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Simplicity and symmetry in physics

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Abstract. Theoretical physics is largely the mathematical development of simple ideas guided by experiments. The uncertainty principle, local symmetry, the standard model, path integrals and general relativity are examples as will be illustrated in this pedagogical paper.

1. The uncertainty principle

In a lecture at Brookhaven National Laboratory in the summer of 1963, Robert Oppenheimer explained the simple ideas that underlie the uncertainty principle:

- To use an instrument to measure the position of something, the instrument must be attached to a known position.
- To use the recoil of an instrument to measure the momentum of something, the instrument must be free to recoil.

Since an instrument can not simultaneously be both attached and free, one can't simultaneously measure position and momentum to arbitrary accuracy. The limit is

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (1)$$

One can use the uncertainty principle to estimate microscopic quantities. For example, setting the product of the average values of r and p equal to \hbar , one has $1/r = p/\hbar$ which gives the energy of the ground state of hydrogen as

$$E_0 = \frac{p^2}{2m} - \frac{e^2}{r} = \frac{p^2}{2m} - \frac{e^2 p}{\hbar}. \quad (2)$$

Its minimum is at $p = me^2/\hbar$. So

$$E_0 = \frac{(me^2)^2}{2m\hbar^2} - \frac{(me^2)^2}{m\hbar^2} = -\frac{mc^2}{2} \frac{e^4}{\hbar^2 c^2} = -\frac{mc^2}{2} \alpha^2 \quad (3)$$

which is the nonrelativistic result.

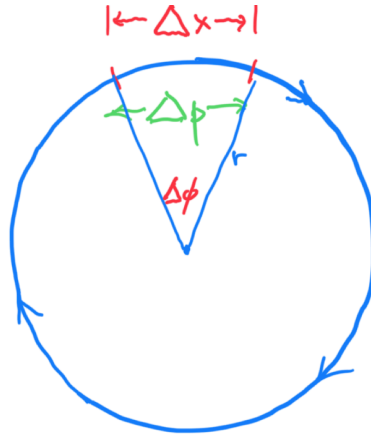


Figure 1: How $\Delta x \Delta p \geq \frac{\hbar}{2}$ suggests $\Delta\phi \Delta L \geq \frac{\hbar}{2}$.

One may apply the uncertainty principle as in Fig. 1 and so derive an uncertainty relation for angle and angular-momentum:

$$\frac{\hbar}{2} \leq \Delta x \Delta p = \frac{\Delta x}{r} (r \Delta p) = \Delta\phi \Delta L \approx \Delta L. \quad (4)$$

This says that particles must have spins greater than $\hbar/2$. Apart from the spinless Higgs particle, the known elementary particles obey this inequality having as they do spins $\hbar/2$, \hbar and $2\hbar$. But why are they spinning? Why isn't the Higgs?

For a system of several degrees of freedom q_n, p_n , the uncertainty principle (1) is generalized to

$$[q_n, p_m] = i\hbar \delta_{nm}. \quad (5)$$

In quantum field theory, these relations become

$$\begin{aligned} [\phi_n(t, \mathbf{x}), \dot{\phi}^\dagger(t, \mathbf{y})] &= i\hbar \delta_{nm} \delta(\mathbf{x} - \mathbf{y}) \quad \text{for bosons and} \\ \{\psi_n(t, \mathbf{x}), \psi_m^\dagger(t, \mathbf{y})\} &= \hbar \delta_{nm} \delta(\mathbf{x} - \mathbf{y}) \quad \text{for fermions.} \end{aligned} \quad (6)$$

2. Simplicity \iff symmetry

The complexity of a thing is the length of its shortest description. The complexity of a circle of radius r is at most the length of its equation $x^2 + y^2 = r^2$. The complexity of an arbitrary squiggle is at least twice the number of its pixels. The $O(2)$ symmetry of the circle makes it simpler than a squiggle.

The complexity of a sphere of radius r is less than the length of its equation

$$x^2 + y^2 + z^2 = r^2. \quad (7)$$

A rambutan has less symmetry and more complexity. The $O(3)$ symmetry of the sphere makes it simpler.

The smallest things are presumably the simplest and the most symmetrical. Elementary particles are the smallest things we know of. They must be simple. They must be the simplest because they have the least room for complexity. They must be symmetrical.

3. Symmetry and conservation

Symmetry implies conservation. In mechanics, symmetry under rotations and under translations in time and space implies conservation of angular momentum, energy and momentum.

In field theory, if an action density $L(\phi_\alpha, \partial_i \phi_\alpha)$ depends upon the spacetime coordinates x^i only through the fields $\phi_\alpha(x)$ and their derivatives $\partial_i \phi_\alpha(x)$, then the energy-momentum tensor

$$T^i_k = \sum_i \frac{\partial L}{\partial \partial_i \phi_\alpha} \frac{\partial \phi_\alpha}{\partial x^k} - \delta^i_k L \quad \text{is conserved} \quad \partial_i T^i_k = 0. \quad (8)$$

4. Global symmetry

A **global** symmetry transformation changes fields in the same way at every spacetime point x . In the global $U(1)$ symmetry transformation

$$\psi(x) \rightarrow \psi'(x) = e^{iq\theta} \psi(x) \quad (9)$$

or

$$\begin{pmatrix} \psi'_1(x) \\ \psi'_2(x) \end{pmatrix} = \begin{pmatrix} \cos(q\theta) & \sin(q\theta) \\ -\sin(q\theta) & \cos(q\theta) \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad (10)$$

the charge q and the angle θ are **independent** of x .

This symmetry conserves charge.

5. Local $U(1)$ symmetry

The electric and magnetic fields of Maxwell and Faraday

$$E_i = c \left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0} \right) \quad \text{and} \quad B_i = \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) = (\nabla \times \mathbf{A})_i \quad (11)$$

have a special symmetry. They are the 4-curl of the vector potential A_k

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} = -F_{ki} \quad (12)$$

and so are unchanged by the gauge transformation

$$A_k(x) \rightarrow A'_k(x) = A_k(x) + \partial_k \theta(x) \quad (13)$$

in which the angle $\theta(x)$ is an **arbitrary** function of x .

The action density $\psi^\dagger \gamma^k \partial_k \psi$, in which $\{\gamma^i, \gamma^k\} = \eta^{ik}$ and $\eta = \text{diag}(-1, 1, 1, 1)$, is not invariant under the local, i.e., spacetime dependent, transformation

$$\begin{aligned}\psi'(x) &= e^{-iq\theta(x)} \psi(x) \\ A'_i(x) &= A_i(x) + \partial_i \theta(x).\end{aligned}\tag{14}$$

But the Dirac action density $\psi^\dagger \gamma^k (\partial_k + iqA_k) \psi$ is invariant because the derivative of the exponential $e^{-iq\theta(x)}$ is canceled by the change in the vector potential $A_i(x)$

$$-iq(\partial_k \theta) e^{-iq\theta} \psi + iq(\partial_k \theta) e^{-iq\theta} \psi = 0.\tag{15}$$

This cancellation happens because the ordinary derivative $\partial_k \psi$ in the action density $\psi^\dagger \gamma^k \partial_k \psi$ has been replaced by the ‘‘covariant’’ derivative $(\partial_k + iqA_k) \psi$. Thus the local symmetry (14) requires $\psi(x)$ to **interact** with $A_i(x)$.

This local $U(1)$ symmetry of classical electrodynamics allows the angle $\theta(x)$ to vary arbitrarily with the spacetime point x and that in turn requires the field $\psi(x)$ to interact with the gauge field $A_k(x)$. This symmetry foreshadowed other local symmetries which require other interactions. In fact, all the known interactions are exactly those required by the known local symmetries.

6. Local nonabelian symmetry

Yang Chen-Ning and Robert Mills generalized local symmetry from the $U(1)$ of electrodynamics (14) to $SU(2)$. That was a big step forward.

Steven Weinberg, Sheldon Lee Glashow and Abdus Salam explained the electroweak interactions in terms of the group $SU_L(2) \otimes U_Y(1)$. They replaced the simple derivative $\partial_k \Psi$ of a complex doublet Ψ such as

$$\Psi = \begin{pmatrix} \nu \\ e \end{pmatrix} \quad \text{or} \quad \Psi = \begin{pmatrix} u \\ d \end{pmatrix}\tag{16}$$

with the ‘‘covariant’’ derivative

$$D_k \Psi(x) = \left(\partial_k + ig \frac{\boldsymbol{\sigma} \cdot \mathbf{W}(x)}{2} + ig' \frac{Y B_k(x)}{2} \right) \Psi(x)\tag{17}$$

in which $\boldsymbol{\sigma}$ is the triplet of Pauli matrices.

The subscript L on $SU_L(2)$ means that the gauge bosons $\mathbf{W}(x)$ interact with the left-handed quark and lepton fields and not with the right-handed quark and lepton fields. Thus the electroweak interactions are lopsided and are not invariant under reflections, $\mathbf{x} \rightarrow -\mathbf{x}$, i.e., they do not conserve parity.

7. Quantum chromodynamics

Murray Gell-Mann and George Zweig explained the strong interactions in terms of the group $SU(3)$ which acts on “color” triplets of quarks. For example, the local $SU_c(3)$ transformation of the u quark triplet (u_r, u_g, u_b) is

$$\Psi'(x) = \begin{pmatrix} u_r(x) \\ u_g(x) \\ u_b(x) \end{pmatrix}' = U(x) \begin{pmatrix} u_r(x) \\ u_g(x) \\ u_b(x) \end{pmatrix} = U(x) \Psi(x) \quad (18)$$

in which $U(x)$ is a 3×3 unitary matrix of unit determinant, i.e., $U^\dagger(x)U(x) = 1$ and $\det U(x) = 1$.

This local symmetry requires that the ordinary derivative ∂_k in the action density $\Psi^\dagger \gamma^k \partial_k \Psi$ be replaced by the covariant derivative $D_k \Psi = (\partial_i + iA_i(x)) \Psi(x)$ in which $A_i(x)$ is the linear combination

$$A_i(x) = \sum_{a=1}^8 t_a A_i^a(x). \quad (19)$$

The resulting action density $\Psi^\dagger(x) \gamma^k (\partial_i + iA_i(x)) \Psi(x)$ is invariant under the local $SU_c(3)$ transformation (18), and that invariance requires that the quarks $\Psi(x)$ interact with the gluons $A_i^a(x)$.

The interactions of the standard model are exactly those required by local gauge invariance.

8. General relativity

Einstein realized that we have no way of knowing which system of coordinates is the “true” or “best” system of coordinates. He concluded that we should write the equations of physics in such a way that they are the same in all coordinate systems. His implementation of these simple ideas was general relativity.

He required the equations of physics to remain the same under arbitrary transformations of the four coordinates of spacetime

$$x^i \rightarrow x'^i(t, x, y, z). \quad (20)$$

To achieve this symmetry, he replaced the simple derivative $\partial_\ell V^k$ of a vector V^k with the covariant derivative

$$\partial_\ell V^k \rightarrow D_\ell V^k = \partial_\ell V^k + \Gamma^k_{i\ell} V^i \quad (21)$$

in which the Christoffel symbol is

$$\Gamma^k_{i\ell} = \frac{1}{2} g^{kn} (\partial_\ell g_{ni} + \partial_i g_{n\ell} - \partial_n g_{i\ell}) \quad (22)$$

and $g_{ik}(x)$ is the metric of spacetime. The replacement (21) requires the vector V^k (and eventually every other field) to interact with the gravitational field $g_{ik}(x)$.

In empty space, particles fall along geodesics independently of their masses

$$0 = \delta S = -m \delta \int \sqrt{-g_{ik} dx^i dx^k} \implies 0 = \frac{d^2 x^k}{d\tau^2} + \Gamma^k_{i\ell} \frac{dx^i}{d\tau} \frac{dx^\ell}{d\tau}. \quad (23)$$

General relativity was verified by measurements of the precession of Mercury and of the bending of light by the Sun as observed by Eddington during the eclipse of 1919.

It is said that when a journalist asked Eddington if it were true that only three people understood general relativity, Eddington replied, “Who’s the other one?”

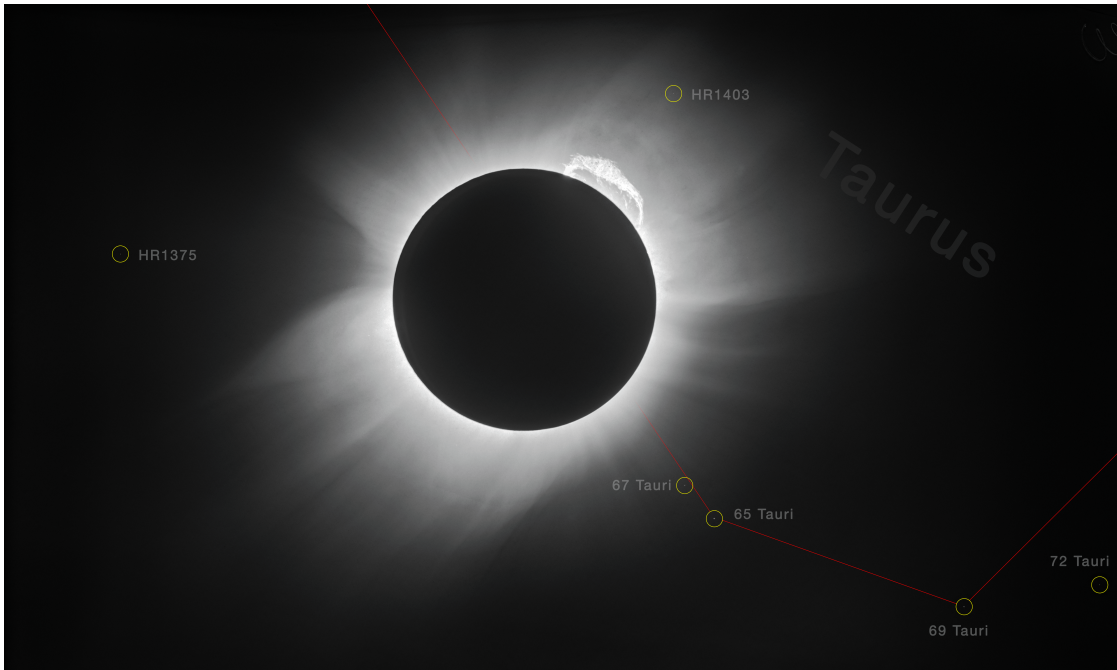


Figure 2: Eclipse of 1919 (ESO/Landessternwarte Heidelberg-Königstuhl/F. W. Dyson, A. S. Eddington, & C. Davidson).

It can be helpful to visualize the points $p(x) = p(x^0, x^1, x^2, x^3)$ of our curved spacetime as embedded in a flat Minkowski spacetime \mathbb{M}^n of $n \geq 4$ dimensions. The four derivatives $\partial_i p(x)$ are then n -dimensional vectors in \mathbb{M}^n tangent to our curved spacetime. The displacement of a point $p(x)$ due to changes dx^i in the four coordinates is $dp = e_i dx^i$, and the invariant squared displacement ds^2 is the Minkowski dot-product

$$ds^2 = dp \cdot dp = (e_i dx^i) \cdot (e_k dx^k) \equiv g_{ik} dx^i dx^k \quad (24)$$

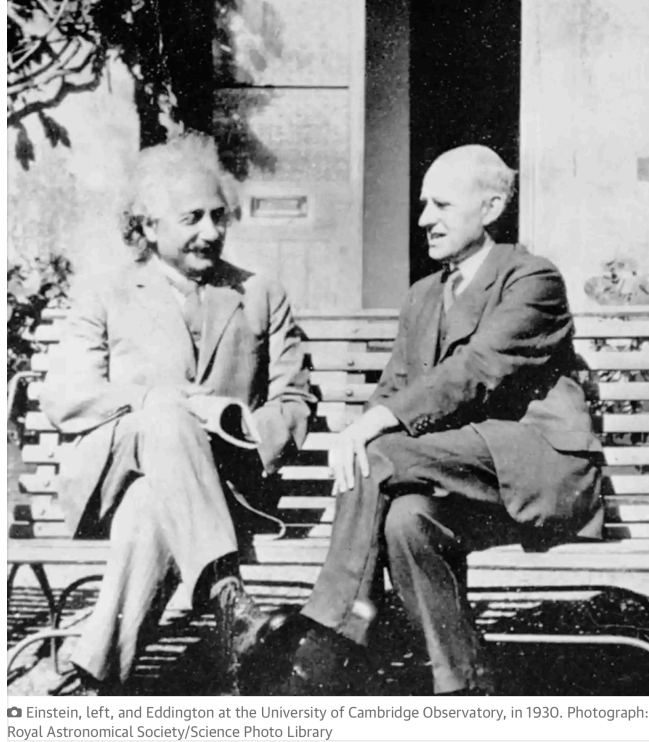


Figure 3: Albert Einstein and Arthur Eddington in 1930 (Royal Astronomical Society/Science Photo Library).

in which the metric of spacetime $g_{ik}(x)$ appears as the dot-product of the tangent vectors $e_i(x)$ and $e_k(x)$

$$g_{ik}(x) = e_i(x) \cdot e_k(x) = \sum_{a,b} e_i^a(x) \zeta_{ab} e_k^b(x) = \sum_a \sigma_a e_i^a(x) e_k^a(x). \quad (25)$$

Here $\sigma_a = \pm 1$, and ζ is the diagonal metric of \mathbb{M}^n .

One may visualize the Friedmann-Lemaître-Robinson-Walker metric for a finite, closed universe

$$g_{ik}(t, r, \theta, \phi) = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2/(1 - r^2/L^2) & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2 \theta \end{pmatrix} \quad (26)$$

as the points

$$p(t, r, \theta, \phi) = \left(ct, a\sqrt{L^2 - r^2}, ar \sin \theta \cos \phi, ar \sin \theta \sin \phi, ar \cos \theta \right) \quad (27)$$

in \mathbb{M}^5 with metric

$$\zeta = \text{diag}(-1, 1, 1, 1, 1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (28)$$

Similarly, one may visualize the FLRW metric for an infinite, open universe

$$g_{ik}(t, r, \theta, \phi) = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2/(1 + r^2/L^2) & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2 \theta \end{pmatrix} \quad (29)$$

as the points

$$p(t, r, \theta, \phi) = \left(ct, a\sqrt{L^2 + r^2}, ar \sin \theta \cos \phi, ar \sin \theta \sin \phi, ar \cos \theta \right) \quad (30)$$

in \mathbb{M}^5 with metric $\zeta = \text{diag}(-1, -1, 1, 1, 1)$. In terms of the tangent vectors e_i and $e^k = g^{k\ell}e_\ell$, the Christoffel symbol (22) is $\Gamma^k_{i\ell} = e^k \cdot \partial_\ell e_i$.

9. Einstein the humanist

Einstein had a way with words:

- I never think of the future. It comes soon enough.
- Once we accept our limits, we go beyond them.
- Any fool can know. The point is to understand.
- Imagination is more important than knowledge.
- Two things are infinite. The universe and human stupidity — I'm not so sure about the universe.
- I never teach my pupils; I only provide the conditions in which they can learn.
- Everybody is a genius. But if you judge a fish by its ability to climb a tree, it will live its whole life believing that it is stupid.
- I know not with what weapons World War III will be fought, but World War IV will be fought with sticks and stones.
- The most beautiful thing we can experience is the Mysterious.

10. Path integrals

Feynman's path integrals are based on two simple ideas. One is that for an infinitesimal interval dt the time-evolution operator $\exp(-iHt/\hbar)$ for a non-relativistic particle is approximately

$$e^{-iHdt/\hbar} = e^{-i(\mathbf{p}^2/2m + V(q))dt/\hbar} \approx e^{-i\mathbf{p}^2 dt/2m\hbar} e^{-iV(q)dt/\hbar}. \quad (31)$$

The other is that the amplitude for the particle to go from position q_a to position q_1 in time dt is

$$\begin{aligned} \langle q_1 | e^{-i\mathbf{p}^2 dt/2m\hbar} e^{-iV(q)dt/\hbar} | q_a \rangle &= \langle q_1 | e^{-i\mathbf{p}^2 dt/2m\hbar} \int_{-\infty}^{\infty} |p'\rangle \langle p'| dp' e^{-iV(q)dt/\hbar} | q_a \rangle \\ &= e^{-iV(q_a)dt/\hbar} \int_{-\infty}^{\infty} e^{-ip'^2 dt/2m\hbar} e^{i(q_1 - q_a)p'/\hbar} \frac{dp'}{2\pi\hbar} \\ &= \left(\frac{m}{2\pi i \hbar dt} \right)^{1/2} \exp \left\{ i \left[\frac{m}{2} \left(\frac{q_1 - q_a}{dt} \right)^2 - V(q_a) \right] \frac{dt}{\hbar} \right\}. \end{aligned} \quad (32)$$

To link two of these matrix elements together, one integrates over the complete set of outer products $|q_1\rangle\langle q_1|$

$$\begin{aligned} \langle q_2 | \left(e^{-i\mathbf{p}^2 dt/2m\hbar} e^{-iV(q)dt/\hbar} \right)^2 | q_a \rangle \\ = \frac{m}{2\pi i \hbar dt} \int_{-\infty}^{\infty} \exp \left[i \left(\frac{m\dot{q}_a^2}{2} - V(q_1) + \frac{m\dot{q}_a^2}{2} - V(q_a) \right) \frac{dt}{\hbar} \right] dq_1 \end{aligned}$$

where now $\dot{q}_a = (q_1 - q_a)/dt$ and $\dot{q}_1 = (q_2 - q_1)/dt$. By stitching together $n = (t_b - t_a)/dt$ time intervals each of length dt and letting $n \rightarrow \infty$, we get the path integral for a particle of mass m to go non-relativistically from q_a to q_b in time t

$$\begin{aligned} \langle q_b | e^{-iHt/\hbar} | q_a \rangle &= \left(\frac{m}{2\pi i \hbar dt} \right)^{n/2} \int \exp \left(\frac{i}{\hbar} \int \left[\frac{m}{2} \dot{q}^2(t) - V(q(t)) \right] dt \right) \\ &\equiv \int e^{iS[q]/\hbar} Dq. \end{aligned} \quad (33)$$

Here the action is the integral $S[q] = \int L(q, \dot{q}) dt$. The amplitude is the integral over all paths from q_a to q_b each weighted by the phase $\exp(iS[q]/\hbar)$.

The amplitude for any state $|\Psi\rangle$ to become $|\Phi\rangle$ in time t is the integral over the fields A that represent all processes that go from $|\Psi\rangle$ to $|\Phi\rangle$ weighted by the phase $\exp(iS/\hbar)$ in which $S[A]$ is the action of the process

$$\langle \Phi | e^{-itH/\hbar} | \Psi \rangle = \int e^{iS[A]/\hbar} DA. \quad (34)$$

11. Supersymmetry

All the predictions of the standard model and of general relativity have been verified experimentally.

In contrast, supersymmetry predicts that every particle of spin s has a superpartner of spin $s \pm \hbar/2$. None have been observed, although they could be lurking at *much* higher energies.

12. Mysteries

The most beautiful thing we can experience is the Mysterious. We now have two huge mysteries:

1. Dark matter, i.e., invisible matter, is 84% of all matter.
2. Dark energy is 68% of the energy of the universe.

What are they?

And we have other mysteries. Why the Higgs? Why does it couple so weirdly? Why are the electroweak interactions so lopsided?

13. Interactions with right-handed fields

One way to balance the gauge group $SU_L(2)$ is to add the gauge group $SU_R(2)$ with a triplet of gauge bosons (W_R^+ , W_R^0 , W_R^-) that interact only with right-handed quarks and leptons. One also would add a new Higgs field H_R that transforms under $SU_R(2) \otimes U_{Y'}(1)$ in which the new weak hypercharge Y' has the eigenvalues of Y but with left and right interchanged.

There would be a total of seven new fields — three W_R 's and the four components of the $SU_R(2)$ doublet H_R . They would have to be very heavy or couple very weakly or both.

14. Summary

The equations of the standard model are invariant under the local symmetries $SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$ and also under the general coordinate transformations of general relativity. All the known interactions—the strong, weak, electromagnetic, and gravitational interactions—are required by these local symmetries.