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LEARNING LOW-DEGREE FUNCTIONS FROM A LOGARITHMIC NUMBER OF RANDOM QUERIES

ALEXANDROS ESKENAZIS AND PAATA IVANISVILI

Abstract. We prove that every bounded function $f : [-1, 1]^n \to [-1, 1]$ of degree at most $d$ can be learned with $L_2$-accuracy $\epsilon$ and confidence $1 - \delta$ from $\log \left(\frac{n}{d}\right) e^{-d - 1} C^{d/2} \sqrt{\log d}$ random queries, where $C > 1$ is a universal finite constant.

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1. Introduction

Every function $f : [-1, 1]^n \to \mathbb{R}$ admits a unique Fourier–Walsh expansion of the form

$$\forall x \in \{-1, 1\}^n, \quad f(x) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) w_S(x),$$

where $w_S(x) = \prod_{i \in S} x_i$ and the Fourier coefficients $\hat{f}(S)$ are given by

$$\forall S \subseteq \{1, \ldots, n\}, \quad \hat{f}(S) = \frac{1}{2^n} \sum_{y \in \{-1, 1\}^n} f(y) w_S(y).$$

We say that $f$ has degree at most $d \in \{1, \ldots, n\}$ if $\hat{f}(S) = 0$ for every subset $S$ with $|S| > d$.

1.1. Learning functions on the hypercube. Let $\mathcal{C}$ be a class of functions $f : [-1, 1]^n \to \mathbb{R}$ on the $n$-dimensional discrete hypercube. The problem of learning the class $\mathcal{C}$ can be described as follows: given a source of examples $(x, f(x))$, where $x \in [-1, 1]^n$, for an unknown function $f \in \mathcal{C}$, compute a hypothesis function $h : [-1, 1]^n \to \mathbb{R}$ which is a good approximation of $f$ up to a given error in some prescribed metric. In this paper we will be interested in the random query model with $L_2$-error, in which we are given $N$ independent examples $(x, f(x))$, each chosen uniformly at random from the discrete hypercube $[-1, 1]^n$, and we want to efficiently construct a (random) function $h : [-1, 1]^n \to \mathbb{R}$ such that $\|h - f\|_2^2 < \epsilon$ with probability at least $1 - \delta$, where $\epsilon, \delta \in (0, 1)$ are given accuracy and confidence parameters. The goal is to construct a randomized algorithm which produces the hypothesis function $h$ from a minimal number $N$ of examples.

The above very general problem has been studied for decades in computational learning theory and many results are known\(^1\), primarily for various classes $\mathcal{C}$ of structured Boolean functions $f : [-1, 1]^n \to [-1, 1]$. Already since the late 1980s, researchers used the Fourier–Walsh expansion (1) to design such learning algorithms (see the survey [14]). Perhaps the most classical of these is the Low-Degree Algorithm of Linial, Mansour and Nisan [12] who showed that for the class $\mathcal{C}_b^d$ of all bounded functions $f : [-1, 1]^n \to [-1, 1]$ of degree at most $d$ there exists an algorithm which produces an $\epsilon$-approximation of $f$ with probability at least $1 - \delta$ using $N = \frac{2 \mu^d}{\epsilon^2} \log \frac{2 \mu^d}{\epsilon}$ samples. In this generality, the $O_{\epsilon, \delta, d}(n^d \log n)$ estimate of [12] was the state of the art until the recent work [11] of Iyer, Rao, Reis, Rothvoss and Yehudayoff who employed analytic techniques to derive new bounds on the $\ell_1$-size of the Fourier spectrum of bounded

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\(^1\)We will by no means attempt to survey this (vast) field, so we refer the interested reader to the relevant chapters of O’Donnell’s book [15] and the references therein.

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functions (see also Section 3) and used these estimates to show that \( N = O_{\epsilon, \delta, d}(n^{d-1} \log n) \) examples suffice to learn \( \ell^1 \). The goal of the present paper is to further improve this result and show that in fact \( N = O_{\epsilon, \delta, d}(\log n) \) samples suffice for this purpose.

**Theorem 1.** Fix \( \epsilon, \delta \in (0, 1) \), \( n \in \mathbb{N} \), \( d \in [1, \ldots, n] \) and a bounded function \( f : [-1, 1]^n \to [-1, 1] \) of degree at most \( d \). If \( N \in \mathbb{N} \) satisfies

\[
N \geq \min \left\{ \frac{\exp(C_d^{3/2} \sqrt{\log d})}{\epsilon^{d+1}}, \frac{4d! n^{d}}{\epsilon} \right\} \log \left( \frac{n}{\delta} \right),
\]

where \( C \in (0, \infty) \) is a large numerical constant, then \( N \) uniformly random independent queries of pairs \( (x, f(x)) \), where \( x \in [-1, 1]^n \), suffice for the construction of a random function \( h : [-1, 1]^n \to \mathbb{R} \) satisfying the condition \( \|h - f\|_{L^2} \leq \epsilon \) with probability at least \( 1 - \delta \).

The proof of Theorem 1 relies on some important approximation theoretic estimates going back to the 1930s which we shall now describe (see also [9]). To the best of our knowledge, these tools had not yet been exploited in the computational learning theory literature.

### 1.2. The Fourier growth of Walsh polynomials in \( \ell^2 \). Estimates for the growth of coefficients of polynomials as a function of their degree and their maximum on compact sets go back to the early days of approximation theory (see [5]). A seminal result of this nature is Littlewood’s celebrated \( \frac{2}{3} \)-inequality [13] for bilinear forms which was later generalized by Bohnenblust and Hille [4] for multilinear forms on the torus \( T^n \) or the unit square \([-1, 1]^n \). By means of polarization, one can use this multilinear estimate to derive an inequality for polynomials which reads as follows\(^2\). For every \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \) and \( d \in \mathbb{N} \), there exists \( B_d^{\mathbb{K}} \in (0, \infty) \) such that for every \( n \in \mathbb{N} \) and every coefficients \( c_n \in \mathbb{K} \), where \( a \in (\mathbb{N} \cup \{0\})^n \) with \( |a| \leq d \), we have

\[
\left( \sum_{|a| \leq d} |c_a|^2 \right)^{d+1} d^{d+1} \leq B_d^{\mathbb{K}} \max \left\{ \left\| \sum_{|a| \leq d} c_a x^a \right\|_{\mathbb{K}} : x \in \mathbb{K}^n \text{ with } \|x\|_{\ell^2(\mathbb{K})} \leq 1 \right\}.
\]

Moreover, \( \frac{2}{d+1} \) is the smallest exponent for which the optimal constant in (4) is independent of the number of variables \( n \) of the polynomial. The exact asymptotics of the constants \( B_d^{\mathbb{R}} \) and \( B_d^{\mathbb{C}} \) remain unknown, however it is known that there is a significant gap between \( B_d^{\mathbb{R}} \) and \( B_d^{\mathbb{C}} \), namely that \( \limsup_{d \to \infty} (B_d^{\mathbb{R}})^{1/d} = 1 + \sqrt{2} \) whereas \( B_d^{\mathbb{C}} \leq C \sqrt{d \log d} \) for a finite constant \( C > 1 \) (see [7, 1, 9, 6, 8] for these and other important advances of the last decade). Restricting inequality (4) to real multilinear polynomials, convexity shows that the maximum on the right-hand side is attained at a point \( x \in [-1, 1]^n \), which, in view of (1), makes (4) an estimate for the Fourier-Walsh growth of functions on the discrete hypercube. We shall denote by \( B_d^{[1+]} \) the corresponding optimal constant (first explicitly investigated by Blei in [3, p. 175]), that is, the least constant such that for every \( n \in \mathbb{N} \) and every function \( f : [-1, 1]^n \to \mathbb{R} \) of degree at most \( d \),

\[
\left( \sum_{S \subseteq [1, \ldots, n]} \| f(S) \|_{\ell^2}^{2d+1} \right)^{d+1} \leq B_d^{[1+]} \| f \|_{L^\infty}.
\]

The best known quantitative result in this setting is due to Defant, Mastylo and Pérez [8] who showed that \( B_d^{[1+]} \leq \exp(\kappa \sqrt{d \log d}) \) for a universal constant \( \kappa \in (0, \infty) \). The main contribution of this work is the following theorem relating the growth of the constant \( B_d^{[1+]} \) and learning.

**Theorem 2.** Fix \( \epsilon, \delta \in (0, 1) \), \( n \in \mathbb{N} \), \( d \in [1, \ldots, n] \) and a bounded function \( f : [-1, 1]^n \to [-1, 1] \) of degree at most \( d \). If \( N \in \mathbb{N} \) satisfies

\[
N \geq \frac{e^{8d^2} \epsilon d^{2d} (B_d^{[1+]})^{2d} \log \left( \frac{n}{\delta} \right)},
\]

For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \), we use the standard notations \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).
then given $N$ uniformly random independent queries of pairs $(x, f(x))$, where $x \in \{-1, 1\}^n$, one can construct a random function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ satisfying $\|h - f\|_{L_2}^2 < \epsilon$ with probability at least $1 - \delta$.

In Section 2 we will prove Theorem 2 and use it to derive Theorem 1. In Section 3 we will present some additional remarks on Boolean analysis and learning, in particular showing that the dependence on $n$ in Theorem 1 is optimal for $\delta \approx \frac{1}{n}$. Moreover, we shall improve the recent bounds of [11] on the $\ell_1$-Fourier growth of bounded functions of low degree.

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2. Proofs

Proof of Theorem 2. Fix a parameter $b \in (0, \infty)$ and denote by

$$N_b \overset{\text{def}}{=} \left\lceil \frac{2}{b^2} \log \left( \frac{2d}{\delta} \sum_{k=0}^d \binom{n}{k} \right) \right\rceil. \quad \text{(7)}$$

Let $X_1, \ldots, X_{N_b}$ be independent random vectors, each uniformly distributed on $\{-1, 1\}^n$. For a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq d$ consider the empirical Walsh coefficient of $f$, given by

$$\alpha_S = \frac{1}{N_b} \sum_{j=1}^{N_b} f(X_j) w_S(X_j). \quad \text{(8)}$$

As $\alpha_S$ is a sum of bounded i.i.d. random variables and $\mathbb{E}[\alpha_S] = \hat{f}(S)$, the Chernoff bound gives

\[ \forall \ S \subseteq \{1, \ldots, n\}, \quad \mathbb{P}\{ |\alpha_S - \hat{f}(S)| > b \} \leq 2 \exp(-N_b b^2/2). \quad \text{(9)} \]

Therefore, using the union bound and taking into account that $f$ has degree at most $d$, we get

\[ \mathbb{P}\{ |\alpha_S - \hat{f}(S)| \leq b, \ \forall \ S \subseteq \{1, \ldots, n\} \} \geq 1 - \sum_{k=0}^d \binom{n}{k} \exp(-N_b b^2/2) \geq 1 - \delta. \quad \text{(10)} \]

Fix an additional parameter $a \in (b, \infty)$ and consider the random collection of sets given by

$$S_a \overset{\text{def}}{=} \{ S \subseteq \{1, \ldots, n\} : |\alpha_S| \geq a \}. \quad \text{(11)}$$

Observe that if the event $G_b$ of equation (10) holds, then

\[ \forall \ S \not\in S_a, \quad |\hat{f}(S)| \leq |\alpha_S - \hat{f}(S)| + |\alpha_S| \leq a + b \quad \text{(12)} \]

and

\[ \forall \ S \in S_a, \quad |\hat{f}(S)| \geq |\alpha_S| - |\alpha_S - \hat{f}(S)| \geq a - b. \quad \text{(13)} \]

Finally, consider the random function $h_{a,b} : \{-1, 1\}^n \rightarrow \mathbb{R}$ given by

\[ \forall \ x \in \{-1, 1\}^n, \quad h_{a,b}(x) \overset{\text{def}}{=} \sum_{S \in S_a} \alpha_S w_S(x). \quad \text{(14)} \]

Combining (13) with inequality (5), we deduce that

\[ |S_a| \overset{(13)}{\leq} (a - b)^{-\frac{2d}{d+1}} \sum_{S \in S_a} |\hat{f}(S)|^{\frac{2d}{d+1}} \leq (a - b)^{-\frac{2d}{d+1}} \sum_{S \subseteq \{1, \ldots, n\}} |\hat{f}(S)|^{\frac{2d}{d+1}} \overset{(5)}{\leq} (a - b)^{-\frac{2d}{d+1}} (B_d^{[\pm1]})^{\frac{2d}{d+1}}. \quad \text{(15)} \]

Therefore, on the event $G_b$ we have

\[ \|h_{a,b} - f\|_{L_2}^2 = \sum_{S \subseteq \{1, \ldots, n\}} \left| h_{a,b}(S) - \hat{f}(S) \right|^2 = \sum_{S \in S_a} |\alpha_S - \hat{f}(S)|^2 + \sum_{S \not\in S_a} |\hat{f}(S)|^2 \overset{(12)}{\leq} |S_a| b^2 + (a + b)^{\frac{2d}{d+1}} \sum_{S \in S_a} |\hat{f}(S)|^{\frac{2d}{d+1}} \overset{(5) \land (15)}{\leq} (B_d^{[\pm1]})^{\frac{2d}{d+1}} \left( (a - b)^{-\frac{2d}{d+1}} b^2 + (a + b)^{\frac{2d}{d+1}} \right). \quad \text{(16)} \]
Choosing $a = b(1 + \sqrt{d+1})$, we deduce that

$$
\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < (B_d^{[\pm 1]})^{2\sqrt{d+1}} b^{\frac{2}{\sqrt{d+1}}} ((d+1)^{-\frac{2}{\sqrt{d+1}}} + 2 + d^{\frac{1}{\sqrt{d+1}}})^{\frac{2}{\sqrt{d+1}}}.
$$

(17)

Next, we need the technical inequality

$$
(d+1)^{-\frac{2}{\sqrt{d+1}}} + 2 + d^{\frac{1}{\sqrt{d+1}}} \leq (e^{d+1})^{\frac{2}{\sqrt{d+1}}} \quad \text{for all } d \geq 1.
$$

(18)

Rearranging the terms, it suffices to show that $(2 + \sqrt{d+1})^{\frac{2}{\sqrt{d+1}}} \leq (d+1)^{\frac{1}{\sqrt{d+1}}}(e^{\frac{1}{\sqrt{d+1}}} - \frac{1}{d+1})$, which is equivalent to $(\frac{2}{d+1} + 1)^{\frac{2}{\sqrt{d+1}}} \leq (\sqrt{2} + 1)^{\frac{2}{\sqrt{d+1}}} \leq 1 + \frac{3}{d+1} \leq e^{\frac{1}{\sqrt{d+1}}} - \frac{1}{d+1},$

(19)

where inequality (*) holds because the left hand side is convex in the variable $\lambda \overset{\text{def}}{=} \frac{2}{d+1}$ whereas the right hand side is linear and since (*) holds at the endpoints $\lambda = 0, 1$.

Combining (17) and (18) we see that $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < \varepsilon$ holds for $b^2 \leq e^{-5}d^{-1}e^{d+1}(B_d^{[\pm 1]})^{-2d}$.

Plugging this choice of $b$ in (7) shows that given $N$ random queries, where

$$
N = \left\lceil \frac{e^{\delta d}(B_d^{[\pm 1]})^{2d}}{\varepsilon^{d+1}} \log \left( \frac{2d}{\delta} \sum_{k=0}^{d} \binom{n}{k} \right) \right\rceil,
$$

(20)

the random function $h_{b(1+\sqrt{d+1}),b}$ satisfies $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < \varepsilon$ with probability at least $1 - \delta$ and the conclusion of the theorem follows from elementary estimates, such as

$$
\sum_{k=0}^{d} \binom{n}{k} \leq \sum_{k=0}^{d} \frac{n^k}{k!} = \sum_{k=0}^{d} \frac{d^k}{k!} \binom{n}{d} \leq \left( \frac{e\sqrt{n}}{d} \right)^d.
$$

\[\square\]

Theorem 1 is a straightforward consequence of Theorem 2.

**Proof of Theorem 1.** Theorem 2 combined with the bound $B_d^{[\pm 1]} \leq \exp(k\sqrt{d\log d})$ of [8] imply the conclusion of Theorem 1 for $\varepsilon \geq \frac{\exp(C\sqrt{d\log d})}{n}$, where $C \in (0, \infty)$ is a large universal constant.

The case $\varepsilon < \frac{\exp(C\sqrt{d\log d})}{n}$ follows from the Low-Degree Algorithm of [12].

\[\square\]

3. **Concluding remarks**

We conclude with a few additional remarks on the spectrum of bounded functions defined on the hypercube and corresponding learning algorithms. For a function $f : \{-1,1\}^n \to \mathbb{R}$, its Rademacher projection on level $\ell \in \{1, \ldots, n\}$ is defined as

$$
\forall x \in \{-1,1\}^n, \quad \text{Rad}_\ell f(x) = \sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} \hat{f}(S)\omega_S(x).
$$

(21)

1. The first main theorem of [11] asserts that if $f : \{-1,1\}^n \to \mathbb{R}$ is a function of degree $d$, then

$$
\forall \ell \in \{1, \ldots, n\}, \quad \|\text{Rad}_\ell f\|_{L^2} \leq \begin{cases}
\frac{|T_d^1(0)|}{T_d^1(\ell)} \cdot \|f\|_{L^2}, & \text{if } (d-\ell) \text{ is even} \\
\frac{|T_d^1(0)|}{T_d^1(\ell)} \cdot \|f\|_{L^2}, & \text{if } (d-\ell) \text{ is odd}
\end{cases},
$$

(22)

where $T_d(t)$ is the $d$-th Chebyshev polynomial of the first kind, that is, the unique real polynomial of degree $d$ such that $\cos(\theta) = T_d(\cos(\theta))$ for every $\theta \in \mathbb{R}$. Moreover, Iyer, Rao, Reis, Rothvoss and Yehudayoff observed in [11, Proposition 2] that this estimate is asymptotically sharp. We present a simple proof of their inequality (22) (see also [10] for related arguments).
Proof of (22). For any \( f : \{-1,1\}^n \to \mathbb{R} \) consider its harmonic extension on \([-1,1]^n\),
\[
\forall (x_1, \ldots, x_n) \in [-1,1]^n, \quad \tilde{f}(x_1, \ldots, x_n) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) \prod_{j \in S} x_j.
\] (23)
By convexity \( \|\tilde{f}\|_{L^\infty([-1,1]^n)} = \|f\|_{L^\infty([-1,1]^n)} \). In particular, the restriction of \( \tilde{f} \) on the ray \( t(x_1, \ldots, x_n), \ t \in [-1,1], \) i.e.
\[
\forall \ t \in \mathbb{R}, \quad h_x(t) \overset{\text{def}}{=} \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) w_S(x)t^{|S|}
\] (24)
satisfies \( \max_{t \in \{-1,1\}} |h_x(t)| \leq \|f\|_{L^\infty} \) for all \( (x_1, \ldots, x_n) \in \{-1,1\}^n \). Therefore, since \( \deg h_x \leq d \), a classical inequality of Markov (see e.g. [5, p. 248]) gives
\[
|\Rad_{\ell}f(x)| = \left| \frac{h_x^{(\ell)}(0)}{\ell!} \right| \leq \left\{ \begin{array}{ll}
\frac{\mathcal{T}^{(\ell)}(0)}{\ell!} \cdot |f|_{L^\infty}, & \text{if } (d - \ell) \text{ is even} \\
\frac{\mathcal{T}^{(\ell)}(0)}{\ell!} \cdot |f|_{L^\infty}, & \text{if } (d - \ell) \text{ is odd}
\end{array} \right. \] (25)
and (22) follows by taking a maximum over all \( x \in \{-1,1\}^n \). \( \square \)

In particular, as observed in [11], inequality (22) implies that if \( f \) has degree at most \( d \) then
\[
\forall \ \ell \in \{1, \ldots, d\}, \quad \|\Rad_{\ell} f\|_{L^\infty} \leq \frac{d^{\ell}}{\ell!} \cdot |f|_{L^\infty}.
\] (26)

2. The second main theorem of [11] asserts that if \( f : [-1,1]^n \to [-1,1] \) is a bounded function of degree at most \( d \), then for every \( \ell \in \{1, \ldots, d\} \) we have
\[
\sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} \left| \Rad_{\ell} f(S) \right| = \sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq n^\frac{\ell}{\ell+1} d^\ell e^{\log \ell}.
\] (27)
The Bohnenblust–Hille-type inequality of [8] implies the following improved bound.

**Corollary 3.** Let \( n \in \mathbb{N} \) and \( d \in \{1, \ldots, n\} \). Then, every bounded function \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \) satisfies
\[
\forall \ \ell \in \{1, \ldots, d\}, \quad \sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{\ell+1}} e^{\kappa \sqrt{\log \ell}} \frac{d^\ell}{\ell!} \leq n^{\frac{\ell}{\ell+1}} d^\ell e^{\log \ell},
\] (28)
for some universal constant \( c \in (0,1) \).

**Proof.** Combining Hölder’s inequality with the estimate of [8] and (26) we get
\[
\sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{\ell+1}} \left( \sum_{S \subseteq \{1, \ldots, n\}} |\Rad_{\ell} f(S)|^{\frac{2\ell}{\ell+1}} \right)^{\frac{\ell+1}{2\ell}} \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{\ell+1}} \exp(\kappa \sqrt{\log \ell}) \|\Rad_{\ell} f\|_{L^\infty} \overset{(26)}{\leq} \left( \frac{n}{\ell} \right)^{\frac{\ell}{\ell+1}} \exp(\kappa \sqrt{\log \ell}) \frac{d^\ell}{\ell!}.
\] (29)
The last inequality of (28) follows from (22) and the elementary bound \( (\ell)_\ell \leq \left( \frac{n}{\ell} \right)^{\ell} \). \( \square \)

We refer to the recent work [2] for a systematic study of inequalities relating the Fourier growth with various well-studied properties of Boolean functions.

3. It is straightforward to observe (see also [15, Proposition 3.31]) that if \( f : [-1,1]^n \to [-1,1] \) is a Boolean function and \( h : [-1,1]^n \to \mathbb{R} \) is an arbitrary function, then
\[
\|\operatorname{sign}(h) - f\|^2_{L^2} = 4\mathbb{P}\{\operatorname{sign}(h) \neq f\} \leq 4 \mathbb{P}\{|h - f| \geq 1\} \leq 4\|h - f\|^2_{L^2},
\] (30)
where we define \( \text{sign}(0) \) as \( \pm 1 \) arbitrarily. Therefore, applying Theorem 1 to a Boolean function, the above algorithm produces a Boolean function \( \hat{f} = \text{sign}(h) \) which is a \( 4\epsilon \)-approximation of \( f \).

4. In Theorem 1 we showed that bounded functions \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \) can be learned with accuracy at most \( \epsilon \) and confidence at least \( 1 - \delta \) from \( N = O_{\epsilon,d} \left( \log(n/\delta) \right) \) random queries. We will now show that this estimate is sharp for small enough values of \( \delta \).

**Proposition 4.** Suppose that bounded linear functions \( \ell : [-1,1]^n \to [-1,1] \) can be learned with accuracy at most \( \frac{1}{2} \) and confidence at least \( 1 - \frac{1}{2n} \) from \( N \) random queries. Then \( N > \log_2 n \).

**Proof.** By the assumption, for any input \( (X_1, y_1), \ldots, (X_N, y_N) \in [-1,1]^n \times [-1,1] \), there exists a function \( h(X_1, \ldots, X_N) : [-1,1]^n \to \mathbb{R} \) such that if \( X_1, \ldots, X_N \) are chosen independently and uniformly from \([-1,1]^n\) and there exists a linear function \( \ell : [-1,1]^n \to [-1,1] \) such that \( y_j = \ell(X_j) \) for every \( j \in \{1, \ldots, N\} \), then \( \mathbb{P}(\Omega_\ell) > 1 - \frac{1}{2n} \), where \( \Omega_\ell \) is the event

\[
\Omega_\ell = \left\{ \mathbb{E}(h(X_1, \ell(X_1)), \ldots, (X_N, \ell(X_N)) - \ell)^2 < \frac{1}{2} \right\}.
\]

Let \( X_j = (X_j(1), \ldots, X_j(n)) \) for \( j \in \{1, \ldots, N\} \) and consider the event

\[
\mathcal{W} = \left\{ X_j(1) = X_j(2), \ \forall j \in \{1, \ldots, N\} \right\}.
\]

By the independence of the samples, we have \( \mathbb{P}(\mathcal{W}) = \frac{1}{2^n} \). Therefore, if \( N \leq \log_2 n \) and we consider the linear functions \( r_i : [-1,1]^n \to [-1,1] \) given by \( r_i(x) = x_i \), then

\[
\mathbb{P}(\Omega_{r_1} \cap \Omega_{r_2}) > 1 - \frac{1}{n} \geq 1 - \frac{1}{2N} = 1 - \mathbb{P}(\mathcal{W}),
\]

which implies that \( \Omega_{r_1} \cap \Omega_{r_2} \cap \mathcal{W} \neq \emptyset \). Choosing \( X_1, \ldots, X_N \) from this event and denoting by \( h = h(X_1, X_2, \ldots, X_N) \), we deduce from the triangle inequality that

\[
2 = \mathbb{E}(r_1 - r_2)^2 \leq 2\mathbb{E}(h - r_1)^2 + 2\mathbb{E}(h - r_2)^2 \leq 2
\]

which is clearly a contradiction. Therefore \( N > \log_2 n \). \( \square \)

**References**


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