



HAL
open science

Welded graphs, Wirtinger groups and knotted punctured spheres

Benjamin Audoux, Jean-Baptiste Meilhan, Akira Yasuhara

► **To cite this version:**

Benjamin Audoux, Jean-Baptiste Meilhan, Akira Yasuhara. Welded graphs, Wirtinger groups and knotted punctured spheres. 2023. hal-04272362

HAL Id: hal-04272362

<https://hal.science/hal-04272362>

Preprint submitted on 6 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

WELDED GRAPHS, WIRTINGER GROUPS AND KNOTTED PUNCTURED SPHERES

BENJAMIN AUDOUX, JEAN-BAPTISTE MEILHAN, AND AKIRA YASUHARA

ABSTRACT. We develop a general diagrammatic theory of welded graphs, and provide an extension of Satoh’s Tube map from welded graphs to ribbon surface-links. As a topological application, we obtain a complete link-homotopy classification of so-called *knotted punctured spheres* in 4-space, by means of the 4-dimensional Milnor invariants introduced by the authors in [7]. On the algebraic side, we show that the theory of welded graphs can be reinterpreted as a theory of Wirtinger group presentations, up to a natural set of transformations; these groups arise as the fundamental group of the exterior of the surface-link obtained from the given welded graph by the extended Tube map. Finally, we address the injectivity question for the Tube map, identifying a new family of local moves on welded links, called Υ moves, under which the (non extended) Tube map is invariant.

INTRODUCTION

A *surface-link* in 4-space is the image of an embedding of a compact oriented surface in \mathbb{R}^4 up to ambient isotopy. The first examples of non trivial surface-links were spun links, described in the mid-twenties by Artin [1]; these are obtained as the trace of the 2π -rotation of a tangle in a half 3-dimensional (3D) subspace of \mathbb{R}^4 around the boundary of this subspace. Such links happen to be *ribbon* surface-links, meaning that they bound immersed 3D handlebodies in \mathbb{R}^4 with only *ribbon singularities*, locally shaped as the 3D thickening of an arc passing transversally through a 3-ball. Not all surface-links are ribbon, but they form an important subclass which has been extensively studied in the late sixties [30, 31, 32, 33], when the systematic study of surface-links began. In several ways, ribbon surface-links are easier to handle than general surface-links. Indeed, they admit *simple projections*, which are projections in \mathbb{R}^3 with only double points — while a generic projection of a surface-link contains in general triple and branch points; for embedded spheres, being ribbon is actually equivalent to having a simple projection [30]. Ribbon surface-links also admit 3D *ribbon presentations*, see e.g. [20], which are certain ribbon-immersed disks in 3-space. But despite their simplicity, ribbon surface-links can have a central role in the study of surface-links, such as in [6] where 2-dimensional string links are classified up to link-homotopy using the fact that any such surface-link is link-homotopic to a ribbon one (a result which is generalized by Proposition 3.3, see below).

In 2000, ribbon surface-links received renewed interest when Satoh extended Yajima’s Tube map [29] — a construction which ‘inflates’ classical knot diagrams into ribbon surface-link diagrams — to *welded knotted objects*, and showed that this yields a surjective map from welded knots and knotoids to ribbon tori and spheres [26]; he also showed that the Tube map coincides with Artin’s spun map on classical knots. Welded knotted objects were first introduced by Fenn–Rimanyi–Rourke [14] for the 4-dimensional study of motions of unknotted circles in \mathbb{R}^3 , and can be seen as a quotient of the virtual (or, equivalently, Gauss) diagram theory. Via Satoh’s Tube map, welded knotted objects provide a 2-dimensional description of ribbon surface-links, which has been used effectively in several recent works [4, 6, 22, 23].

However, welded links and string links cannot describe all type of ribbon surface-links, as it restricts to surfaces with connected components of genus 0 or 1, with 0 or 2 boundary components. To overcome this issue, the present paper develops a theory of *welded graphs* which generalize welded knots, links and string links. We observe that the Tube map extends naturally to welded graphs (Proposition 1.14), thus allowing for the study of the whole class of ribbon surface-links, of any topological type. As a concrete topological application, we obtain the following classification result.

Theorem (Theorem 3.4). *Knotted punctured spheres are classified, up to link-homotopy, by non repeated 4-dimensional Milnor invariants.*

In this statement, knotted punctured spheres are properly and smoothly embedded unions of punctured spheres whose boundary is a fixed unlink (or slice link, see Remark 3.6), forming a vast class of surface-links which contains Le Dimet’s linked disks [19] and 2-string links studied in [4, 6]. Link-homotopy is

Milnor’s classical notion of continuous deformation leaving distinct components disjoint at all time [24], and the classification is given using the 4–dimensional Milnor invariants of surface-links developed in [7], where this result was first announced; see Section 3 for details.

In a different direction, we use welded graphs to address the injectivity problem for the Tube map on welded knotted objects. An important open problem is indeed to determine the kernel of this map. The Tube map is known to be injective for welded braids [11] and to have a non-trivial kernel in the link case [28, 16], but the general case remains widely open. As developed in this paper, one advantage of the notion of welded graphs is that it naturally encompasses all the known moves on welded objects for which the Tube map is invariant. Actually, using this setting, we identify a large family of local moves on welded (string) links, the Υ moves (Definition 1.7), under which the Tube map is invariant:

Proposition (Proposition 1.16). *The Tube map is invariant under Υ moves.*

These Υ moves do not seem to be induced by usual welded moves, which leads us to conjecture that they provide general obstructions for the injectivity of the Tube map on welded (string) links.

Welded graphs already appeared in the literature [18, 9], in closely related forms. Our approach is in comparison deliberately more combinatorial, in terms of oriented graphs with edges labelled by elements in the free group generated by the vertices. This highlights a striking relationship between welded objects and *Wirtinger presentations*, which are finite presentations where each relation identifies a generator with a conjugate of another one. A celebrated result of Wirtinger provides such a presentation for the fundamental group of the complement of a (string-)link described by a diagram, and it is well-known that this procedure extends to welded knots and (string) links. One of the main claim of the present paper is that the theory of welded graphs can actually be recasted as a theory of Wirtinger presentations up to some natural transformations which preserve the associated group; see Theorem 2.2 for a precise statement. Moreover, this correspondence is naturally compatible with the extended Tube map, providing a nice connection between combinatorial, algebraic and topological objects, illustrated by the following result:

Proposition (Proposition 2.3). *For any welded graph Γ , the associated Wirtinger group $G(\Gamma)$ is isomorphic to the fundamental group $\pi_1(X^4 \setminus \text{Tube}(\Gamma))$ of the exterior of the surface-link $\text{Tube}(\Gamma)$.*

More generally, these Wirtinger presentations associated with welded graphs, yield a whole *peripheral system*, which also corresponds to a topological peripheral system for the associated ribbon surface-links. As a matter of fact, the above-mentioned 4–dimensional Milnor invariants are extracted from this peripheral system, and the link-homotopy classification of knotted punctured spheres actually follows from a more general diagrammatic result. Specifically, we introduce a notion of *self-virtualization* equivalence (sv) for welded graphs, which is the welded counterpart of link-homotopy for ribbon surface-links, and we prove the following, where *reduced* refers to a quotient of the peripheral system (see Section 2.3):

Theorem (Theorem 2.22). *Two welded graphs are sv-equivalent if and only if they have equivalent reduced peripheral systems.*

The paper is organized as follows. In Section 1, welded graphs are introduced and the relationship with welded links and string links is clarified; the connection with ribbon surface-links via an extension of Satoh’s Tube map is also presented. In Section 2, we introduce Wirtinger presentations and their associated peripheral systems, and the relation to welded graphs is discussed. We also introduce the notion of self-virtualization, and prove our classification result for welded graphs up to self-virtualization. Finally, in Section 3 we return to topology; we prove that every knotted punctured spheres is link-homotopic to a ribbon one, which leads to the classification of these objects up to link-homotopy.

Notation. For every element a of a group G , the inverse of a can be denoted by \bar{a} or a^{-1} ; and for every $a, b \in G$, we set $a^b = \bar{b}ab$ and $[a, b] = \bar{b}ab\bar{a}$. We denote by $F(a_1, \dots, a_r)$ the free group on the generators a_1, \dots, a_r .

Acknowledgments. The first, resp. second, author is partially supported by the project SyTriQ (ANR-20-CE40-0004), resp. the project AIMaRe (ANR-19-CE40-0001-01), of the ANR. The third author is supported by the JSPS KAKENHI grant 21K03237.

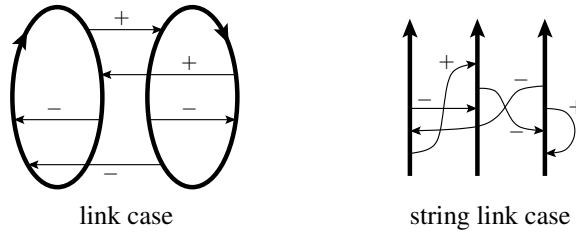


FIGURE 1. Some Gauss diagrams



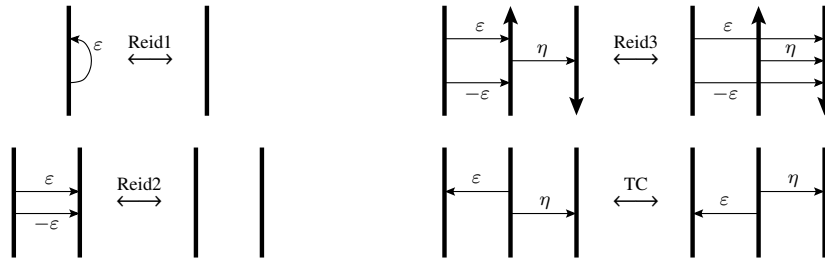
FIGURE 2. Exchanging arrow endpoints

1. WELDED GRAPHS

1.1. **Welded knotted objects.** Let us start with a quick review of the “usual” welded theory. This theory was initially given in terms of virtual diagrams [17], but we will consider here the Gauss diagram point of view; see for instance [4] for a correspondance between the two approaches.

Gauss diagrams were first introduced as a pictorial form for the Gauss code representation of knots. They are abstract and signed arrows whose endpoints are embedded in a compact oriented 1–dimensional manifold that shall be denoted by M hereinafter; see Figure 1 for examples.

Welded knotted objects are defined as equivalence classes of Gauss diagrams up to the following *welded moves* (here, and in subsequent figures, ϵ and η are two arbitrary signs):



We stress that a TC move allows one to freely exchange¹ the relative position of two adjacent arrow tails on M . Figure 2 represents the *exchange moves*, which are consequences of Reid2 and Reid3 moves (see [4, Fig. 4.1], and which similarly allow to exchange the relative positions of two adjacent arrow endpoints, up to the addition of a pair of oppositely-signed arrows.

Depending on the topological nature of the 1–dimensional manifold M , one can define various types of welded knotted objects:

- *welded knots* correspond to the case where M is a single copy of S^1 , and *welded links* to the case where M is a finite union of (ordered) disjoint copies of S^1 ;
- *welded string links* correspond to the case where M is a finite union of (ordered) disjoint intervals;
- *welded knotoids* are the quotient of 1–component welded string links, by the move which deletes or creates an arrow whose tail is adjacent to an endpoint of the interval $[0, 1]$.

¹TC actually stands for *Tails Commute*.

We next give some notation and observations that will be useful in Section 1.3, and that shall also smoothly drag the above definitions towards the notion of welded graphs.

Observe that, up to ambient isotopy and TC moves, the relevant information for an arrow tail is not its precise position, but rather the sub-arc of M , bounded by two arrow heads, where this tail is located. This means that the information carried by an arrow can be summarized by the position of its head, together with a label indicating the sub-arc of M where the tail is located; the arrow sign can in turn be given by an overlining, or a “-1” exponent, on this label if the sign is negative. More generally, a bunch of arrows with adjacent heads can be described by a word obtained by concatenating the corresponding labels. Moreover, observe that, thanks to the TC move, bunches of adjacent arrow tails can be thought of as gathered in a single point. Such a point corresponds to the sub-arc of M containing only these tails and comprised between two arrow heads. This allows to think of Gauss diagrams as a collection of such “tails localization” points, connected by oriented arcs decorated with words describing the sequence of arrow heads met on this portion of the Gauss diagram.

These rough observations shall be formalized in the next subsection, through the notion of welded graph.

Finally, given an arrow A of a Gauss diagram G , and assigning a letter to every sub-arc of M comprised between two heads and containing no head in its interior, we define the *head (resp. tail) conjugation of A by a word w* , as the insertion of the bunch of arrow heads associated to w on one side of the head (resp. tail) of A , and of the bunch of arrow heads associated to \bar{w} on the other side. Notice that inserting such extra arrows splits some of the sub-arcs of M , hence changes the alphabet. For example, Figure 2 can be summarized by saying that an endpoint of some arrow A_1 can be pushed through the adjacent head of an arrow A_2 , at the cost of conjugating the other endpoint of A_1 by the one-letter word corresponding to A_2 .

1.2. Definition of welded graphs.

Definition 1.1. A w -graph is an oriented graph Γ with edges decorated by elements of the free group generated by the vertices of Γ , and with a subset of marked vertices. In other words, a w -graph is the data of

- a set V_Γ of *vertices*,
- a set E_Γ of *edges*, made of triplets (a, b, w) where $a, b \in V_\Gamma$, and $w \in F(V_\Gamma)$ is the edge decoration,
- a (possibly empty) subset $V_\Gamma^\circ \subset V_\Gamma$ of *marked vertices*.

The connected components of the graph are ordered, and so are the marked vertices on each connected component. An edge decorated by the unit element 1 of $F(V_\Gamma)$ shall be called an *empty edge*.

In figures, we always represent marked vertices by white dots \circ , while unmarked vertices are represented by black dots \bullet .

We say that a w -graph is of *type* $((m_1, b_1), \dots, (m_\ell, b_\ell))$ if the underlying graph Γ has ℓ (ordered) connected components, and the i -th component has exactly m_i marked vertices and first Betti number b_i ; we simply say that it has *type* (m, b) if $m_1 = m_2 = \dots = m_\ell = m$ and $b_1 = b_2 = \dots = b_\ell = b$.

We define the following moves on w -graphs, illustrated in Figure 3:

- *contraction* (C), which removes an empty edge $(a, b, 1)$ from E_Γ and replaces V_Γ by $V_\Gamma/a \sim b$, where a and b are identified, under the assumption that $b \notin V_\Gamma^\circ$;
- *orientation reversal* (OR), which replaces an edge (a, b, w) by (b, a, \bar{w}) ;
- *stabilization* (S), which adds a common prefix a to all the edges connected to a given vertex b , under the assumption that all the edges are oriented outwards b , that $b \notin V_\Gamma^\circ$ and that all edge decorations of Γ are in $F(V \setminus \{b\})$;
- *Reidemeister I* (R1), which replaces an edge (a, b, w) by $(a, b, a^\varepsilon w)$ for some $\varepsilon \in \{\pm 1\}$;
- *Reidemeister III* (R3), which replaces by wb the decoration aw of an edge $e \in E_\Gamma$, under the assumption that $(a, b, w) \in E_\Gamma \setminus \{e\}$.

Remark 1.2. The contraction move induces its inverse move, the *expansion* move (E), which duplicates a vertex v into v_1 and v_2 , and creates a new empty edge between them. All the edges which were adjacent to v , have to be distributed between v_1 and v_2 , and any occurrence of v in edge decorations can be freely chosen to be replaced by v_1 or v_2 .

Note that the number of marked vertices, as well as the first Betti number, are preserved by all the above moves.

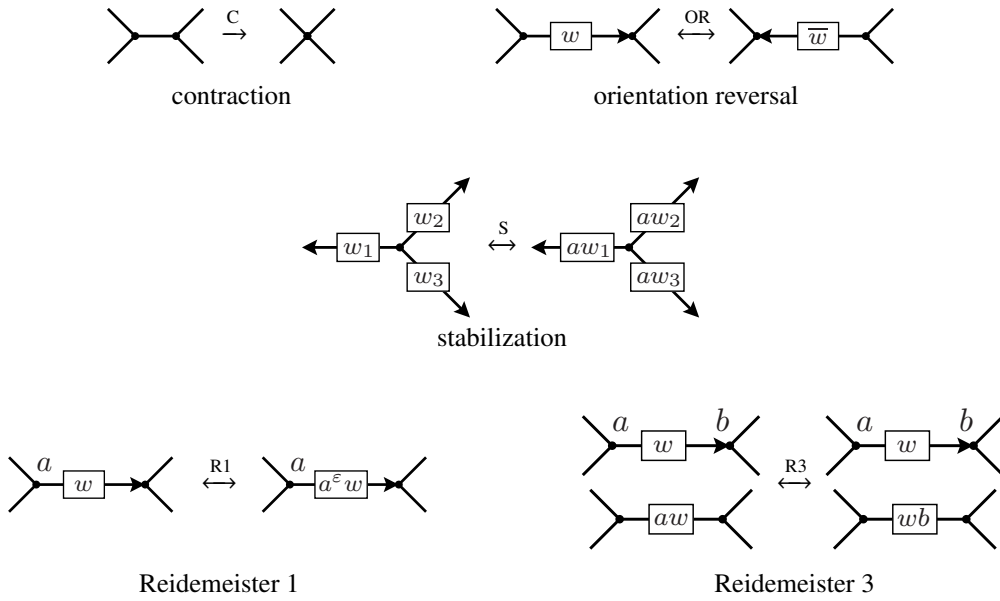


FIGURE 3. Moves on w-graphs

Here, and in the next figure, all depicted trivalent vertices represent vertices of any valency

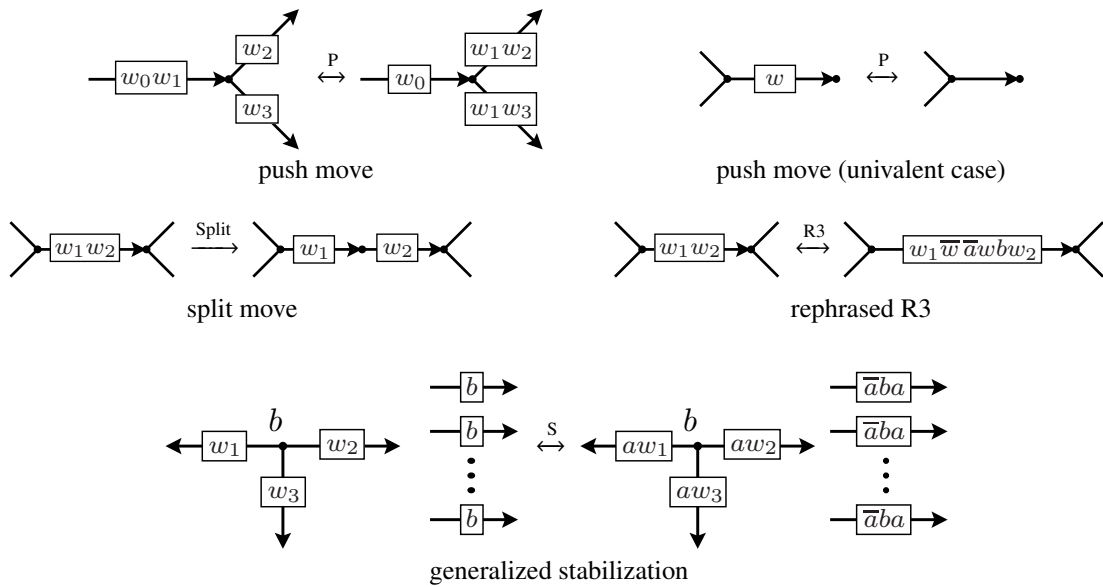


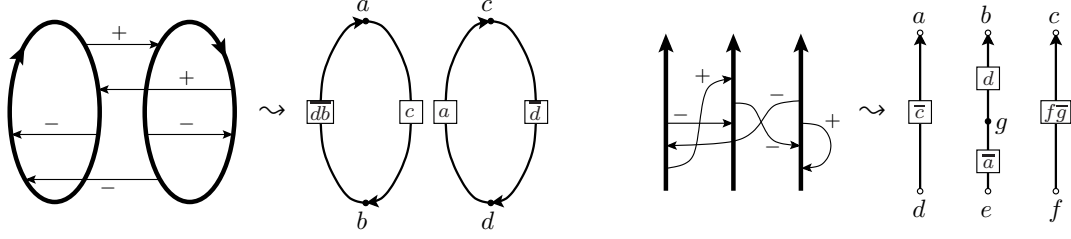
FIGURE 4. Extra moves on w-graphs

Definition 1.3. A welded graph is an element of the set

$$w\text{Graph} := \{w\text{-graphs}\} / C, \text{OR}, S, \text{R1}, \text{R3},$$

and its type is defined as the type of any of its representatives.

More admissible moves can be obtained by combining moves C, OR, S, R1 and R3. We gather below some of these extra moves, illustrated in Figure 4:

FIGURE 5. From Gauss diagrams to w -graphs

- *push move* (P), which pushes, according to the edges orientations, a word through an unmarked vertex that does not appear in any edge decoration; note that, in the case where the involved vertex is univalent, the push move simply erases the word on the incident edge;
- *split move* (Split), which splits an edge in two and separates the decoration between them. Note that it induces its inverse move, which deletes a bivalent vertex that does not appear in any edge decoration, merges the two adjacent edges and concatenates their edge decorations.

Also:

- the Reidemeister III move can be rephrased as follows: given an edge (a, b, w) , insert the word $\bar{w}awb$ (or any cyclic permutation of it, or of its inverse) within the decoration of any other edge;
- in a stabilization move, the condition that the central vertex b does not appear in any edge decoration can also be relaxed, at the cost of conjugating by the prefix a every occurrence of b in the edge decorations.

We postpone to Appendix A the proof that each of these extra moves are indeed consequences of moves C, OR, S and R3.

1.3. Relation to welded links and string links. As mentioned earlier, welded graphs are closely related to welded links and string links. This is made explicit in Theorem 1.8 below, which is the main result of this subsection.

1.3.1. *Statement.* We define two maps

$$\begin{aligned} \psi_L &: \{\text{welded links}\} \rightarrow \{\text{welded graphs of type } (0, 1)\} \\ \psi_{SL} &: \{\text{welded string links}\} \rightarrow \{\text{welded graphs of type } (2, 0)\} \end{aligned}$$

as follows. To a Gauss diagram D for a welded link, we associate a w -graph Γ_D . The vertices of Γ_D are all unmarked, and are in bijection with the arrow tails of D . The edges correspond to the oriented pieces of circle bounded by two ‘adjacent’ arrow tails, which are such that they contain no arrow tail in their interior. When running according to its orientation, each such piece of circle contains an ordered (possibly empty) sequence $h_1 \cdots h_k$ of arrow heads; the decoration of the corresponding edge of Γ_D is obtained by replacing each h_i by $t_i^{\varepsilon_i}$, where t_i and ε_i are respectively the tail and the sign of the arrow which contains h_i . Finally, in order to reduce the number of vertices, empty edges can be contracted. See the left-hand side of Figure 5 for an example.

For a Gauss diagram D of welded string link, we define Γ_D in a similar way; each component of the w -graph Γ_D now contains two marked vertices, which are in bijection with the endpoints of the intervals, and the edges correspond to the oriented pieces of interval between every pair of adjacent arrow tails and/or interval endpoints. See the right-hand side of Figure 5.

It is a straightforward exercise to check that this assignment $D \mapsto \Gamma_D$ is well-defined on welded links and string links, hence induces the requested maps ψ_L and ψ_{SL} .

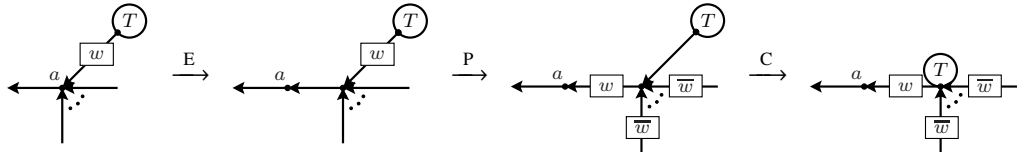
Moreover, as Figure 5 illustrates, the w -graphs obtained from the above procedure, have a very specific shape. This motivates the following definitions.

Definition 1.4. A w -graph of type $(0, 1)$ is called *cyclic* if each component is isomorphic to a circle, decomposed into bivalent unmarked vertices and oriented edges, such that each vertex has one inward and one outward oriented incident edges; see the left-hand side of Figure 5.

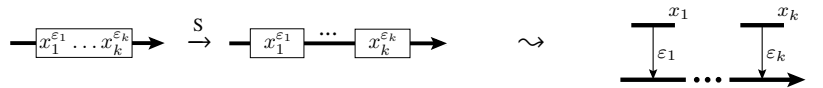
A w -graph of type $(2, 0)$ is called *linear* if each component is isomorphic to an interval, decomposed into two univalent marked vertices, bivalent unmarked vertices and oriented edges, such that each unmarked vertex has one inward and one outward oriented incident edge; see the right-hand side of Figure 5.

Lemma 1.5. *The maps ψ_L and ψ_{SL} are surjective.*

Proof. Since the link case is strictly similar, we only give the argument in the string link case. We first observe that, up to push and contraction moves, any w -graph of type $(2, 0)$ is equivalent to a linear w -graph. Indeed, each component of a w -graph of type $(2, 0)$ is a tree with two marked vertices; we consider the shortest path of edges between these two marked vertices. Next we pick some edge e which is not in this path, but which has an endpoint a on it (and a tree T attached to its other endpoint). If a is not used in any edge decoration, we can simply apply a push move to turn e into an empty edge, and then a contraction move to delete it; otherwise, we need to first push a inside the shortest path using an expansion move, which creates an empty edge, so that we can remove e as in the former case:²



Applying this procedure recursively, we can delete all edges which are not in the shortest path. Then it only remains to apply OR moves to orient all edges coherently. Now, it remains to note that a linear w -graph Γ , admits a preimage for the map ψ_{SL} . The process is summarized in the figure below:



First, using S moves, we split each word label on a linear w -graph into letter labels so that each edge is decorated by the, possibly overlined, name of a vertex. Next, we replace each (non empty) edge decoration by the head of an arrow, whose tail is attached in a neighborhood of the vertex corresponding to the letter decoration; the sign of this arrow being $-$ if the letter is overlined, and $+$ otherwise. Note that the relative position of the tails corresponding to a same letter label, is irrelevant thanks to the TC move. \square

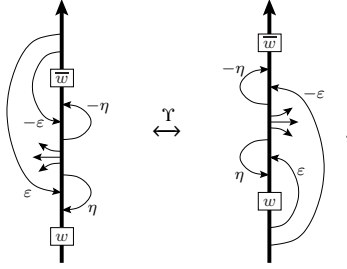
Remark 1.6. As roughly outlined at the end of Section 1.1, and formalized by the map ψ_{SL} , word labelings on the edges of a welded graph can locally be thought of as some Gauss diagram arrows. In subsequent figures, we shall use implicitly this correspondence. Concretely, this means that we shall sometimes represent bunches of adjacent arrow heads in Gauss diagrams by some edge decoration words, whose signed letters correspond to the arrows signs and tails. A typical example follows in the next definition.

We now show that these maps are not injective. More precisely, we prove that the injectivity defect is controlled by the following moves.

Definition 1.7.

²Another issue would be that the vertex we are willing to push through is a marked vertex, but this can be safely avoided by pulling first the initial and final vertices away by using C moves.

- On welded links and string links, we define the Υ move as the following local move:



where the three open arrows represent any number of tails whose associated heads are located outside of the picture.³

- On welded links, we define the *global reversal* (GR) move, as the move which reverses the orientation of a component, and reverses the sign of all arrows whose head is attached to this component.

The following is the main result of this section. As we shall see in Subsection 1.4, it is also relevant from the topological point of view.

Theorem 1.8.

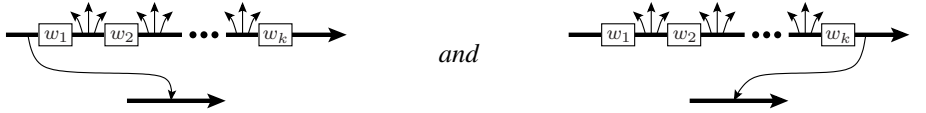
- The map ψ_L induces a bijection between welded graphs of type $(0, 1)$ and welded links up to Υ and global reversal moves.
- The map ψ_{SL} induces a bijection between welded graphs of type $(2, 0)$ and welded string links up to Υ moves.

In particular, the set of welded links and string links, up to Υ and global reversal moves, embed into the set of welded graphs.

The proof of Theorem 1.8 occupies the rest of Section 1.3.

1.3.2. *Three technical lemmas.* We prepare below three technical lemmas that will be used in the proof of Theorem 1.8.

Lemma 1.9. *Two Gauss diagrams which differ only locally as*



where each triad of open arrows stands for any bunch of tails, are equivalent up to welded moves, if the concatenated word $w_1 \cdots w_k$ is trivial as a free group element.

Proof. The leftmost tail in the left-hand diagram, can be pushed to the right at the cost of conjugating its head by the word $w_1 \cdots w_k$ by the exchange moves of Figure 2. But since this word is trivial as a free group element, all the conjugating arrows can be removed pairwise using Reid2 moves. \square

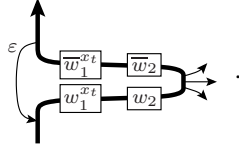
Lemma 1.10. *Up to welded moves, two Gauss diagrams which differ only locally as*



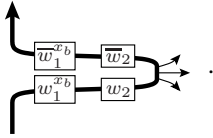
where none of the right-pointing tails are connected to w_1 , are equivalent.

³Since the two diagrams are identical outside the represented portion, this means that the two bunches are “identical” for both diagrams; it also means that they are not connected to any of the arrows associated to w .

Proof. We start from the left-hand diagram. We denote by a_ε and $a_{-\varepsilon}$ the depicted ε and $(-\varepsilon)$ -signed arrows, and by x_t and x_b the depicted vertical top and bottom sub-arcs of the diagram. Then \bar{w}_1 can be pushed through the head of $a_{-\varepsilon}$, so that $a_{-\varepsilon}$ can be removed with a Reid1 move. Pushing \bar{w}_1 through $a_{-\varepsilon}$ in this way, has the effect of conjugating all tails of arrows in \bar{w}_1 by x_t , and we denote this by replacing the label \bar{w}_1 by $\bar{w}_1^{x_t}$. Next, w_1 can be pushed through the head of a_ε , at the cost of conjugating w_1 by x_t .⁴ We thus obtain the local diagram

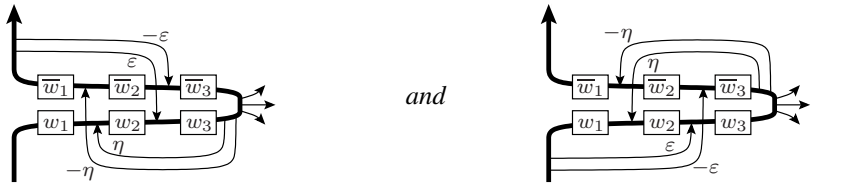


Now, observe that each letter in w_1 corresponds to two arrows, one from w_1 and one from \bar{w}_1 , which can be assumed to have adjacent tails by move TC. When conjugated by x_t , these tails are now separated by two x_t -tailed arrows with opposite signs which can be removed using a Reid2 move. By this observation, we have that $w_1^{x_t}$ and $\bar{w}_1^{x_t}$ indeed represent inverse elements in the free group: we are therefore in the situation of Lemma 1.9. We apply hence Lemma 1.9, not only to the tail of a_ε , but also to all x_t -tailed arrows conjugating $w_1^{x_t}$ and $\bar{w}_1^{x_t}$: all these tails are pulled in this way down to the bottom interval x_b . After this deformation, the arrow a_ε can be deleted thanks to a Reid1 move, while all conjugating x_t -tailed arrows are turned into conjugating x_b -tailed arrows, since their tails now sit on x_b . As a result, we obtain the following local diagram:



We now turn to the right-hand diagram in the statement. We can perform a similar sequence of moves as above, except that, at the final step, we only apply Lemma 1.9 to the arrow $a_{-\varepsilon}$ (that is, we leave all conjugating x_b -tailed arrows on x_b). This yields the very same diagram as above, which concludes the proof. \square

Lemma 1.11. *Up to Υ and welded moves, two Gauss diagrams which differ only locally as:*



where none of the right-pointing tails are connected to w_1 , w_2 or w_3 , are equivalent.

Proof. We denote by a_α the depicted α -signed arrow (for $\alpha \in \{\pm\varepsilon, \pm\eta\}$), and by x_t and x_b the depicted vertical sub-arcs of the Gauss diagram. We also denote by x_r the rightmost sub-arc of the diagram, which contains the tails of arrows a_η and $a_{-\eta}$.

We first note that in the case where $w_2 = w_3 = \emptyset$, the local deformation of the statement is nothing but the Υ move; hence the case $w_2 = w_3 = \emptyset$ is immediate.

Now let us show how the case where $w_3 = \emptyset$, reduces to this $w_2 = w_3 = \emptyset$ case. This is done by pushing, in the left-hand side diagram, w_2 and \bar{w}_2 across the head of a_η and $a_{-\eta}$, respectively. As in the proof of Lemma 1.10, this deformation is achieved at the cost of conjugating these words w_2 and \bar{w}_2 by x_r . We can then apply the $w_2 = w_3 = \emptyset$ case, and then push back the x_r -conjugated word w_2 , resp. \bar{w}_2 , across the head

⁴Conjugation is also by x_t since a_ε and $a_{-\varepsilon}$ have opposite signs and we are pushing w_1 and \bar{w}_1 in opposite directions according to the diagram orientation.

of a_η , resp. $a_{-\eta}$. Being the reverse process of the initial deformation, this simply unconjugates these words, and yields the right-hand-side diagram.

Finally, we observe that the general case similarly reduces to the $w_3 = \emptyset$ case. The argument is the same as above, except for one additional subtlety. Starting with the left-hand diagram, we push w_3 and \bar{w}_3 across the head of a_ε and $a_{-\varepsilon}$, respectively. We can then apply the $w_3 = \emptyset$ case of the statement. The point is that the initial deformation was achieved at the cost of conjugating the words w_3 and \bar{w}_3 by x_t (and not x_r as above). Hence we cannot immediately push back these conjugated words to obtain the desired move. But each of the x_t -conjugating arrows introduced at this initial stage, has its tail on the top vertical sub-arc x_t , and, by assumption, its head is located away from the depicted part of the diagram. Hence Lemma 1.9 can be applied to each of these arrows, whose tail can be moved down to the bottom vertical part x_b . Indeed, the concatenation of the words separating x_t and x_b is equivalent to the empty word, so that the lemma applies. After applying Lemma 1.9 in this way, w_3 and \bar{w}_3 can be unconjugated by pushing them back in place, and the proof is complete. \square

1.3.3. *Proof of Theorem 1.8.* Before proceeding with the proof of Theorem 1.8, we introduce some terminology.

Each connected component of a w -graph of type $(2, 0)$ has two marked vertices, which are ordered according to the orientation: the first one is called *initial vertex* and the second one is called *final vertex*. There is also a unique shortest path from the initial to the final vertex: every edge which is not on this path is called an *extra edge*. Note that the w -graph is linear if it has no extra edge.

Every vertex v of a w -graph of type $(2, 0)$ which is not the final vertex, has a unique shortest path to the final vertex: the first edge on this path is called the *out edge* of v . For vertices distinct from the initial vertex, the *in edge* is similarly defined as the first edge on the shortest path to the initial vertex. Note that the in and out edges of a vertex are distinct if and only if the vertex is on the shortest path from the initial to the final vertex.

In the same way, each connected component of a w -graph of type $(0, 1)$ has a unique shortest non contractible path: edges which are not on this path are also called *extra edges*. Observe that the w -graph is cyclic if it has no extra edge.

Proof of Theorem 1.8. Let us consider the string link case. First, we observe that the maps ψ_{SL} is invariant under Υ moves. Indeed, on one hand we have by definition that

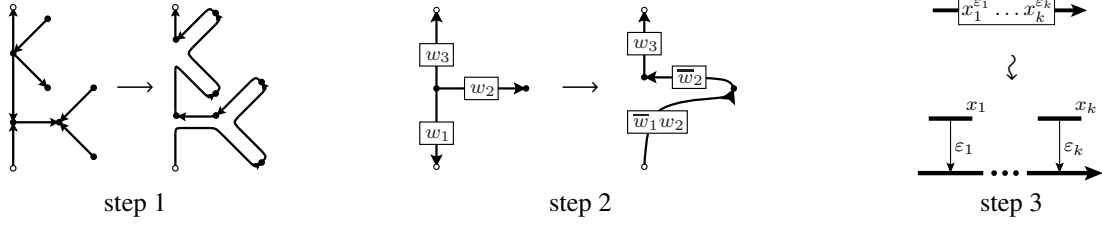
$$\psi_{SL} \left(\begin{array}{c} \uparrow \\ \text{---} \uparrow \text{---} \\ \text{---} \uparrow \text{---} \\ \downarrow \end{array} \right) = \begin{array}{c} \uparrow a \\ \boxed{a^{-\varepsilon} b^{-\eta} \bar{w}} \\ \uparrow b \\ \boxed{wb^\eta a^\varepsilon} \\ \downarrow c \end{array},$$

and applying a sequence of expansion, contraction and push moves to the resulting w -graph, shows that

$$\begin{array}{c} \uparrow a \\ \boxed{a^{-\varepsilon} b^{-\eta} \bar{w}} \\ \uparrow b \\ \boxed{wb^\eta a^\varepsilon} \\ \downarrow c \end{array} \xrightarrow{E} \begin{array}{c} \uparrow a \\ \boxed{a^{-\varepsilon} b^{-\eta} \bar{w}} \\ \uparrow \text{---} \rightarrow b \\ \boxed{wb^\eta a^\varepsilon} \\ \downarrow c \end{array} \xrightarrow{P} \begin{array}{c} \uparrow a \\ \uparrow \text{---} \rightarrow b \\ \boxed{wb^\eta a^\varepsilon} \\ \downarrow c \end{array} \xrightarrow{C} \begin{array}{c} \uparrow a \\ \uparrow \text{---} \rightarrow b \\ \uparrow \text{---} \rightarrow b \\ \downarrow c \end{array} \xrightarrow{C} \begin{array}{c} \uparrow a \\ \uparrow \text{---} \rightarrow b \\ \uparrow \text{---} \rightarrow b \\ \downarrow c \end{array},$$

On the other hand, applying the ψ_{SL} map and a similar sequence of expansion, contraction and push moves, gives:

$$\psi_{SL} \left(\begin{array}{c} \uparrow \\ \text{---} \uparrow \text{---} \\ \text{---} \uparrow \text{---} \\ \downarrow \end{array} \right) = \begin{array}{c} \uparrow a \\ \boxed{c^{-\varepsilon} b^{-\eta} \bar{w}} \\ \uparrow b \\ \boxed{wb^\eta c^\varepsilon} \\ \downarrow c \end{array} \xrightarrow{E,P,C} \begin{array}{c} \uparrow a \\ \uparrow \text{---} \rightarrow b \\ \uparrow \text{---} \rightarrow b \\ \downarrow c \end{array} = \begin{array}{c} \uparrow a \\ \uparrow \text{---} \rightarrow b \\ \uparrow \text{---} \rightarrow b \\ \downarrow c \end{array}.$$


 FIGURE 6. Definition of $\bar{\xi}_{SL}$

Therefore $\bar{\psi}_{SL}$ induces a well-defined map

$$\bar{\psi}_{SL} : \{\text{welded string links up to } \Upsilon \text{ moves}\} \rightarrow \{\text{welded graphs of type } (2, 0)\}.$$

We now define an inverse map $\bar{\xi}_{SL}$ for $\bar{\psi}_{SL}$. Let Γ be a w -graph of type $(2, 0)$. The description of $\bar{\xi}_{SL}(\Gamma)$ will be given in three steps, summarized in Figure 6. The first step provides an algorithm for turning Γ into a linear w -graph, by a canonical sequence of push and contraction moves; the second step specifies the edge labels on this linear w -graph, induced by those of Γ ; the third step associates a welded string link to the resulting welded graph.

Step 1. For every vertex of Γ , we fix an arbitrary ordering on the adjacent edges such that the out edge is last and, if it exists, the in edge is penultimate. The first edge of a vertex is called the *leading* edge of the vertex, and the second one is called the *co-leading* edge. Using these orderings clock-wisely, we obtain a planar way of embedding Γ in \mathbb{R}^2 ; we can then ‘contour’ these planar trees (see step 1 in Figure 6) to get a union I of intervals, connecting initial vertices to final ones. We orient these intervals from the initial to the final vertex. Each vertex of Γ corresponds to at least one, possibly more, points on I : we keep only the point which is closest to the final vertex. This turns I into a linear graph, whose vertices are in one-to-one correspondence with those of Γ , and whose edges are unions of edges of Γ .

Step 2. Each edge e of I is a union of edges e_1, \dots, e_k of Γ with the same or the reversed orientation. Let w_i be the decoration of e_i in Γ . Then we decorate e by the concatenation $w_1^{e_1} \dots w_k^{e_k}$ where $e_i \in \{\pm 1\}$ is 1 if and only if the orientations of e and e_i agree; see step 2 in Figure 6 for an illustration.

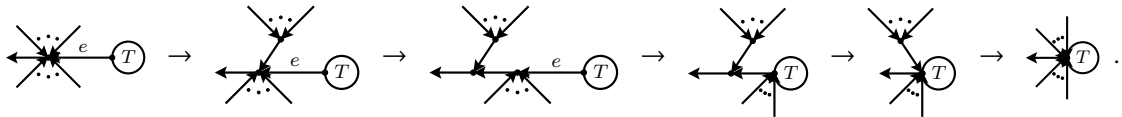
At this point, we obtain a linear w -graph Γ_l which is equivalent to Γ . Indeed, Γ_l is the result of iteratively contracting, as outlined in the proof of Lemma 1.5, all extra edges with a univalent unmarked endpoint, starting with the one which is closest to the final vertex of the corresponding component of I .

Step 3. We apply, on the linear w -graph Γ_l , the exact same procedure as in the latter half of the proof of Lemma 1.5: we split each edge by S moves to obtain a linear w -graph labeled by letters, and we replace each of these letter labels by an arrow as illustrated in step 3 of Figure 6. The result is the desired welded string link $\bar{\xi}_{SL}(\Gamma)$; as observed in the proof of Lemma 1.5, this is indeed a preimage for Γ_l by the map $\bar{\psi}_{SL}$.

We now have to prove that $\bar{\xi}_{SL}(\Gamma)$ is well defined, that is $\bar{\xi}_{SL}(\Gamma)$ is invariant under all moves on w -graphs, and does not depend on the choice of planar embedding for Γ (that is, on the chosen orderings at each vertex).

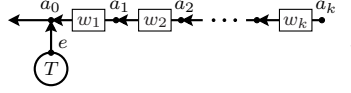
First, it is easily checked that OR moves do not change the resulting Gauss diagram, so that we can freely assume that all edges are oriented toward the final vertex. Invariance under S moves (hence under push moves) is also easy to check, as they introduce pairs of arrows which can be deleted by R2 moves.

Contraction moves turn out to be the most intricate moves to check. Contraction/expansion of a leading edge, which we shall call *leading contraction/expansion* below, has no impact, so that invariance is directly checked. But contraction of a non leading edge may change drastically the position of some vertex in Γ_l . First, let us observe that contraction of an arbitrary (non leading) edge e , can always be realized by a sequence of expansion and contractions on either leading edges, or co-leading edges at a trivalent vertex; this is illustrated in the following figure, where T is any tree attached to the edge e :

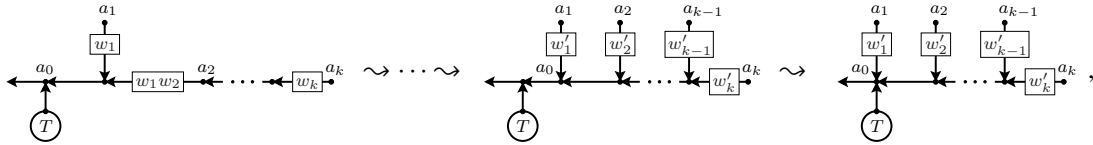


There, the first step is a leading expansion, followed by an expansion on a co-leading edge at a trivalent vertex; this turns edge e into a leading edge, so that contraction can be performed; it then only remains to perform the inverse of the first two operations.

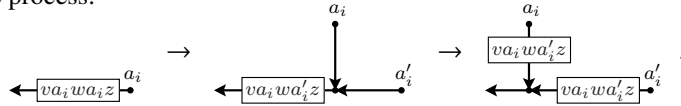
Hence we are left with proving the case of a contraction of a co-leading edge e at a trivalent vertex. Using push moves and leading contractions, the tree attached via the leading edge can be assumed to be linear, that is we consider a w -graph of the form:



where as above, T is any tree, attached to the edge e that we would like to contract. By recursively applying push moves and leading expansions, this w -graph can be turned into the following:



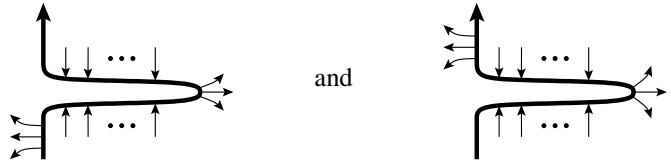
where the final step is achieved by a leading contraction. Furthermore, we may freely assume that for each $i = 1, \dots, k$, the letter a_i appears at most once in the word w'_i . This is done by using iteratively the following trick during the above process:



Here, the first operation is a leading expansion (which allows for replacing each label a_i by either a_i or a'_i), followed by a push move.

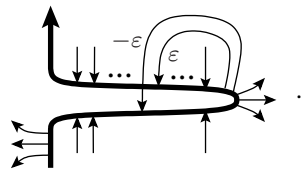
Now, we show the invariance under the contraction of e by induction on k . For $k = 1$, there are two cases:

- Suppose that a_1 does not appear in w_1 . The figure below gives the image by $\bar{\xi}_{SL}$ of the w -graph before (left) and after (right) contraction on the edge e :



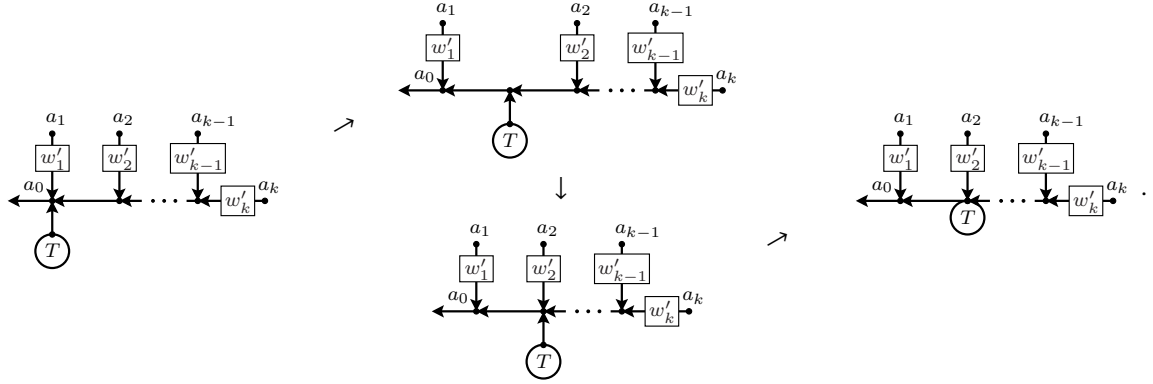
where each pair of opposed vertical arrows have adjacent tails and opposite signs. Note that the bunch of right-pointing arrows represent all occurrences a_1 in the edge labelings, so that we know by assumption that none of these tails is connected to the vertical arrow heads of the figure. Starting from the left-hand side figure above, we aim at moving all left-pointing arrow tails from the bottom to the top vertical strand. This is achieved by Lemma 1.9 for any such arrow that is not connected to the vertical arrow heads; left-pointing arrows that are connected to the vertical arrow heads, come by pairs with opposite signs, so that Lemma 1.10 can be applied to perform the desired move.

- Suppose that a_1 appears once in w_1 . Concretely this means that, in the above left-hand figure, a pair of vertical arrows corresponds to two of the right-pointing arrows:



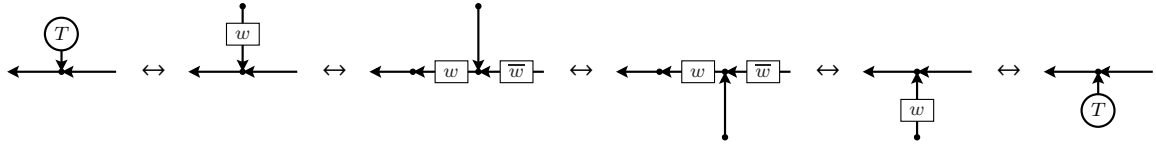
This case is handled as the previous one, except that we might use Lemma 1.11 instead of Lemma 1.10.

The inductive step is illustrated below:



The first operation is an expansion in the “k=1 case”; the second step is a leading contraction, and the final operation is a contraction using the induction hypothesis. Finally, we undo all the previous steps to get back to the initial configuration with the desired co-leading edge contracted.

Invariance under a change of ordering of the edges at some vertex is now easily handled using push, expansion and contraction moves. First, we note that, up to expansion and contraction moves, the graph can be assumed to be uni-trivalent. Then the procedure is schematically illustrated as follows:



There, T denote a tree that we would like to ‘move down’ by reversing the cyclic order at the trivalent vertex. With contraction and push moves, we first replace T by a linear graph T_L : this is schematically denoted by a w -labeled edge in the above figure. Next we push down all the decorations of T_L , so that the vertical edges are empty and can be contracted. The reversed procedure can then be performed with the cyclic order reversed.

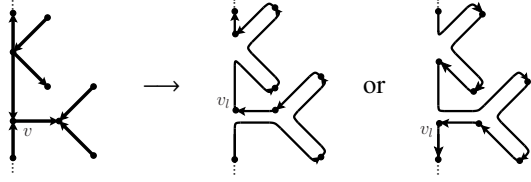
We next check the invariance under an R1 move. At the level of the images by $\bar{\xi}_{SL}$, such a move creates two or one arrow(s), depending on whether the edge where the move is performed, is an extra edge or not. The head h and tail t of such an arrow are separated by tails and sequences of heads such that the concatenation of the corresponding words is trivial as a free group element. Lemma 1.9 can hence be applied to pull t next to h . The arrow can then be removed using a Reid1 move.

For R3 moves, consider in some welded graph Γ , an edge in Γ between two vertices a and b , and decorated by a word w , and suppose that Γ contains some other edge decorated by wb . Let Γ' be the result of the R3 move that replaces the latter label by $wa^w = aw$. By construction, the linear welded graph Γ_L associated to Γ (step 2 of the definition of $\bar{\xi}_{SL}(\Gamma)$) also contains a w -labeled edge between vertices a and b . In the welded string link $\bar{\xi}_{SL}(\Gamma)$, this yields bunches of tails corresponding, respectively, to all occurrences of a and b in the edge decorations, separated by a sequence of heads which correspond to the word w . Pulling one tail from the b -bunch through the w -sequence of heads will conjugate the associated head by w . The resulting welded string link is precisely $\bar{\xi}_{SL}(\Gamma')$.

This concludes the proof that the map $\bar{\xi}_{SL}$ is well-defined. Moreover, it clearly satisfies $\bar{\xi}_{SL} \circ \bar{\psi}_{SL} = \text{Id}$. This shows that $\bar{\psi}_{SL}$ is injective, hence bijective, and the proof is complete in the string link case.

The case of welded links is handled in the same way, except that extra edges are contracted in order to get a cyclic w -graph instead of a linear one. There is however one extra issue: the resulting cyclic w -graph has no canonical orientation. This affects the 3-step procedure defining $\bar{\xi}_L$, since in step 3 we need to choose an arbitrary orientation on each circle component; this is precisely why one needs to introduce the Global Reversal move in the link case, which identifies two Gauss diagrams that differ by this choice of orientation.

A more subtle point is that reversing the orientation on some component also affects step 1, as follows:



As the figure illustrates, in the linearization algorithm given in step 1, the chosen orientation is used at each vertex v of a w -graph, to define the corresponding vertex v_l in the linear w -graph. This vertex v_l represents, in the resulting Gauss diagrams, a bunch of arrow tails: Lemma 1.9 ensures that the Gauss diagrams resulting from the above two situations, are actually equivalent. This concludes the proof of Theorem 1.8 \square

1.4. Relation to ribbon surface-links. One important fact about welded links and string links, is that they are closely related to ribbon surface-links whose components are, respectively, tori and annuli, in the sense that there exists a surjective map from the former to the latter. Welded graphs are similarly related to all kind of ribbon surface-links, thus providing a more flexible combinatorial theory for the study of general ribbon surface-links. In what follows, X^4 denotes a given ambient 4-manifold in $\{S^4, \mathbb{R}^4, B^4\}$.

Definition 1.12.

- A *ribbon handlebody* is an immersion in X^4 of a union H of oriented solid handlebodies up to ambient isotopy, such that
 - the singular set is a finite union of *ribbon disks*; here a ribbon disk is a disk D of transverse double points with one preimage D_{con} , called *contractible*, embedded in the interior of H , and one preimage D_{ess} , called *essential*, properly embedded in H ;
 - if $X^4 = B^4$, $H \cap \partial B^4$ is a disjoint union of embedded disks.
- A *ribbon surface-link* is an oriented surface-link which bounds⁵ a ribbon handlebody; the latter is called a *ribbon filling* of the ribbon surface-link.

We note that a given ribbon surface-link, may admit several different ribbon fillings.

By definition, the essential preimages of the ribbon disks of a ribbon handlebody H , cut H into 3-dimensional pieces, that we call *chambers*. Each chamber has the topology of a 3-ball with a finite number of handles attached, and a finite number of disks removed from its boundary. These removed disks correspond to essential preimages, and each of them is shared by two chambers (possibly twice the same). Moreover, each contractible preimage of a ribbon disk of H embeds into the interior of one of these chambers.

We say that a w -graph is *reduced* if all empty edges are loops, all non empty edges are decorated by one-letter words, and all marked vertices are univalent. We can define a map

$$C : \{\text{ribbon handlebodies}\} \rightarrow \{\text{reduced } w\text{-graphs}\} / \text{OR}$$

as follows. If H is a ribbon handlebody, then $C(H)$ has:

- one unmarked vertex for each chamber and, if $X^4 = B^4$, one marked vertex for each connected component of $H \cap \partial B^4$;
- as many empty loops attached to each vertex as the genus of the corresponding chamber;
- one arbitrarily-oriented edge e_D for each ribbon disk D : the edge connects the two vertices corresponding to the two chambers that are adjacent to D_{ess} , and it is decorated by the letter corresponding to the (vertex associated to the) chamber containing D_{con} ; this letter is overlined if and only if, for any given $x_0 \in D$, the concatenation of (the push-forward of) a positive basis of $T_{x_0^{\text{con}}}H$, where x_0^{con} is the preimage of x_0 in D_{con} , with (the push-forward of) a normal vector for D_{ess} at x_0^{ess} , where x_0^{ess} is the preimage of x_0 in D_{ess} , pointing into the chamber corresponding to the target vertex of e_D , is a negative basis of X^4 .

Lemma 1.13. *The map C is a bijection.*

⁵In case $X^4 = B^4$, the boundary has to be understood as the closure of $\partial H \setminus (H \cap \partial B^4)$.

Proof. This follows from the exact same argument as in the proof of [3, Prop. 3.7], which deals with the string link case. Roughly speaking, the proof is based on the fact that, given embedded 4–balls in S^4 with a finite number of paired embedded 3–balls on their boundary, there is a unique way, up to ambient isotopy, to connect these 3–balls with disjoint 1–handles. These 1–handles are indeed characterized by their 1–dimensional framed core; and since the framing takes values in S^2 , the unit sphere of the normal bundle of the core, which is simply connected, all framings are isotopic. \square

Next, consider the map

$$\partial : \{\text{ribbon handlebodies}\} \rightarrow \{\text{ribbon surface-links}\},$$

which sends a ribbon handlebody to its boundary.

Proposition 1.14. *The composite $\partial \circ C^{-1}$, induces a well-defined, surjective map*

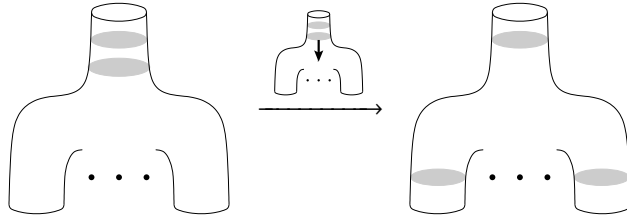
$$\text{Tube} : \{\text{welded graphs}\} \rightarrow \{\text{ribbon surface-links}\},$$

sending a type $((m_1, b_1), \dots, (m_\ell, b_\ell))$ welded graph to an embedded $\sqcup_{i=1}^\ell \Sigma_i$, where Σ_i is a genus b_i surface with m_i disjoint disks removed.

We shall sometimes call this map, the *generalized Tube map*, since it naturally coincides with the known Tube map for welded (string) links, see Remark 1.15 below.

Proof. Denote by \tilde{T} the composite $\partial \circ C^{-1}$. Note that, thanks to contraction and split moves, every w –graph is canonically equivalent to a reduced w –graph. Hence \tilde{T} naturally extends to all w –graphs. Let us check now that \tilde{T} is invariant under C, S, R1 and R3 moves.

Invariance under a contraction move is clear, and since the moves are local, invariance under R1 and R3 moves follow from the same arguments as for the original Tube map; see for instance the explicit ribbon filling changes argument given in the proof of [2, Prop. 2.5]. Let us now consider the stabilization move, which inserts a letter x_0 on the edges incident to a vertex v . Via \tilde{T} , the letter x_0 corresponds to a chamber C_0 , and v corresponds locally to a punctured sphere S filled by a 3–ball C ; up to ambient isotopy, C and C_0 can be assumed to stand close to each other. Now, we can create⁶ two oppositely-signed ribbon disks between C and C_0 , such that the boundaries of the essential preimages are both parallel to a given puncture of S – see the left-hand side of the figure below, which represents the chamber C . Then we push one of these ribbon disks through C , until it splits into ribbon disks parallel to every other punctures of S :



The result is the image by \tilde{T} , of the stabilization move. This completes the proof that \tilde{T} is invariant under all moves on welded graphs, so that the map \tilde{T} induces the desired map Tube.

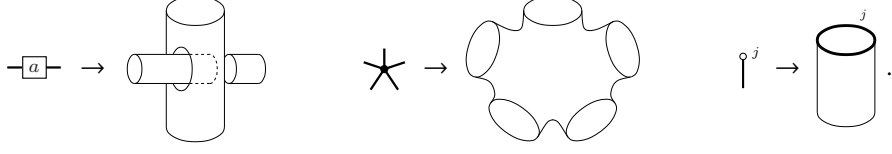
Surjectivity of Tube is clear, since any ribbon surface-link L admits at least one ribbon filling R_L by definition, $C(R_L)$ provides then a welded graph Γ_L such that $\text{Tube}(\Gamma_L) = L$. The latter part of the statement, concerning the types of welded graphs, just follows by construction. \square

Remark 1.15. As the name suggests, the definition of our generalized Tube map follows closely the original construction, as given in [2, 3]. Specifically, by construction we have that $\text{Tube} \circ \psi_L$ and $\text{Tube} \circ \psi_{SL}$, coincide with the original Tube maps for welded links and string links, respectively.

As a matter of fact, the generalized Tube map can be defined directly, in terms of broken surface diagrams, as in [26] (we shall not recall here the language of broken surface diagrams, but refer instead the reader to

⁶See, for instance the local ribbon filling change given in [2, Fig. 16].

[12]). This is illustrated by the following local pictures:



In general, for an arbitrary welded graph Γ , we obtain a diagram for $\text{Tube}(\Gamma)$ by connecting the above pieces of broken surface diagrams with arbitrary embedded annuli, as prescribed by the welded graph Γ .

The Tube map, as defined by Satoh on welded knotoids and knots [26], and extended later to welded links and string links [4], served as a tool for studying ribbon 2–knots, tori and 2–string links. By Proposition 1.14, welded graphs now provide a general setting for studying the whole class of ribbon surface-links.

In contrast, injectivity of the Tube map has remained an intriguing question. In [11], the Tube map has been proved to be injective on welded braids, but Global Reversal moves were proved to be in its kernel for links [28, 16], and the question remains widely open for welded string links. In view of Theorem 1.8, we have the following.

Corollary 1.16. *The Tube map is invariant under Υ moves.*

We conjecture that, except for some very specific cases (see Appendix A.3), the Υ moves are not induced by the usual welded moves, and hence provide a new large class of non trivial moves in the kernel of the Tube map for welded links and string links. As a matter of fact, welded graphs intrinsically contain all the known moves under which the Tube map is invariant.

2. WIRTINGER GROUPS, PERIPHERAL SYSTEMS AND REDUCED QUOTIENT

2.1. Wirtinger groups. The fundamental group of the complement is an essential invariant in the topological study of embedded manifolds in codimension 2. In the case of knots and links, any diagram provides a so-called Wirtinger presentation for this group. This diagrammatical point of view on the fundamental group generalizes to other situations, such as surface-links [12] or welded knot theory [17]. The goal of the present section is to show that the theory of welded graphs can in fact be interpreted as a theory of Wirtinger presentations.

Definition 2.1. A *Wirtinger presentation* is a group presentation

$$\langle x_1, \dots, x_k \mid R_1, \dots, R_s \rangle$$

with a finite number of generators, and a finite number of relations of the form $R = \bar{x}_j x_i^w$, where x_i and x_j are two generators and w is an element of the free group $F(x_1, \dots, x_k)$. A *Wirtinger group* is a group admitting a Wirtinger presentation.

One way to describe a Wirtinger presentation is by a decorated oriented graph with one vertex per generator, and one w -decorated edge from x_i to x_j for each relation $\bar{x}_j x_i^w$. Such a decorated graph is nothing but a w -graph, with no marked vertex. Note that performing an R1 move on this graph does not change the Wirtinger presentation since in $F(x_1, \dots, x_k)$, we have $\bar{x}_j x_i^{x_i^w} = \bar{x}_j x_i^{\bar{x}_i^w} = \bar{x}_j x_i^w$. More generally, we have the following.

Proposition 2.2. *Welded graphs with no marked vertex are in one-to-one correspondence with Wirtinger presentations up to the following operations:*

- $\langle x_1, \dots, x_k \mid R_1, \dots, R_s, \bar{x}_{i_2} x_{i_1} \rangle \leftrightarrow \langle x_1, \dots, x_{i_2-1}, x_{i_2+1}, \dots, x_k \mid R_1|_{x_{i_2} \rightarrow x_{i_1}}, \dots, R_s|_{x_{i_2} \rightarrow x_{i_1}} \rangle$;
- $\langle x_1, \dots, x_k \mid R_1, \dots, R_s, \bar{x}_{i_1} x_{i_2}^w \rangle \leftrightarrow \langle x_1, \dots, x_k \mid R_1, \dots, R_s, \bar{x}_{i_2} x_{i_1}^w \rangle$;
- $\langle x_1, \dots, x_k \mid R_1, \dots, R_s \rangle \leftrightarrow \langle x_1, \dots, x_k \mid R_1|_{x_{i_1} \rightarrow x_{i_1}^\alpha}, \dots, R_s|_{x_{i_1} \rightarrow x_{i_1}^\alpha} \rangle$;
- $\langle x_1, \dots, x_k \mid R_1, \dots, R_s, \bar{x}_{i_2} x_{i_1}^w, \bar{x}_{i_4} x_{i_3}^{x_{i_1}^w} \rangle \leftrightarrow \langle x_1, \dots, x_k \mid R_1, \dots, R_s, \bar{x}_{i_2} x_{i_1}^w, \bar{x}_{i_4} x_{i_3}^{w x_{i_2}^{i_1}} \rangle$

where $R_{|a \rightarrow b}$ denotes the word obtained from some word R by replacing all occurrences of a by b , and where α is any generator or inverse of a generator.

Proof. It is straightforwardly checked that the first operation corresponds to a contraction move, the second one to an orientation reversal, the third one to a generalized stabilization, and the last one to a Reidemeister 3 move. \square

Observe that the four operations in Proposition 2.2, preserve the underlying group up to isomorphism. For any welded graph Γ , there is hence an associated Wirtinger group, which we denote by $G(\Gamma)$, given by the presentation associated to Γ , after unmarking all marked vertices. As a matter of fact, this group is the fundamental group of the complement of the associated ribbon surface-link via the generalized Tube map:

Proposition 2.3. *For every welded graph Γ , the group $G(\Gamma)$ is isomorphic to $\pi_1(X^4 \setminus \text{Tube}(\Gamma))$.*

Proof. Let Γ be a w -graph representative of a given welded graph. As noted in the proof of Proposition 1.14, this representative Γ can be assumed to be reduced, so that we can consider the associated ribbon handlebody $H = C(\Gamma)$. Recall that H is cut into chambers by ribbon disks, and that $\text{Tube}(\Gamma) = \partial H$. We observe that the desired presentation for $\pi_1(X^4 \setminus \text{Tube}(\Gamma))$, can be given in terms of the ribbon disks and the chambers of H . This follows from the three following facts:

- H retracts to a 1-dimensional graph in X^4 , so that $X^4 \setminus H$ is connected and simply connected;
- since H is 3-dimensional, and since ribbon disks are 2-dimensional, a generic loop in $X^4 \setminus \text{Tube}(\Gamma)$ avoids all ribbon disks and intersects H in a finite number of transverse points. Moreover, two loops intersecting a same chamber once with same sign, are homotopic. Hence $\pi_1(X^4 \setminus \text{Tube}(\Gamma))$ is generated by a set of loops which is in bijection with chambers, a chamber c corresponding to the class of a loop intersecting H exactly once, in the interior of c ;
- a generic homotopy of loops in $X^4 \setminus \text{Tube}(\Gamma)$ intersects the ribbon disks in a finite number of transverse points. Each intersection corresponds to a Wirtinger relation between the two generators associated to the chambers on either side of the intersected ribbon disk, one being equal to the conjugate of the other by the generator associated to the chamber containing the contractible preimage of the ribbon disk.

Since ribbon disks and chambers of H are in bijection with edges and vertices of Γ , this presentation is nothing but the Wirtinger presentation of $G(\Gamma)$. \square

In the rest of this section, we shall see that a Wirtinger presentation carries more information than just the associated group.

2.2. Peripheral system. The fundamental group of a codimension 2 embedding complement can be endowed with the data of so-called peripheral elements, which are given by loops staying essentially close to the embedded manifold. In the case of links, these elements are meridians and longitudes, and they form, together with the group, the peripheral system. Implicitly used by Dehn in 1914 to distinguish the right and the left-handed trefoil, peripheral systems were only formally introduced by Fox [15] in the early fifties and proved to be a complete invariant for links by Waldhausen [27] in 1968.

This notion of peripheral system has a combinatorial counterpart for welded graphs, which we introduce now. Let Γ be a w -graph, with connected components $\Gamma_1, \dots, \Gamma_\ell$. Denote by b_i the first Betti number of Γ_i and by m_i its number of marked vertices.

Notation 2.4. A combinatorial path between two vertices v_1 and v_2 on Γ , is a finite sequence of adjacent edges starting at v_1 and ending at v_2 . Note that, on each edge, the orientation given by the w -graph and the orientation given by the path may differ. To any combinatorial path $p = (e_1, \dots, e_k)$, one can associate the element $w_p \in G(\Gamma)$ defined by the concatenated word $w_1^{\varepsilon_1} \dots w_k^{\varepsilon_k}$, where w_i is the label of e_i and $\varepsilon_i \in \{\pm 1\}$ is 1 if and only if the orientations of e_i as an edge of Γ and as a path segment of p coincide.

A combinatorial path corresponds in particular to a continuous path between two vertices and, reciprocally, any continuous path γ between two vertices is homotopic relative to the endpoints, to a combinatorial path p_γ . Note that two path representatives for γ differ only by some back and forth along some edges, so that the associated element in $G(\Gamma)$ is well-defined. We shall hence use simply the notation w_γ for the element $w_{p_\gamma} \in G(\Gamma)$.

Remark 2.5. It follows from iterated uses of Wirtinger relations that, if γ is a path between two vertices $v_1, v_2 \in \Gamma$, then $w_\gamma v_2 = v_1 w_\gamma$ in $G(\Gamma)$. This implies, in particular, that, for every $i \in \{1, \dots, \ell\}$, any two

vertices of Γ_i , seen as elements of $G(\Gamma)$, are conjugate, and that if γ is a loop based at some vertex v , then v and w_γ commute.

Definition 2.6. *Meridians* of Γ are defined as the generators of $G(\Gamma)$, that is, the vertices of Γ . Vertices on the i -th component of Γ are sometimes more precisely called *i -th meridians*.

A *basing* for Γ is the data $((\mu_i)_i, \{l_1^i, \dots, l_{m_i}^i\}_{i,j}, (a_1^i, \dots, a_{n_i}^i)_i)$ where, for every $i \in \{1, \dots, \ell\}$:

- μ_i is a choice of a meridian on Γ_i ;
- $l_1^i, \dots, l_{b_i}^i$ are b_i loops on Γ_i based at μ_i , that generate $\pi_1(\Gamma_i, \mu_i)$;
- $a_1^i, \dots, a_{m_i}^i$ are m_i paths that connect μ_i to each marked vertex of Γ_i .

Note that, since connected components and marked points of Γ are ordered, meridians and a_i^j paths are also ordered, whereas l_i^j loops are not.

Remark 2.7. If Γ has a marked vertex on each component, then there is a canonical choice for the chosen meridians, namely the minimal marked vertex on each component.

Definition 2.8. Let Γ be endowed with a basing $((\mu_i)_i, \{l_j^i\}_{i,j}, (a_j^i)_{i,j})$.

For every $i \in \{1, \dots, \ell\}$, we define

- *preferred i -th loop-longitudes*, as the elements $\mu_i^{-|\lambda_j^i|} w_{l_j^i} \in G(\Gamma)$, for $j \in \{1, \dots, b_i\}$;
- *preferred i -th arc-longitudes*, as the elements $\mu_i^{-|\alpha_j^i|} w_{a_j^i} \in G(\Gamma)$, for $j \in \{1, \dots, m_i\}$;

where λ_j^i , resp. α_j^i , denote the element $w_{l_j^i}$, resp. $w_{a_j^i}$, of $G(\Gamma)$ and where $|w|_i$ is the algebraic number of letters in some $w \in G(\Gamma)$ that correspond to vertices in Γ_i . A *longitude* is either a (preferred) loop or an arc-longitude.

As noted above, meridians and arc-longitudes are gathered in ordered tuples, whereas loop-longitudes are elements of unordered sets.

Proposition 2.9. *Let Γ' be a w -graph obtained from Γ by a welded move. There is a canonical isomorphism $\varphi : G(\Gamma) \rightarrow G(\Gamma')$, which sends any basing of Γ to a basing for Γ' .*

Proof. The statement is clear for contraction, orientation reversal, stabilization and Reidemeister 3 moves. A Reidemeister 1 move applied on an edge e introduces some letter $v^{\pm 1}$ in the longitude word w_γ associated to any path γ passing through e , and changes $|w_\gamma|_i$ by ± 1 . Using Remark 2.5, $v^{\pm 1}$ can be replaced by a $\mu_i^{\pm 1}$ factor at the beginning of the longitude word w_γ , so that it compensates the change of $|w_\gamma|_i$. \square

Definition 2.10. A *peripheral system* for a welded graph Γ is the data

$$(G(\Gamma), (\mu_i)_i, \{w_{l_j^i}\}_{i,j}, (w_{a_j^i})_{i,j}),$$

where $((\mu_i)_i, \{l_1^i, \dots, l_{m_i}^i\}_{i,j}, (a_1^i, \dots, a_{n_i}^i)_i)$ is a basing for Γ . Two peripheral systems for Γ are *equivalent* if they are related by a finite sequence of the following operations:

- replacing $(G, (\mu_i)_i, \{l_j^i\}_{i,j}, (a_j^i)_{i,j})$ by $(G', (\mu'_i)_i, \{\lambda_j^i\}_{i,j}, (\alpha_j^i)_{i,j})$, when there exists an isomorphism $\varphi : G \rightarrow G'$, some permutations $\sigma_i \in \mathfrak{S}_{b_i}$ and some elements $w_1, \dots, w_\ell \in G'$ such that:
 - $\mu'_i = \varphi(\mu_i)^{w_i}$ for all $i \in \{1, \dots, \ell\}$;
 - $\lambda_{\sigma_i(j)}^i = \varphi(l_j^i)^{w_i}$ for all $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, m_i\}$;
 - $\alpha_j^i = \overline{w_i} \varphi(a_j^i)$ for all $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, n_i\}$;
- replacing a loop-longitude by its inverse;
- precomposing an i -th arc or loop-longitude by some other i -th loop-longitude.

We emphasize that peripheral systems depend on a choice of a basing, which corresponds to the choice of a vertex v_i for each connected component Γ_i , together with a choice of a generating set for $\pi_1(\Gamma_i, v_i)$ and of paths from v_i to each marked vertex. But, since $\pi_1(\Gamma_i, v_i)$ is a free group, it follows from Nielsen transformations that equivalence classes are precisely defined to encompass any change of such choices. By Proposition 2.9, we thus have:

Corollary 2.11. *Peripheral systems up to equivalence, are well-defined for welded graphs.*

From now on, and by abuse of notation, peripheral systems shall be identified with their equivalence classes.

Remark 2.12. For a ribbon surface-link S , given with a ribbon filling, meridians are loops that meet the ribbon filling only once, and longitudes are defined by pushing curves on S off the surface. In light of the proof of Proposition 2.3, longitudes can be characterized by their intersections with the ribbon filling. But curves on $S = \text{Tube}(\Gamma)$ induce curves on Γ , and the intersections of the curves push-outs with the ribbon filling, can be read out of the curves on Γ . These precisely coincide with the longitudes of Γ . It follows that the Tube map preserves the whole peripheral system, in the sense that the peripheral system of a welded graph Γ is equal to the peripheral system of $\text{Tube}(\Gamma)$.

Remark 2.13. Suppose that a welded graph has only simply connected components, each having at least one marked vertex. This is for example the case for type $(2, 0)$ welded graphs, which are intimately related to welded string links by Theorem 1.8. In this situation, there is a preferred representative in the equivalence class of peripheral systems, corresponding to meridians being the minimal marked vertices on each component (see Remark 2.7). Since there are no loop-longitudes in this case, the peripheral system reduces to the data of the associated group, together with $(\sum_{i=1}^{\ell} m_i) - \ell$ elements, corresponding to paths from the minimal marked vertices to each of other marked vertices on the same component.

In light of Remarks 2.12 and 2.13, it is clear that, for a string link L , the notion of peripheral system for L and the one for $\psi_{SL}(L)$ coincide. Since by Waldhausen's theorem, peripheral systems are classifying for classical knots, links and string links; this implies the following result.

Corollary 2.14. *Classical string links embed into welded graphs.*

By Theorem 1.8, this means that, should Υ moves be non trivial, they would not be able change the isotopy class of classical string links.

Remark 2.15. For links, the situation is slightly more intricate. Indeed, classical peripheral systems provide an orientation for each circular component, whereas welded graph peripheral systems do not. For instance, if a link L_2 is obtained from another link L_1 by mirror image and reversing all orientations, then $\psi_L(L_1) = \psi_L(L_2)$ even if L_2 and L_1 are not isotopic. Is it thus an interesting question to determine when reversing some longitudes in the peripheral system of a classical links, corresponds to the peripheral system of a distinct classical link.

2.3. Reduced quotients and self-virtualization. By definition, $G(\Gamma)$ has as many generators as vertices in Γ . Nevertheless, since any two meridians on a same component are conjugate, $G(\Gamma)$ is normally generated by any choice of one generator per component (as, for example, specified by choice of a basing). In this situation, the following classical notion, due to Milnor [24], has proved useful:

Definition 2.16. Let G be a group normally generated by elements x_1, \dots, x_p . We denote by

$$RG := G / \{ [x_i, x_i^g] \mid i \in \{1, \dots, p\}; g \in G \},$$

the *reduced quotient* of G , which is the smallest quotient where each generator commutes with any of its conjugates.

This leads to the following definition.

Definition 2.17. We define a *reduced peripheral system* for a welded graph Γ as the data

$$\left(RG(\Gamma), ([\mu_i])_i, \{ [\lambda_j^i] \cdot N_i \}_{i,j}, ([\alpha_j^i] \cdot N_i)_{i,j} \right)$$

where

- $(G(\Gamma), (\mu_i)_i, \{\lambda_j^i\}_{i,j}, \{\alpha_j^i\}_{i,j})$ is a peripheral system for Γ ,
- $RG(\Gamma)$ is the reduced quotient of $G(\Gamma)$, and $[x]$ denotes images in $RG(\Gamma)$ of $x \in G(\Gamma)$,
- $w \cdot N_i$ is the coset of w in $RG(\Gamma)$ relatively to the normal subgroup N_i of $RG(\Gamma)$ generated by $[\mu_i]$.

Two reduced peripheral systems for Γ are *equivalent* if they arise from equivalent peripheral systems for Γ .

It should be noted that, since two i -th meridians are always conjugate, the reduced quotient $RG(\Gamma)$ does not depend on the choice of one meridian per component. Equivalence classes of reduced peripheral systems are clearly invariant under welded moves.

Reduced quotients were introduced by Milnor in [24] for the study links up to link-homotopies, which are homotopies that keep distinct components disjoint. In the welded context, the combinatorial counterpart of link-homotopy is the notion of sv -equivalence, defined in [4] for welded string links, and that we generalize now to welded graphs:

Definition 2.18. Two welded graphs differ by a *self-virtualization* (SV) if they differ by the replacement of an empty edge e , by an edge decorated by a letter v or \bar{v} , where v is a vertex that belongs to the same component as e . Two welded graphs are *sv-equivalent* if they are related by a sequence of self-virtualizations and welded moves.

Remark 2.19. In terms of Gauss diagrams, an SV move is the insertion or deletion of a self-arrow, that is, an arrow which has both endpoints on a same component. In terms of virtual diagrams, this translates as replacing a self-crossing by a virtual one, or vice versa.

Proposition 2.20. *Reduced peripheral systems are invariant under self-virtualization.*

Proof. An empty edge e yields a relation $\bar{v}_2 v_1$ in the group, where v_1 and v_2 are the two endpoints of e . Adding a decoration v on e , turns the relation into $\bar{v}_2 \bar{v} v_1 v = \bar{v}_2 [v_1, v] v_1$, which is equal to $\bar{v}_2 v_1$ in the reduced quotient since v and v_2 belong to the same component and are hence conjugate one to another (see Remark 2.5). The reduced group is hence invariant.

Moreover, longitudes passing through e are modified from $w_1 w_2$ to $w_1 v w_2 = w_1 w_2 v^{w_2}$, hence define the same coset. \square

Proposition 2.21 (Chen–Milnor presentation). *Let $(G(\Gamma), (\mu_i)_i, \{\lambda_j^i\}_{i,j}, (\alpha_j^i)_{i,j})$ be a peripheral system for a welded graph Γ . There exists $w_j^i \in F(\mu_1, \dots, \mu_\ell)$ for $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, b_i\}$ such that $w_j^i = [\lambda_j^i]$ in $RG(\Gamma)$ and*

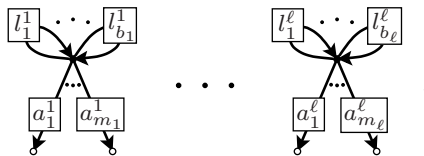
$$RG(\Gamma) \simeq R \left\langle \mu_1, \dots, \mu_\ell \mid [\mu_i, w_j^i] \text{ for all } i \in \{1, \dots, \ell\} \text{ and } j \in \{1, \dots, b_i\} \right\rangle.$$

Proof. The core of the proof is to show that Γ is sv -equivalent to a welded graph with a unique unmarked vertex (and possibly several marked vertices) per component. This is done component per component. Consider some component Γ_i , and pick an unmarked vertex: our goal is to eliminate all other unmarked vertices of Γ_i by contraction moves. First we remove from the edge labels, all letters corresponding to i -th meridians, using Split and self-virtualization moves. Having done so, we can successively apply generalized stabilization moves to those edges that we wish to contract, to turn them into empty edges: since no i -th meridian appear in the label that we are sweeping out, those generalized stabilization moves will only modify decorations on other components; note that only the labels of other components are affected in the process, so that previously-treated components remain of the desired form.

It remains to observe that, for the resulting welded graph with a unique unmarked vertex per component, the associated Wirtinger presentation provides the desired presentation for $RG(\Gamma)$. \square

We stress that the proof of Proposition 2.21 provides not only a presentation for $RG(\Gamma)$, but also a ‘normal form’ for a welded graph Γ endowed with a basing (hence with a reduced peripheral system).

Specifically, if $((\mu_i)_i, \{l_1^i, \dots, l_{b_i}^i\}_{i,j}, (a_1^i, \dots, a_{m_i}^i)_i)$ is a basing for Γ , then the i -th component of this normal form has one unmarked vertex μ_i , b_i loops based at μ_i and decorated by $l_1^i, \dots, l_{b_i}^i$, and m_i edges connecting μ_i to each marked vertex, and decorated by $a_1^i, \dots, a_{m_i}^i$:

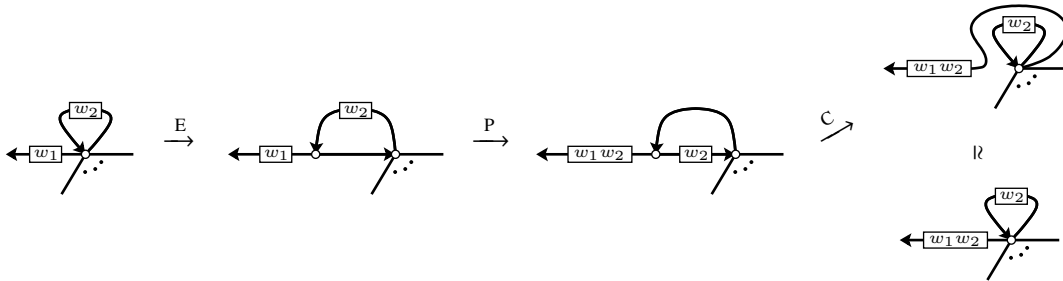


This yields the following classification of welded graphs up to self-equivalence by the reduced peripheral system:

Theorem 2.22. *Two welded graphs are sv-equivalent if and only if they have equivalent reduced peripheral systems.*

Proof. The proof essentially follows the proof of [5, Thm. 2.1], which addresses the link case. Consider two welded graphs Γ_1 and Γ_2 with equivalent reduced peripheral systems, and suppose that Γ_1 and Γ_2 are given in normal forms as above.

The first step is to modify Γ_2 into a welded graph having the ‘same’ reduced peripheral system as Γ_1 ; by this we means that the isomorphism φ of Definition 2.10, sends directly meridians to meridians and longitudes to longitudes. As stated in this same definition, equivalence of reduced peripheral systems is generated by three elementary operations. The first one corresponds to conjugation, by a same word w , of a meridian and the associated loop-longitudes, and precomposition by \bar{w} of the associated arc-longitudes; at the level of w -graphs, this can be realized by generalized stabilizations. The second operation reverses a loop-longitude; this can be realized by an orientation reversal move on w -graphs. The last operation precomposes a longitude by a loop-longitude from the same component; this can be realized on w -graphs by the following sequence of moves:



Now, Γ_1 and Γ_2 only differ by the representative words for the longitudes. Thanks to Proposition 2.21, we know that, for a given i -th longitude, the representative words differ by a sequence of insertion/deletion of

- conjugates of μ_i ,
- commutators $[\mu_j, w_j]$ for some $j \in \{1, \dots, \ell\}$, where w_j is a j -th loop-longitude,
- commutators $[\mu_j, \mu_j^w]$ for some $j \in \{1, \dots, \ell\}$, where w is any word.

All these operations are handled in a local way in the proof of [5, Thm. 2.1], and the argument can be transposed directly to the welded graph case. \square

Remark 2.23. Theorem 2.22 can be seen as a generalization of the main theorem of [5], which deals with the link case. It provides an equivalence between reduced peripheral systems and welded graphs up to sv-equivalence. Other quotients of the peripheral system, such as q -nilpotent [13, Thm. 2] or k -reduced [8, Thm. 4.3] quotients have also been investigated and have led to similar correspondances for welded links up to, respectively, w_q and self w_k -concordance. The notion of (self) w_q -concordance refers, on one hand, to a combinatorial notion of concordance for welded objects which generalizes the classical notion of concordance for topological objects and, on the other hand, the notion of (self) w_q -equivalence based on the *arrow calculus* developed by the second and third authors. This arrow calculus relies on surgery operations on welded diagrams, along certain oriented univalent trees, filtered by the size of the associated tree.

It turns out that arrow calculus can be translated in terms of welded graph. A degree q surgery corresponds to inserting a length q iterated commutator in some edge decoration, and a degree k self-surgery likewise corresponds to inserting an iterated commutator which involves at least $k + 1$ times the same component. The diagrammatic formulation of concordance can also be extended to all welded graph in a straightforward way. All the arguments extend to this setting, so that two welded graphs are w_q -concordant (resp. self w_k -concordant) if and only if they have equivalent q -nilpotent (resp. k -reduced) peripheral systems.

Note that, all together, this implies that, up sv-equivalence, w_q or self w_k -concordance, the sets of welded links and string links embed in welded graphs. As a matter of fact, and compared with Theorem 1.8, it can indeed be directly checked by hand that Υ moves are trivial up to sv-equivalence, w_q -equivalence or self w_k -equivalence.

2.4. Milnor invariants. The peripheral system and its reduced quotient are strong invariants, but they are also very difficult to compute and compare. In [25], Milnor introduced numerical invariant extracted from the nilpotent peripheral system, which are algorithmically computable, and which are still powerful link invariants. We define below similar invariants for welded graphs. A comprehensive account on Milnor invariants can be found in [7, Sec. 4], see also the survey [21].

For simplicity of exposition, here, we will restrict ourselves to the so-called ‘non repeated’ Milnor invariants, that are extracted from the reduced peripheral system. Moreover, with our main topological result in sight (see Section 3), we shall further restrict to the following class of welded graphs.

Definition 2.24. A *welded forest*, is a welded graph of type $((m_1, 0), \dots, (m_\ell, 0))$ such that $m_1, \dots, m_\ell \neq 0$. In other word, each connected component is simply connected and contains at least one marked vertex.

Let Γ be a welded forest. Following Remark 2.7, Γ has a canonical set of meridians μ_1, \dots, μ_ℓ , which corresponds to the minimal marked vertices on each component. Moreover, Γ has no non-trivial loop-longitude, so that by Proposition 2.21, we have that $RG(\Gamma)$ is isomorphic to the reduced free group $RF(\mu_1, \dots, \mu_\ell)$. Hence, following Remark 2.13, the reduced peripheral system of Γ is simply the data of $(\sum_{i=1}^\ell m_i) - \ell$ elements of $RF(\mu_1, \dots, \mu_\ell)$, representing all arc-longitudes. In what follow, we shall simply denote by $\alpha_j^i \in F(\mu_1, \dots, \mu_\ell)$, the element representing the arc-longitude running to the j -th marked point on the i -th component of Γ ($i \in \{1, \dots, \ell\}$, $j \in \{2, \dots, n_i\}$).

Next we recall the *reduced Magnus Expansion*

$$\begin{aligned} RF(\mu_1, \dots, \mu_\ell) &\rightarrow \mathbb{Z}\langle\langle X_1, \dots, X_\ell \rangle\rangle / \mathcal{R} \\ \mu_i^\varepsilon &\mapsto 1 + \varepsilon X_i \end{aligned}$$

taking values in the quotient of the ring $\mathbb{Z}\langle\langle X_1, \dots, X_\ell \rangle\rangle$ of formal power series in ℓ non-commuting variables, by the ideal \mathcal{R} generated by monomials containing at least twice a same variable. The reduced Magnus expansion is known to be injective, see for instance [34].

Definition 2.25. For every $i \in \{1, \dots, \ell\}$, $j \in \{2, \dots, m_i\}$, and for every sequence $I = i_1 \dots i_r$ of distincts integer in $\{1, \dots, \ell\} \setminus \{i\}$, we define $\mu_\Gamma(I; i, j)$ as the coefficient of $X_{i_1} \dots X_{i_r}$ in $E(\alpha_j^i)$. These coefficients are called the *non repeated Milnor invariants* of Γ .

Remark 2.26. The well-definedness of these non repeated Milnor invariants $\mu_\Gamma(I; i, j)$ is clear. Indeed, as discussed above, the welded forest Γ has a preferred set of meridians (Remark 2.13), and the $(\sum_{i=1}^\ell m_i) - \ell$ arc-longitudes forming its reduced peripheral system, induce well-defined elements α_j^i in $RG(\Gamma) = RF(\mu_1, \dots, \mu_\ell)$ (see Notation 2.4). Their invariance under self-virtualization follows from Proposition 2.20.

We have the following classification result.

Proposition 2.27. *Two welded forests are sv-equivalent if and only they have same non repeated Milnor invariants.*

Proof. This is a direct consequence of Theorem 2.22, combined with the injectivity of the reduced Magnus expansion [34]. \square

3. APPLICATION TO KNOTTED PUNCTURED SPHERES UP TO LINK-HOMOTOPY

We can now get back to 4-dimensional topology, and deal with the case of knotted punctured spheres.

Definition 3.1. Let ℓ be some positive integer and $(m_1, \dots, m_\ell) \in \mathbb{N}$. A union of ℓ *knotted punctured spheres* is the image of a proper embedding in B^4 of $\Sigma = \sqcup_{i=1}^\ell S_i$, where S_i is a 2-sphere with m_i disjoint disks removed and such that $\partial\Sigma$ is sent to a fixed (oriented and ordered) unlink U in $\partial B^4 = S^3$, up to isotopy fixing the boundary.

As noted in the introduction, this large class of surface-links contains in particular Le Dimet’s linked disks [19] (case $m_i = 1$ for all i), and 2-string links [4, 6] (case $m_i = 2$ for all i). We stress that the boundary components of each punctured sphere are ordered, so that there is a preferred meridian for each component of a union S of knotted punctured spheres, defined as a loop in $\partial B^4 \setminus S$ which is a meridian of the first boundary component.

Our purpose is to classify unions of knotted punctured spheres up to link-homotopy.

Definition 3.2. Two unions of knotted punctured spheres are *link-homotopic* if they are images of embeddings which are homotopic through proper immersions that keep distinct components disjoint, and that send $\partial\Sigma$ to U .

Our first result in this direction, is a generalization of [6, Thm. 3.5].

Proposition 3.3. *Any union of knotted punctured spheres is link-homotopic to a ribbon one.*

Proof. We can use the very same strategy as used in [6, Sec. 3.2] in the 2–string link case. Pick some union S of knotted punctured spheres. The strategy goes in three steps, which can be roughly summarized as follows. First, using the fact that ∂S is an unlink, one can shrink and stretch a neighborhood of this boundary; this allows to regard S as a 2–link (*i.e.* smoothly embedded disjoint copies of the 2–sphere) with thin, unknotted and unlinked tubes attached. Next, by a theorem of Bartels and Teichner [10, Thm. 1], there is a link-homotopy which takes this 2–link to a disjoint union of unknotted spheres. The proof then amounts to showing that, in the process of this link-homotopy, only ribbon-type linking will be created among the attached tubes, so that the final union of knotted punctured spheres is indeed ribbon. This last step uses broken surface diagrams of immersed surfaces, and consists in a systematic study of all Roseman moves and singular Roseman moves on these diagrams, which are local moves that generate link-homotopy of surface-links [6, Prop. 2.4]; the arguments in [6] being local, they apply verbatim to our more general situation. \square

In [7, Sec. 5.2.3], Milnor link-homotopy invariants for surface-links were defined; they come in three flavors: Milnor maps, Milnor loop-invariants and Milnor arc-invariants. Milnor maps and Milnor loop-invariants are extracted from the loop-longitudes; in the case of a union S of ℓ knotted punctured spheres, the latter are given by curves parallel to the boundary components, which are copies of the unknot, so that Milnor maps and Milnor loop-invariants of S are trivial. Milnor arc-invariants are defined by pushing in $B^4 \setminus S$, and closing in a canonical way, the $(\sum_{i=1}^{\ell} m_i) - \ell$ boundary-to-boundary arcs given by the basing. These define a collection of elements of the reduced fundamental group of $B^4 \setminus S$, which by [6, Thm. 5.15] is isomorphic to the reduced free group generated by ℓ elements m_1, \dots, m_{ℓ} , represented by the preferred meridians of each component of S . Taking the coefficients in the reduced Magnus expansion of these elements, defines non-repeated Milnor invariants of S . We refer to [7] for details.

We can now prove the classification result announced in [7, Thm. 6.10].

Theorem 3.4. *Unions of knotted punctured spheres are classified, up to link-homotopy, by non repeated Milnor invariants.*

Proof. By [7, Prop. 5.23], non repeated Milnor invariants are invariant under link-homotopy. Now, let S_1 and S_2 be two unions of knotted punctured spheres with same non repeated Milnor invariants. By Proposition 3.3, we may assume that S_1 and S_2 are ribbon, so that by Proposition 1.14, they can be described as $\text{Tube}(\Gamma_1)$ and $\text{Tube}(\Gamma_2)$ for two welded forests Γ_1 and Γ_2 . It follows from Remark 2.12 that the non-repeated Milnor arc-invariants of S_i coincide with those of Γ_i ($i = 1, 2$). Hence Γ_1 and Γ_2 have same non-repeated Milnor invariants, and Proposition 2.27 implies that Γ_1 and Γ_2 are therefore sv–equivalent. Now, in the same way as it was observed in [4] in the welded string link case, an SV move on a welded graph can be realized as a link-homotopy on its image under Tube ; more precisely, the image by Tube of an SV move is a ‘self-circle crossing change’ as illustrated in [4, Fig. 6]. Thus S_1 and S_2 are link-homotopic, and the proof is complete. \square

Note that, in particular, Theorem 3.4 implies that linked disks (that is, unions of once-punctured knotted spheres) are all link-homotopically trivial, since linked disks admit no non-trivial longitudes. The following consequence, using results of [23], might be well-known to the experts.

Corollary 3.5. *Up to link-homotopy, a slice link bounds a unique union of slice disks.*

Proof. Let L be a slice link in S^3 , and let D_1 and D_2 be two unions of slice disks for L in the 4–ball. Pick in $S^3 \times [0, 1]$ a concordance C from L to the unlink U , and let \bar{C} be the mirror image of C . By [23, lem. 1.1], the stacking product $C \cdot \bar{C}$ is link-homotopic to the product $L \times [0, 1]$. Since, for $i = 1, 2$, a collar neighborhood of the boundary of D_i is isotopic to $L \times [0, 1]$, we have that D_i is link-homotopic to a product $C \cdot S_i$, where S_i is a union of slice disks for the unlink U . But as noted above, by Theorem 3.4 we have that S_1 and S_2 are link-homotopic. It follows that D_1 and D_2 are link-homotopic, as desired. \square

Remark 3.6. The trick used in the proof of Corollary 3.5 above, relying on [23, Lemma 1.1], applies more generally to all knotted punctured spheres. This allows to extend the classification of Theorem 3.4 to a larger class of surface-links, which are images of proper embeddings in B^4 of a union of punctured spheres $\sqcup_{i=1}^{\ell} S_i$, such that the boundary is sent to a fixed *slice* link in $\partial B^4 = S^3$.

For the classification given in Theorem 3.4, we considered link-homotopies sending the boundary to the fixed unlink U at all time. This condition can be relaxed to define *weak link-homotopies*, as homotopies through proper immersions that may not preserve the image of the boundary.

Corollary 3.7. *Any two unions of knotted punctured spheres are weakly link-homotopic.*

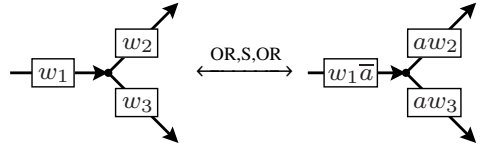
Proof. Let us call *unknotted punctured spheres*, a union of punctured spheres which admits a ribbon filling with no ribbon disk. It follows from the same arguments as in the proof of Lemma 1.13, that such an unknotted punctured spheres is unique. We denote by S_0 these unknotted punctured spheres. Let us prove that any union S of ℓ knotted punctured spheres is weakly link-homotopic to S_0 .

The strategy is to provide a union of ℓ knotted punctured spheres S' which has, on one hand, the same non repeated Milnor invariants as S , so that it is link-homotopic to S by Theorem 3.4, and is on the other hand ‘nice enough’ to be unknotted by an ambient isotopy that does not fix ∂B^4 . For that, we consider the arc-longitudes of S , seen as words in the reduced free group on ℓ generators, and build an ℓ -component welded string link L realizing these words as its longitudes: see for example [23, Fig. 2.1] or [4, Rem.4.23]. The image of L by the Tube map is an $(\sum_{i=1}^{\ell} m_i)$ -component 2-string link which is ribbon. By [4, Theorem 2.30], $\text{Tube}(L)$ is link-homotopic to a *monotone* ribbon 2-string link, the latter being the 4-dimensional trace of an ambient isotopy $(f_t)_{t \in [0,1]}$ of an unlink in S^3 . On the unlink U forming the ‘upper’ boundary of this monotone ribbon 2-string link, we glue a copy of S_0 : this yields the desired union of knotted punctured spheres S' . By construction, it has the same Milnor invariants as S , and performing backward $(f_t)_{t \in [0,1]}$ on a collar neighborhood of ∂B^4 does provide an ambient isotopy which unknot S' into S_0 . \square

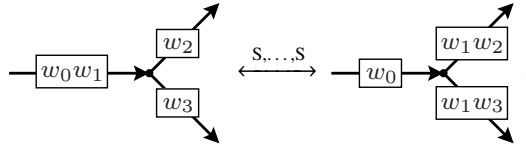
APPENDIX A. EXTRA MOVES ON WELDED GRAPHS

In this appendix, we explain how the moves of Figure 4 on welded graphs, follow from the C, OR, S and R3 moves of Figure 3, as claimed at the end of Section 1. We also show how one version the Υ move is implied by usual welded moves.

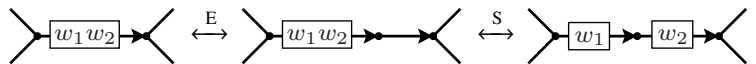
A.1. Push and split moves. We first observe that the extra push (P) and split (Split) moves, are consequences of the C, OR and S moves. For this purpose, we first note that, up to OR moves, the S move can be generalized to the case where some edge is oriented toward the vertex where the move is performed; in that case, the prefix is then reversed and added at the end of the corresponding decoration:



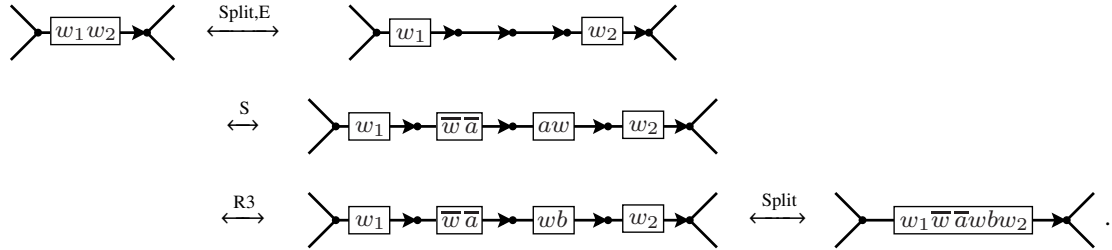
Applying this variant of the S move iteratively, provides a P move:



Moreover, combining this variant of the S move with a C move, allows to ‘split’ edge decorations, thus providing a Split move:

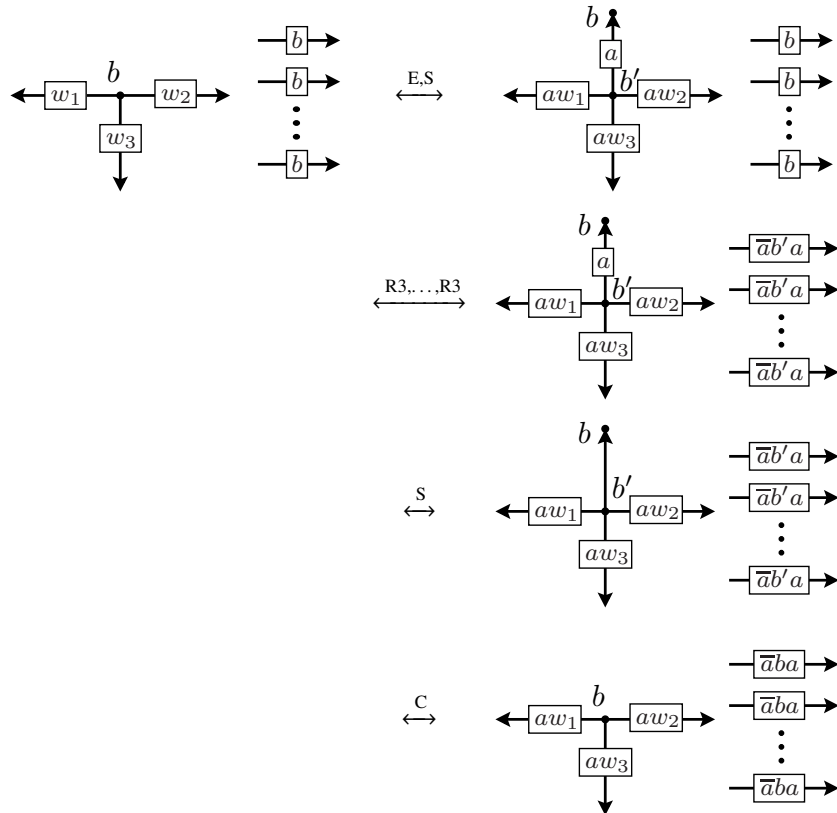


A.2. Rephrased R3 and generalized stabilization move. Next, we show that the Split, C, S and R3 moves on welded graphs, realize a rephrased R3 move. More precisely, we illustrate below how some edge label w_1w_2 can be replaced by $w_1\bar{w}\bar{a}wbw_2$, where (a, b, w) is some other edge in the graph:



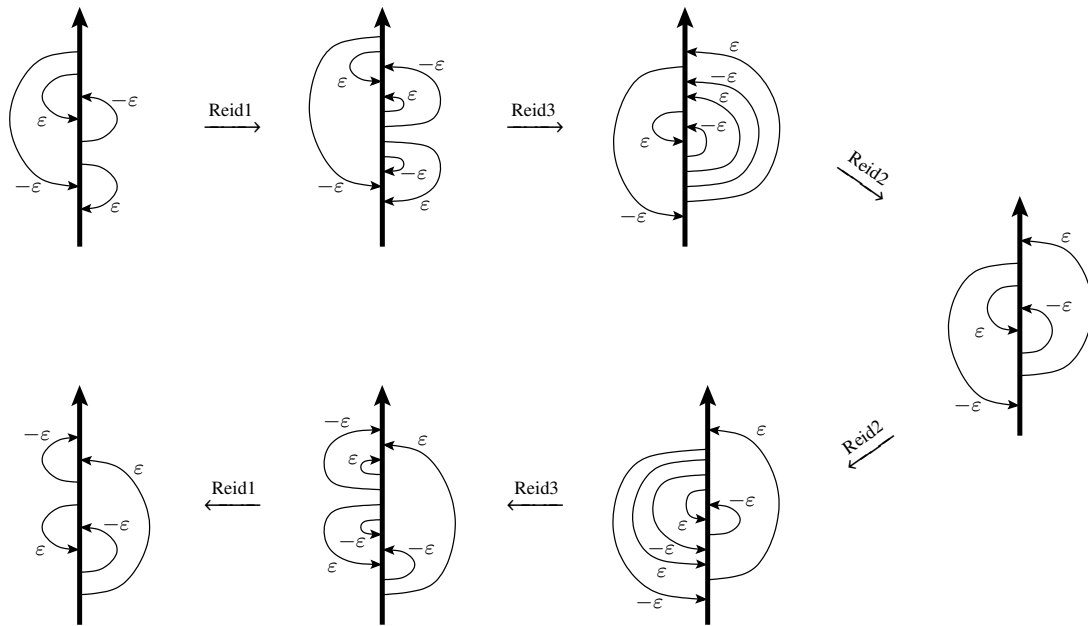
The other versions of this move, that insert any cyclic permutation of $\bar{w}\bar{a}wb$, or any cyclic permutation of it, are proved in a similar way.

Finally, the figure below illustrates how the generalized stabilization move, is achieved by a sequence of C,S and R3 moves:



A.3. A trivial Y move. We finally show that, in the special case where $\eta = -\varepsilon$ and involving only the four self-arrows shown in the figure of Definition 1.7, the Y move follows from Reid1,2,3 moves. This is

illustrated in the following figure:



The strategy is to connect the two sides of the move to a common symmetric configuration. For both, the deformation starts by inserting two (self) arrows with Reid1 moves, then combines two Reid3 moves with TC moves to rearrange the relative positions of the arrow tails, and deletes a pair of arrows with a Reid2 move. To the author's knowledge, this is the only version of the Υ move that follows from the usual welded moves.

REFERENCES

- [1] E. Artin. Zur isotopie zweidimensionalen flächen im R_4 . *Abh. Math. Sem. Univ. Hamburg*, 4:174–177, 1926.
- [2] B. Audoux. On the welded tube map. In *Knot theory and its applications*, volume 670 of *Contemp. Math.*, pages 261–284. Amer. Math. Soc., Providence, RI, 2016.
- [3] B. Audoux. *Applications de modèles combinatoires issus de la topologie*. Habilitation, Aix–Marseille University, 2018.
- [4] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. Homotopy classification of ribbon tubes and welded string links. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 17(1):713–761, 2017.
- [5] B. Audoux and J.-B. Meilhan. Characterization of the reduced peripheral system of links. arXiv:1904.04763, 2019.
- [6] B. Audoux, J.-B. Meilhan, and E. Wagner. On codimension two embeddings up to link-homotopy. *J. Topol.*, 10(4):1107–1123, 2017.
- [7] B. Audoux, J.-B. Meilhan, and A. Yasuhara. Milnor concordance invariant for surface-links and beyond, 2021. arXiv:2109.14578.
- [8] B. Audoux, J.-B. Meilhan, and A. Yasuhara. k -reduced groups and milnor invariants, 2022. arXiv:2205.04768.
- [9] D. Bar-Natan and Z. Dancso. Finite type invariants of w -knotted objects II: tangles, foams and the Kashiwara-Vergne problem. *Math. Ann.*, 367(3-4):1517–1586, 2017.
- [10] A. Bartels and P. Teichner. All two dimensions links are null homotopic. *Geom. Topol.*, 3:235–252, 1999.
- [11] T. E. Brendle and A. Hatcher. Configuration spaces of rings and wickets. *Comment. Math. Helv.*, 88(1):131–162, 2013.
- [12] J. S. Carter, S. Kamada, and M. Saito. *Surfaces in 4-space*, volume 142 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Low-Dimensional Topology, III.
- [13] B. Colombari. A diagrammatical characterization of milnor invariants, 2022. arXiv:2201.01499.
- [14] R. Fenn, R. Rimányi, and C. Rourke. The braid-permutation group. *Topology*, 36(1):123–135, 1997.
- [15] R. H. Fox. A quick trip through knot theory. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pages 120–167. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.
- [16] A. Ichimori and T. Kanenobu. Ribbon torus knots presented by virtual knots with up to four crossings. *Journal of Knot Theory and Its Ramifications*, 21(13):1240005, 2012.
- [17] L. H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.
- [18] L. H. Kauffman and J. a. Faria Martins. Invariants of welded virtual knots via crossed module invariants of knotted surfaces. *Compos. Math.*, 144(4):1046–1080, 2008.
- [19] J.-Y. Le Dimet. Cobordisme d'enlacements de disques. *Mém. Soc. Math. France (N.S.)*, (32):ii+92, 1988.
- [20] Y. Marumoto. Stable equivalence of ribbon presentations. *J. Knot T. Ram.*, 1:241–251, 1992.

- [21] J.-B. Meilhan. Linking number and Milnor invariants. In *Encyclopedia of knot theory*, pages 817–829. Boca Raton, FL: CRC Press, 2021.
- [22] J.-B. Meilhan and A. Yasuhara. Arrow calculus for welded and classical links. *Alg. Geom. Topol.*, 19(1):397–456, 2019.
- [23] J.-B. Meilhan and A. Yasuhara. Link concordances as surfaces in 4–space and the 4–dimensional milnor invariants. *Indiana Univ. Math. J.*, 71:2647–2669, 2022.
- [24] J. Milnor. Link groups. *Ann. of Math. (2)*, 59:177–195, 1954.
- [25] J. Milnor. Isotopy of links. Algebraic geometry and topology. In *A symposium in honor of S. Lefschetz*, pages 280–306. Princeton University Press, Princeton, N. J., 1957.
- [26] S. Satoh. Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications*, 9(4):531–542, 2000.
- [27] F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. Math. (2)*, 87:56–88, 1968.
- [28] B. Winter. The classification of spun torus knots. *J. Knot Theory Ramifications*, 18(9):1287–1298, 2009.
- [29] T. Yajima. On the fundamental groups of knotted 2-manifolds in the 4-space. *J. Math. Osaka City Univ.*, 13:63–71, 1962.
- [30] T. Yajima. On simply knotted spheres in R^4 . *Osaka J. Math.*, 1:133–152, 1964.
- [31] T. Yanagawa. On ribbon 2-knot: The 3-manifold bounded by the 2-knots. *Osaka J. Math.*, 6:447–464, 1969.
- [32] T. Yanagawa. On ribbon 2-knots II: The second homotopy group of the complementary domain. *Osaka J. Math.*, 6:465–473, 1969.
- [33] T. Yanagawa. On Ribbon 2-knots III: On the unknotting Ribbon 2-knots in S^4 . *Osaka J. Math.*, 7:165–172, 1970.
- [34] E. Yurasovskaya. *Homotopy String Links Over Surfaces*. Phd, University of British Columbia, 2008.

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
Email address: benjamin.audoux@univ-amu.fr

UNIV. GRENOBLE ALPES, CNRS, INSTITUT FOURIER, F-38000 GRENOBLE, FRANCE
Email address: jean-baptiste.meilhan@univ-grenoble-alpes.fr

FACULTY OF COMMERCE, WASEDA UNIVERSITY, 1-6-1 NISHI-WASEDA, SHINJUKU-KU, TOKYO 169-8050, JAPAN
Email address: yasuhara@waseda.jp