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LEARNING LOW-DEGREE FUNCTIONS FROM A LOGARITHMIC NUMBER OF RANDOM QUERIES

ALEXANDROS ESKENAZIS AND PAATA IVANISVILI

ABSTRACT. We prove that every bounded function $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d can be learned with L_2 -accuracy ε and confidence $1 - \delta$ from $\log(\frac{n}{\delta}) \varepsilon^{-d-1} C^{d^{3/2}} \sqrt{\log d}$ random queries, where $C > 1$ is a universal finite constant.

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1. INTRODUCTION

Every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ admits a unique Fourier–Walsh expansion of the form

$$\forall x \in \{-1, 1\}^n, \quad f(x) = \sum_{S \subseteq \{1, \dots, n\}} \hat{f}(S) w_S(x), \quad (1)$$

where $w_S(x) = \prod_{i \in S} x_i$ and the Fourier coefficients $\hat{f}(S)$ are given by

$$\forall S \subseteq \{1, \dots, n\}, \quad \hat{f}(S) = \frac{1}{2^n} \sum_{y \in \{-1, 1\}^n} f(y) w_S(y). \quad (2)$$

We say that f has degree at most $d \in \{1, \dots, n\}$ if $\hat{f}(S) = 0$ for every subset S with $|S| > d$.

1.1. Learning functions on the hypercube. Let \mathcal{C} be a class of functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ on the n -dimensional discrete hypercube. The problem of learning the class \mathcal{C} can be described as follows: given a source of *examples* $(x, f(x))$, where $x \in \{-1, 1\}^n$, for an unknown function $f \in \mathcal{C}$, compute a *hypothesis* function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ which is a good approximation of f up to a given error in some prescribed metric. In this paper we will be interested in the *random query model* with L_2 -error, in which we are given N independent examples $(x, f(x))$, each chosen uniformly at random from the discrete hypercube $\{-1, 1\}^n$, and we want to efficiently construct a (random) function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ such that $\|h - f\|_{L_2}^2 < \varepsilon$ with probability at least $1 - \delta$, where $\varepsilon, \delta \in (0, 1)$ are given accuracy and confidence parameters. The goal is to construct a randomized algorithm which produces the hypothesis function h from a minimal number N of examples.

The above very general problem has been studied for decades in computational learning theory and many results are known¹, primarily for various classes \mathcal{C} of structured Boolean functions $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$. Already since the late 1980s, researchers used the Fourier–Walsh expansion (1) to design such learning algorithms (see the survey [14]). Perhaps the most classical of these is the *Low-Degree Algorithm* of Linial, Mansour and Nisan [12] who showed that for the class \mathcal{C}_b^d of all *bounded* functions $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d there exists an algorithm which produces an ε -approximation of f with probability at least $1 - \delta$ using $N = \frac{2n^d}{\varepsilon} \log(\frac{2n^d}{\delta})$ samples. In this generality, the $O_{\varepsilon, \delta, d}(n^d \log n)$ estimate of [12] was the state of the art until the recent work [11] of Iyer, Rao, Reis, Rothvoss and Yehudayoff who employed analytic techniques to derive new bounds on the ℓ_1 -size of the Fourier spectrum of bounded

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¹We will by no means attempt to survey this (vast) field, so we refer the interested reader to the relevant chapters of O’Donnell’s book [15] and the references therein.

functions (see also Section 3) and used these estimates to show that $N = O_{\varepsilon, \delta, d}(n^{d-1} \log n)$ examples suffice to learn \mathcal{C}_b^d . The goal of the present paper is to further improve this result and show that in fact $N = O_{\varepsilon, \delta, d}(\log n)$ samples suffice for this purpose.

Theorem 1. Fix $\varepsilon, \delta \in (0, 1)$, $n \in \mathbb{N}$, $d \in \{1, \dots, n\}$ and a bounded function $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d . If $N \in \mathbb{N}$ satisfies

$$N \geq \min \left\{ \frac{\exp(Cd^{3/2} \sqrt{\log d})}{\varepsilon^{d+1}}, \frac{4dn^d}{\varepsilon} \right\} \log \left(\frac{n}{\delta} \right), \quad (3)$$

where $C \in (0, \infty)$ is a large numerical constant, then N uniformly random independent queries of pairs $(x, f(x))$, where $x \in \{-1, 1\}^n$, suffice for the construction of a random function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ satisfying the condition $\|h - f\|_{L_2}^2 < \varepsilon$ with probability at least $1 - \delta$.

The proof of Theorem 1 relies on some important approximation theoretic estimates going back to the 1930s which we shall now describe (see also [9]). To the best of our knowledge, these tools had not yet been exploited in the computational learning theory literature.

1.2. The Fourier growth of Walsh polynomials in $\ell_{\frac{2d}{d+1}}$. Estimates for the growth of coefficients of polynomials as a function of their degree and their maximum on compact sets go back to the early days of approximation theory (see [5]). A seminal result of this nature is Littlewood's celebrated $\frac{4}{3}$ -inequality [13] for bilinear forms which was later generalized by Bohnenblust and Hille [4] for multilinear forms on the torus \mathbb{T}^n or the unit square $[-1, 1]^n$. By means of polarization, one can use this multilinear estimate to derive an inequality for polynomials which reads as follows². For every $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $d \in \mathbb{N}$, there exists $B_d^{\mathbb{K}} \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and every coefficients $c_\alpha \in \mathbb{K}$, where $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq d$, we have

$$\left(\sum_{|\alpha| \leq d} |c_\alpha|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}} \leq B_d^{\mathbb{K}} \max \left\{ \left| \sum_{|\alpha| \leq d} c_\alpha x^\alpha \right| : x \in \mathbb{K}^n \text{ with } \|x\|_{\ell_\infty^n(\mathbb{K})} \leq 1 \right\}. \quad (4)$$

Moreover, $\frac{2d}{d+1}$ is the smallest exponent for which the optimal constant in (4) is independent of the number of variables n of the polynomial. The exact asymptotics of the constants $B_d^{\mathbb{R}}$ and $B_d^{\mathbb{C}}$ remain unknown, however it is known that there is a significant gap between $B_d^{\mathbb{R}}$ and $B_d^{\mathbb{C}}$, namely that $\limsup_{d \rightarrow \infty} (B_d^{\mathbb{R}})^{1/d} = 1 + \sqrt{2}$ whereas $B_d^{\mathbb{C}} \leq C^{\sqrt{d \ln d}}$ for a finite constant $C > 1$ (see [7, 1, 9, 6, 8] for these and other important advances of the last decade). Restricting inequality (4) to real *multilinear* polynomials, convexity shows that the maximum on the right-hand side is attained at a point $x \in \{-1, 1\}^n$, which, in view of (1), makes (4) an estimate for the Fourier–Walsh growth of functions on the discrete hypercube. We shall denote by $B_d^{\{\pm 1\}}$ the corresponding optimal constant (first explicitly investigated by Blei in [3, p. 175]), that is, the least constant such that for every $n \in \mathbb{N}$ and every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ of degree at most d ,

$$\left(\sum_{S \subseteq \{1, \dots, n\}} |\hat{f}(S)|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}} \leq B_d^{\{\pm 1\}} \|f\|_{L_\infty}. \quad (5)$$

The best known quantitative result in this setting is due to Defant, Mastyló and Pérez [8] who showed that $B_d^{\{\pm 1\}} \leq \exp(\kappa \sqrt{d \log d})$ for a universal constant $\kappa \in (0, \infty)$. The main contribution of this work is the following theorem relating the growth of the constant $B_d^{\{\pm 1\}}$ and learning.

Theorem 2. Fix $\varepsilon, \delta \in (0, 1)$, $n \in \mathbb{N}$, $d \in \{1, \dots, n\}$ and a bounded function $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d . If $N \in \mathbb{N}$ satisfies

$$N \geq \frac{e^8 d^2}{\varepsilon^{d+1}} (B_d^{\{\pm 1\}})^{2d} \log \left(\frac{n}{\delta} \right), \quad (6)$$

²For $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, we use the standard notations $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

then given N uniformly random independent queries of pairs $(x, f(x))$, where $x \in \{-1, 1\}^n$, one can construct a random function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ satisfying $\|h - f\|_{L_2}^2 < \varepsilon$ with probability at least $1 - \delta$.

In Section 2 we will prove Theorem 2 and use it to derive Theorem 1. In Section 3 we will present some additional remarks on Boolean analysis and learning, in particular showing that the dependence on n in Theorem 1 is optimal for $\delta \asymp \frac{1}{n}$. Moreover, we shall improve the recent bounds of [11] on the ℓ_1 -Fourier growth of bounded functions of low degree.

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2. PROOFS

Proof of Theorem 2. Fix a parameter $b \in (0, \infty)$ and denote by

$$N_b \stackrel{\text{def}}{=} \left\lceil \frac{2}{b^2} \log \left(\frac{2}{\delta} \sum_{k=0}^d \binom{n}{k} \right) \right\rceil. \quad (7)$$

Let X_1, \dots, X_{N_b} be independent random vectors, each uniformly distributed on $\{-1, 1\}^n$. For a subset $S \subseteq \{1, \dots, n\}$ with $|S| \leq d$ consider the empirical Walsh coefficient of f , given by

$$\alpha_S = \frac{1}{N_b} \sum_{j=1}^{N_b} f(X_j) w_S(X_j). \quad (8)$$

As α_S is a sum of bounded i.i.d. random variables and $\mathbb{E}[\alpha_S] = \hat{f}(S)$, the Chernoff bound gives

$$\forall S \subseteq \{1, \dots, n\}, \quad \mathbb{P}\{|\alpha_S - \hat{f}(S)| > b\} \leq 2 \exp(-N_b b^2/2). \quad (9)$$

Therefore, using the union bound and taking into account that f has degree at most d , we get

$$\underbrace{\mathbb{P}\{|\alpha_S - \hat{f}(S)| \leq b, \text{ for every } S \subseteq \{1, \dots, n\} \text{ with } |S| \leq d\}}_{G_b} \geq 1 - 2 \sum_{k=0}^d \binom{n}{k} \exp(-N_b b^2/2) \stackrel{(7)}{\geq} 1 - \delta. \quad (10)$$

Fix an additional parameter $a \in (b, \infty)$ and consider the random collection of sets given by

$$\mathcal{S}_a \stackrel{\text{def}}{=} \{S \subseteq \{1, \dots, n\} : |\alpha_S| \geq a\}. \quad (11)$$

Observe that if the event G_b of equation (10) holds, then

$$\forall S \notin \mathcal{S}_a, \quad |\hat{f}(S)| \leq |\alpha_S - \hat{f}(S)| + |\alpha_S| < a + b \quad (12)$$

and

$$\forall S \in \mathcal{S}_a, \quad |\hat{f}(S)| \geq |\alpha_S| - |\alpha_S - \hat{f}(S)| \geq a - b. \quad (13)$$

Finally, consider the random function $h_{a,b} : \{-1, 1\}^n \rightarrow \mathbb{R}$ given by

$$\forall x \in \{-1, 1\}^n, \quad h_{a,b}(x) \stackrel{\text{def}}{=} \sum_{S \in \mathcal{S}_a} \alpha_S w_S(x). \quad (14)$$

Combining (13) with inequality (5), we deduce that

$$|\mathcal{S}_a| \stackrel{(13)}{\leq} (a-b)^{-\frac{2d}{d+1}} \sum_{S \in \mathcal{S}_a} |\hat{f}(S)|^{\frac{2d}{d+1}} \leq (a-b)^{-\frac{2d}{d+1}} \sum_{S \subseteq \{1, \dots, n\}} |\hat{f}(S)|^{\frac{2d}{d+1}} \stackrel{(5)}{\leq} (a-b)^{-\frac{2d}{d+1}} (B_d^{\{\pm 1\}})^{\frac{2d}{d+1}}. \quad (15)$$

Therefore, on the event G_b we have

$$\begin{aligned} \|h_{a,b} - f\|_{L_2}^2 &= \sum_{S \subseteq \{1, \dots, n\}} |\hat{h}_{a,b}(S) - \hat{f}(S)|^2 = \sum_{S \in \mathcal{S}_a} |\alpha_S - \hat{f}(S)|^2 + \sum_{S \notin \mathcal{S}_a} |\hat{f}(S)|^2 \\ &\stackrel{(12)}{<} |\mathcal{S}_a| b^2 + (a+b)^{\frac{2}{d+1}} \sum_{S \notin \mathcal{S}_a} |\hat{f}(S)|^{\frac{2d}{d+1}} \stackrel{(5) \wedge (15)}{\leq} (B_d^{\{\pm 1\}})^{\frac{2d}{d+1}} \left((a-b)^{-\frac{2d}{d+1}} b^2 + (a+b)^{\frac{2}{d+1}} \right). \end{aligned} \quad (16)$$

Choosing $a = b(1 + \sqrt{d+1})$, we deduce that

$$\|h_{b(1+\sqrt{d+1}),b} - f\|_{L_2}^2 < (B_d^{\{\pm 1\}})^{\frac{2d}{d+1}} b^{\frac{2}{d+1}} ((d+1)^{-\frac{d}{d+1}} + (2 + \sqrt{d+1})^{\frac{2}{d+1}}). \quad (17)$$

Next, we need the technical inequality

$$(d+1)^{-\frac{d}{d+1}} + (2 + \sqrt{d+1})^{\frac{2}{d+1}} \leq (e^4(d+1))^{\frac{1}{d+1}} \quad \text{for all } d \geq 1. \quad (18)$$

Rearranging the terms, it suffices to show that $(2 + \sqrt{d+1})^{\frac{2}{d+1}} \leq (d+1)^{\frac{1}{d+1}} (e^{\frac{4}{d+1}} - \frac{1}{d+1})$, which is equivalent to $(\frac{2}{\sqrt{d+1}} + 1)^{\frac{2}{d+1}} \leq e^{\frac{4}{d+1}} - \frac{1}{d+1}$. We have

$$\left(\frac{2}{\sqrt{d+1}} + 1\right)^{\frac{2}{d+1}} \leq (\sqrt{2} + 1)^{\frac{2}{d+1}} \stackrel{(*)}{\leq} 1 + \frac{3}{d+1} \leq e^{\frac{4}{d+1}} - \frac{1}{d+1}, \quad (19)$$

where inequality $(*)$ holds because the left hand side is convex in the variable $\lambda \stackrel{\text{def}}{=} \frac{2}{d+1}$ whereas the right hand side is linear and since $(*)$ holds at the endpoints $\lambda = 0, 1$.

Combining (17) and (18) we see that $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L_2}^2 < \varepsilon$ holds for $b^2 \leq e^{-5} d^{-1} \varepsilon^{d+1} (B_d^{\{\pm 1\}})^{-2d}$. Plugging this choice of b in (7) shows that given N random queries, where

$$N = \left\lceil \frac{e^6 d (B_d^{\{\pm 1\}})^{2d}}{\varepsilon^{d+1}} \log \left(\frac{2}{\delta} \sum_{k=0}^d \binom{n}{k} \right) \right\rceil, \quad (20)$$

the random function $h_{b(1+\sqrt{d+1}),b}$ satisfies $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L_2}^2 < \varepsilon$ with probability at least $1 - \delta$ and the conclusion of the theorem follows from elementary estimates, such as

$$\sum_{k=0}^d \binom{n}{k} \leq \sum_{k=0}^d \frac{n^k}{k!} = \sum_{k=0}^d \frac{d^k}{k!} \left(\frac{n}{d}\right)^k \leq \left(\frac{en}{d}\right)^d. \quad \square$$

Theorem 1 is a straightforward consequence of Theorem 2.

Proof of Theorem 1. Theorem 2 combined with the bound $B_d^{\{\pm 1\}} \leq \exp(\kappa \sqrt{d \log d})$ of [8] imply the conclusion of Theorem 1 for $\varepsilon \geq \frac{\exp(C \sqrt{d \log d})}{n}$, where $C \in (0, \infty)$ is a large universal constant. The case $\varepsilon < \frac{\exp(C \sqrt{d \log d})}{n}$ follows from the Low-Degree Algorithm of [12]. \square

3. CONCLUDING REMARKS

We conclude with a few additional remarks on the spectrum of bounded functions defined on the hypercube and corresponding learning algorithms. For a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$, its Rademacher projection on level $\ell \in \{1, \dots, n\}$ is defined as

$$\forall x \in \{-1, 1\}^n, \quad \text{Rad}_\ell f(x) = \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} \hat{f}(S) w_S(x). \quad (21)$$

1. The first main theorem of [11] asserts that if $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ is a function of degree d , then

$$\forall \ell \in \{1, \dots, d\}, \quad \|\text{Rad}_\ell f\|_{L_\infty} \leq \begin{cases} \frac{|T_d^{(\ell)}(0)|}{\ell!} \cdot \|f\|_{L_\infty}, & \text{if } (d - \ell) \text{ is even} \\ \frac{|T_{d-1}^{(\ell)}(0)|}{\ell!} \cdot \|f\|_{L_\infty}, & \text{if } (d - \ell) \text{ is odd} \end{cases}, \quad (22)$$

where $T_d(t)$ is the d -th Chebyshev polynomial of the first kind, that is, the unique real polynomial of degree d such that $\cos(d\theta) = T_d(\cos \theta)$ for every $\theta \in \mathbb{R}$. Moreover, Iyer, Rao, Reis, Rothvoss and Yehudayoff observed in [11, Proposition 2] that this estimate is asymptotically sharp. We present a simple proof of their inequality (22) (see also [10] for related arguments).

Proof of (22). For any $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ consider its harmonic extension on $[-1, 1]^n$,

$$\forall (x_1, \dots, x_n) \in [-1, 1]^n, \quad \tilde{f}(x_1, \dots, x_n) = \sum_{S \subseteq \{1, \dots, n\}} \hat{f}(S) \prod_{j \in S} x_j. \quad (23)$$

By convexity $\|\tilde{f}\|_{L^\infty([-1, 1]^n)} = \|f\|_{L^\infty(\{-1, 1\}^n)}$. In particular, the restriction of \tilde{f} on the ray $t(x_1, \dots, x_n)$, $t \in [-1, 1]$, i.e.

$$\forall t \in \mathbb{R}, \quad h_x(t) \stackrel{\text{def}}{=} \sum_{S \subseteq \{1, \dots, n\}} \hat{f}(S) w_S(x) t^{|S|} \quad (24)$$

satisfies $\max_{t \in [-1, 1]} |h_x(t)| \leq \|f\|_{L^\infty}$ for all $(x_1, \dots, x_n) \in \{-1, 1\}^n$. Therefore, since $\deg h_x \leq d$, a classical inequality of Markov (see e.g. [5, p. 248]) gives

$$|\text{Rad}_\ell f(x)| = \frac{|h_x^{(\ell)}(0)|}{\ell!} \leq \begin{cases} \frac{|T_d^{(\ell)}(0)|}{\ell!} \cdot \|f\|_{L^\infty}, & \text{if } (d - \ell) \text{ is even} \\ \frac{|T_{d-1}^{(\ell)}(0)|}{\ell!} \cdot \|f\|_{L^\infty}, & \text{if } (d - \ell) \text{ is odd} \end{cases} \quad (25)$$

and (22) follows by taking a maximum over all $x \in \{-1, 1\}^n$. \square

In particular, as observed in [11], inequality (22) implies that if f has degree at most d then

$$\forall \ell \in \{1, \dots, d\}, \quad \|\text{Rad}_\ell f\|_{L^\infty} \leq \frac{d^\ell}{\ell!} \cdot \|f\|_{L^\infty}. \quad (26)$$

2. The second main theorem of [11] asserts that if $f : \{-1, 1\}^n \rightarrow [-1, 1]$ is a bounded function of degree at most d , then for every $\ell \in \{1, \dots, d\}$ we have

$$\sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} |\widehat{\text{Rad}_\ell f}(S)| = \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} |\hat{f}(S)| \leq n^{\frac{\ell-1}{2}} d^\ell e^{\binom{\ell+1}{2}}. \quad (27)$$

The Bohnenblust–Hille-type inequality of [8] implies the following improved bound.

Corollary 3. *Let $n \in \mathbb{N}$ and $d \in \{1, \dots, n\}$. Then, every bounded function $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d satisfies*

$$\forall \ell \in \{1, \dots, d\}, \quad \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} |\hat{f}(S)| \leq \binom{n}{\ell}^{\frac{\ell-1}{2\ell}} e^{\kappa \sqrt{\ell \log \ell}} \frac{d^\ell}{\ell!} \leq n^{\frac{\ell-1}{2}} d^\ell \ell^{-c\ell}, \quad (28)$$

for some universal constant $c \in (0, 1)$.

Proof. Combining Hölder's inequality with the estimate of [8] and (26) we get

$$\begin{aligned} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} |\hat{f}(S)| &\leq \binom{n}{\ell}^{\frac{\ell-1}{2\ell}} \left(\sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = \ell}} |\widehat{\text{Rad}_\ell f}(S)|^{\frac{2\ell}{\ell+1}} \right)^{\frac{\ell+1}{2\ell}} \\ &\leq \binom{n}{\ell}^{\frac{\ell-1}{2\ell}} \exp(\kappa \sqrt{\ell \log \ell}) \|\text{Rad}_\ell f\|_{L^\infty} \stackrel{(26)}{\leq} \binom{n}{\ell}^{\frac{\ell-1}{2\ell}} \exp(\kappa \sqrt{\ell \log \ell}) \frac{d^\ell}{\ell!}. \end{aligned} \quad (29)$$

The last inequality of (28) follows from (22) and the elementary bound $\binom{n}{\ell} \leq \left(\frac{ne}{\ell}\right)^\ell$. \square

We refer to the recent work [2] for a systematic study of inequalities relating the Fourier growth with various well-studied properties of Boolean functions.

3. It is straightforward to observe (see also [15, Proposition 3.31]) that if $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is a Boolean function and $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ is an arbitrary function, then

$$\|\text{sign}(h) - f\|_{L_2}^2 = 4\mathbb{P}\{\text{sign}(h) \neq f\} \leq 4\mathbb{P}\{|h - f| \geq 1\} \leq 4\|h - f\|_{L_2}^2, \quad (30)$$

where we define $\text{sign}(0)$ as ± 1 arbitrarily. Therefore, applying Theorem 1 to a Boolean function, the above algorithm produces a *Boolean* function $\tilde{h} = \text{sign}(h)$ which is a 4ε -approximation of f .

4. In Theorem 1 we showed that bounded functions $f : \{-1, 1\}^n \rightarrow [-1, 1]$ of degree at most d can be learned with accuracy at most ε and confidence at least $1 - \delta$ from $N = O_{\varepsilon, d}(\log(n/\delta))$ random queries. We will now show that this estimate is sharp for small enough values of δ .

Proposition 4. *Suppose that bounded linear functions $\ell : \{-1, 1\}^n \rightarrow [-1, 1]$ can be learned with accuracy at most $\frac{1}{2}$ and confidence at least $1 - \frac{1}{2n}$ from N random queries. Then $N > \log_2 n$.*

Proof. By the assumption, for any input $(X_1, y_1), \dots, (X_N, y_N) \in \{-1, 1\}^n \times [-1, 1]$, there exists a function $h_{(X_1, y_1), \dots, (X_N, y_N)} : \{-1, 1\}^n \rightarrow \mathbb{R}$ such that if X_1, \dots, X_N are chosen independently and uniformly from $\{-1, 1\}^n$ and there exists a linear function $\ell : \{-1, 1\}^n \rightarrow [-1, 1]$ such that $y_j = \ell(X_j)$ for every $j \in \{1, \dots, N\}$, then $\mathbb{P}(\Omega_\ell) > 1 - \frac{1}{2n}$, where Ω_ℓ is the event

$$\Omega_\ell \stackrel{\text{def}}{=} \left\{ \mathbb{E} \left(h_{(X_1, \ell(X_1)), \dots, (X_N, \ell(X_N))} - \ell \right)^2 < \frac{1}{2} \right\}. \quad (31)$$

Let $X_j = (X_j(1), \dots, X_j(n))$ for $j \in \{1, \dots, N\}$ and consider the event

$$\mathcal{W} = \left\{ X_j(1) = X_j(2), \forall j \in \{1, \dots, N\} \right\}. \quad (32)$$

By the independence of the samples, we have $\mathbb{P}(\mathcal{W}) = \frac{1}{2^N}$. Therefore, if $N \leq \log_2 n$ and we consider the linear functions $r_i : \{-1, 1\}^n \rightarrow \{-1, 1\}$ given by $r_i(x) = x_i$, then

$$\mathbb{P}(\Omega_{r_1} \cap \Omega_{r_2}) > 1 - \frac{1}{n} \geq 1 - \frac{1}{2^N} = 1 - \mathbb{P}(\mathcal{W}), \quad (33)$$

which implies that $\Omega_{r_1} \cap \Omega_{r_2} \cap \mathcal{W} \neq \emptyset$. Choosing X_1, \dots, X_N from this event and denoting by $h = h_{(X_1, X_1(1)), \dots, (X_N, X_N(1))} = h_{(X_1, X_1(2)), \dots, (X_N, X_N(2))}$, we deduce from the triangle inequality that

$$2 = \mathbb{E}(r_1 - r_2)^2 \leq 2\mathbb{E}(h - r_1)^2 + 2\mathbb{E}(h - r_2)^2 \stackrel{(31)}{<} 2 \quad (34)$$

which is clearly a contradiction. Therefore $N > \log_2 n$. \square

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