rightarrow G_{\text{der}}$ that induces an isomorphism of Shimura data $(G_{1, \text{ad}}, h_{1, \text{ad}}) \xrightarrow{\sim}$

$(G_{\text{ad}}, h_{\text{ad}})$, where $h_{1,\text{ad}}$ (resp. h_{ad}) is the composition of h_1 (resp. h) and of the quotient morphism $G_1 \rightarrow G_{1,\text{ad}}$ (resp. $G \rightarrow G_{\text{ad}}$). In this case, we also say that the corresponding Shimura varieties are of abelian type.

Another way to formulate the definition is to say that a Shimura variety $M_K(G, h)(\mathbb{C})$ is of abelian type if all its connected components are finite quotients of connected components of Shimura varieties of Hodge type (see Section 9 of [37]). Deligne has classified all connected Shimura varieties of abelian type in 2.3 of [37]. The very rough upshot is that all Shimura data (G, h) with G of type A , B and C are of abelian type; if G is of type D , it's complicated, and if G is of type E_6 or E_7 , then the Shimura datum is never of abelian type. See page 61 of [68] for more details.

THEOREM 2.26 (Deligne, see Corollaire 2.7.21 of [37]). *Let (G, h) be a Shimura datum of abelian type. Then the Shimura variety of (G, h) admits a canonical model.*

In fact, Deligne reduces the construction of a canonical model of $M_K(G, h)(\mathbb{C})$ to that of canonical models of its connected components (over finite extensions of $F(G, h)$). See for example Corollaire 2.7.18 of [37].

REMARK 2.27. Shimura varieties of abelian type are *not* moduli spaces of abelian varieties in general. However, Milne proved in [86] that they are moduli spaces of motives if $h \circ w : \text{GL}_{1,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is defined over \mathbb{Q} , and this leads to a more direct proof of existence of their canonical models even without that condition.

THEOREM 2.28 (Milne, see Theorem 3.31 of [86]; see also Brylinski's paper [26]). *Let (G, h) be a Shimura datum of abelian type such that $h \circ w : \text{GL}_{1,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is defined over \mathbb{Q} . Then each $M_K(G, h)$ is a moduli space of abelian motives (over the reflex field of (G, h)).*

By reducing to the case of Shimura varieties of Hodge type, Kisin was able to prove the existence of canonical integral models of Shimura varieties of abelian type for $p > 2$. The case $p = 2$ was then settled by Kim and Madapusi.

THEOREM 2.29 (Corollary 3.4.14 of [54] and Theorem 1 of [53]). *Let (G, h) be a Shimura datum of abelian type, let p be a prime number such that $G_{\mathbb{Q}_p}$ extends to a reductive group scheme \mathcal{G} over \mathbb{Z}_p , and let $K_p = G(\mathbb{Z}_p)$. Then $M_{K_p} := \varprojlim_{K^p \subset G(\mathbb{A}_F^p)} M_{K_p K^p}(G, h)$ admits a canonical integral model over $\mathcal{O}_{F,(p)}$, where $F = F(G, h)$.*

2.3. General Shimura varieties. Let (G, h) be a Shimura datum that is not of abelian type. Then we know very little, but we do know that canonical models exist. This result was first announced by Milne (based on earlier results of Kazhdan and Borovoi), but there was a gap in the proof, which was fixed by Moonen.

THEOREM 2.30 (Deligne, Borovoi, Milne-Shih, Milne, Moonen, cf. [90] Section 2.). *Let (G, h) be a Shimura datum. Then the Shimura variety of (G, h) admits a canonical model.*

In fact, Section 2 of [90] contains a good summary of the different construction methods of canonical models. The proof in the general case does not proceed by reduction to the case of Siegel modular varieties (unlike the previous proofs in the abelian type case), but uses results of Borovoi (see [22] and [23]), Deligne, Milne-Shih (Chapter V of the book [38]) and Milne ([83]) on a conjecture of Langlands,

that says that a conjugate of a Shimura variety over \mathbb{C} by an automorphism of \mathbb{C} is still a Shimura variety (Langlands's conjecture is much more precise than this, see for example Theorem 2.14 of [90]).

2.4. Kottwitz's simple Shimura varieties. After considering more and more complicated Shimura varieties in the previous subsections, we will now introduce a very simple PEL family, that has been studied by Kottwitz in [63]; their PEL datum already appeared in Example 2.6. These Shimura varieties are simple for several reasons:

- they are compact;
- they are PEL of type A , hence moduli spaces of abelian schemes with extra structures;
- their reductive group “has no endoscopy” (see Proposition 2.40 for a precise statement).

2.4.1. *Inner forms of unitary groups.* We fix a totally real extension F_0 of \mathbb{Q} and a totally imaginary quadratic extension F of F_0 . Such an extension F of \mathbb{Q} is called a *CM extension*. We denote by $z \mapsto \bar{z}$ the nontrivial element of $\text{Gal}(F/F_0)$.

Let n be a positive integer. The quasi-split unitary group $U^*(n)$ over F_0 is defined to be the unitary group of the Hermitian F_0 -space F^n , with the form

$$((x_1, \dots, x_n), (y_1, \dots, y_n)) \mapsto \sum_{i=1}^n x_i \overline{y_{n+1-i}}.$$

In other words, for every commutative F_0 -algebra R , we have

$$U^*(n) = \{g \in \text{GL}_n(F \otimes_{F_0} R) \mid {}^t \bar{g} J g = J\},$$

where J is the $n \times n$ antidiagonal matrix with all nonzero coefficients equal to 1.

We want to describe all inner forms of $U^*(n)$. Remember that, if G and H are algebraic groups over a field k , we say that they are *inner forms* of each other if there exists an isomorphism $\varphi : G_{\bar{k}} \xrightarrow{\sim} H_{\bar{k}}$ such that, for every $\sigma \in \text{Gal}(\bar{k}/k)$, the automorphism $\varphi^{-1} \circ {}^\sigma \varphi$ of $G(\bar{k})$ is inner (${}^\sigma \varphi$ is the isomorphism $\sigma \varphi \sigma^{-1} : G_{\bar{k}} \xrightarrow{\sim} H_{\bar{k}}$, where σ acts on $G_{\bar{k}}$ and $H_{\bar{k}}$ via its action on \bar{k}). Inner forms of G are in bijection with elements of $H^1(k, G_{\text{ad}}) := H^1(\text{Gal}(\bar{k}/k), G_{\text{ad}})$, where $G_{\text{ad}} = G/Z(G)$ and $Z(G)$ is the center of G . See Section III.1 of Serre's book [110] and Section 3 of Springer's Corvallis notes [113] for more about (inner) forms.

Here, observing that $U^*(n)_{\text{ad}}$ is the group of automorphisms of the couple formed by the central simple algebra $M_n(F)$ over F and the involution $g \mapsto J {}^t \bar{g} J$, we see that inner forms of $U^*(n)$ are all of the form $U(B, *)$, where B is a central simple algebra of dimension n^2 over F and $*$ is an involution on B extending the involution $z \mapsto \bar{z}$ on F , and $U(B, *)$ is the F_0 -group defined by

$$U(B, *) (R) = \{g \in B \otimes_{F_0} R \mid g g^* = 1\},$$

for every commutative F_0 -algebra R . As the notation indicates, the group $U(B, *)$ is a unitary group. More precisely, we can write $B = M_{n/m}(D)$ where m is a divisor of n and D is a division algebra of dimension m^2 over F ; if we choose a simple B -module V , then we can take $D = \text{End}_B(V)^{\text{op}}$ and see V as a D^{op} -module. By Theorem 4.1 of the lecture notes [117] of Tignol, there exists an involution on D^{op} that restricts to $z \mapsto \bar{z}$ on F and a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on the D^{op} -module V , Hermitian with respect to that involution, such that $*$ is equivalent

to the involution \dagger on $B \simeq \text{End}_{D^{\text{op}}}(V)$ corresponding to the Hermitian form $\langle \cdot, \cdot \rangle$, i.e. defined by the condition $\langle bx, y \rangle = \langle x, b^\dagger y \rangle$ for $b \in B$ and $x, y \in V$.

On the other hand, if G is an inner form of $U^*(n)$, then $G_{F_{0,v}}$ is an inner form of $U^*(n)_{F_{0,v}}$ for every place v of F_0 , and $G_{F_{0,v}}$ is isomorphic to $U^*(n)_{F_{0,v}}$ itself for all but finitely many v . Inner forms over local fields are easier to classify (because the absolute Galois groups of local fields are simpler); but then we must be able to decide when a family of inner forms of the $U^*(n)_{F_{0,v}}$ comes from a “global” inner form of $U^*(n)$ (defined over F_0). Using Galois cohomology calculations and the calculation of the Brauer groups of local and global fields, we get the following two propositions (see for example Section 2 of Clozel’s paper [30], and see Gille and Szamuely’s book [43] for more information on central simple algebras and Brauer groups):

PROPOSITION 2.31. *Let v be a place of F_0 .*

- (i) *Suppose that v is finite and does not split in F . If n is odd, then the only inner form of $U^*(n)_{F_{0,v}}$ is $U^*(n)_{F_{0,v}}$ itself (up to isomorphism). If n is even, then there are two isomorphism classes of inner forms of $U^*(n)_{F_{0,v}}$.*
- (ii) *Suppose that v splits in F (in particular, v is finite), and let w be a place of F above v . Then $U^*(n)_{F_{0,v}} \simeq \text{GL}_{n, F_{0,v}} \simeq \text{GL}_{n, F_w}$, and its inner forms are (up to isomorphism) the groups $\text{GL}_m(D)$, for m dividing n and D a central division algebra over F_w of dimension $(n/m)^2$.*
- (iii) *Suppose that v is infinite, hence a real place of F_0 . Then the inner forms of $U^*(n)_{F_{0,v}}$ are (up to isomorphism) the real unitary groups $U_{p,q}$ of signature (p, q) , for $p + q = n$, and we have $U_{p,q} \simeq U_{r,s}$ if and only if $(r, s) = (p, q)$ or $(r, s) = (q, p)$.*

Suppose that n is even, let v be a place of F_0 and let G be an inner form of $U^*(n)_{F_{0,v}}$.

- If v is finite and does not split in F , set $\epsilon(G) = 1$ if $G \simeq U^*(n)_{F_{0,v}}$ and $\epsilon(G) = -1$ otherwise;
- If v is finite and splits in F , set $\epsilon(G) = (-1)^m$ if $G \simeq \text{GL}_m(D)$ with m dividing n and D a central division algebra over F_w of dimension $(n/m)^2$ (note that m only depends on v , not on w);
- If v is infinite, set $\epsilon(G) = (-1)^{n/2-p}$ if $G \simeq U_{p,q}$.

PROPOSITION 2.32. *For every place v of F_0 , let G_v be an inner form of $U^*(n)_{F_{0,v}}$. Suppose that $G_v \simeq U^*(n)_{F_{0,v}}$ for all but finitely many v .*

- (i) *If n is odd, there exists an inner form G of $U^*(n)$ such that $G_{F_{0,v}} \simeq G_v$ for every v .*
- (ii) *If n is even, there exists an inner form G of $U^*(n)$ such that $G_{F_{0,v}} \simeq G_v$ for every v if and only if $\prod_v \epsilon(G_v) = 1$.*

REMARK 2.33. Let G be an inner form of $U^*(n)$. We know that $G \simeq U(B, *)$, with B a central simple algebra over F and $*$ an involution on B extending the involution $z \mapsto \bar{z}$ on F . In the next subsection, it will be of interest to us to know when B is a division algebra. Let v be a place of F_0 , and let $G_v = G_{F_{0,v}}$.

- If v does not split in F , let w be a place of F above v . By the discussion at the beginning of 2.4.1 (i.e. Theorem 4.1 of [117], which applies to any field), there exists a division algebra D_v over F_w with an involution extending the nontrivial element of $\text{Gal}(F_w/F_{0,v})$ and a Hermitian module

$(V, \langle \cdot, \cdot \rangle)$ over D_v such that G_v is the unitary group of $(V, \langle \cdot, \cdot \rangle)$. As F_w is a local field, we must have $D_v = F_w$, and then $V \simeq F_w^n$ and the Hermitian form $\langle \cdot, \cdot \rangle$ is characterized by its determinant (up to isomorphism).

- If v splits in F , then we have $G_v \simeq M_{m_v}(D_v)$, with m_v dividing n and D_v a central division algebra of dimension $(n/m_v)^2$ over $F_{0,v}$.

We can now deduce from the classification of central simple algebras over F that B is a division algebra if the gcd of the family (m_v) is equal to 1, because it forces the order of the class of B in the Brauer group of \mathbb{Q} to be equal to n . (See Corollary 6.5.4 of [43] for the classification of central simple algebras over a number field.) The simplest way to make sure that this condition is satisfied is to take one of the m_v equal to 1, for example to take G_v of the form D_v^\times for a place v of F_0 split in F , where D_v is a central division algebra over $F_{0,v}$.

2.4.2. Simple Shimura varieties.

DEFINITION 2.34 (See §1 of [63].). A *Kottwitz simple Shimura variety* is a Shimura variety defined by the Shimura datum $(G_{\mathbb{Q}}, h)$ associated to a PEL datum $(\mathcal{O}, *, \Lambda, \langle \cdot, \cdot \rangle, h)$ as in Example 2.6.

In particular:

- (a) $D := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra with center a CM extension F of \mathbb{Q} ;
- (b) $*$ extends the nontrivial automorphism of F/F_0 , where F_0 is the maximal totally real subextension of F ;
- (c) $\Lambda = \mathcal{O}$;
- (d) $\langle x, y \rangle = \text{Tr}_{\mathcal{O}/\mathbb{Z}}(xby^*)$ for all $x, y \in \Lambda$, where $b \in D^\times$ is such that $b^* = -b$.

We have $\text{End}_{\mathcal{O}}(\Lambda) = \mathcal{O}^{\text{op}}$ (acting by right multiplication), so, for every commutative ring R ,

$$G(R) = \{g \in (\mathcal{O}^{\text{op}} \otimes_{\mathbb{Z}} R)^\times \mid gg^* \in R^\times\}.$$

We have a morphism $c : G \rightarrow \text{GL}_1$ sending $g \in G(R)$ to $gg^* \in \text{GL}_1(R)$, and, if we denote its kernel by G_0 , then $G_{0,\mathbb{R}}$ is a product of unitary groups of the form $\text{U}(p, q)$. We gave the formula for h in Example 2.6: if $z \in \mathbb{C}$ has absolute value 1, then the projection of $h(z)$ to the $\text{U}(p, q)$ factor of $G_0(\mathbb{R})$ is $\begin{pmatrix} zI_p & 0 \\ 0 & \bar{z}I_q \end{pmatrix}$. (There is a choice here, as the couple (p, q) is only determined up to order.)

Note that we can choose the signatures of $G_{0,\mathbb{R}}$ arbitrarily, by manipulating what happens at finite places. On the other hand, the Shimura varieties of $(G_{\mathbb{Q}}, h)$ are always compact, because of the following lemma. Here we use the fact that the Shimura varieties defined by a Shimura datum (G, h) are compact if and only if G_{der} has \mathbb{Q} -rank 0, which was recalled in 1.1 (see for example [51, Theorem 5.10]).

LEMMA 2.35. *The group G_{der} is of \mathbb{Q} -rank 0.*

PROOF. Let $N : D^\times \rightarrow F^\times$ be the reduced norm, seen as a morphism of algebraic groups over F . We have $G_{\text{der}} = \text{Res}_{F_0/\mathbb{Q}} \text{SU}$, where $\text{SU} = \{g \in \text{U} \mid N(g) = 1\}$,²⁴ so $G_{\text{der},F} \simeq (\text{Ker } N)^{[F_0:\mathbb{Q}]}$. Let T be a maximal torus of G_{der} . Then T_F is a maximal torus of $(\text{Ker } N)^{[F_0:\mathbb{Q}]}$, hence a product of maximal tori of $\text{Ker } N$. Let T' be a maximal torus of $\text{Ker } N$. As D is a division algebra, there exists a degree n extension F' of F such that $T' = \{x \in F'^\times \mid N_{F'/F}(x) = 1\}$, and so the

²⁴We can do the calculation after base changing to \mathbb{C} , where it reduces to the fact that $\text{GL}_{n,\text{der}} = \text{SL}_n$.

maximal split subtorus of T' is trivial. This implies that the maximal split subtorus of T is trivial. \square

We now discuss endoscopy.

2.4.3. *Endoscopy.* In its simplest form, endoscopy is the following phenomenon: let G be an algebraic group over a field k . Then the $G(\bar{k})$ -conjugacy classes in $G(k)$ can be larger than the $G(k)$ -conjugacy classes. Some vocabulary: $G(\bar{k})$ -conjugacy classes in $G(k)$ are often called *stable conjugacy classes*, and $G(\bar{k})$ -conjugate elements are called *stably conjugate*. Actually we are cheating here, and our definition is only correct if the semisimple elements of G have connected centralizers; the correct definition is in Section 3 of Kottwitz's paper [58]. Note that semisimple elements of G have connected centralizers if G_{der} is simply connected: this is a result of Steinberg (see Corollary 8.5 of [115]) for G semisimple, and Kottwitz explains in Section 3 of [58] how to extend it to the case of a reductive group.

- EXAMPLES 2.36. (1) Suppose that $G = \text{GL}_n$. It is then a classical exercise that any two elements of $G(k)$ that are $G(\bar{k})$ -conjugate are actually $G(k)$ -conjugate. We say that GL_n has no endoscopy.
- (2) We can generalize (1) to inner forms of GL_n , i.e. algebraic groups of the form B^\times , where B is a central simple algebra over k .
- (3) Take $k = \mathbb{R}$ and $G = \text{SL}_2$. As $\text{GL}_2(\mathbb{C}) = \mathbb{C}^\times \cdot \text{SL}_2(\mathbb{C})$, elements of $G(\mathbb{R})$ are $G(\mathbb{C})$ -conjugate if and only if they are $\text{GL}_2(\mathbb{C})$ -conjugate. For example, the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are in the same $\text{SL}_2(\mathbb{C})$ -conjugacy class, but we can check by a direct calculation that they are not in the same $\text{SL}_2(\mathbb{R})$ -conjugacy class. So SL_2 has endoscopy, and we can generalize that example to SL_n for $n \geq 2$.
- (4) If $k = \mathbb{R}$ and $G = \text{U}_n := \{g \in \text{GL}_n(\mathbb{C}) \mid g^*g = I_n\}$, then again it is a classical exercise to check that $\text{U}_n(\mathbb{C})$ -conjugacy classes in $\text{U}_n(\mathbb{R})$ coincide with $\text{U}_n(\mathbb{R})$ -conjugacy classes.

REMARK 2.37. We care about stable conjugacy classes because, in the Langlands philosophy, groups are related via their L -groups (see 3.2.1 and 3.3.2). As inner forms have the same L -groups, this means that we should be able to move information between inner forms; but we cannot compare conjugacy classes in two inner forms, only stable conjugacy classes. More generally, if G and H are algebraic groups over k and there is a morphism ${}^L H \rightarrow {}^L G$ between their L -groups, then we can use this to transport stable conjugacy classes of regular semisimple elements from H to G .

It quickly becomes tiring to calculate conjugacy classes by hand, so we need more efficient methods to check for endoscopy. Let $\gamma \in G(k)$, and let $G_\gamma \subset G$ be the centralizer of γ .

Let $\delta \in G(k)$ be stably conjugate to γ , and let $g \in G(\bar{k})$ such that $\delta = g\gamma g^{-1}$. For every $\sigma \in \text{Gal}(\bar{k}/k)$, we have

$$g\gamma g^{-1} = \delta = \sigma(\delta) = \sigma(g)\sigma(\gamma)\sigma(g)^{-1} = \sigma(g)\gamma\sigma(g)^{-1},$$

hence $g^{-1}\sigma(g) \in G_\gamma(\bar{k})$. So we get a 1-cocycle $c : \text{Gal}(\bar{k}/k) \rightarrow G_\gamma(\bar{k})$, $\sigma \mapsto g^{-1}\sigma(g)$, and we can check that the image $\text{inv}(\gamma, \delta)$ of this 1-cocycle in $H^1(k, G_\gamma)$ does not depend on the choice of g . Moreover, the image of $\text{inv}(\gamma, \delta)$ in $H^1(k, G)$ is trivial.

PROPOSITION 2.38. *(See Section 3 of [58].) Suppose that γ is semisimple and that G_γ is connected. Then the map $\delta \mapsto \text{inv}(\gamma, \delta)$ gives a bijection from the set of stable conjugacy classes in the $G(k)$ -conjugacy class of γ to $\text{Ker}(H^1(k, G_\gamma) \rightarrow H^1(k, G))$.*

This is particularly useful when G_{der} is simply connected and γ is regular and semisimple, as the centralizer of γ is then a maximal torus (because it is connected, as explained before Example 2.36).

EXAMPLE 2.39. (1)²⁵ Let $p \geq q \geq 0$ be integers. Take $k = \mathbb{R}$ and $G = U_{p,q}$, the unitary group of the Hermitian form on \mathbb{C}^{p+q} with matrix $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, where I_r denotes as usual the identity matrix in $\text{GL}_r(\mathbb{Z})$. One can show that any maximal torus T of G is isomorphic to $\mathbb{S}^r \times U_1^s$, with $2r + s = p + q$. (Remember that $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \text{GL}_{1,\mathbb{C}}$, i.e. it is \mathbb{C}^\times regarded as an algebraic group over \mathbb{R} .) By Shapiro's lemma, we have

$$H^1(\mathbb{R}, \mathbb{S}) = H^1(\mathbb{C}, \text{GL}_{1,\mathbb{C}}) = 0.$$

Using the exact sequence

$$1 \rightarrow U_1 \rightarrow \mathbb{S} \xrightarrow{\text{Nm}} \text{GL}_{1,\mathbb{R}} \rightarrow 1$$

where $\text{Nm} : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ is the map $z \mapsto z\bar{z}$, we see that $H^1(\mathbb{R}, U_1) = \mathbb{R}^\times / \text{Nm}(\mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$. So

$$H^1(\mathbb{R}, T) = (\mathbb{Z}/2\mathbb{Z})^s.$$

Let T_0 be the group of diagonal matrices in G . Then T_0 is a maximal torus of G and $T_0 \simeq U(1)^{p+q}$, so we get that $H^1(\mathbb{R}, T_0) = (\mathbb{Z}/2\mathbb{Z})^{p+q}$. On the other hand, by the main result (Theorem 3.1) of Borovoi's paper [24], we have

$$H^1(\mathbb{R}, G) = H^1(\mathbb{R}, T_0)/W_0(\mathbb{R}),$$

and W_0 is the Weyl group scheme $W_{T_0} = N_G(T_0)/Z_G(T_0)$. It is also proved in Section 3 of the same paper that $W_0(\mathbb{R}) = W_0(\mathbb{C})$, so $W_0(\mathbb{R})$ is isomorphic to the symmetric group \mathfrak{S}_{p+q} . However, as permutation matrices are not always inside $U_{p,q}$, the group \mathfrak{S}_{p+q} does not act on T_0 by permuting the entries.

We calculate the quotient $H^1(\mathbb{R}, T_0)/\mathfrak{S}_{p+q}$. Note that $H^1(\mathbb{R}, T_0) \simeq T_0(\mathbb{R})_2$, where $T_0(\mathbb{R})_2 = \{\text{diag}(z_1, \dots, z_{p+q}) \in T_0(\mathbb{R}) \mid \forall i, z_i \in \{\pm 1\}\}$, so we need to understand the action of \mathfrak{S}_{p+q} on $T_0(\mathbb{R})_2$. We have $T_0(\mathbb{R})_2 = \{c_\Xi, \Xi \subset \{1, 2, \dots, p+q\}\}$, where, for every subset Ξ of $\{1, 2, \dots, p+q\}$, we write c_Ξ for the matrix $\text{diag}(z_1, \dots, z_{p+q})$ such that $z_i = -1$ for $i \in \Xi$ and $z_i = 1$ for $i \notin \Xi$. We regard $\mathfrak{S}_p \times \mathfrak{S}_q$ as a subgroup of \mathfrak{S}_{p+q} by identifying \mathfrak{S}_q to the permutation group on $\{p+1, p+2, \dots, p+q\}$. Then \mathfrak{S}_{p+q} is generated by $\mathfrak{S}_p \times \mathfrak{S}_q$ and by the transposition $\sigma := (1, p+1)$. Also, elements of $\mathfrak{S}_p \times \mathfrak{S}_q$ act on $T_0(\mathbb{R})_2$ by permuting the coordinates,

²⁵I am very grateful to Mikhail Borovoi for patiently explaining how to deduce this example from Example 4.4 of his paper [24]. Any remaining mistakes are all mine.

because the corresponding permutation matrices are in $U_{p,q}$. So two elements $c_\Xi, c_{\Xi'}$ of $T_0(\mathbb{R})_2$ are in the same $\mathfrak{S}_p \times \mathfrak{S}_q$ -orbit if and only if $\text{card}(\Xi \cap \{1, \dots, p\}) = \text{card}(\Xi' \cap \{1, \dots, p\})$ and $\text{card}(\Xi \cap \{p+1, \dots, p+q\}) = \text{card}(\Xi' \cap \{p+1, \dots, p+q\})$.

It remains to understand the action of σ . It will be represented by an element of $N_{T_0}(\mathbb{C})$, so we need a convenient representation of $G(\mathbb{C})$. For this we consider the \mathbb{R} -algebra embedding $\mathcal{M} : \mathbb{C} \rightarrow M_2(\mathbb{R})$, $a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Applying this to every matrix entry, we get an embedding of real algebraic groups $G \subset \text{GL}_{2(p+q), \mathbb{R}}$, hence also $G_{\mathbb{C}} \subset \text{GL}_{2(p+q), \mathbb{C}}$.

Suppose that $(p, q) = (1, 1)$. Then the matrix $n = i \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \in \text{GL}_4(\mathbb{C})$ is in $N_{T_0}(\mathbb{C}) \subset G(\mathbb{C})$ and represents σ , so the action of σ on $T_0(\mathbb{R})_2$ is given by $c \mapsto n^{-1}c\bar{n}$. It is then easy to see that σ fixes $c_{\{1\}}$ and $c_{\{2\}}$, and exchanges $c_{\emptyset} = 1$ and $c_{\{1,2\}} = -1$. So $H^1(\mathbb{R}, G)$ has three elements.

The general case is similar: we write an explicit representative $n \in N_{T_0}(\mathbb{C}) \subset \text{GL}_{2(p+q)}(\mathbb{C})$ of σ by putting the I_2 blocks in the correct positions. For every $\Xi \subset \{1, 2, \dots, p+q\}$, write $p_\Xi = \text{card}(\Xi \cap \{1, \dots, p\})$ and $q_\Xi = \text{card}(\Xi \cap \{p+1, \dots, p+q\})$. By a calculation similar to that of the previous paragraph, we get that c_Ξ and $c_{\Xi'}$ are in the same \mathfrak{S}_{p+q} -orbit if and only if Ξ, Ξ' satisfy the following three conditions:

$$\begin{cases} p_\Xi = p_{\Xi'} \\ q_\Xi = q_{\Xi'} \\ p_\Xi - q_\Xi = p_{\Xi'} - q_{\Xi'}. \end{cases}$$

So a set of representations for the \mathfrak{S}_{p+q} -orbits in $T_0(\mathbb{R})_2$ is given by $\{c_{\emptyset}\} \cup \{c_r^1, 1 \leq r \leq p\} \cup \{c_s^2, 1 \leq s \leq q\}$, where $c_r^1 = c_{\{1,2,\dots,r\}}$ and $c_s^2 = c_{\{p+1,p+2,\dots,p+s\}}$.

We deduce that $H^1(\mathbb{R}, G)$ is of cardinality $p+q+1$, and that

$$\text{Ker}(H^1(\mathbb{R}, T_0) \rightarrow H^1(\mathbb{R}, G)) = \{c_\Xi \mid p_\Xi = q_\Xi\}.$$

Alternatively, to calculate $H^1(\mathbb{R}, G)$, one can use the fact that $H^1(\mathbb{R}, G)$ for $G = U_{p,q}$ classifies isomorphism classes of non-degenerate Hermitian forms on \mathbb{C}^{p+q} (see Serre [109], Section X.2, Proposition 4), and one can classify the isomorphism classes of such forms using the Hermitian version of Sylvester's law of inertia.

- (2) We take k to be a finite extension of \mathbb{Q}_p , E to be an unramified quadratic extension of k , and $G = \text{U}^*(n)_{E/k}$ to be the quasi-split unitary group defined by that extension, i.e. the unitary group of the Hermitian k -space E^n , with the form

$$((x_1, \dots, x_n), (y_1, \dots, y_n)) \mapsto \sum_{i=1}^n x_i \overline{y_{n+1-i}},$$

where $x \mapsto \bar{x}$ is the nontrivial element of $\text{Gal}(E/k)$. Any maximal torus T of G is isomorphic to $(\text{Res}_{E/k} \text{GL}_1)^r \times \text{U}(1)_{E/k}^s$, where $\text{U}(1)_{E/k}$ is the subgroup of norm one elements in $\text{Res}_{E/k} \text{GL}_1$ and $2r+s = n$. As before, we have $H^1(k, \text{Res}_{E/k} \text{GL}_1) = 0$ and $H^1(k, \text{U}(1)_{E/k}) = k^\times / N_{E/k}(E^\times) = \mathbb{Z}/2\mathbb{Z}$.

But now we have $H^1(k, G) = \mathbb{Z}/2\mathbb{Z}$, and the map $H^1(k, T) \rightarrow H^1(k, G)$ is the sum map. Indeed, the derived group $G_{\text{der}} = \text{SU}^*(n)_{E/k}$ of G is simply connected, so $H^1(k, G_{\text{der}}) = 0$ by a theorem of Kneser (see [57]), and then we use the fact that $G = G_{\text{der}} \rtimes \text{U}(1)_{E/k}$.

So we see that, for some maximal tori T , the set $\text{Ker}(H^1(k, T) \rightarrow H^1(k, G))$ is not a singleton, hence there are stable conjugacy classes containing more than one conjugacy class. As in (1), the more anisotropic factors the torus T has, the bigger its H^1 .

The examples show that, if G is the general unitary group of a Kottwitz simple Shimura variety as in Definition 2.34, then it is not reasonable to expect that G will behave as GL_n and have absolutely no endoscopy over any field. In fact, what Kottwitz actually proved about these groups is the following:

PROPOSITION 2.40 (Lemma 2 of [63] and Theorem 6.6 of [61]). *Let $\overline{\mathbb{A}}$ be the restricted product of the $\overline{\mathbb{Q}}_v$, for v a place of \mathbb{Q} . Let γ be a semisimple element of $G(\mathbb{Q})$, and let δ be an element of $G(\mathbb{A})$ that is $G(\overline{\mathbb{A}})$ -conjugate to γ . Then there exists an element of $G(\mathbb{Q})$ that is $G(\mathbb{A})$ -conjugate to δ .²⁶*

This implies that in the trace formula for G (see 3.4.4), we will be able to group the orbital integrals on the geometric side by stable conjugacy class and obtain an expression that is easier to transfer between groups. See Section 4 of [63].

The proof of Proposition 2.40 is a more complicated global version of the Galois cohomology calculations of Example 2.39. First Kottwitz reduces to the case of $G_{\text{der}} = \text{Res}_{F_0/\mathbb{Q}} \text{U}$, where U is the unitary group (over F_0) given by $\text{U}(R) = \{g \in (D \otimes_{F_0} R)^\times \mid gg^* = 1\}$. Then he reduces to a similar result for U , now involving F_0 -rational points and the ring of adèles of F_0 . The main point is that, if T is a maximal torus of U , then T_F is a maximal torus of D^\times , and maximal tori of D^\times are all of the form $\text{Res}_{K/F} \text{GL}_1$, for K a degree n extension of F . This allows us to control the Galois cohomology of T .

3. Lecture 3: the cohomology of Shimura varieties

In this lecture, we discuss techniques to calculate the cohomology of Shimura varieties, concentrating on the compact non-endoscopic case (see 3.5 for a short survey of the general case).

We fix a connected reductive group G over \mathbb{Q} ; let $Z(G)$ be the center of G , and let S_G be the maximal \mathbb{Q} -split torus in $Z(G)$. Except in Subsection 3.5, we will often assume that G_{der} is of \mathbb{Q} -rank 0, which means that G has no proper parabolic subgroup.

3.1. Matsushima's formula.

3.1.1. *Discrete automorphic representations.* Let $A_G = S_G(\mathbb{R})^0$. If G_{der} is of \mathbb{Q} -rank 0, then the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_G$ is compact; this follows from the theorem on page 461 of the paper [91] by Mostow and Tamagawa and from the fact that, with the notation of that paper, we have $G(\mathbb{A}) = A_G \cdot G(\mathbb{A})^1$ (which is easy to prove). Without this assumption, the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_G$ is still of finite volume.²⁷ We denote by L_G^2 the space of complex L^2 functions on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_G$.

²⁶Note however that this new element of $G(\mathbb{Q})$ may not be $G(\mathbb{Q})$ -conjugate to γ .

²⁷This is due to Borel in this generality once we know about the relation to locally symmetric spaces (cf 1.1); see [16] 8.4 and 13.2.

The group $G(\mathbb{A})$ acts on L_G^2 by right translation on the argument, and this defines a continuous unitary representation of $G(\mathbb{A})$.

If $G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_G$ is compact, the representation L_G^2 decomposes as a Hilbertian sum of irreducible representations π of $G(\mathbb{A})$, with finite multiplicities $m(\pi)$.²⁸ This is not true in general, but we can still consider the part $L_{G,\text{disc}}^2$ of L_G^2 that decomposes discretely, and we still denote by $m(\pi)$ the multiplicity of an irreducible representation π of $G(\mathbb{A})$ in $L_{G,\text{disc}}^2$.

We denote by Π_G the set of equivalence classes of irreducible representations π of $G(\mathbb{A})$ such that $m(\pi) \neq 0$. Elements of Π_G are called *discrete automorphic representations* of $G(\mathbb{A})$ (or just of G). If G_{der} is of rank 0, then the concepts of automorphic representations, discrete automorphic representations and cuspidal automorphic representations coincide.

If $\pi \in \Pi_G$, then, as $G(\mathbb{A}) = G(\mathbb{A}_f) \times G(\mathbb{R})$, we can write $\pi = \pi_f \otimes \pi_\infty$, where π_f (resp. π_∞) is an irreducible representation of $G(\mathbb{A}_f)$ (resp. $G(\mathbb{R})$).

3.1.2. The theorem. To state Matsushima's formula, we first need some definitions. Let \mathfrak{g} be the Lie algebra of $G_{\mathbb{R}}$, let K'_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and set $K_\infty = A_G \cdot K'_\infty$. If $\pi \in \Pi_G$, then we denote by $H^*(\mathfrak{g}, K_\infty; \pi_\infty)$ the (\mathfrak{g}, K_∞) -cohomology of π_∞ , i.e. the cohomology of the complex $C^q(\mathfrak{g}, K_\infty; \pi_\infty) = \text{Hom}_{K_\infty}(\wedge^q(\mathfrak{g}/\mathfrak{k}), (\pi_\infty)^\infty)$, where $\mathfrak{k} = \text{Lie}(K_\infty)$ and $(\pi_\infty)^\infty$ is the space of smooth vectors in π_∞ (which is stable by K_∞ because π_∞ is K'_∞ -finite and A_G acts trivially).²⁹

The following theorem for connected components of complex Shimura varieties is Corollary VII.3.4 of [21], and the adelic reformulation can be found in Section 2 of Arthur's paper [6].

THEOREM 3.1 (Matsushima's formula). *Let (G, h) be a Shimura datum with G_{der} of \mathbb{Q} -rank 0. Then we have a $G(\mathbb{A}_f)$ -equivariant isomorphism of graded \mathbb{C} -vector spaces*

$$\varinjlim_K H^*(M_K(G, h)(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{\pi \in \Pi_G} \pi_f \otimes H^*(\mathfrak{g}, K_\infty; \pi_\infty)^{m(\pi)},$$

where $G(\mathbb{A}_f)$ acts on the factors π_f on the right hand side.

Here $H^*(M_K(G, h)(\mathbb{C}), \mathbb{C})$ is Betti cohomology with coefficients in \mathbb{C} .

The theorem is equivalent to the following corollary: Let K be an open compact subgroup of $G(\mathbb{A}_f)$, and remember from page 2 of Lecture 1 that the *Hecke algebra* $\mathcal{H}_{G,K}$ at level K is the space of bi- K -invariant functions from $G(\mathbb{A}_f)$ to \mathbb{Q} with compact support, with the convolution product as multiplication.³⁰ If $\pi \in \Pi_G$, then π_f^K is a finite-dimensional representation of $\mathcal{H}_{G,K} \otimes_{\mathbb{Q}} \mathbb{C}$.

COROLLARY 3.2. *Let (G, h) be as in Theorem 3.1, and let K be an open compact subgroup of $G(\mathbb{A}_f)$. Then we have an isomorphism of graded $\mathcal{H}_{G,K} \otimes_{\mathbb{Q}} \mathbb{C}$ -modules*

$$H^*(M_K(G, h)(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{\pi \in \Pi_G} \pi_f^K \otimes H^*(\mathfrak{g}, K_\infty; \pi_\infty)^{m(\pi)}.$$

²⁸This is because nice enough functions on $G(\mathbb{A})$ act on L_G^2 by compact operators, see for example Proposition 2.2.1 and Corollary 2.2.2 of Booher's notes [15].

²⁹See Section I.5 of Borel and Wallach's book [21] for more about (\mathfrak{g}, K) -cohomology. The main point for us is that this is something that can in theory be calculated.

³⁰Here we fixed any Haar measure on $G(\mathbb{A}_f)$ such that open compact subgroups of $G(\mathbb{A}_f)$ have rational volume.

3.2. Étale cohomology of canonical models: the Kottwitz conjecture.

Let ℓ be a prime number.

If (G, h) is a Shimura datum, then the projective system $(M_K(G, h)(\mathbb{C}))_K$ with its action of $G(\mathbb{A}_f)$ has a model over the reflex field $F = F(G, h)$. So the ℓ -adic étale cohomology $H_{\text{ét}}^*(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$ has commuting actions of $\mathcal{H}_{G, K}$ and $\text{Gal}(\overline{F}/F)$. For every isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, we have comparison isomorphisms

$$H_{\text{ét}}^*(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell) \simeq H^*(M_K(G, h)(\mathbb{C}), \mathbb{C})$$

equivariant for the action of $\mathcal{H}_{G, K}$, because this action comes from the geometric Hecke correspondences (see Subsection 1.1). So, when G_{der} is of \mathbb{Q} -rank 0, Matushima's formula tells us that the action of $\mathcal{H}_{G, K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$ on the cohomology groups $H_{\text{ét}}^i(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$ is semi-simple, that the only representations of $\mathcal{H}_{G, K}$ that appear are the π_f^K for $\pi \in \Pi_G$, and that the corresponding π_f^K -multiplicity space $H_K^i(\pi_f) := \text{Hom}_{\mathcal{H}_{G, K}}(\pi_f^K, H_{\text{ét}}^i(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell))$ is of dimension

$$\sum_{\pi' \in \Pi_G | \pi_f' \simeq \pi_f} m(\pi') \dim H^i(\mathfrak{g}, K_\infty; \pi'_\infty).$$

We would like to calculate the action of $\text{Gal}(\overline{K}/K)$ on $H_K^i(\pi_f)$. Kottwitz has a very precise conjecture describing $H_K^i(\pi_f)$, and we want to state that conjecture in the simplest case. We need some preparation.

3.2.1. The Langlands group of F . The Langlands group \mathcal{L}_F of F is a conjectural group scheme over \mathbb{C} whose irreducible representations on n -dimensional vector spaces should classify the cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$. Remember that we defined discrete automorphic representations of $\text{GL}_n(\mathbb{A}_F) = \text{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{A})$ in 3.1.1; roughly speaking, cuspidal automorphic representations are discrete automorphic representations that don't arise from an automorphic representation of a Levi subgroup of GL_n via parabolic induction. Langlands conjectures that there is a bijection $\pi \mapsto \phi_\pi$ from the set of cuspidal automorphic representations π of $\text{GL}_n(\mathbb{A}_F)$ to the set of equivalence classes of representations $\phi_\pi : \mathcal{L}_F \rightarrow \text{GL}_n(\mathbb{C})$, and that this correspondence is determined by local compatibilities. We call ϕ_π the *Langlands parameter* of π .

More precisely, if π is a discrete automorphic representation of $\text{GL}_n(\mathbb{A}_F)$, then we can write π as a restricted tensor product $\bigotimes'_v \pi_v$ over all places of F , where π_v is an irreducible admissible representation of $\text{GL}_n(F_v)$. On the other hand, for each place v of F , we have the (non conjectural) Langlands groups \mathcal{L}_{F_v} of F_v , with a (conjectural) embedding $\mathcal{L}_{F_v} \subset \mathcal{L}_F$, and the (non conjectural) local Langlands correspondence relates irreducible admissible representations of $\text{GL}_n(F_v)$ and n -dimensional representations of \mathcal{L}_{F_v} ("local Langlands parameters"). The local Langlands correspondence over \mathbb{R} and \mathbb{C} was proved by Langlands in [71] (for all groups, not just GL_n). The local Langlands correspondence over p -adic local fields was proved independently by Harris-Taylor ([49]), Henniart ([50]) and Scholze ([105]). For our purposes it is enough to understand the unramified local Langlands correspondence (over p -adic fields), which is just given by the Satake isomorphism; see Dick Gross's notes [47] for an introduction to the Satake isomorphism. If π is a cuspidal automorphic representation and we write $\pi = \bigotimes'_v \pi_v$, then we expect that, for every place v of F , the restriction $\phi_{\pi|_{\mathcal{L}_{F_v}}}$ corresponds to π_v by the local Langlands correspondence, and that this uniquely determines ϕ_π . In fact, it should

be enough to know ϕ_π at the finite places v such that π_v is unramified (hence the corresponding representation of \mathcal{L}_{F_v} is given by the Satake isomorphism).

It is also expected that \mathcal{L}_F canonically surjects to $\mathcal{G}_{F,\mathbb{C}}$, where \mathcal{G}_F is the motivic Galois group of F , a group scheme over \mathbb{Q} defined as the Tannakian group of the conjectural category of mixed motives over F .³¹ The irreducible representations of \mathcal{L}_F factoring through $\mathcal{G}_{F,\mathbb{C}}$ are supposed to correspond to automorphic representations satisfying a certain condition at the infinite places of F , called *algebraic automorphic representations*. On the other hand, for every prime number ℓ , the étale ℓ -adic realization functor defines a continuous morphism of groups $\mathrm{Gal}(\overline{F}/F) \rightarrow \mathcal{G}_F(\overline{\mathbb{Q}}_\ell)$, that is supposed to be injective (by the conservativity conjecture) and have dense image (by the Tate conjecture).

REMARK 3.3. The philosophy behind the Langlands group of a number field, and its relation to the motivic Galois group, are explained much better in Clozel's paper [29]. The extension of Clozel's ideas to groups other than GL_n was worked out in the paper [27] by Buzzard and Gee.

Now we come back to the case of a connected reductive group G . We need to define the L -group of G ; a good reference for this is Part I of [18] or Section 1 of [60]. Let T be a maximal torus of $G_{\overline{\mathbb{Q}}}$ and B be a Borel subgroup of $G_{\overline{\mathbb{Q}}}$ containing T . The *based root datum* of G is the family $\mathfrak{R}(B, T) := (X^*, \Phi, \Delta, X_*, \Phi^\vee, \Delta^\vee)$, where $X^* = X^*(T_{\overline{\mathbb{Q}}})$, $\Phi \subset X^*$ is the set of roots of $T_{\overline{\mathbb{Q}}}$ in $G_{\overline{\mathbb{Q}}}$, Δ is the set of simple roots corresponding to B , $X_* = X_*(T_{\overline{\mathbb{Q}}})$ and $\Delta^\vee \subset \Phi^\vee \subset X_*$ are the set of simple coroots and coroots. The based root datum is independent of the choice of (T, B) in the following sense: if T' is a maximal torus of $G_{\overline{\mathbb{Q}}}$ and $B' \supset T'$ is a Borel subgroup of $G_{\overline{\mathbb{Q}}}$, then there exists $g \in G(\overline{\mathbb{Q}})$ such that $B' = gTg^{-1}$ and $T' = gTg^{-1}$, and the isomorphism $\mathfrak{R}(B, T) \xrightarrow{\sim} \mathfrak{R}(B', T')$ induced by g is independent of the choice of g . So we just write $\mathfrak{R}(G)$ instead of $\mathfrak{R}(B, T)$. In particular, we get a morphism of groups from $\mathrm{Aut}(G_{\overline{\mathbb{Q}}})$ to the group of automorphisms of the based root datum of G , which turns out to be surjective with kernel the group of inner automorphisms; any pinning of G gives a splitting of the surjective morphism $\mathrm{Aut}(G_{\overline{\mathbb{Q}}}) \rightarrow \mathrm{Aut}(\mathfrak{R}(G))$.

The *dual group* \widehat{G} of G is the complex connected reductive group with based root datum $(X_*, \Phi^\vee, \Delta^\vee, X^*, \Phi, \Delta)$.

Until now, we just needed G to be a connected reductive group over $\overline{\mathbb{Q}}$. But as G is defined over \mathbb{Q} , we get an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on its based root datum $\mathfrak{R}(G)$, and if we fix a pinning of \widehat{G} , then this defines an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on \widehat{G} . The L -group of G is ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}}$, where $W_{\mathbb{Q}} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Weil group of \mathbb{Q} .³²

REMARK 3.4. If ℓ is a prime number, we could define \widehat{G} to be the connected reductive group over $\overline{\mathbb{Q}}_\ell$ with root datum $(X_*, \Phi^\vee, X^*, \Phi)$, and we would get a group ${}^L G$ over $\overline{\mathbb{Q}}_\ell$. We write $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ and ${}^L G(\overline{\mathbb{Q}}_\ell)$ for the resulting groups, when we want to distinguish them from the complex versions. Which form of the L -group we use depends on the context: for Langlands parameters defined on \mathcal{L}_F , we use the complex form, and for Langlands parameters defined on $\mathrm{Gal}(\overline{F}/F)$, we use the ℓ -adic form.

³¹To make this precise, we need a fiber functor. We fix an embedding of F into \mathbb{C} and take the fiber functor given by the corresponding Betti realization with \mathbb{Q} -coefficients.

³²For the Weil group, see for example Tate's Corvallis notes [116].

EXAMPLE 3.5. If G is of type A , D , E , F or G , then \widehat{G} is of the same type as G . If G is of type B_n (resp. C_n), then \widehat{G} is of type C_n (resp. B_n). We can also relate other properties of G and \widehat{G} : for example, the derived group G_{der} is simply connected if and only if $Z(\widehat{G})$ is connected, and in that case $Z(\widehat{G})$ is the dual group of G/G_{der} .

Here are some examples of dual groups:

- $\widehat{\text{GL}}_n = \widehat{\text{U}(p, q)} = \text{GL}_n(\mathbb{C})$ if $p + q = n$;
- $\widehat{\text{SL}}_n = \widehat{\text{SU}(p, q)} = \text{PGL}_n(\mathbb{C})$ if $p + q = n$;
- $\widehat{\text{PGL}}_n = \text{SL}_n(\mathbb{C})$;
- $\widehat{\text{Sp}}_{2n} = \text{SO}(2n + 1, \mathbb{C})$, $\widehat{\text{GSp}}_{2n} = \text{GSpin}_{2n+1}(\mathbb{C})$;
- $\widehat{\text{GU}}(p, q) = \text{GL}_1(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$.

We see that different groups can have isomorphic dual groups. In fact, if G' is an inner form of G , then the L -groups ${}^L G$ and ${}^L G'$ are isomorphic. However, as $\text{U}(p, q)$ is not an inner form of GL_n , the actions of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on their dual groups are not the same, and we get non-isomorphic L -groups. If $G = \text{GL}_n$ (or more generally if G is split over \mathbb{Q}), then $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on \widehat{G} . On the other hand, if $G = \text{U}(p, q)$ and if E is the imaginary quadratic extension of \mathbb{Q} that we used to define G , then G splits over E , so $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G} = \text{GL}_{p+q}(\mathbb{C})$ via its quotient $\text{Gal}(E/\mathbb{Q})$, and the nontrivial element of $\text{Gal}(E/\mathbb{Q})$ acts as a non-inner automorphism of $\text{GL}_{p+q}(\mathbb{C})$, i.e. a conjugate of the automorphism $g \mapsto {}^t g^{-1}$ (which conjugate depends on the choice of the pinning).

Coming back to the Langlands correspondence for G , there are several complications:

- A cuspidal automorphic representation π of $G(\mathbb{A})$ should now have a Langlands parameter ϕ_π with values not in $\text{GL}_n(\mathbb{C})$ but in ${}^L G$;
- There is still a characterization of algebraic automorphic representations (conjecturally corresponding to the parameters that factors through $\mathcal{G}_{F, \mathbb{C}}$), but it is more complicated, see Buzzard and Gee's paper [27];
- Distinct cuspidal automorphic representations can have the same Langlands parameter. We say that they are in the same L -packet.

If π is an algebraic cuspidal automorphic representation of G , then we expect $\phi_\pi : \mathcal{G}_{\mathbb{Q}, \mathbb{C}} \rightarrow {}^L G$ to be defined over a finite extension L of \mathbb{Q} in \mathbb{C} . Choosing a finite place λ of L over a prime number ℓ , we get a morphism from $\mathcal{G}_{\mathbb{Q}}(\overline{\mathbb{Q}}_\ell)$ into the ℓ -adic version of ${}^L G$, and this gives a morphism $\sigma_\pi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$, also called the Langlands parameter of π , and whose value on the Frobenius elements at almost all prime numbers p is predicted by the Satake parameter of π_p . Now the conjecture only involves well-defined objects, and we can actually try to prove it !

REMARK 3.6. If we are very brave and want to classify all discrete automorphic representations of $G(\mathbb{A})$, then there is an extension of the Langlands conjecture due to Arthur. Now a discrete automorphic representation π should have a parameter $\psi_\pi : \mathcal{L}_F \times \text{SL}_2(\mathbb{C}) \rightarrow {}^L G$, satisfying a long list of properties (in particular compatibility with a local version of the Arthur conjectures), and there is a somewhat explicit formula to calculate the multiplicity $m(\pi)$. For a quick review of Arthur's conjectures, see Section 8 of Kottwitz's paper [62]. Warning: if π is cuspidal, then we can recover ϕ_π from ψ_π and vice versa, but the two parameters are not equal.

3.2.2. *The Kottwitz conjecture.* Let (G, h) be a Shimura datum, and let $F = F(G, h)$ be its reflex field. We assume that G_{der} is of \mathbb{Q} -rank 0, so that the Shimura varieties $M_K(G, h)(\mathbb{C})$ are compact.

The conjugacy class of the cocharacter $\mu := h_{\mathbb{C}} \circ r : \text{GL}_{1, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ is defined over F (see 1.4.3), hence it defines a finite-dimensional representation r_{μ} of ${}^L G_F(\overline{\mathbb{Q}}_{\ell}) := \widehat{G}(\overline{\mathbb{Q}}_{\ell}) \rtimes W_F$ in the following way (see Lemma 2.1.2 of Kottwitz's paper [59]): choose a maximal torus \widehat{T} of \widehat{G} and a Borel subgroup \widehat{B} containing \widehat{T} that are part of a splitting fixed by W_F . The cocharacter $h_{\mathbb{C}} \circ r$ corresponds to a unique dominant character μ of \widehat{T} , and we denote by V_{μ} the corresponding highest weight representation of $\widehat{G}(\overline{\mathbb{Q}}_{\ell})$. The action of $\widehat{G}(\overline{\mathbb{Q}}_{\ell})$ on V_{μ} extends to a unique action r_{μ} of ${}^L G_F(\overline{\mathbb{Q}}_{\ell})$ such that W_F acts trivially on the highest weight subspace.

EXAMPLES 3.7. (1) If $G = \text{GSp}_{2d}$ and (G, h) is the Shimura datum of Example 1.35, then $\widehat{G} = \text{GSpin}_{2d+1}(\mathbb{C})$ and $r_{\mu} : \widehat{G} \rightarrow \text{GL}_{2d}(\mathbb{C})$ is the spin representation.

If $d = 1$, then $\text{GSp}_{2d} = \text{GL}_2$ and r_{μ} is the standard representation if $\widehat{\text{GL}}_2 = \text{GL}_2(\mathbb{C})$.

If $d = 2$, then we have an exceptional isomorphism $\text{GSpin}_5(\mathbb{C}) \simeq \text{GSp}_4(\mathbb{C})$, and r_{μ} is isomorphic to the standard representation of $\text{GSp}_4(\mathbb{C})$.

(2) If $G = \text{GU}(p, q)$ and (G, h) is the Shimura datum of Example 1.41, then $\widehat{G} \simeq \text{GL}_1(\mathbb{C}) \times \text{GL}_{p+q}(\mathbb{C})$ and r_{μ} is, up to twists by characters, the q th exterior power of the standard representation of $\text{GL}_{p+q}(\mathbb{C})$.

Let K be an open compact subgroup of $G(\mathbb{A}_f)$, let $d = \dim M_K(G, h)$, and let π be a cuspidal automorphic representation of $G(\mathbb{A})$ such that the π_f^K -multiplicity space $\text{H}_K^i(\pi_f)$ of the i th cohomology group $\text{H}^i(M_K(G, h)(\mathbb{C}), \mathbb{C})$ is nonzero for at least one $i \in \mathbb{Z}$. Then π should be algebraic, so its Langlands parameter should give rise to a Galois representation $\sigma_{\pi} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow {}^L G(\overline{\mathbb{Q}}_{\ell})$ as above.

Here we give a simplified version of the conjecture (from Section 1 of Kottwitz's paper [63]); the general version of the conjecture, also due to Kottwitz, can be found in Section 10 of [62], and we will discuss it in 3.5.

CONJECTURE 3.8. *Under some assumptions on the Shimura datum (see Remark 3.9(1)), there is an explicitly defined integer $a(\pi_f)$ such that we have an equality of virtual representations of $\text{Gal}(\overline{F}/F)$:*

$$\sum_{i=0}^{2d} (-1)^i [\text{H}_K^i(\pi_f)(d/2)] = a(\pi_f) [(r_{\mu} \circ \sigma_{\pi})].$$

Moreover, the integers i such that $\text{H}_K^i(\pi_f^K) \neq 0$ all have the same parity.

REMARK 3.9. (1) The conjecture as stated is in a very naive form and false for general Shimura varieties. In fact, we only expect it to be true when G_{der} has \mathbb{Q} -rank 0 and G has no endoscopy, i.e. satisfies Proposition 2.40 (the correct technical condition is that G should admit no nontrivial elliptic endoscopic triple).

As the Shimura varieties are then projective (and smooth for K small enough), by the Weil conjectures (proved by Deligne, see [36]), the representation $\text{H}_K^i(\pi_f)$ of $\text{Gal}(\overline{F}/F)$ is pure of weight i for every $i \in \mathbb{Z}$, so we can separate the degrees in the formula of Conjecture 3.8 by using Frobenius weights.

- (2) While we cannot hope to prove the Kottwitz conjecture without first constructing the Langlands parameter of σ_π , we do know what the image by σ_π of the Frobenius element Frob_\wp at a place \wp of F over almost every prime number p should be, so we can try to prove that Frob_\wp has the correct characteristic polynomial on the $H_K^i(\pi_f)$. For the simple Shimura varieties of Subsection 2.4, Kottwitz proved this consequence of his conjecture in [63, Theorem 1].
- (3) If we know the local Langlands correspondence for G , we can also try to check that the restriction to all local Galois groups of the representations $H_K^i(\pi_f)$ are as predicted by the Kottwitz conjecture. This is a much harder problem and we won't discuss it here; see for example the book [49] of Harris-Taylor for an example among many of this kind of calculation.

3.3. Applications of the Kottwitz conjecture.

3.3.1. The zeta function of a Shimura variety.

DEFINITION 3.10. Let X be a smooth proper variety over a finite field \mathbb{F}_q . The *Hasse-Weil zeta function* of X is the following formal power series in q^{-s} :

$$Z(X, s) = \exp\left(\sum_{n \geq 1} \frac{\text{card}(X(\mathbb{F}_{q^n}))}{n} q^{-sn}\right).$$

Using the Grothendieck-Lefschetz fixed point formula (cf. Theorem 3.15), we get the following result.

THEOREM 3.11 (Grothendieck, see Theorem 3.1 of [32]). *Let $\text{Frob}_q \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius (the inverse of the arithmetic Frobenius $a \mapsto a^q$). Then*

$$Z(X, s) = \prod_{i=0}^{2 \dim(X)} \det(1 - q^{-s} \text{Frob}_q, H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

In particular, the formal power series $Z(X, s)$ is actually a rational function in q^{-s} . (This last corollary was already known by work of Dwork, see [40].)

REMARK 3.12. We can define the zeta function of any algebraic variety over \mathbb{F}_q by the same formula, and Theorem 3.11 still holds providing we use étale cohomology with proper supports.

Now let X be a proper smooth algebraic variety over a number field F . For all but finite many finite places \wp of F , the variety X has a proper smooth model \mathcal{X} over $\mathcal{O}_{F, \wp}$ (we say that X has *good reduction* at \wp), and we set

$$\zeta_{X, \wp}(s) = \zeta_{\mathcal{X}_{\kappa(\wp)}}(\text{card}(\kappa(\wp))^{-s}),$$

where $\kappa(\wp) = \mathcal{O}_F/\wp$ is the residue field of \wp . By Theorem 3.11 and the specialization theorem for étale cohomology,³³ this does not depend on the choice of the model.

If v is a finite place of F where X does not have good reduction or an infinite place, we will not give the definition of $\zeta_{X, v}(s)$; we will just say that $\zeta_{X, v}(s)$ is a rational function of $\text{card}(\kappa(v))^{-s}$ if v is finite and a product of Γ functions if v

³³Which follows from the proper and smooth base change theorems, see [1] Exposés XII and XVI.

is infinite. See Serre's notes [108] for the definitions and for the conjectures on Hasse-Weil zeta functions.

DEFINITION 3.13. The *Hasse-Weil zeta function* of X is the infinite product

$$\zeta_X(s) = \prod_{v \text{ place of } F} \zeta_{X,v}(s).$$

EXAMPLE 3.14. If $X = \text{Spec } \mathbb{Q}$, then ζ_X is the Riemann zeta function.

A priori this product only makes sense for $\Re(s)$ big enough. The *Hasse-Weil conjecture* predicts that $\zeta_X(s)$ has a meromorphic continuation to \mathbb{C} and a functional equation similar to the one of the Riemann zeta function.

This conjecture seems to be out of reach in general, but for Shimura varieties we can approach it using the Kottwitz conjecture. The general idea goes as follows:

- (1) Essentially by Theorem 3.11, we have an equality

$$\zeta_X(s) = \prod_{i=0}^{2 \dim(X)} L(\mathrm{H}_{\text{ét}}^i(X_{\overline{F}}, \mathbb{Q}_\ell), s)^{(-1)^{i+1}},$$

where, for every continuous representation ρ of $\text{Gal}(\overline{F}/F)$, we denote by $L(\rho, s)$ the *L-function* of ρ . (See Section 3 of Tate's Corvallis notes [116].)

- (2) Suppose that (G, h) is a Shimura datum such that G_{der} has \mathbb{Q} -rank 0, that $F = F(G, h)$ and $X = M_K(G, h)$. For every i , we have up to semi-simplification

$$\mathrm{H}_{\text{ét}}^i(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \Pi_G} \mathrm{H}_K^i(\pi_f)^{\dim(\pi_f^K)}$$

as representations of $\text{Gal}(\overline{F}/F)$.

- (3) The Kottwitz conjecture predicts that $\mathrm{H}_K^i(\pi_f)$ is a sum of copies of $r_\mu \circ \sigma_\pi$, where $\sigma_\pi : \text{Gal}(\overline{F}/F) \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ is the Langlands parameter of π and r_μ is the algebraic representation of ${}^L G$ defined in 3.2.2. But the local compatibility between π and σ_π implies immediately that

$$L(r_\mu \circ \sigma_\pi, s) = L(\pi, s, r_\mu),$$

where the *L-function* $L(\pi, s, r_\mu)$ is defined in Borel's survey [18].

- (4) In theory we understand the analytic properties of *L-functions* of automorphic representations better, so we get some information on the zeta function of $M_K(G, h)$.

In practice the automorphic *L-functions* that appear are usually not standard *L-functions* and so our understanding of them is still limited. However, this method can still go through when r_μ is the standard representation of a classical group, such as in the case of modular curves (when $G = \text{GL}_2$) or Picard modular surfaces (when $G = \text{GU}(2, 1)$), or more generally when $G_{\mathbb{R}}$ is a unitary group of signature $(n-1, 1) \times (n, 0)^{r-1}$, as can be the case for Kottwitz's simple Shimura varieties (see 2.4.2).

3.3.2. *The global Langlands correspondence.* Let (G, h) be a Shimura datum, and let $F = F(G, h)$. In a way, the Kottwitz conjecture says that the cohomology $\varinjlim_K \mathrm{H}_{\text{ét}}^*(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$ realizes the global Langlands correspondence for those automorphic representations of $G(\mathbb{A})$ that contribute to Matsushima's formula for $M_K(G, h)(\mathbb{C})$. So we could try to use this cohomology to construct the global

Langlands correspondence in that case, and then use results like the main theorem of Kottwitz’s [63] to check that this does satisfy the desired compatibility with the local correspondence (at least in the unramified case).

For GL_2 and its inner forms, this is older than Kottwitz’s conjecture; see for example the papers [65] and [34] of Kuga-Shimura and Deligne, respectively. For GL_n with $n \geq 3$, we have no Shimura variety, so we have to use other groups, and we run into several problems:

- (1) The representation of $\mathrm{Gal}(\overline{F}/F)$ that appears in the cohomology of $M_K(G, h)$ is not σ_π (this would not even make sense, as σ_π is a morphism into ${}^L G(\overline{\mathbb{Q}}_\ell)$) but $r_\mu \circ \sigma_\pi$;
- (2) There are multiplicities (the integer $a(\pi_f)$ in the Kottwitz conjecture);
- (3) We want to construct the Langlands correspondence for GL_n , not some strange unitary or symplectic group;
- (4) We want to get a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, not $\mathrm{Gal}(\overline{F}/F)$;
- (5) This will only ever work for “cohomological” automorphic representations, i.e. those π that appear in the cohomology of Shimura varieties. This is a condition on π_∞ : roughly, we need it to have nontrivial (\mathfrak{g}, K_∞) -cohomology.

All of these can be somewhat addressed, at some cost. For point (1), we can choose the group such that \widehat{G} is classical and r_μ is the standard representation. This will for example be the case if $G = \mathrm{GU}(n-1, 1)$, although in practice we will rather want G to be a more complicated unitary group (defined by a CM extension of \mathbb{Q} of degree $2r > 2$) of signature $(n-1, 1) \times (n, 0)^{r-1}$ at infinity, so that we can get a simple Shimura variety. Of course, this solution puts even greater restrictions on the groups that we can use, so it seems that we are making problems (3) and (4) worse. For problem (2), we can sometimes calculate the multiplicities, and they tend to be equal to 1 for nice unitary groups.

Problem (3) can be attacked using the *Langlands functoriality principle*. The idea is that, if discrete automorphic representations of $G(\mathbb{A})$ are parametrized by morphisms $\mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G$, then, if H is another connected reductive algebraic group over \mathbb{Q} and if we have a morphism ${}^L G \rightarrow {}^L H$, then we should be able to “transfer” discrete automorphic representations from G to H . As with the global Langlands correspondence or the Arthur conjectures, this principle can be made very precise at “good” primes (i.e. primes where both groups and all automorphic representations we consider are unramified) using the Satake isomorphism, so we can pin down the conjectural transfer using a local-global compatibility principle. Of course, things are not so simple: the conjectural Arthur parametrization is not bijective in general so we can only expect to transfer L -packets, and the heuristic does not tell us how to actually construct the transfer.

One favorable case is when \widehat{G} is the set of fixed points of an automorphism of \widehat{H} , because then we can use the (twisted) Arthur-Selberg trace formula to construct the transfer map, although this is very technically difficult; see 3.4.4 for a very simple instance of the (untwisted) trace formula. This is for example the case if $G = \mathrm{Sp}_{2d}$ and $H = \mathrm{GL}_{2d+1}$ (see Arthur’s book [8]), if G is the group $\mathrm{U}(p, q)$ constructed using a quadratic imaginary extension E of \mathbb{Q} and $H = \mathrm{GL}_{p+q}(E)$, seen as an algebraic group over \mathbb{Q} (see the works by Mok [89] and Kaletha-Minguez-Shin-White [52]), or if $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_n(E)$ (seen as an algebraic group over \mathbb{Q}) for E/\mathbb{Q} a cyclic extension (see the book [9] of Arthur-Clozel). Using trace formula techniques,

we can transfer discrete automorphic representations of $\mathrm{GL}_{p+q}(\mathbb{A})$ to $\mathrm{GL}_{p+q}(\mathbb{A}_E)$, then back down to $\mathrm{U}(p, q)(\mathbb{A})$, and then use the cohomology of Shimura varieties for this unitary group to construct Galois representations. One caveat is that this only works for representations of $\mathrm{GL}_{p+q}(\mathbb{A}_E)$ that are conjugate self-dual (because their parameter should be stable by the involution of $\widehat{\mathrm{GL}_{p+q}(E)}$ whose set of fixed points is equal to $\widehat{\mathrm{U}(p, q)}$). This kind of construction will also work for more general unitary groups defined using other CM fields or division algebras over them, and in fact we want to use some of them rather than $\mathrm{U}(p, q)$ because of problem (1). Again, this is difficult and there are many technical problems, including annoying congruence conditions on $p + q$. The first results that we are aware of on the construction of parts of the Langlands correspondence using the cohomology of Shimura varieties, outside of the case of GL_2 , are due to Clozel (see [30], which rests on the results of Kottwitz from [63]). More recent and more powerful results can be found in the papers [111] of Shin and [107] of Scholze-Shin; these feature unitary Shimura varieties that are not simple, so they use a more complicated form of the Kottwitz conjecture (see Subsection 3.5), as well as the fundamental lemma. See Dat’s Bourbaki notes [31] for a statement of the fundamental lemma and some context; the fundamental lemma is known by work of Laumon-Ngo ([76]), Ngo ([94]) and Waldspurger ([119], [120]).

Problem (4) can be addressed by “gluing” Galois representations constructed using different Shimura varieties. More precisely, suppose that you have a discrete automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, and that you know that it is self-dual and that π_∞ satisfies all the required conditions for π to transfer to cohomological automorphic representations of unitary groups. By varying the CM field E_i (and the corresponding unitary group), we get a family of representations σ_i of $\mathrm{Gal}(\overline{E}_i/E_i)$, and σ_i should be the restriction of σ_π . We can check that these representations are compatible by looking at what happens at “nice” prime numbers, and then glue them if we chose the E_i disjoint enough.

Problem (5) is already an issue for GL_2 and requires new techniques, as in the paper [39] of Deligne and Serre.

An additional problem, mentioned in the discussion of (3), is that we can only ever transfer self-dual automorphic representations of GL_n to groups that have Shimura varieties, but we will not try to explain the solution to this problem here. See for example the papers [28] of Chenevier-Harris, [48] of Harris-Lan-Taylor-Thorne, [106] of Scholze, or [25] of Boxer.

3.4. Proving the Kottwitz conjecture. We will present the original approach to the Kottwitz conjecture, due to Ihara, Langlands and Kottwitz. We will not talk about the more refined approaches through Igusa varieties or through Scholze’s methods (see for example the book [49], or the papers [111] and [107] already cited above), that also allow us to understand the cohomology at ramified primes.

The situation is the following: we have a Shimura datum (G, h) such that G_{der} is of \mathbb{Q} -rank 0 (so that the Shimura varieties $M_K(G, h)$ are projective), and a discrete automorphic representation π of $G(\mathbb{A})$. We fix a small enough open compact subgroup K of $G(\mathbb{A}_f)$, and we write $F = F(G, h)$. We are trying to understand the representation of $\mathrm{Gal}(\overline{F}/F)$ on the π_f^K -multiplicity spaces $H_K^i(\pi_f)$ in the cohomology groups $H_{\mathrm{ét}}^i(M_K(G, h)_{\overline{F}}, \overline{\mathbb{Q}}_\ell)$.

3.4.1. *The specialization theorem.* We restrict our attention to the action of the local Galois group at finite places φ of F that are “nice enough”, i.e. such that $M_K(G, h)$ has a proper smooth integral model \mathcal{M}_K over $\mathcal{O}_{F, \varphi}$. Remember from 2.2.2 that, if (G, h) is of abelian type, then we have control (in terms of G and K) over the places φ where we have a smooth integral model; when (G, h) is of Hodge type and G_{der} has \mathbb{Q} -rank 0, we also know that this integral model is projective by the work of Madapusi Pera (see [80]). However, in general we only know that we have smooth projective integral models for all but finitely many φ .

Fix φ as in the previous paragraph, and let $\kappa(\varphi)$ be the residue field of φ . Then we have an exact sequence

$$1 \rightarrow I_\varphi \rightarrow \text{Gal}(\overline{F}_\varphi/F_\varphi) \rightarrow \text{Gal}(\overline{\kappa(\varphi)}/\kappa(\varphi)) \rightarrow 1,$$

where I_φ is the *inertia group at φ* . Moreover, as $\kappa(\varphi)$ is a finite field, its absolute Galois group is topologically generated by the geometric Frobenius Frob_φ , which is the inverse of the arithmetic Frobenius $a \mapsto a^{\text{card}(\kappa(\varphi))}$.

The specialization theorem for étale cohomology (which follows from the proper and smooth base change theorems, see [1] Exposés XII and XVI) tells us that the representations $\text{H}_{\text{ét}}^i(M_K(F, h)_{\overline{F}_\varphi}, \overline{\mathbb{Q}}_\ell)$ of $\text{Gal}(\overline{F}_\varphi/F_\varphi)$ are *unramified*, i.e. that I_φ acts trivially on them, and that we have isomorphisms of representations of $\text{Gal}(\overline{\kappa(\varphi)}/\kappa(\varphi))$:

$$(3.1) \quad \text{H}_{\text{ét}}^i(M_K(F, h)_{\overline{F}_\varphi}, \overline{\mathbb{Q}}_\ell) \simeq \text{H}_{\text{ét}}^i(\mathcal{M}_{K, \kappa(\varphi)}, \overline{\mathbb{Q}}_\ell).$$

If we have a Hecke correspondence defined by $g \in G(\mathbb{A}_f)$ and $K' \subset K \cap gKg^{-1}$, then these isomorphisms will be compatible with the corresponding Hecke operator, provided that $M_{K'}(G, h)$ also has a proper smooth model over $\mathcal{O}_{F, \varphi}$ and that the Hecke correspondence extends to the models over $\mathcal{O}_{F, \varphi}$.

So we can now work over the finite field $\kappa(\varphi)$.

3.4.2. *The Grothendieck-Lefschetz fixed point formula and Deligne’s conjecture.* Suppose that we were only trying to understand the representation of $\text{Gal}(\overline{F}_\varphi/F_\varphi)$ on $\text{H}_{\text{ét}}^*(M_K(G, h)_{\overline{F}_\varphi}, \overline{\mathbb{Q}}_\ell)$ (and not on the multiplicity spaces $\text{H}_K^i(\pi_f^K)$), and that we only cared about semisimplifications. Then, by the isomorphism (3.1), it would suffice to calculate the characteristic polynomial of Frob_φ acting on $\text{H}_{\text{ét}}^*(\mathcal{M}_{K, \kappa(\varphi)}, \overline{\mathbb{Q}}_\ell)$. We can do this thanks to the Grothendieck-Lefschetz fixed point formula (already mentioned before Theorem 3.11):

THEOREM 3.15 (Grothendieck, cf. Théorème 3.2 of [32]). *Let \mathbb{F}_q be a finite field, $\overline{\mathbb{F}}_q$ be an algebraic closure of \mathbb{F}_q , ℓ be a prime number different from the characteristic of \mathbb{F}_q and Frob_q be the geometric Frobenius automorphic of $\overline{\mathbb{F}}_q$. Then, for every separated \mathbb{F}_q -scheme of finite type X and every positive integer r , we have*

$$\sum_{i=0}^{2 \dim(X)} (-1)^i \text{Tr}(\text{Frob}_q^r, \text{H}_{\text{ét}, c}^i(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)) = \text{card}(X(\mathbb{F}_{q^r})),$$

where \mathbb{F}_{q^r} is the unique extension of \mathbb{F}_q of degree r in $\overline{\mathbb{F}}_q$.

If we want instead to understand the action of $\text{Gal}(\overline{F}_\varphi/F_\varphi)$ on the Hecke isotypic components, we need to calculate the traces of Hecke operators multiplied by powers of Frob_φ on $\text{H}_{\text{ét}}^*(M_K(G, h)_{\overline{F}_\varphi}, \overline{\mathbb{Q}}_\ell)$. For this, we use a generalization of

the Grothendieck-Lefschetz fixed point formula. We will not state the most general version here, but just the consequence that we need.

We use the notation of Theorem 3.15. Let X, X' be separated \mathbb{F}_q -schemes of finite type, and let $a, b : X' \rightarrow X$ be finite morphisms. Suppose that the trace morphism $\mathrm{Tr}_b : b_*b^* \rightarrow \mathbf{id}$ exists.³⁴ Let u be the endomorphism of $H_{\acute{e}t,c}^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ that is the composition of the pullback by a map $H_{\acute{e}t,c}^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \rightarrow H_{\acute{e}t,c}^*(X'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ and of the map $H_{\acute{e}t,c}^*(X'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \rightarrow H_{\acute{e}t,c}^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ induced by Tr_b . Finally, we denote by $F_X : X \rightarrow X$ the Frobenius morphism (which is identity on the underlying topological spaces and raises functions to the q th power).

THEOREM 3.16. *For any big enough positive integer r , we have*

$$\sum_{i=0}^{2 \dim(X)} (-1)^i \mathrm{Tr}(\mathrm{Frob}_q^r \cdot u, H_{\acute{e}t,c}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \mathrm{card}(\{x' \in X'(\overline{\mathbb{F}}_q) \mid a(x') = F_X^r \circ b(x')\}).$$

Although it is still often referred to as ‘‘Deligne’s conjecture’’, this statement is a theorem: if X, X' are reductions of Shimura varieties and a, b are Hecke operators, it was proved by Pink in [98]. In the general case, it was proved independently by Fujiwara ([42]) and Varshavsky ([118]).

So we now need to understand the set of points of the Shimura varieties $M_K(G, h)$ and $M_{K'}(G, h)$ (or rather of their integral models) over $\kappa(\wp)$, as well as the action of the Frobenius morphism and of Hecke correspondences on these points.

3.4.3. The Langlands-Rapoport conjecture. The Langlands-Rapoport conjecture gives a purely group-theoretical description of the set of points of a Shimura variety over the algebraic closure of the residue field at a good place \wp of the reflex field, as well as a description of the action of the Frobenius at \wp and of Hecke operators on this set. We only give a rough statement here. We actually present a corollary of the original conjecture, that is, a description of the points of a Shimura variety over a finite field (not its algebraic closure); the derivation is explained in Section 5 of Milne’s paper [85]. As a consequence, our notation is not the same as in the paper of Langlands and Rapoport or the book of Reimann (see the next paragraph), in fact we follow Kottwitz’s notation in [62]. We will also restrict to the case of groups with simply connected derived subgroups.

The original paper of Langlands and Rapoport on their conjecture is [72]; another good source is Milne’s survey paper [85]. A more compact presentation of the conjecture can be found in Appendix B of Reimann’s book [102], where the author also fixes some technical mistakes in the previous references.

Let (G, h) be a Shimura datum with reflex field F , let p be a prime number and \wp a place of F above p . Suppose that G and F are unramified at p , and let $K = K^p K_p$ be a level with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial. Then we expect the Shimura variety $M_K(G, h)$ to have a ‘‘nice’’ model \mathcal{M}_K over $\mathcal{O}_{F,\wp}$, and the Langlands-Rapoport conjecture gives a description of $\mathcal{M}_K(\mathbb{F}_q)$, where \mathbb{F}_q is an extension of \mathcal{O}/\wp . The description has the rough shape

$$(3.2) \quad \mathcal{M}_K(\mathbb{F}_q) = \coprod_{\varphi} I_{\varphi}(\mathbb{Q}) \backslash (X^p(\varphi) \times X_p(\varphi)),$$

³⁴This holds for example if b is flat or if X, X' are normal; this is due to Deligne, see SGA 4 XVII 6.2 in [1].

where φ is in a certain set of parameters, I_φ is an algebraic group over \mathbb{Q} (the centralizer of φ), $X^p(\varphi)$ involves the finite adeles outside of p and $X_p(\varphi)$ is a purely p -adic object. The conjecture also includes a description of the actions of the Frobenius morphism and of the Hecke operators.

It would take too long to explain what the parameters φ are, but we can say that they give rise to triples $(\gamma_0, \gamma, \delta)$, where:

- γ_0 is a semisimple element of $G(\mathbb{Q})$, given up to $G(\overline{\mathbb{Q}})$ -conjugacy;
- $\gamma = (\gamma_\ell)_{\ell \neq p}$ is an element of $G(\mathbb{A}_f^p)$, given up to $G(\mathbb{A}_f^p)$ -conjugacy, and such that γ_ℓ and γ_0 are conjugated under $G(\overline{\mathbb{Q}}_\ell)$ for every $\ell \neq p$;
- δ is an element of $G(L)$, where L is an unramified extension of degree $r = [\mathbb{F}_q : \mathcal{O}/\mathfrak{o}]$ of F_φ , such that, if we denote by $\sigma \in \text{Gal}(L/\mathbb{Q}_p)$ the lift of the (arithmetic) Frobenius, then $N(\delta) = \delta\sigma(\delta) \dots \sigma^{r-1}(\delta)$ is $G(\overline{\mathbb{Q}}_p)$ -conjugate to γ_0 .

There are some more conditions on the triple $(\gamma_0, \gamma, \delta)$, see for example Section 2 of [62]. One of these conditions is that a certain invariant $\alpha(\gamma_0; \gamma, \delta)$ defined by Kottwitz should vanish, and it implies that there exists an inner form I of G_{γ_0} , the centralizer of γ_0 in G , such that $I_{\mathbb{Q}_v} \simeq G_{\mathbb{Q}_v, \gamma_v}$ for $v \neq p, \infty$, $I_{\mathbb{Q}_p}$ is isomorphic to the twisted centralizer of δ (see page 169 of [62]) and $I_{\mathbb{R}}$ is anisotropic modulo the center of G . We then take $I_\varphi = I$,

$$X^p(\varphi) = \{g \in G(\mathbb{A}_f^p)/K^p \mid g^{-1}\gamma g \in K^p\}$$

and

$$X_p(\varphi) = \{g \in G(L)/G(\mathcal{O}_L) \mid g^{-1}\delta\sigma(g) \in G(\mathcal{O}_L)\mu_h(\varpi_L)G(\mathcal{O}_L)\},$$

where we extended G to a reductive group scheme over \mathbb{Z}_p , ϖ_L is a uniformizer of L , and μ_h is the morphism $h_{\mathbb{C}} \circ r : \text{GL}_{1, \mathbb{C}} \rightarrow G_{\mathbb{C}}$, seen as a conjugacy class of morphisms $\text{GL}_{1, \overline{\mathbb{Q}}_p} \rightarrow G_{\overline{\mathbb{Q}}_p}$ that is stable by $\text{Gal}(\overline{\mathbb{Q}}_p/L)$; as G is quasi-split over L for r big enough, we may assume up to taking r big enough that μ_h is defined over L (see Lemma 1.1.3 of [59]).

If the Shimura datum (G, h) is PEL of type A or C , so that \mathcal{M}_K has a modular description as in Definition 2.10, then the triples $(\gamma_0, \gamma, \delta)$ should parametrize the \mathbb{Q} -isogeny classes of triples (A, λ, ι) as in the moduli problem. This parametrization rests on Honda-Tate theory, which classifies abelian varieties A over finite fields using their Frobenius, seen as a central element of $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. See Part III of [62] for the case of Siegel modular varieties, and [64] for the PEL cases of type A and C .

The Langlands-Rapoport conjecture is not known for general Shimura varieties, because we do not even have integral models for general Shimura varieties. It is also not known for Shimura varieties of abelian type, except in some particular cases such as the case of quaternionic Shimura varieties (see the paper [102] of Reimann), but there are some weaker versions that are proved and suffice for the application to the Kottwitz conjecture. Here is my understanding of the current situation:

- If (G, h) is of PEL type and G_{der} is simply connected (i.e. G is of type A or C), then Kottwitz formulated a (strictly) weaker form of the Langlands-Rapoport conjecture in [62] and proved this reformulation in [64]. This version is enough to provide a counting-point formula that can be compared to the Arthur-Selberg trace formula (see below).

- Dong Uk Lee (see [77]) generalized Kottwitz’s results for Shimura varieties of Hodge type. The methods, and even the formulation itself, do not seem to be generalizable to Shimura varieties of abelian type.
- Before that, Kisin has proved his own weaker version of the Langlands-Rapoport conjecture in [55] for Shimura varieties of abelian type. This version of the conjecture is not obviously weaker or stronger than Kottwitz’s version. One of its advantages is that it makes it easier to go from Shimura varieties of Hodge type to Shimura varieties of abelian type, and one drawback is that it does not imply a counting-point formula that we can use with the Arthur-Selberg trace formula.
- In [56], Kisin, Shin and Zhu, working with Shimura varieties of abelian type, formulated and proved a version of the Langlands-Rapoport conjecture that is weaker than the original conjecture but stronger than Kottwitz’s and Kisin’s previous versions, and that is enough for the application to the Kottwitz conjecture. (In fact they did more than that, as they also “stabilized” the expression that they got; see Remark 3.17(1).)

The upshot is that, if $g \in G(\mathbb{A}_f)$ has a trivial component at the prime number p under φ and $f^\infty = \mathbb{1}_{KgK} \in \mathcal{H}_K$, we get a formula for the trace of $\text{Frob}_\varphi^r \cdot f^\infty$ on the ℓ -adic cohomology of $M_K(G, h)$ involving terms such as orbital integrals for f^∞ (i.e. integrals of f over $G(\mathbb{A}_f)$ -conjugacy orbits of elements of $G(\mathbb{Q})$) and twisted orbital integrals of a function (depending on r) at p ; a priori we only get this for r big enough, but then the identity can be extended to all non-negative r .³⁵ This is the kind of input that we can plug into the geometric side of the Arthur-Selberg trace formula, see 3.4.4. The spectral side of the trace formula will then give us an expression that can be massaged into what we want, that is, in cases when the simplest form of the Kottwitz conjecture applies, the trace of $\text{Frob}_\varphi^r \cdot f^\infty$ on the virtual representation

$$\sum_{i \geq 0} (-1)^i \sum_{\pi \in \Pi_G} a(\pi_f) \pi_f^K \otimes (r_\mu \circ \sigma_\pi)$$

of $\mathcal{H}_K \times \text{Gal}(\overline{F}/F)$. So we win.

- REMARK 3.17. (1) The sentence “This is the kind of input that we can plug into the geometric side of the Arthur-Selberg trace formula” in the previous paragraph is sweeping a lot of difficulties under the rug. If we are looking at cases where G has no endoscopy (such as the simple Shimura varieties of Kottwitz), then the formula given by the Langlands-Rapoport conjecture is not too far from the geometric side of the trace formula (see Section 4 of [63]). In general, we must first perform a complicated process known as “stabilization”, which uses difficult results such as the fundamental lemma, cf. 3.3.2. See Section 4 of Kottwitz’s paper [62] for an explanation of stabilization in the simpler case when G_{der} is simply connected, and the book [56] of Kisin-Shin-Zhu for the case of general Shimura data.
- (2) Note that we never used Matsushima’s formula in our outline of the proof of the Kottwitz conjecture. In fact, though it serves as a guide, Matsushima’s formula is not logically necessary to the proof.

³⁵This does require some work. We need to see that both sides are finite sums of terms of the form $a \cdot b^r$ with $a, b \in \mathbb{C}$, which is not obvious for the twisted orbital integrals.

3.4.4. *The Arthur-Selberg trace formula for a compact quotient.* We consider the situation of 3.1.1, so G is a connected reductive group over \mathbb{Q} , S_G is the maximal \mathbb{Q} -split torus in the center of G and $A_G = S_G(\mathbb{R})^0$. We write L_G^2 for $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/A_G, \mathbb{C})$, with the action of $G(\mathbb{A})$ given by right translations.

We also assume that G_{der} is of \mathbb{Q} -rank 0, so that the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})/A_G$ is compact. Then the representation L_G^2 is semi-simple, and we have

$$(3.3) \quad L_G^2 \simeq \widehat{\bigoplus_{\pi \in \Pi_G} \pi^{m(\pi)}}.$$

Fix a Haar measure on $G(\mathbb{A})$. If f is a smooth function with compact support on $G(\mathbb{A})$, then we write $R(f)$ for the action of f on L_G^2 by right convolution. Thanks to our hypothesis on G , the operator $R(f)$ is of trace class, and the goal of the Arthur-Selberg trace formula is to give two expressions of its trace.

THEOREM 3.18 (See Section 1 of [7] or [15]). *We have*

$$\begin{aligned} \text{Tr}(R(f)) &= \sum_{\pi \in \Pi_G} m(\pi) \text{Tr}(\pi(f)) \\ &= \sum_{\gamma \in G(\mathbb{Q})/\sim} \text{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})/A_G) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx, \end{aligned}$$

where, in the first formula, $\pi(f)$ is the operator $\int_G f(x)\pi(x)dx$ acting on the space of π and, in the second formula, the sum is over all elements of $G(\mathbb{Q})$ modulo conjugation and, if $\gamma \in G(\mathbb{Q})$, we denote by G_γ the centralizer of γ in G .

Note that we need to choose Haar measures on the groups $G_\gamma(\mathbb{A})$ to make sense of the second formula for $R(f)$ (we also use the counting measure on $G_\gamma(\mathbb{Q})$), but that the result does not depend on that choice.

The first formula for $R(f)$ is called the *spectral side*. It follows from the isomorphism of (3.3).

The second formula for $R(f)$ is called the *geometric side*. We can deduce it by noting that $R(f)$ is an integral operator with kernel

$$K(x, y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y),$$

so the trace of $R(f)$ must be equal to $\int_G K(x, x)dx$. See Section 1 of [7] for more details.

3.5. The general Kottwitz conjecture. Let us finally explain briefly how the Kottwitz conjecture generalizes to arbitrary Shimura varieties. There are two sources of difficulty:

- The fact that general Shimura varieties are not compact, which introduces complications in the geometry: we will have to deal with compactifications and choose which cohomology theory to use.
- Endoscopy, which will make the Arthur-Selberg trace formula harder to analyze.

REMARK 3.19. Although we are concentrating on the case of constant coefficients for simplicity, we could also consider cohomology with coefficients in a local system defined by a representation of the group G , as in Section 5 of Pink's paper [97]. In fact, it is necessary to work in this more general situation to make some

inductive arguments work, but it does not complicate the calculation of cohomology in an essential way.

3.5.1. *Compactifications.* There are many ways to compactify locally symmetric spaces, see for example the book [20] of Borel and Ji for a detailed presentation. However, in these notes we are interested in compactifications of Shimura varieties that have the structure of an algebraic variety, and then there are two choices:

- (1) The Baily-Borel compactification, also called minimal Satake compactification, Satake-Baily-Borel compactification or just minimal compactification. Its main advantage is that it is canonical, so for example Hecke correspondences will extend to the Baily-Borel compactifications of Shimura varieties, and that its boundary is stratified by lower-dimensional Shimura varieties.³⁶ Its main drawback is that it is very singular in general; for example, the boundary of the Baily-Borel compactification of the Siegel modular variety $\mathcal{M}_{d,n}$ has codimension d .
- (2) The toroidal compactifications, which map to the Baily-Borel compactification. As the plural indicates, they are not canonical and depend on extra data, so we cannot extend the action of the Hecke algebra to the cohomology of any fixed toroidal compactification. On the other hand, we can always find toroidal compactifications which are smooth projective and whose boundary is a divisor with normal crossings, and the boundary of a toroidal compactification is stratified by so-called *mixed Shimura varieties*, which are a generalization of Shimura varieties to certain non-reductive groups. Even if we were only interested in the Baily-Borel compactification, the toroidal compactifications would be a very useful tool to study it; they are also interesting in their own right.

Sections I.4, I.5 and III.4 of the book [20] contain a review of Satake and Baily-Borel compactifications of locally symmetric spaces as topological spaces and complex analytic spaces. The fact that the Baily-Borel compactification is a projective algebraic variety over \mathbb{C} was proved by Satake for Siegel modular varieties, and by Baily and Borel in general, see the paper [12]. As for toroidal compactifications of complex Shimura varieties, they are studied in the book [11] of Ash, Mumford, Rapoport and Tai. In his thesis [96], Pink constructed canonical models of the toroidal compactifications over the reflex field, and deduced that the Baily-Borel compactification also descends to a projective algebraic variety over that field. As in the case of Shimura varieties, the story of integral models is not yet complete,³⁷ and as for canonical models, the crucial part is to construct models of the toroidal compactifications (we then get a model of the Baily-Borel compactification as a byproduct). The case of modular curves is treated in the paper [33] of Deligne-Rapoport, and that of Siegel modular varieties in the book [41] of Chai and Faltings. The next case to be handled systematically was that of PEL type Shimura varieties, in Lan's thesis [67]. Then Madapusi Pera treated the case of Hodge type Shimura varieties in his paper [80]. The case of abelian type Shimura varieties seems to be within reach, but as far as I know it has not been written yet.

³⁶Almost: the boundary strata are actually quotients of Shimura varieties by finite groups.

³⁷And as in the case of Shimura varieties, we only discuss places of good reduction here.

3.5.2. *Intersection cohomology and Zucker's conjecture.* It is perfectly possible to try calculating compact support cohomology of noncompact Shimura varieties with the methods outlined in 3.4, this is for example what Laumon does in [73], [75]. The issue is that the Arthur-Selberg trace formula will be harder to use, because there is no simple spectral description of the cohomology such as the one given by Matsushima's formula in the compact case. Indeed, the generalization of Matsushima's formula, which is a theorem of Borel and Casselman (see [19]) gives a spectral description of the L^2 cohomology of the Shimura variety, so it would be simpler to study that. The next problem is that L^2 cohomology has a very non-algebraic definition (it is the cohomology of the complex of L^2 differential forms, for a certain metric on the locally symmetric space); it has an action of the Hecke algebra, but it is not clear at all a priori how to make the absolute Galois group of the reflex field act on it.

Fortunately, this problem is solved by Zucker's conjecture, which says that there should be a Hecke-equivariant isomorphism between L^2 cohomology of a Shimura variety and intersection cohomology of its Baily-Borel compactification. Intersection cohomology is a cohomology theory adapted to singular spaces. It was first introduced by Goresky and MacPherson (see [45]) as a modification of Betti cohomology for stratified spaces, and then further studied by them in [46] using a sheaf-theoretic interpretation due to Deligne. Then, in the book [14], Beilinson, Bernstein, Deligne and Gabber put intersection cohomology into the wider context of perverse sheaves: if X is a possibly singular space, its intersection cohomology is the cohomology of a particular complex of sheaves on it, called the *intersection complex*, which is a simple object in the abelian category of perverse sheaves (for the middle perversity). We have been deliberately vague about the precise nature of X , as one of the advantages of the sheaf-theoretic point of view is that it can be adapted to many different situations; for example, X could be an algebraic variety over a field and the intersection complex could be an étale complex with \mathbb{Q}_ℓ -coefficients; or X could be an algebraic variety over \mathbb{C} and the intersection complex could be a complex of \mathbb{Q} -vector spaces for the analytic topology, or a mixed Hodge module.³⁸ In any case, the intersection cohomology of X satisfies some of the properties that we would expect from the cohomology of a nonsingular variety:

- Poincaré duality if X is proper;
- the hard Lefschetz theorem if X is projective;
- purity if X is proper (in the appropriate sense, e.g. in the sense of Deligne (see [36]) if X is a variety over a finite field or as a Hodge structure if X is a variety over \mathbb{C} and we are using mixed Hodge modules).

It also satisfies comparison theorems between the different cohomology theories. (See for example Chapter 6 of [14] for the comparison between the étale and Betti theories.)

3.5.3. *The Borel-Casselman theorem.* We come back to Shimura varieties. Let (G, h) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ be a small enough level, X be the Shimura variety $M_K(G, h)$, X^* be the Baily-Borel compactification of X and $IH^*(X^*)$ be its intersection cohomology. We have seen that Hecke correspondences extend to

³⁸We are avoiding the cases of non-algebraic X or integral coefficients, as there is no unique “middle perversity” there. But the theory still makes sense, it is just slightly more complicated.

Baily-Borel compactifications. Although intersection cohomology is not as functorial as usual cohomology, we can still make the Hecke algebra at level K act on $IH^*(X^*)$ (because Hecke operators are finite morphisms between Baily-Borel compactifications and become finite étale when restricted to Shimura varieties). Zucker's conjecture (see [125, (6.20)]) says that, if we use the Betti version of $IH^*(X^*)$ with complex coefficients, then there is a Hecke-equivariant isomorphism between $IH^*(X^*)$ and L^2 cohomology of X . Also, Zucker's conjecture is actually a theorem, as it has been proved (at least) four times: by Looijenga ([78]), Looijenga-Rapoport ([79]), Saper-Stern ([104]) and Saper ([103]). If we combine it with the generalization of Matsushima's formula due to Borel and Casselman ([19]), we finally get the following theorem (note the similarity with Theorem 3.1):

THEOREM 3.20. *Let (G, h) be a Shimura datum. Then we have a $G(\mathbb{A}_f)$ -equivariant isomorphism of graded \mathbb{C} -vector spaces*

$$\varinjlim_K IH^*(M_K(G, h)^*(\mathbb{C})) \simeq \bigoplus_{\pi \in \Pi_G} \pi_f \otimes H^*(\mathfrak{g}, K_\infty; \pi_\infty)^{m(\pi)},$$

where $G(\mathbb{A}_f)$ acts on the factors π_f on the right hand side, $M_K(G, h)^*$ is the Baily-Borel compactification of $M_K(G, h)$ and we are using the Betti version of intersection cohomology (with constant coefficients for simplicity).

3.5.4. The Kottwitz conjecture. Let F be the reflex field of (G, h) . If we use the étale ℓ -adic version of intersection cohomology, then we get cohomology groups $\varinjlim_K IH^*(M_K(G, h)_{\overline{F}}^*, \overline{\mathbb{Q}}_\ell)$ equipped with commuting actions of $G(\mathbb{A}_f)$ and $\text{Gal}(\overline{F}/F)$, which become isomorphic as $G(\mathbb{A}_f)$ -modules to the cohomology groups of Theorem 3.20 if we choose any isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We introduce the following virtual representation of $G(\mathbb{A}_f) \times \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$:

$$IH = \sum_{i \geq 0} (-1)^i [\varinjlim_K IH^i(M_K(G, h)_{\overline{F}}^*, \overline{\mathbb{Q}}_\ell)],$$

where the brackets denote the class of a representation in the group of virtual representations. The sum is actually finite, because the intersection cohomology group vanishes for $i \geq 2 \dim(M^K(G, h))$, and this dimension is independent of the level K . We have a decomposition

$$IH = \sum_{\pi_f} [\pi_f] \otimes \sigma(\pi_f),$$

where π_f goes through the set of equivalence classes of irreducible representations of $G(\mathbb{A}_f)$ and each $\sigma(\pi_f)$ is a virtual representation of $\text{Gal}(\overline{F}/F)$. By Theorem 3.20, we know that the only π_f that can occur are the finite parts of discrete automorphic representations of G , and we have a formula for the dimension of $\sigma(\pi_f)$. The Kottwitz conjecture, stated on page 201 of [62], gives a precise description of $\sigma(\pi_f)$.

We will not state a precise form of the conjecture, as it would require introducing too much notation, but we will try to give its flavor. The most naive guess is that we should have something like

$$\sigma(\pi_f) = \pm \sum_{\pi' \in \Pi_G, \pi'_f \simeq \pi_f} m(\pi') [(r_\mu \circ \sigma_{\pi'})],$$

where r_μ and σ_π are as in Conjecture 3.8, but this is not true because of endoscopy. What we actually expect is as follows. As mentioned in Remark 3.6,

discrete automorphic representations of G should be classified by *Arthur parameters* $\psi : \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G$. The parameters corresponding to the representations appearing in the cohomology of Shimura varieties are expected to factor through the motivic Galois group of \mathbb{Q} and to be defined over a finite extension of \mathbb{Q} in \mathbb{C} , and there is an explicit recipe giving σ_{π} from the parameter corresponding to π ; we denote by V_{ψ} the space of the representation r_{μ} with the action of $\mathrm{Gal}(\overline{F}/F)$ coming from σ_{π} .³⁹ Let S_{ψ} be the centralizer of ψ in \widehat{G} ; this group, or rather its quotient $S_{\psi}/S_{\psi}^0 Z(\widehat{G})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$, should control the structure of the Arthur packet of representations associated to ψ and the multiplicities of these representations. We can decompose V_{ψ} as a direct sum

$$V_{\psi} = \bigoplus_{\nu} V_{\psi, \nu},$$

where ν runs over characters of S_{ψ} and $V_{\psi, \nu}$ is the part where S_{ψ} acts by ν^{-1} . Then the Kottwitz conjecture says that

$$\sigma(\pi_f) = \sum_{\psi} \sum_{\nu} n(\pi_f, \nu) [V_{\psi, \nu}],$$

where ψ runs over Arthur parameters whose packets contains a representation with finite part π_f , and $n(\pi_f, \nu)$ is the product of a sign and of a multiplicity (which is the multiplicity of ν in some finite-dimensional representation of S_{ψ} depending on π_f).

3.5.5. *Proving the Kottwitz conjecture.* The general strategy of 3.4 still applies with some adaptations:

- (1) The specialization theorem still holds for intersection cohomology, as long as we have toroidal compactifications that are proper and smooth over the base ring (which is the case for Hodge type Shimura varieties at places of good reduction). This follows from the specialization theorem for the cohomology of the toroidal compactifications and from the decomposition theorem (see Corollaires 5.4.2 and 5.4.6 of [14]), which says that intersection cohomology of the Baily-Borel compactification is a direct factor of this cohomology.
- (2) Deligne's conjecture makes sense (and is known) for general complexes and cohomological correspondences, not just for constant coefficients. It reduces the calculation of the trace of a Hecke operator twisted by a power of the Frobenius on intersection cohomology to counting fixed points (in the Baily-Borel compactification) and calculating the stalks of the intersection complex at these fixed points.
- (3) The Langlands-Rapoport conjecture makes no distinction between compact and noncompact Shimura varieties. It also theoretically gives a description of points of the Baily-Borel compactification over a finite field, as this compactification is stratified by Shimura varieties (or explicit finite quotients thereof).

³⁹We are cheating here: if we want an action of $\mathrm{Gal}(\overline{F}/F)$ and not of the motivic Galois group of F , we have to take as coefficients an ℓ -adic completion of the field of definition of ψ .

- (3') The intersection complex can be quite difficult to calculate, but in the case of Shimura varieties there is an explicit description given by the theory of weighted cohomology of Goresky-Harder-MacPherson (see [44]); in particular, it is a complex of local systems on each stratum of the Baily-Borel compactification. Once the Langlands-Rapoport conjecture (or an appropriate weakening) is known, this reduces the calculation of the cohomology to a complicated combinatorial problem.
- (4) As for the Arthur-Selberg trace formula, we now need to use the stable trace formula for a noncompact quotient, fortunately in a somewhat simple form (see Section 29 of Arthur's overview [7] for an introduction to the stable trace formula, and Arthur's paper [6] for the simplifications that occur when calculating the cohomology of Shimura varieties). The stabilization process that we must apply to the point-counting formula coming from (3) and (3') is also more complicated, because we need to stabilize the terms coming from the boundary.⁴⁰ For an example of this process, see Yihang Zhu's thesis [124].

In the end, we expect to get the following formula for $h \in \mathcal{H}_K$ a Hecke operator trivial at p , Frob_φ the geometric Frobenius at a place φ of F above p and $j \in \mathbb{N}$:

$$(3.4) \quad \text{Tr}(h \times \text{Frob}_\varphi, IH^*(M_K(G, h)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)) = \sum_H \iota(G, H) ST_H((h^p)^H \varphi_{p,j} \varphi_\infty),$$

where H runs over elliptic endoscopic groups of G (to be more precise, it should run over elliptic endoscopic triples), $\iota(G, H)$ is a positive rational number, ST_H is the stable trace formula for H , $(h^p)^H$ is the so-called *endoscopic transfer* to $H(\mathbb{A}_f^p)$ of the prime-to- p part of h , $\varphi_{p,j}$ is a function in the spherical Hecke algebra of $G(\mathbb{Q}_p)$ that only depends on j (not on h), and φ_∞ is a function on $G(\mathbb{R})$ that depends only on the coefficients of the cohomology (here we are taking constant coefficients, but we could take coefficients in any algebraic representation of G).

- (5) The last step is to transform the right-hand side of the previous formula into an expression of the form $\sum_{\pi_f} \text{Tr}(h, \pi_f) \text{Tr}(\text{Frob}_\varphi^j, \sigma(\pi_f))$. Kottwitz explains a way to do this in Section 10 of [62], but his method requires Arthur's conjectural classification of discrete automorphic representations, which is only available for classical and unitary groups (see [8], [89] and [52]) so far, while the groups of Shimura varieties of PEL type (i.e. the best understood Shimura varieties) tend to be *general* symplectic, orthogonal or unitary groups (i.e. GSp, GO and GU); there are some results for general symplectic groups, see Bin Xu's thesis [121] and his paper [122], though as far as I understand they are not sufficient to completely prove the Kottwitz conjecture. It also requires the agreement of the various constructions of the classification of representations of real groups, that was proved for classical and unitary groups in papers

⁴⁰Some of these complications already arise in the compact case with endoscopy. Indeed, even when the group G has no nontrivial parabolic subgroups, its endoscopic groups almost certainly do because they are quasi-split, and this produces "boundary terms" in their stable trace formulas on the right-hand side of identity (3.4). These terms must cancel each other out, as there are no corresponding terms on the left-hand side of (3.4).

of Arancibia-Moeglin-Renard ([5]), Arancibia ([3]), Arancibia-Mezo ([4]) and Adams-Arancibia-Mezo ([2]).

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