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Linear orbital stability of discrete shock profiles for systems of conservation laws

Lucas Coeuret[1]

Abstract

We prove the linear orbital stability of spectrally stable stationary discrete shock profiles for conservative finite difference schemes applied to systems of conservation laws. The proof relies on a precise description of the pointwise asymptotic behavior of the Green's function associated with those discrete shock profiles, improving on the result of Godillon [God03]. The main novelty of this stability result is that it applies for a fairly large family of schemes that introduce some artificial viscosity and most importantly, that we do not impose any weakness assumption on the shock.

AMS classification: 35L65, 65M06

Keywords: systems of conservation laws, finite difference scheme, discrete shock profiles, semigroup estimates, linear stability

Contents

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Notations

Throughout this article, we define the following sets:

$$\mathbb{U} := \{z \in \mathbb{C}, |z| > 1\}, \quad \mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\},$$
$$\overline{\mathbb{U}} := \mathbb{S}^1 \cup \mathbb{U}, \quad \overline{\mathbb{D}} := \mathbb{S}^1 \cup \mathbb{D}.$$

For $z \in \mathbb{C}$ and $r > 0$, we let $B(z,r)$ denote the open ball in $\mathbb{C}$ centered at z with radius r. We also introduce the Kronecker symbol $\delta_{i,j}$ which is equal 1 if $i = j$ and 0 when $i \neq j$.

For E a Banach space, we denote $\mathcal{L}(E)$ the space of bounded operators acting on E and $\|\cdot\|_{\mathcal{L}(E)}$ the operator norm. For T in $\mathcal{L}(E)$, the notation $\sigma(T)$ stands for the spectrum of the operator T and $\rho(T)$ denotes the resolvent set of T.

We let $\mathcal{M}_{n,k}(\mathbb{C})$ denote the space of complex valued $n \times k$ matrices and we use the notation $\mathcal{M}_n(\mathbb{C})$ when $n = k$. For an element M of $\mathcal{M}_{n,k}(\mathbb{C})$, the notation M^T stands for the transpose of M. For a square matrix M, $\mathrm{com}(M)$ corresponds to the cofactor matrix associated with M.

We use the notation $\lesssim$ to express an inequality up to a multiplicative constant. Eventually, we let C (resp. c) denote some large (resp. small) positive constants that may vary throughout the text (sometimes within the same line). Furthermore, we use the usual Landau notation $O(\cdot)$ to introduce a term uniformly bounded with respect to the argument. For more clarity, we will also occasionally use the notation $O_s(\cdot)$ to precise the fact that the term is a complex scalar and will try to reserve the notation $O(\cdot)$ for vectors/matrices.

We let $\mathrm{Res}(f,a)$ denote the residue of a meromorphic function f at the point a.

1 Introduction

1.1 Context

A fundamental issue on the subject of systems conservation laws is to understand how discontinuities that can arise in solutions are handled by conservative finite difference schemes. At the center of this question stands the notion of discrete shock profiles which correspond to solutions of the numerical scheme which are traveling waves linking two states. They are numerical approximations of shocks and the overarching goal is thus to prove that for any admissible and physically relevant shock of the system of conservation laws, there exists a discrete shock profile or a family of them that verifies satisfying stability properties. For a general introduction on the questions of existence and stability of discrete shock profiles, we highly encourage the interested reader to take a closer look at [Ser07].

In the present paper, we will consider conservative finite difference schemes which introduce numerical viscosity and will focus on the study of the discrete shock profiles associated with standing Lax shocks. We assume that there exists a continuous one-parameter family of discrete shock profiles associated with such a shock. Such an existence result has been proved for instance in [MR79, Mic84] under a weakness assumption on the shock, i.e. when the difference between the two states is sufficiently small. Let us introduce two notions of stability for the family of discrete shock profiles:

- Spectral stability amounts to asking for the operators obtained by linearizing the numerical scheme about the discrete shock profiles to have no unstable or marginally stable eigenvalues except for 1 which is always an eigenvalue because of the existence of the continuous one-parameter family of discrete shock profiles. Furthermore, we ask for 1 to be a simple eigenvalue of the linearized operator. This corresponds to Hypotheses 6 and 7 below.

- Nonlinear orbital stability signifies that for initial conditions of the numerical scheme which are suitably small perturbations of one of the discrete shock profiles, then the solutions of the numerical scheme that ensue stay close to the manifold of the discrete shock profiles. This is a stronger stability property.

There are some results surrounding nonlinear stability that have been proven. Most of them introduce a weakness assumption on the underlying shocks and/or focus on fairly specific schemes or situations. For instance, [LX93a, LX93b, Yin97] focus on proving a nonlinear orbital stability result on discrete shock profiles moving with rational speeds associated with weak Lax shocks for the Lax-Friedrichs scheme. Two of the main results which we can point out are the following ones:

- In [Mic02], Michelson proves nonlinear orbital stability of the family of discrete shock profiles associated with weak standing Lax shocks for schemes of any odd order under an assumption of stability of the viscous shock profiles associated with some scalar problem.

- In [Jen74], Jennings focuses on the particular case of *monotone* schemes for *scalar* conservation laws. The main results are the existence and uniqueness of continuous one-parameter family of discrete shock profiles with rational speeds and a proof of nonlinear orbital stability for them when they are associated with Lax shocks. In this paper, no weakness assumption on the associated shocks is introduced.

Compared with the nonlinear stability theory for viscous shock profiles [ZH98] or for semi-discrete shock profiles [BGHR03, BHSZ10], we hope to prove that spectrally stable discrete shock profiles verify nonlinear orbital stability. This new result would generalize the previously cited article by proving a result of nonlinear stability for systems of conservation laws, for a fairly large family of finite difference schemes, whilst avoiding to introduce a weakness assumption on the shocks.

Just like in [ZH98, BGHR03, BHSZ10], proving that spectral stability implies nonlinear orbital stability relies on an accurate description of the Green's function (defined below by (1.24)) associated with the operator obtained by linearizing the numerical scheme about the discrete shock profiles. The main result of the present paper provides such an accurate description (see Theorem 1). We have not yet proven that specral stability implies nonlinear stability, however the description of the semi-group associated with the linearized operator deduced by Theorem 1 already allows us to quite simply prove linear orbital stability (see Theorem 2). We hope to prove the nonlinear stability result in a future paper.

Let us now focus on the study of the Green's function. Theorem 1 can be seen as an improvement on the result of [God03] that highly influenced the analysis performed in the present paper. In [God03, LG01], Pauline Lafitte generalizes in the fully discrete setting several tools introduced in [ZH98] necessary to study the Green's function for the linearized operator. More precisely, she constructs the Evans function for this problem and introduces in her thesis [LG01] the notion of geometric dichotomies (an equivalent version of the exponential dichotomies in the discrete dynamical systems). Those tools will be redefined and used intensively in the present paper. Lafitte then attempts to obtain precise estimates on the Green's function of the linearized operator. However, the result of [God03] has two limitations:

- The proof is done specifically for the modified Lax-Friecrichs scheme. This is not a strong limitation as it is quite clear that the content of the paper [God03] can be generalized for a larger class of numerical schemes (at least for odd ordered schemes).

- The estimates on the Green's function proved in [God03, Theorem 1.1] are not sufficient to conclude on the nonlinear stability as they are only local with respect to the initial localization of the Dirac mass associated with the Green's function (the parameter l in [God03, Theorem 1.1] which corresponds to the parameter j_0 in Theorem 1). This is a consequence of the analysis on the so-called spatial Green's function (defined below by (1.25)) done in [God03] which is not precise enough.

In the present paper, we solve those issues by describing precisely the leading order of the Green's function and proving sharp and uniform estimates on the remainder. We also consider schemes of any odd order, in particular with only few restrictions on the size of the stencil of the scheme.

1.2 Definition of stationary discrete shock profiles (SDSP)

We consider a mono dimensional system of conservation laws

$$\begin{gathered}\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U},\end{gathered} \tag{1.1}$$

where $d \in \mathbb{N}\setminus\{0\}$ corresponds to the number of unknown $u =: (u_1, \dots, u_d)$ of (1.1), the space of states $\mathcal{U}$ is an open set of $\mathbb{R}^d$ and the flux $f : \mathcal{U} \to \mathbb{R}^d$ is a smooth function. We will suppose that the system of conservation laws is hyperbolic, meaning that for all $u \in \mathcal{U}$, $df(u)$ is diagonalisable with real eigenvalues.

We fix two states $u^-, u^+ \in \mathcal{U}$ such that

$$f(u^-) = f(u^+). \tag{1.2}$$

This is the well-known Rankine-Hugoniot condition which allows to state that the standing shock defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{else,} \end{cases} \tag{1.3}$$

is a weak solution of (1.1).

Since the system of conservation laws we consider is hyperbolic at the states $u^\pm$, we introduce the eigenvalues $\boldsymbol{\lambda}_1^\pm, \dots, \boldsymbol{\lambda}_d^\pm \in \mathbb{R}$ and a basis of nonzero eigenvectors $\boldsymbol{r}_1^\pm, \dots, \boldsymbol{r}_d^\pm \in \mathbb{R}^d$ of $df(u^\pm) \in \mathcal{M}_d(\mathbb{C})$ associated with those eigenvalues. We also define the invertible matrix

$$\boldsymbol{P}^\pm := \begin{pmatrix} \boldsymbol{r}_1^\pm & |\dots| & \boldsymbol{r}_d^\pm \end{pmatrix} \in \mathcal{M}_d(\mathbb{R}) \tag{1.4}$$

and the dual basis $\boldsymbol{l}_1^\pm, \dots, \boldsymbol{l}_d^\pm \in \mathbb{R}^d$ associated with the eigenvectors $\boldsymbol{r}_1^\pm, \dots, \boldsymbol{r}_d^\pm$ defined by

$$\begin{pmatrix} \boldsymbol{l}_1^\pm & |\dots| & \boldsymbol{l}_d^\pm \end{pmatrix}^T := \left(\boldsymbol{P}^\pm\right)^{-1}. \tag{1.5}$$

The vectors $\boldsymbol{l}_l^\pm$ are then eigenvectors of $df(u^\pm)^T$ associated with the eigenvalues $\boldsymbol{\lambda}_l^\pm$. We organize the eigenvalues so that

$$\boldsymbol{\lambda}_1^\pm \le \dots \le \boldsymbol{\lambda}_d^\pm.$$

In this paper, we focus our attention on Lax shocks.

Hypothesis 1 (Lax shock)**.** *We assume that* $0 \notin \sigma(df(u^\pm)) = \{\boldsymbol{\lambda}_1^\pm, \dots, \boldsymbol{\lambda}_d^\pm\}$ *(i.e. the shock is non-characteristic). Furthermore, we assume that there exists an index* $I \in \{1, \dots, d\}$ *such that*

$$\begin{aligned} \boldsymbol{\lambda}_I^+ < 0 < \boldsymbol{\lambda}_{I+1}^+, \\ \boldsymbol{\lambda}_{I-1}^- < 0 < \boldsymbol{\lambda}_I^-, \end{aligned}$$

where $\boldsymbol{\lambda}_0^\pm := -\infty$ *and* $\boldsymbol{\lambda}_{d+1}^\pm := +\infty$.

We fix a constant $\boldsymbol{\nu} > 0$ and introduce a space step $\Delta x > 0$ and a time step $\Delta t := \boldsymbol{\nu}\Delta x > 0$. The constant $\boldsymbol{\nu}$ then corresponds to the ratio between the space and time steps. We introduce the discrete evolution operator $\mathcal{N} : \mathcal{U}^{\mathbb{Z}} \to \mathcal{U}^{\mathbb{Z}}$ defined for $u = (u_j)_{j\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ as

$$\forall j \in \mathbb{Z}, \quad (\mathcal{N}u)_j := u_j - \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\boldsymbol{\nu}; u_{j-p}, \dots, u_{j+q-1}\right)\right), \tag{1.6}$$

where $p, q \in \mathbb{N}\setminus\{0\}$ and the numerical flux $F : (\nu; u_{-p}, \dots, u_{q-1}) \in]0, +\infty[\times\mathcal{U}^{p+q} \to \mathbb{R}^d$ is a $\mathcal{C}^1$ function. We are interested in solutions of the conservative one-step explicit finite difference scheme defined by

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}u^n \tag{1.7}$$

where $u^0 \in \mathcal{U}^{\mathbb{Z}}$.

We assume that the numerical scheme satisfies the following consistency condition with regards to the PDE (1.1)

$$\forall \nu \in]0, +\infty[, \forall u \in \mathcal{U}, \quad F(\nu; u, \dots, u) = f(u). \tag{1.8}$$

We also suppose that the following CFL condition is verified[2]

$$\forall u \in \mathcal{U}, \quad \boldsymbol{\nu} \min \sigma(df(u)) \ge -q \quad \text{and} \quad \boldsymbol{\nu} \max \sigma(df(u)) \le p \tag{1.9}$$

which is required to have linear ℓ^2-stability at constant states.

Traveling waves solutions of the numerical scheme (1.7) linking two states of some shocks are the so-called discrete shock profiles. Since we are considering stationary shocks (1.3) in the present paper, the discrete shock profiles associated will also be stationary and will thus correspond to fixed points of the operator $\mathcal{N}$.

[2] Up to considering that the space of state $\mathcal{U}$ is a close neighborhood of the SDSP defined underneath in Hypothesis 2, we should be able to satisfy such a condition.

Hypothesis 2 (Existence of a stationary discrete shock profile (SDSP))**.** *We suppose that there exists a sequence* $\overline{u}^s = (\overline{u}^s_j)_{j\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ *that satisfies*

$$\mathcal{N}(\overline{u}^s) = \overline{u}^s \quad \text{and} \quad \overline{u}^s_j \underset{j\to\pm\infty}{\to} u^\pm.$$

Let us point out that in [Ser07], it is proved that the existence of a SDSP implies that the Rankine-Hugoniot (1.2) is verified. However, the existence of a SDSP for all admissible and physically significant standing shock is not fully answered. Existence results tend to actually prove the existence of a continuous one-parameter family of discrete shock profiles. The main results tackling the issue of existence of SDSP would be [MR79, Mic84, Jen74]:

- In [Jen74], Jennings focuses on discrete shock profiles for monotone conservative schemes applied to scalar conservation laws. In this context, he proves the existence and uniqueness of a continuous one-parameter family of discrete shock profiles associated with shocks of any strength for rational speeds. He also proves nonlinear orbital stability for such DSPs.
- In [MR79], Majda and Ralston tackle the case of system of conservation laws and prove the existence of a continuous one-parameter family of DSPs with rational speeds. They introduce two limitations though: They consider schemes of order 1 (this corresponds to the case where $\mu = 1$ in Hypothesis 5 below) and they only consider weak shocks, i.e. shocks where the difference between the two states must be small enough. The result is generalized in [Mic84] for schemes of order 3 (i.e. $\mu = 2$ in Hypothesis 5 below).

The following assumption on the convergence of the SDSP $\overline{u}^s$ towards its limit state is important in the article as it it used to construct some of the main tools needed to carry the analysis of this paper (for instance to prove the geometric dichotomy in Section 3.3 or for the proof of Lemma 4.3).

Hypothesis 3 (Exponential convergence of the SDSP towards its limit states)**.** *There exist some constants* $C, c > 0$ *such that*

$$\forall j \in \mathbb{N}, \qquad \begin{array}{c} |\overline{u}^s_j - u^+| \leq Ce^{-cj}, \\ |\overline{u}^s_{-j} - u^-| \leq Ce^{-cj}. \end{array} \tag{1.10}$$

Hypothesis 3 can most likely be proved to be a consequence of the shock being non-characteristic (Hypothesis 1). We refer to [ZH98, Corollary 1.2] for a proof of this fact in the continuous setting and [BHSZ10, Lemma 1.1] in the semi-discrete case.

1.3 Linearized scheme about the end states $u^\pm$

Let us now introduce some hypotheses on the end states u^+ and u^-. To summarize briefly the main assumptions, we mainly ask for the numerical schemes we consider to introduce numerical viscosity and to have linear ℓ^r-stability at the states u^+ and u^-.

We linearize the discrete evolution operator $\mathcal{N}$ about the constant states u^- and u^+ and thus define the bounded operators $\mathscr{L}^\pm$ acting on $\ell^r(\mathbb{Z}, \mathbb{C}^d)$ with $r \in [1, +\infty]$ defined by

$$\forall h \in \ell^r(\mathbb{Z}, \mathbb{C}^d), \forall j \in \mathbb{Z}, \quad (\mathscr{L}^\pm h)_j := \sum_{k=-p}^{q} A^\pm_k h_{j+k}, \tag{1.11}$$

where for $k \in \{-p, \dots, q-1\}$, we have

$$B^\pm_k := \boldsymbol{\nu}\partial_{u_k} F\left(\boldsymbol{\nu}; u^\pm, \dots, u^\pm\right) \in \mathcal{M}_d(\mathbb{C}) \tag{1.12}$$

and for $k \in \{-p, \dots, q\}$

$$A^\pm_k := \begin{cases} -B^\pm_{q-1} & \text{if } k = q, \\ B^\pm_{-p} & \text{if } k = -p, \\ \delta_{k,0} Id + B^\pm_k - B^\pm_{k-1} & \text{else.} \end{cases} \tag{1.13}$$

We start by introducing the following assumption on the matrices $B^\pm_k$ and $A^\pm_k$.

Hypothesis 4. *The eigenvectors* $\boldsymbol{r}^\pm_1, \dots, \boldsymbol{r}^\pm_d$ *of* $df(u^\pm)$ *are also eigenvectors of* $B^\pm_k$ *and thus also of* $A^\pm_k$*. Equivalently, the matrices* $df(u^\pm)$ *and* $B^\pm_k$ *must commute for all* $k \in \{-p, \dots, q-1\}$*.*

Hypothesis 4 is fairly usual and is not that far fetched since the consistency condition (1.8) links the numerical flux F and the flux f and that, most of the time, the matrices $B_k^\pm$ defined by (1.12) are expressed using $df(u^\pm)$. If you consider for example the modified Lax-Friedrichs scheme, this hypothesis is satisfied. We can then introduce the notation for $k \in \{-p, \dots, q\}$

$$\begin{pmatrix} \lambda_{1,k}^\pm & & \\ & \ddots & \\ & & \lambda_{d,k}^\pm \end{pmatrix} := \boldsymbol{P}^{\pm -1} A_k^\pm \boldsymbol{P}^\pm. \tag{1.14}$$

For $l \in \{1, \dots, d\}$, we define the meromorphic function $\mathcal{F}_l^\pm$ on $\mathbb{C}\backslash\{0\}$ by

$$\forall \kappa \in \mathbb{C}\backslash\{0\}, \quad \mathcal{F}_l^\pm(\kappa) := \sum_{k=-p}^{q} \lambda_{l,k}^\pm \kappa^k \in \mathbb{C}. \tag{1.15}$$

The functions $\mathcal{F}_l^\pm$ allow us to characterize the spectrum of the operators $\mathscr{L}^\pm$. We refer for instance to [CF22, Coe22] for a study in the scalar case of similar convolution operators as $\mathscr{L}^\pm$. Fourier analysis and in particular the well-known Wiener theorem [New75] imply that

$$\sigma(\mathscr{L}^\pm) = \bigcup_{\kappa \in \mathbb{S}^1} \sigma\left(\sum_{k=-p}^{q} \kappa^k A_k^\pm \right) = \bigcup_{l=1}^{d} \mathcal{F}_l^\pm(\mathbb{S}^1). \tag{1.16}$$

The definition (1.13) of the matrices $A_k^\pm$ and the consistency condition (1.8) imply that

$$\sum_{k=-p}^{q} A_k^\pm = Id \quad \text{and} \quad \sum_{k=-p}^{q} k A_k^\pm = -\boldsymbol{\nu} df(u^\pm)$$

which translates into having

$$\forall l \in \{1, \dots, d\}, \quad \mathcal{F}_l^\pm(1) = 1 \quad \text{and} \quad \alpha_l^\pm := -\mathcal{F}_l^{\pm\prime}(1) = \boldsymbol{\nu} \boldsymbol{\lambda}_l^\pm \neq 0. \tag{1.17}$$

The following assumption is linked to the linear ℓ^r-stability of the numerical scheme (1.7) at the end state which corresponds to the ℓ^r-power boundedness of the operators $\mathscr{L}^\pm$.

Hypothesis 5. *For all $l \in \{1, \dots, d\}$, we have*

$$\forall \kappa \in \mathbb{S}^1\backslash\{1\}, \quad |\mathcal{F}_l^\pm(\kappa)| < 1. \quad \textit{(Dissipativity condition)}$$

Moreover, we suppose that there exists an integer $\mu \in \mathbb{N}\backslash\{0\}$ and for all $l \in \{1, \dots, d\}$, there exists a complex number $\beta_l^\pm$ with positive real part such that

$$\mathcal{F}_l^\pm(e^{i\xi}) \underset{\xi \to 0}{=} \exp(-i\alpha_l^\pm \xi - \beta_l^\pm \xi^{2\mu} + O(|\xi|^{2\mu+1})). \quad \textit{(Diffusivity condition)} \tag{1.18}$$

Asking for the ℓ^2-power boundedness of the operator $\mathscr{L}^\pm$ is equivalent to asking that $\sigma(\mathscr{L}^\pm) \subset \overline{\mathbb{D}}$ (Von Neumann condition). The stronger Hypothesis 5 is inspired by the fundamental contribution [Tho65] due to Thomée and has much further consequences, as the asymptotic expansion (1.18) assures the ℓ^r-power boundedness of the operator $\mathscr{L}^\pm$ for every r in $[1, +\infty]$ (see [Tho65, Theorem 1] which focuses in the scalar case on the ℓ^∞-power boundedness but also studies the ℓ^r-power boundedness as a consequence). The diffusivity condition (1.18) can be translated into asking for the numerical scheme $\mathcal{N}$ to introduce numerical viscosity at the end state $u^\pm$.

We conclude this section by defining the open set $\mathcal{O}$ which corresponds to the unbounded connected component of $\mathbb{C}\backslash(\sigma(\mathscr{L}^+) \cup \sigma(\mathscr{L}^-))$ represented on Figure 1. Hypothesis 5 implies that $\overline{\mathbb{U}}\backslash\{1\} \subset \mathcal{O}$.

1.4 Linearized scheme about the SDSP $\overline{u}^s$

We now linearize the discrete evolution operator $\mathcal{N}$ about the discrete shock profile $\overline{u}^s$ and thus define the bounded operator $\mathscr{L}$ acting on $\ell^r(\mathbb{Z}, \mathbb{C}^d)$ with $r \in [1, +\infty]$ defined by

$$\forall h \in \ell^r(\mathbb{Z}, \mathbb{C}^d), \forall j \in \mathbb{Z}, \quad (\mathscr{L}h)_j := \sum_{k=-p}^{q} A_{j,k} h_{j+k}, \tag{1.19}$$

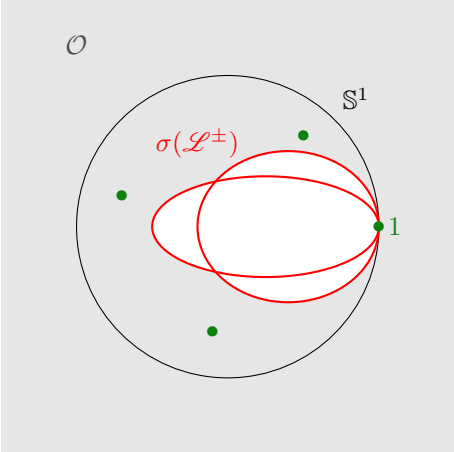

Figure 1: In red, we have the spectrum of the operators $\mathscr{L}^{\pm}$ which corresponds to the union of the curves $\mathcal{F}_l^{\pm}(\mathbb{S}^1)$. In gray, we represent the set $\mathcal{O}$ which corresponds to the unbounded component of $\mathbb{C}\backslash(\sigma(\mathscr{L}^+)\cup\sigma(\mathscr{L}^-))$. The elements of the set $\mathcal{O}$ are either eigenvalues of the operator $\mathscr{L}$ (represented in green) or belong to the resolvent set $\rho(\mathscr{L})$. We know that 1 is an eigenvalue of $\mathscr{L}$ and Hypothesis 6 implies that the eigenvalues of $\mathscr{L}$ in $\mathcal{O}$ are located within the open unit disk.

where for $j\in\mathbb{Z}$ and $k\in\{-p,\dots,q-1\}$, we have

$$B_{j,k}:=\boldsymbol{\nu}\partial_{u_k}F\left(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1}\right)\in\mathcal{M}_d(\mathbb{C})$$

and for $j\in\mathbb{Z}$ and $k\in\{-p,\dots,q\}$

$$A_{j,k}:=\begin{cases}-B_{j+1,q-1} & \text{if } k=q,\\ B_{j,-p} & \text{if } k=-p,\\ \delta_{k,0}Id+B_{j,k}-B_{j+1,k-1} & \text{otherwise.}\end{cases}\tag{1.20}$$

We observe that since the SDSP $(\overline{u}^s_j)_{j\in\mathbb{Z}}$ converges exponentially fast towards its limit states $u^{\pm}$, we have that the matrices $A_{j,k}$ (resp. $B_{j,k}$) converge exponentially fast towards the matrices $A^{\pm}_k$ (resp. $B^{\pm}_k$) defined by (1.13) (resp. (1.12)) as j tends towards $\pm\infty$.

We will now focus on the spectral properties of the operator $\mathscr{L}$ when it acts on $\ell^2(\mathbb{Z},\mathbb{C}^d)$. The following proposition which localizes the essential spectrum of the operator $\mathscr{L}$ is central.

Proposition 1. *We have that*

$$\sigma_{ess}(\mathscr{L})\cap\mathcal{O}=\emptyset.$$

Proposition 1 allows us to conclude that for $z\in\mathcal{O}$, $zId_{\ell^2}-\mathscr{L}$ is a Fredholm operator of index 0 and thus that z either belongs to the resolvent set of $\mathscr{L}$ or is an eigenvalue of $\mathscr{L}$. Proposition 1 is proved for instance in [Ser07, Theorem 4.1] using the so-called geometric dichotomy developped in the thesis of Pauline Godillon [LG01, Section III.1.5]. We will have to reintroduce the geometric dichotomy in Section 3 and we will thus provide the proof of Proposition 1 (see Lemma 3.7).

We will now introduce the spectral stability assumption that we impose on our SDSP $\overline{u}^s$. It can be separated in two parts.

Hypothesis 6. *The operator $\mathscr{L}$ has no eigenvalue of modulus equal or larger than 1 other than 1.*

Combining Hypothesis 6 with Proposition 1, we can then conclude that the set $\overline{\mathbb{U}}\backslash\{1\}$ is included in the resolvent set of $\mathscr{L}$.

The second part of the spectral stability assumption and the last hypothesis we will introduce on the spectrum of the operator $\mathscr{L}$ has to do with the so-called Evans function Ev defined later on in the article by (4.18). This is a complex holomorphic function defined in a neighborhood of 1 that vanishes at the eigenvalues of $\mathscr{L}$. The Evans function plays more or less the role of a characteristic polynomial for the operator $\mathscr{L}$.

We will show that under the previous hypotheses, 1 is an eigenvalue of the operator $\mathscr{L}$ and thus that the Evans function Ev vanishes at 1. This is the consequence of the existence of a continuous one-parameter family of

SDSPs associated with the one we are studying. In continuous and semi-discrete settings as in [ZH98, BHSZ10], there is an underlying regular profile that describes the continuous one-parameter family of traveling waves studied in those papers sd translations of said profile. The derivative of the regular profile belongs to the kernel of the linearized operator in these settings. In our present fully discrete setting, existence of such a regular underlying profile that describes the different SDSPs on one-parameter family has translations of each other is not clear.

Here, we will make a stronger hypothesis on the behavior of the Evans function at 1.

Hypothesis 7. *We have that* 1 *is a simple zero of the Evans function, i.e.*

$$\frac{\partial \mathrm{Ev}}{\partial z}(1) \neq 0.$$

We will show that Hypothesis 7 implies that 1 is actually a simple eigenvalue of the operator $\mathscr{L}$. More precisely, we will prove that there exists a sequence $V \in \ell^2(\mathbb{Z}, \mathbb{C}^d)\setminus\{0\}$ such that

$$\ker(Id_{\ell^2} - \mathscr{L}) = \mathrm{Span} V \tag{1.21}$$

and such that the sequence V converges exponentially fast towards 0 at infinity, i.e. there exist two positive constants C, c such that

$$\forall j \in \mathbb{Z}, \quad |V_j| \leq Ce^{-c|j|}. \tag{1.22}$$

Coming back to the discussion above, if there exists a regular profile that allows us to describe the one-parameter family of SDSPs as translations from one another, the sequence V would then correspond to the "derivative" of said profile.

We finalize this section by introducing two last hypotheses that we one could qualify as more technical.

Hypothesis 8. *The matrices* $A_{j,-p} = B_{j,-p}$, $A_{j,q} = -B_{j+1,q-1}$, $A^{\pm}_{-p} = B^{\pm}_{-p}$ *and* $A^{\pm}_{q} = -B^{\pm}_{q-1}$ *are invertible for all* $j \in \mathbb{Z}$.

This hypothesis is a consequence of the CFL condition (1.9), at least for the modified Lax-Friedrichs scheme. Hypothesis 8 serves us in the article to express the eigenvalue problems associated the operator $\mathscr{L}$, $\mathscr{L}^+$ and $\mathscr{L}^-$ as dynamical systems (see Section 3.1). Finally, we impose for the following assumption to be verified.

Hypothesis 9. *For all* $l \in \{1, \dots, d\}$, *the equation*

$$\mathcal{F}^{\pm}_l(\kappa) = 1 \tag{1.23}$$

has $p+q$ *distinct solutions* $\kappa \in \mathbb{C}\setminus\{0\}$.

Hypothesis 9 will be used to prove that the matrix $M^{\pm}_l(1)$ defined by (3.8) is diagonalizable with simple eigenvalues. This will allow us to study the eigenvalues and eigenvectors of the matrix $M^{\pm}(1)$ defined by (3.1) in Section 4. Let us observe that the expression (1.15) of $\mathcal{F}^{\pm}_l$ implies that searching for solutions $\kappa \in \mathbb{C}^{p+q}\setminus\{0\}$ of (1.23) is equivalent to searching for zeroes of

$$\kappa \in \mathbb{C}\setminus\{0\} \mapsto \kappa^p - \sum_{k=-p}^{q} \lambda^{\pm}_{l,k}\kappa^{k+p}.$$

The function above has $p+q$ zeroes counted with multiplicity and Hypothesis 8 is then just equivalent to asking for the zeroes of the function above to be simple.

1.5 Temporal and spatial Green's functions

For $j_0 \in \mathbb{Z}$, we define the temporal Green's function recursively as

$$\begin{aligned} \mathscr{G}(0, j_0, \cdot) &:= \delta_{j_0} \\ \forall n \in \mathbb{N}, \quad \mathscr{G}(n+1, j_0, \cdot) &:= \mathscr{L}\mathscr{G}(n, j_0, \cdot), \end{aligned} \tag{1.24}$$

where $\delta_{j_0} := (\delta_{j_0,j} Id)_{j\in\mathbb{Z}}$. For $z \in \rho(\mathscr{L})$, we also define the spatial Green's function $G(z, j_0, \cdot)$ as the only element of $\ell^2(\mathbb{Z}, \mathcal{M}_d(\mathbb{C}))$ such that

$$(zId_{\ell^2} - \mathscr{L})G(z, j_0, \cdot) = \delta_{j_0}. \tag{1.25}$$

Remark 1. To be more precise, the previous definitions can seem unclear since $\mathscr{L}$ is defined on $\ell^2(\mathbb{Z},\mathbb{C}^d)$. A way to understand it is that the temporal Green's function is defined as

$$\begin{gathered}\mathscr{G}(0,j_0,\cdot):=\delta_{j_0}\\ \forall n\in\mathbb{N},\forall\vec{e}\in\mathbb{C}^d,\quad \mathscr{G}(n+1,j_0,\cdot)\vec{e}:=\mathscr{L}\mathscr{G}(n,j_0,\cdot)\vec{e}.\end{gathered}$$

Also, $G(z,j_0,\cdot)$ is the only element of $\ell^2(\mathbb{Z},\mathcal{M}_d(\mathbb{C}))$ such that for all $\vec{e}\in\mathbb{C}^d$, $G(z,j_0,\cdot)\vec{e}$ is the only element of $\ell^2(\mathbb{Z},\mathbb{C}^d)$ such that

$$(zId_{\ell^2}-\mathscr{L})G(z,j_0,\cdot)\vec{e}=\delta_{j_0}\vec{e}.$$

The main consequence of the introduction of the temporal Green's function is that for all $h\in\ell^r(\mathbb{Z},\mathbb{C}^d)$ with $r\in[1,+\infty]$, we have

$$\forall n\in\mathbb{N},\forall j\in\mathbb{Z},\quad (\mathscr{L}^n h)_j=\sum_{j_0\in\mathbb{Z}}\mathscr{G}(n,j_0,j)h_{j_0}.$$

Thus, a precise description of the temporal Green's function is sufficient to understand the action of the semi-group $(\mathscr{L}^n)_{n\in\mathbb{N}}$ associated with the operator $\mathscr{L}$.

The following lemma proved via a simple recurrence is a direct consequence of the definition (1.24) of the temporal Green's function and the finite speed propagation of the linearized scheme.

Lemma 1.1. *For all $n\in\mathbb{N}$, $j_0,j\in\mathbb{Z}$, we have that*

$$j-j_0\notin\{-nq,\dots,np\}\Rightarrow\mathscr{G}(n,j_0,j)=0.$$

Our goal is now to describe the behavior of the temporal Green's function when $j-j_0\in\{-nq,\dots,np\}$. To do so, we define the functions $H_{2\mu},E_{2\mu}:\mathbb{R}\to\mathbb{C}$ such that for $\beta\in\mathbb{C}$ with positive real part, we have

$$\begin{aligned}\forall x\in\mathbb{R},\quad H_{2\mu}(\beta;x)&:=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ixu}e^{-\beta u^{2\mu}}du,\\ \forall x\in\mathbb{R},\quad E_{2\mu}(\beta;x)&:=\int_x^{+\infty}H_{2\mu}(\beta;y)dy,\end{aligned}\tag{1.26}$$

where we recall that the integer μ is defined in Hypothesis 5. We call the functions $H_{2\mu}$ generalized Gaussians and the functions $E_{2\mu}$ generalized Gaussian error functions since for $\mu=1$, we have

$$\forall x\in\mathbb{R},\quad H_2(\beta;x)=\frac{1}{\sqrt{4\pi\beta}}e^{-\frac{x^2}{4\beta}}.$$

Noticing that the function $H_{2\mu}$ is an even function and that it is the inverse Fourier transform of $u\mapsto e^{-\beta u^{2\mu}}$, we observe that

$$\lim_{x\to-\infty}E_{2\mu}(\beta;x)=\int_{-\infty}^{+\infty}H_{2\mu}(\beta;y)dy=1\tag{1.27a}$$

$$\forall x\in\mathbb{R},\quad E_{2\mu}(\beta,-x)=1-E_{2\mu}(\beta,x).\tag{1.27b}$$

The following lemma introduces some useful inequalities on the functions $H_{2\mu}$ and $E_{2\mu}$ defined by (1.26).

Lemma 1.2. *Let us consider a compact subset A of $\{z\in\mathbb{C},\Re(z)>0\}$ and integers $\mu,m\in\mathbb{N}\setminus\{0\}$. There exist two positive constants C,c such that for all $\beta\in A$*

$$\forall x\in\mathbb{R},\quad |\partial_x^m H_{2\mu}(\beta;x)|\leq C\exp(-c|x|^{\frac{2\mu}{2\mu-1}}),\tag{1.28a}$$

$$\forall x\in]0,+\infty[,\quad |E_{2\mu}(\beta;x)|\leq C\exp(-c|x|^{\frac{2\mu}{2\mu-1}}),\tag{1.28b}$$

$$\forall x\in]-\infty,0[,\quad |1-E_{2\mu}(\beta;x)|\leq C\exp(-c|x|^{\frac{2\mu}{2\mu-1}}).\tag{1.28c}$$

The interested reader can find a proof of (1.28a) when the subset A is a one point set in [Coe22, Lemma 9] or in [Rob91, Proposition 5.3] for a more general point of view. By observing that the constants C,c constructed in those proofs depend continuously on β, we can then conclude the proof (1.28a) for general sets A. Inequalities (1.28b) and (1.28c) for the function $E_{2\mu}$ are directly deduced by integrating the function $H_{2\mu}$ and using (1.27a) and (1.28a).

We then introduce for $n\in\mathbb{N}\setminus\{0\}$ and $j,j_0\in\mathbb{Z}$ the functions defined by

- For $j_0 \geq 0$ and $l \in \{1, \dots, d\}$

$$S_l^+(n, j_0, j) := \mathbb{1}_{j \geq 0} \mathbb{1}_{\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+, \frac{n\alpha_l^+ + j_0 - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^+ \boldsymbol{l}_l^{+T}, \tag{1.29a}$$

- For $j_0 \geq 0$, $l' \in \{1, \dots, I\}$ and $l \in \{I+1, \dots d\}$ (i.e. such that $\alpha_{l'}^+ < 0$ and $\alpha_l^+ > 0$)

$$R_{l',l}^+(n, j_0, j) := \mathbb{1}_{j \geq 0} \mathbb{1}_{\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^+}\beta_l^+ - \frac{j_0}{n\alpha_{l'}^+}\beta_{l'}^+ \left(\frac{\alpha_l^+}{\alpha_{l'}^+}\right)^{2\mu}, \frac{n\alpha_l^+ + j_0 \frac{\alpha_l^+}{\alpha_{l'}^+} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^+ \boldsymbol{l}_{l'}^{+T}, \tag{1.29b}$$

- For $j_0 \geq 0$, $l' \in \{1, \dots, I\}$ and $l \in \{1, \dots I-1\}$ (i.e. such that $\alpha_{l'}^+ < 0$ and $\alpha_l^- < 0$)

$$T_{l',l}^+(n, j_0, j) := \mathbb{1}_{j < 0} \mathbb{1}_{\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^-}\beta_l^- - \frac{j_0}{n\alpha_{l'}^+}\beta_{l'}^+ \left(\frac{\alpha_l^-}{\alpha_{l'}^+}\right)^{2\mu}, \frac{n\alpha_l^- + j_0 \frac{\alpha_l^-}{\alpha_{l'}^+} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^- \boldsymbol{l}_{l'}^{+T}, \tag{1.29c}$$

- For $j_0 \geq 0$ and $l' \in \{1, \dots, I\}$ (i.e. such that $\alpha_{l'}^+ < 0$)

$$E_{l'}^+(n, j_0) := E_{2\mu}\left(\beta_{l'}^+, \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{l}_{l'}^{+T}, \tag{1.29d}$$

- For $j_0 < 0$ and $l \in \{1, \dots, d\}$

$$S_l^-(n, j_0, j) := \mathbb{1}_{j \leq 0} \mathbb{1}_{\frac{j-j_0}{\alpha_l^-} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^-, \frac{n\alpha_l^- + j_0 - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^- \boldsymbol{l}_l^{-T}, \tag{1.29e}$$

- For $j_0 < 0$, $l' \in \{I, \dots, d\}$ and $l \in \{1, \dots I-1\}$ (i.e. such that $\alpha_{l'}^- > 0$ and $\alpha_l^- < 0$)

$$R_{l',l}^-(n, j_0, j) := \mathbb{1}_{j \leq 0} \mathbb{1}_{\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^-} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^-}\beta_l^- - \frac{j_0}{n\alpha_{l'}^-}\beta_{l'}^- \left(\frac{\alpha_l^-}{\alpha_{l'}^-}\right)^{2\mu}, \frac{n\alpha_l^- + j_0 \frac{\alpha_l^-}{\alpha_{l'}^-} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^- \boldsymbol{l}_{l'}^{-T}, \tag{1.29f}$$

- For $j_0 < 0$, $l' \in \{I, \dots, d\}$ and $l \in \{I+1, \dots d\}$ (i.e. such that $\alpha_{l'}^- > 0$ and $\alpha_l^+ > 0$)

$$T_{l',l}^-(n, j_0, j) := \mathbb{1}_{j > 0} \mathbb{1}_{\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^-} \in \left[\frac{n}{2}, 2n\right]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^+}\beta_l^+ - \frac{j_0}{n\alpha_{l'}^-}\beta_{l'}^+ \left(\frac{\alpha_l^+}{\alpha_{l'}^-}\right)^{2\mu}, \frac{n\alpha_l^+ + j_0 \frac{\alpha_l^+}{\alpha_{l'}^-} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^+ \boldsymbol{l}_{l'}^{-T}, \tag{1.29g}$$

- For $j_0 < 0$, $l' \in \{I, \dots, d\}$ (i.e. such that $\alpha_{l'}^- > 0$)

$$E_{l'}^-(n, j_0) := E_{2\mu}\left(\beta_{l'}^-, \frac{-n\alpha_{l'}^- - j_0}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{l}_{l'}^{-T}. \tag{1.29h}$$

The functions introduced above describe different behaviors that will be observed for the temporal Green's function. To be more precise, the main theorem of the present paper is the following description of the temporal Green's function.

Theorem 1. *Let us assume that Hypotheses 1-9 are verified. There exist a sequence $V \in \ell^2(\mathbb{Z}, \mathbb{C}^d) \setminus \{0\}$ such that (1.21) and (1.22) and families of complex scalars $(C_{l'}^{E^\pm})_{l'}$, $(C_{l',l}^{R}{}^{\pm})_{l',l}$ and $(C_{l',l}^{T}{}^{\pm})_{l',l}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$ such that*

$$\mathscr{G}(n, j_0, j)\vec{e} = \sum_{l=1}^{d} S_l^+(n, j_0, j)\vec{e} + \sum_{l'=1}^{I} \left[\sum_{l=I+1}^{d} C_{l',l}^{R}{}^{+} R_{l',l}^+(n, j_0, j)\vec{e} + \sum_{l=1}^{I-1} C_{l',l}^{T}{}^{+} T_{l',l}^+(n, j_0, j)\vec{e} + C_{l'}^{E^+} E_{l'}^+(n, j_0)\vec{e} V_j \right] + \mathcal{R}(n, j_0, j)\vec{e} \tag{1.30}$$

where $\mathcal{R}(n, j_0, j)$ is a faster decaying residual in comparison to the other terms. There is a similar result when $j_0 \leq 0$ using the families of complex scalars $(C_{l'}^{E^-})_{l'}$, $(C_{l',l}^{R^-})_{l',l}$ and $(C_{l',l}^{T^-})_{l',l}$ and the functions $E_{l'}^-$, S_l^-, $R_{l',l}^-$ and $T_{l',l}^-$.

Let us describe more clearly the result of Theorem 1 conveys for $j_0 \geq 0$. The same description can be done when $j_0 < 0$. The first term on the right hand-side using the function S_l^+ of (1.30) corresponds to d generalized Gaussian waves arising from the Dirac mass at j_0 which travel along the characteristics of the right state u^+. The generalized Gaussian behavior of the different waves originates from the smearing effect caused by the diffusivity condition in Hypothesis 5 which corresponds to the introduction of artificial viscosity at the states $u^\pm$. Recalling that we are considering a Lax shock under Hypotheses 1, we observe the following separation of behavior:

- The first I generalized Gaussian waves follow the characteristics entering the shock since $\alpha_l^+ < 0$ for $l \in \{1, \dots, I\}$ and will reach the shock.
- The last $d - I$ generalized Gaussian waves follow the outgoing characteristics with respect to the shock since $\alpha_l^+ > 0$ for $l \in \{I+1, \dots, d\}$ and travel towards $+\infty$.

When the generalized Gaussian waves following entering characteristics reach the shock, we observe that they are dispersed in three different behaviors:

- There are reflected generalized Gaussian waves along the outgoing characteristics of the state u^+. It corresponds to the second term using the function $R_{l',l}^+$ in (1.30).
- There are transmitted generalized Gaussian waves along the outgoing characteristics of the state u^-. It corresponds to the third term using the function $T_{l',l}^+$ in (1.30).
- Because of the properties of the function $E_{2\mu}$ defined by (1.26), we have that the vectors $E_{l'}^+(n, j_0)$ are closer to 0 for small times n and converge towards $\boldsymbol{l}_{l'}^{+T}$ as n tends towards $+\infty$. Thus, the last term in the decomposition (1.30) could be described as the apparition of a stationary part corresponding the profile V that describes the vector subspace $\ker(Id_{\ell^2} - \mathscr{L})$. Each wave activates this profile as they reach the shock.

One of the main consequence that can be deduced from Theorem 1 corresponds to the so-called linear orbital stability of the stationary discrete shock profile $\overline{u}^s$.

Theorem 2. *Let us assume that Hypotheses 1-9 are verified. For $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, there exists positive constant C such that*

$$\forall h \in \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d), \forall n \in \mathbb{N}, \quad \min_{V \in \ker(Id_{\ell^2} - \mathscr{L})} \|\mathscr{L}^n h - V\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}} \|h\|_{\ell^{r_1}} .$$

We recall that when we introduced the Hypothesis 7 on the Evans function, we discussed on the fact that, when that Hypotheses 1-9 are verified, if there exists a regular profile that allows us to describe the one-parameter family of SDSPs as translations from one another, then the elements of $\ker(Id_{\ell^2} - \mathscr{L})$ correspond to the derivative of said profile and scalar multiples of it.

1.6 Plan of the paper

Firstly, Section 2 will be dedicated to the proof of Theorem 2 using Theorem 1. The main part of the article however (from Sections 3 to 5) will concern the proof of Theorem 1:

- In Section 3, we will prove Proposition 1 which describes the spectrum of the operator $\mathscr{L}$ in the set $\mathcal{O}$ and will allow to define the spatial Green's function defined by (1.25) on $\overline{\mathbb{U}} \setminus \{1\}$. We will then prove Proposition 2 which implies exponential bounds on the spatial Green's function in the neighborhood of any point of $\overline{\mathbb{U}} \setminus \{1\}$.
- In Section 4, we prove that the spatial Green's function can be meromorphically extended in a neighborhood of 1 through the essential spectrum of the operator $\mathscr{L}$. We will show that it has a pole of order 1 at $z = 1$ and find precise expressions (4.45)-(4.48) on it that will be essential in the Section 5.

- In Section 5, we express the temporal Green's function defined by (1.24) with the spatial Green's function using the inverse Laplace Transform. Using the different results proved on the spatial Green's function in Sections 3 and 4, we will conclude the proof of Theorem 1.

Section 6 is the Appendix and contains the proof of some technical lemmas used throughout the paper. Some of those proofs are done in other papers, however, the authors feels like they needed to be reproved either to correct mistakes or because the way they are presented here is fairly different from the statement in other papers.

1.7 Possible further developments

The main development that we hope for is of course that the description of the temporal Green's function we have proved in Theorem 1 would be sufficient to conclude on the nonlinear orbital stability for stationary spectrally stable discrete shock profiles. Such a description of the Green's function were sufficient to prove conclude nonlinear orbital stability results in some instances in continuous settings (in [MZ02] Theorem 1.11 implies Theorem 1.14) and semi-discrete settings (in [BGHR03] Theorem 4.11 implies Theorem 5.1 and in [BHSZ10] Theorem 8 implies Theorem 1).

Possible further developments that could be expected surrounding the question of the stability discrete shock profiles are presented below. Some of the developments presented are discussions on relaxations of restrictions we imposed in the paper.

- The present papers only studies the case of Lax shocks (Hypothesis 1). However, in [God03], the description obtained on the Green's function for the linearized operator also holds for discrete shock profiles associated with stationary under-compressive and over-compressive shocks. Using ideas and calculations performed in [God03], one could hope to adapt the results of the present paper to such cases. For instance, in the under-compressive case, 1 would not be an eigenvalue of the linearized operator and thus the expected description of the Green's function would only be composed of generalized Gaussian waves. On a more technical level, the main modification would be linked to the behavior of the spatial Green's function which could be holomorphically extended on $z = 1$.
- One could hope to extend the analysis performed in the present paper for standing discrete shock profiles to moving ones, at least when the speed is rational. This would rely on studying iterations of the operator $\mathcal{N}$ which would be fairly more difficult.
- An other option of extension that was tackled in the paper [God03] of Godillon would be to study the case of boundary layer profiles for numerical schemes applied to systems of conservation laws on the half-line.
- Looking at the case of dispersive schemes like the Lax-Wendroff scheme would be another interesting direction.

Those are only some ideas of questions surrounding the stability questions on discrete shock profiles. This is a large subject with far more results to discover (existence of spectrally stable DSPs, equivalent results in the multidimensional setting, etc...).

2 Proof of linear orbital stability (Theorem 2)

The goal of this section is to prove Theorem 2 using the description of the temporal Green's function obtained in Theorem 1. We recall that for $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, we have that for all $n \in \mathbb{N}$ and $h \in \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d)$

$$\mathscr{L}^n h = \left(\sum_{j_0 \in \mathbb{Z}} \mathscr{G}(n, j_0, j) h_{j_0} \right)_{j \in \mathbb{Z}} \in \ell^{r_2}(\mathbb{Z}, \mathbb{C}^d). \tag{2.1}$$

We claim and will prove later on that for all couple $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, the operators

$$\begin{array}{ccc} \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d) & \to & \ell^{r_2}(\mathbb{Z}, \mathbb{C}^d) \\ h & \mapsto & \left(\sum_{j_0 \in \mathbb{N}} S_l^+(n, j_0, j) h_{j_0} \right)_{j \in \mathbb{Z}} \end{array} \tag{2.2}$$

for $l \in \{1,\dots,d\}$ and $n \in \mathbb{N}\setminus\{0\}$ are well-defined and that there exists a positive constants C such that

$$\forall l \in \{1,\dots,d\}, \forall n \in \mathbb{N}\setminus\{0\}, \forall h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d), \quad \left\| \left(\sum_{j_0\in\mathbb{N}} S_l^+(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}} \right\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}} \|h\|_{l^{r_1}}. \quad (2.3)$$

We claim that, using an identical proof, for all couple $r_1, r_2 \in [1,+\infty]$ such that $r_1 \leq r_2$, the operators from $\ell^{r_1}(\mathbb{Z},\mathbb{C}^d)$ to $\ell^{r_2}(\mathbb{Z},\mathbb{C}^d)$ defined by

$$h \mapsto \left(\sum_{j_0\in\mathbb{N}} R_{l',l}^+(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}} \qquad h \mapsto \left(\sum_{j_0<0} R_{l',l}^-(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}}$$

$$h \mapsto \left(\sum_{j_0\in\mathbb{N}} T_{l',l}^+(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}} \qquad h \mapsto \left(\sum_{j_0<0} T_{l',l}^-(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}}$$

$$h \mapsto \left(\sum_{j_0<0} S_l^-(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}} \qquad h \in\mapsto \left(\sum_{j_0\in\mathbb{Z}} \mathcal{R}(n,j_0,j)h_{j_0} \right)_{j\in\mathbb{Z}}$$

for $n \in \mathbb{N}\setminus\{0\}$ are well-defined and satisfy similar bounds as (2.3). Furthermore, using the definitions (1.29d) and (1.29h) of the functions $E_{l'}^+$ and $E_{l'}^-$ and the estimates (1.28b) and (1.28c) on the function $E_{2\mu}$, we have that for $h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d)$ and $n \in \mathbb{N}\setminus\{0\}$, the series

$$\sum_{j_0\in\mathbb{N}} \left(\sum_{l'=1}^{I} C_{l'}^{E^+} E_{l'}^+(n,j_0)h_{j_0} \right) + \sum_{j_0<0} \left(\sum_{l'=I}^{d} C_{l'}^{E^-} E_{l'}^-(n,j_0)h_{j_0} \right)$$

converges. Using the observations above, the equality (2.1) and the decomposition of the Green's function given in Theorem 1, we can then conclude that there exists a positive constant $C>0$ such that

$$\forall h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d), \forall n \in \mathbb{N}\setminus\{0\},$$
$$\left\| \mathscr{L}^n h - \left(\sum_{j_0\in\mathbb{N}} \left(\sum_{l'=1}^{I} C_{l'}^{E^+} E_{l'}^+(n,j_0)h_{j_0} \right) + \sum_{j_0<0} \left(\sum_{l'=I}^{d} C_{l'}^{E^-} E_{l'}^-(n,j_0)h_{j_0} \right) \right) V \right\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}} \|h\|_{\ell^{r_1}}$$

where we recall that the sequence $V \in \ell^1(\mathbb{Z},\mathbb{C}^d)$ defined in Theorem 1 verifies that

$$\ker(Id_{\ell^2} - \mathscr{L}) = \mathrm{Span} V.$$

This allows us to conclude the proof of Theorem 2. Therefore, there just remains to prove that the operators (2.2) are well-defined and verify (2.3). For $l \in \{1,\dots,d(p+q)\}$, using the bounds (1.28a) on the function $H_{2\mu}$ and the definition (1.29a) of the functions S_l^+, we easily prove that for each of the couple (r_1,r_2) in $\{(1,1),(1,+\infty),(+\infty,+\infty)\}$, the operators (2.2) are well-defined for $n \in \mathbb{N}\setminus\{0\}$ and verify inequality (2.3). Then, reasoning by interpolation using Riesz-Thorin Theorem, we can conclude that the same statement actually holds for all couple $(r_1,r_2) \in [1,+\infty]^2$ such that $r_1 \leq r_2$.

3 Local exponential bounds on the spatial Green's function for z far from 1

In this section, the goal is twofold:

- In order to determine where the spatial Green's function is defined, we want to study the spectrum of the operator $\mathscr{L}$ in the set $\mathcal{O}$ (i.e. outside of the curves representing the spectrum of the limit operators $\mathscr{L}^{\pm}$). More precisely, we will prove Lemma 3.7 which characterizes the eigenvalues of $\mathscr{L}$ in the set $\mathcal{O}$ and states that there is no essential spectrum of the operator $\mathscr{L}$ which lies in the set $\mathcal{O}$. This result was already proved in [Ser07, Theorem 4.1]. As a direct consequence, we will have proved that the element of the set $\mathcal{O}$ are either in the resolvent set of the operator $\mathscr{L}$ or are eigenvalues of $\mathscr{L}$. Using Hypothesis 6, we can thus deduce that the set $\overline{\mathbb{U}}\setminus\{1\}$ is included in the resolvent set of $\mathscr{L}$ and that the spatial Green's function can be defined in a neighborhood of any point of $\overline{\mathbb{U}}\setminus\{1\}$.

- We will prove Proposition 2 which introduce local uniform exponential bounds on the spatial Green's function $G(z, j_0, \cdot)$ when z belongs to $\overline{\mathbb{U}}\setminus\{1\}$ and $j_0 \in \mathbb{Z}$. We will see later on in Section 4 that the study of the spatial Green's function for z near 1 will require some special care and that it is a more refined analysis of the case where z is in $\overline{\mathbb{U}}\setminus\{1\}$. It might be important to keep in mind that a lot of the tools we will introduce will also be useful in the case of z near 1.

The main ideas of this section will be to characterize the solutions of the eigenvalue problem associated with the operator $\mathscr{L}$ using solutions of a discrete dynamical system of finite dimension.

We will then define a central tool for our analysis : the *geometric dichotomy* introduced by Pauline Lafitte-Godillon in her thesis [LG01] and based on the exponential dichotomy coined by Coppel in [Cop78]. We will take some time to rewrite the proofs of some lemmas even though most of the ideas can already be found in the previously cited texts.

3.1 Expression of the eigenvalue problem as a dynamical system

As we explained, one of our objectives is to study the spectrum of the operator $\mathscr{L}$. In this section, we express the eigenvalue problem $(zId_{\ell^2} - \mathscr{L})u = 0$ as a dynamical system. We will define a few important mathematical objects that will appear often throughout the article.

For $z \in \mathbb{C}$, we define the matrices $\mathbb{A}_{j,k}(z) = z\delta_{k,0}Id - A_{j,k}$ and $\mathbb{A}^{\pm}_k(z) = z\delta_{k,0}Id - A^{\pm}_k$ for all $j \in \mathbb{Z}$ and $k \in \{-p, \dots, q\}$. We then introduce the matrices

$$\forall j \in \mathbb{Z}, \forall z \in \mathbb{C}, \quad M_j(z) := \begin{pmatrix} -\mathbb{A}_{j,q}(z)^{-1}\mathbb{A}_{j,q-1}(z) & \dots & \dots & -\mathbb{A}_{j,q}(z)^{-1}\mathbb{A}_{j,-p}(z) \\ Id & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & 0 & Id & 0 \end{pmatrix} \in \mathcal{M}_{d(p+q)}(\mathbb{C})$$

and

$$\forall z \in \mathbb{C}, \quad M^{\pm}(z) := \begin{pmatrix} -\mathbb{A}^{\pm}_q(z)^{-1}\mathbb{A}^{\pm}_{q-1}(z) & \dots & \dots & -\mathbb{A}^{\pm}_q(z)^{-1}\mathbb{A}^{\pm}_{-p}(z) \\ Id & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & 0 & Id & 0 \end{pmatrix} \mathcal{M}_{d(p+q)}(\mathbb{C}). \tag{3.1}$$

Hypothesis 8 implies that for all $z \in \mathbb{C}$ and $j \in \mathbb{Z}$, the matrices $\mathbb{A}_{j,-p}(z)$, $\mathbb{A}_{j,q}(z)$, $\mathbb{A}^{\pm}_{-p}(z)$ and $\mathbb{A}^{\pm}_q(z)$ are invertible. Thus, the matrices $M_j(z)$ and $M^{\pm}(z)$ are well-defined and invertible. We observe that $M_j(z) \underset{j\to\pm\infty}{\to} M^{\pm}(z)$ for all $z \in \mathbb{C}$. If we define for every $z \in \mathbb{C}$ and $j \in \mathbb{Z}$ the matrices

$$\mathcal{E}^{\pm}_j(z) := M_j(z) - M^{\pm}(z),$$

then Hypothesis 3 implies that there exists a constant $\alpha > 0$ such that for every bounded set U of $\mathbb{C}$, there exists a constant $C > 0$ such that[3]

$$\forall z \in U, \forall j \in \mathbb{N}, \quad \begin{matrix} |\mathcal{E}^{+}_j(z)| \leq Ce^{-\alpha j}, \\ |\mathcal{E}^{-}_{-j}(z)| \leq Ce^{-\alpha j}. \end{matrix} \tag{3.2}$$

We observe that if there exists $u \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$ such that

$$(zId_{\ell^2} - \mathscr{L})u = 0,$$

[3]Since

$$\mathcal{E}^{\pm}_j(z) = \begin{pmatrix} \varepsilon^{\pm}_{j,q-1}(z) & \dots & \varepsilon^{\pm}_{j,-p}(z) \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

where

$$\varepsilon^{\pm}_{j,k}(z) := \begin{cases} (A^{\pm}_q)^{-1}A^{\pm}_k - (A_{j,q})^{-1}A_{j,k} & \text{if } k \in \{-p, \dots, q-1\}\setminus\{0\}, \\ (A^{\pm}_q)^{-1}A^{\pm}_0 - (A_{j,q})^{-1}A_{j,0} - z((A^{\pm}_q)^{-1} - (A_{j,q})^{-1}) & \text{if } k = 0, \end{cases}$$

the constant α can be taken uniformly on $\mathbb{C}$ but the constant C must depend on z.

then, if we define for all $j \in \mathbb{Z}$ the vectors

$$W_j = \begin{pmatrix} u_{j+q-1} \\ \vdots \\ u_{j-p} \end{pmatrix},$$

we have that $W_j \underset{j\to\pm\infty}{\to} 0$ and

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z) W_j. \tag{3.3}$$

To study the solution of the dynamical system (3.3), we define the family of fundamental matrices $(X_j(z))_{j\in\mathbb{Z}} \in \mathcal{M}_{(p+q)d}(\mathbb{C})$ defined by

$$\begin{array}{c} \forall j \in \mathbb{Z}, \quad X_{j+1}(z) = M_j(z) X_j(z), \\ X_0(z) = Id. \end{array} \tag{3.4}$$

We observe that a solution $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (3.3) thus verifies that

$$\forall j \in \mathbb{Z}, \quad W_j = X_j(z) W_0.$$

To find the eigenvalues of the operator $\mathscr{L}$, the assertion above urges us to search for the solutions $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (3.3) which converge towards 0 when j tends to $\pm\infty$. We thus introduce the sets for $z \in \mathbb{C}$

$$E^{\pm}(z) := \left\{ (W_j)_{j\in\mathbb{Z}} \in \left(\mathbb{C}^{(p+q)d}\right)^{\mathbb{Z}} \text{ solution of (3.3) such that } W_j \underset{j\to\pm\infty}{\to} 0 \right\} \tag{3.5a}$$

$$E_0^{\pm}(z) := \left\{ W_0 \in \mathbb{C}^{(p+q)d}, \quad (X_j(z)W_0)_{j\in\mathbb{Z}} \in E^{\pm}(z) \right\} \tag{3.5b}$$

which correspond to the solutions of the dynamical system (3.3) which converge towards 0 as j tends towards $\pm\infty$ and their traces at $j = 0$.

3.2 Spectral splitting: study of the spectrum of $M^{\pm}(z)$

Since $M_j(z) \underset{j\to\pm\infty}{\to} M^{\pm}(z)$, the dynamical system (3.3) can be considered to be perturbations respectively for $j \in \mathbb{N}$ and $j \in -\mathbb{N}$ of the dynamical systems

$$\forall j \in \mathbb{N}, \quad W_{j+1} = M^{+}(z) W_j. \tag{3.6a}$$

$$\forall j \in -\mathbb{N}, \quad W_{j+1} = M^{-}(z) W_j. \tag{3.6b}$$

To study the solutions of (3.3) which converge towards 0 as j tends to $\pm\infty$, we will study solutions converging towards 0 of the dynamical systems (3.6a) and (3.6b). This relies on studying the spectrum of the matrices $M^{\pm}(z)$.

Using the eigenvalues $\lambda_{l,k}^{\pm}$ of the matrix $A_k^{\pm}$ defined by (1.14), we introduce

$$\forall z \in \mathbb{C}, \forall l \in \{1,\dots,d\}, \forall k \in \{-p,\dots,q\}, \quad \Lambda_{l,k}^{\pm}(z) := z\delta_{k,0} - \lambda_{l,k}^{\pm}, \tag{3.7}$$

and

$$\forall z \in \mathbb{C}, \forall l \in \{1,\dots,d\}, \quad M_l^{\pm}(z) := \begin{pmatrix} -\Lambda_{l,q}^{\pm}(z)^{-1}\Lambda_{l,q-1}^{\pm}(z) & \dots & \dots & -\Lambda_{l,q}^{\pm}(z)^{-1}\Lambda_{l,-p}^{\pm}(z) \\ 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.8}$$

Hypothesis 8 implies that the matrices $A_{-p}^{\pm}$ and $A_q^{\pm}$ are invertible so $\Lambda_{l,-p}^{\pm}(z), \Lambda_{l,q}^{\pm}(z) \neq 0$. Thus, the matrices $M_l^{\pm}(z)$ are well-defined and invertible. We have the following result.

Lemma 3.1. *There exists an invertible matrix $Q^{\pm} \in \mathcal{M}_{(p+q)d}(\mathbb{C})$ such that*

$$\forall z \in \mathbb{C}, \quad M^{\pm}(z) = Q^{\pm} \begin{pmatrix} M_1^{\pm}(z) & & \\ & \ddots & \\ & & M_d^{\pm}(z) \end{pmatrix} Q^{\pm -1}.$$

Proof We observe that

$$\begin{pmatrix} P^{\pm} & & \\ & \ddots & \\ & & P^{\pm} \end{pmatrix}^{-1} M^{\pm}(z) \begin{pmatrix} P^{\pm} & & \\ & \ddots & \\ & & P^{\pm} \end{pmatrix} = \begin{pmatrix} -\mathbb{D}_q^{\pm}(z)^{-1}\mathbb{D}_{q-1}^{\pm}(z) & \dots & \dots & -\mathbb{D}_q^{\pm}(z)^{-1}\mathbb{D}_{-p}^{\pm}(z) \\ Id & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & 0 & Id & 0 \end{pmatrix} \tag{3.9}$$

where

$$\forall k \in \{-p, \dots, q\}, \quad \mathbb{D}_k^{\pm}(z) = \begin{pmatrix} \Lambda_{1,k}^{\pm}(z) & & \\ & \ddots & \\ & & \Lambda_{d,k}^{\pm}(z) \end{pmatrix}.$$

Then, in the right hand term's matrix of (3.9), by reassembling the first columns of each blocks, then the second columns, ... and then doing the same for the lines, we prove that the matrix $M^{\pm}(z)$ is similar to the matrix

$$\begin{pmatrix} M_1^{\pm}(z) & & \\ & \ddots & \\ & & M_d^{\pm}(z) \end{pmatrix}.$$

□

The following lemma is due to Kreiss (see [Kre68]) and describes precisely the spectrum of the matrix $M_l^{\pm}(z)$ as z belongs to $\mathcal{O} \cup \{1\}$.

Lemma 3.2 (Spectral Splitting).

- *For $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$ is an eigenvalue of $M_l^{\pm}(z)$ if and only if $\kappa \neq 0$ and*

$$\mathcal{F}_l^{\pm}(\kappa) = z.$$

- *Let $z \in \mathcal{O}$ and $l \in \{1, \dots, d\}$. Then the matrix $M_l^{\pm}(z)$ has*
  - *no eigenvalue on $\mathbb{S}^1$,*
  - *p eigenvalues in $\mathbb{D}\backslash\{0\}$ (that we call stable eigenvalues),*
  - *q eigenvalues in $\mathbb{U}$ (that we call unstable eigenvalues).*
- *We also have that*
  - *if $\alpha_l^{\pm} > 0$, $M_l^{\pm}(1)$ has 1 as a simple eigenvalue, $p-1$ eigenvalues in $\mathbb{D}\backslash\{0\}$ and q eigenvalues in $\mathbb{U}$.*
  - *if $\alpha_l^{\pm} < 0$, $M_l^{\pm}(1)$ has 1 as a simple eigenvalue, p eigenvalues in $\mathbb{D}\backslash\{0\}$ and $q-1$ eigenvalues in $\mathbb{U}$.*

Lemma 3.2 is proved in [CF22, Lemma 1]. Combining the consequences of Lemmas 3.1 and 3.2, for $z \in \mathcal{O}$, the matrix $M^{\pm}(z)$ only has eigenvalues in $\mathbb{D}$ or $\mathbb{U}$. Also, if we define the space $E^s(M^{\pm}(z))$ (resp. $E^u(M^{\pm}(z))$) which is the strictly stable (resp. strictly unstable) subspace of $M^{\pm}(z)$ which corresponds to the subspace spanned by the generalized eigenvectors of $M^{\pm}(z)$ associated with eigenvalues in $\mathbb{D}$ (resp. $\mathbb{U}$), then $\dim E^s(M^{\pm}(z)) = dp$, $\dim E^u(M^{\pm}(z)) = dq$ and

$$\mathbb{C}^{(p+q)d} = E^s(M^{\pm}(z)) \oplus E^u(M^{\pm}(z)).$$

We consider $P_s^{\pm}(z)$ and $P_u^{\pm}(z)$ the associated projectors. Those projectors $P_s^{\pm}(z)$ and $P_u^{\pm}(z)$ can be expressed as a contour integral. For instance, we have

$$P_s^{\pm}(z) = \frac{1}{2i\pi} \int_{\gamma} (tId - M^{\pm}(z))^{-1} dt$$

where γ is a simple closed positively oriented contour which surrounds the stable eigenvalues of $M^{\pm}(z)$ and not the unstable ones (for instance, $\mathbb{S}^1$ is a good candidate). Therefore, the projectors $P_s^{\pm}(z)$ and $P_u^{\pm}(z)$ depend holomorphically on $z \in \mathcal{O}$.

3.3 "Local" geometric dichotomy

The conclusion of the study of the spectrum of the matrices $M^{\pm}(z)$ done in Section 3.2 is that the vector space of solutions of (3.6a) (resp. (3.6b)) converging towards 0 as j tends towards $+\infty$ (resp. $-\infty$) is of dimension dp (resp. dq) and can be characterized using the spectral projector $P_s^+(z)$ (resp. $P_u^-(z)$). We recall that (3.3) is a perturbation of the dynamical systems (3.6a) and (3.6b). Thus, we could expect for the vector spaces $E^+(z)$ and $E_0^+(z)$ (resp. $E^-(z)$ and $E_0^-(z)$) defined by (3.5a) and (3.5b) to be of dimension dp (resp. dq) and we would want some way to characterize their elements.

In the present section, the goal is to construct projectors which will play for the dynamical system (3.3) a similar role as the spectral projectors $P_s^+(z)$ and $P_u^-(z)$ for the dynamical systems (3.6a) and (3.6b). This is the aim of the following lemma.

Lemma 3.3 (Geometric dichotomy)**.** *For all bounded open set U such that $\overline{U} \subset \mathcal{O}$, there exist two holomorphic functions $Q_U^{\pm} : U \to \mathcal{M}_{(p+q)d}(\mathbb{C})$ such that*

- *For all $z \in U$, $Q_U^{\pm}(z)$ is a projector and we have*

$$\dim \Im Q_U^+(z) = \dim \ker Q_U^-(z) = dp \quad \text{and} \quad \dim \Im Q_U^-(z) = \dim \ker Q_U^+(z) = dq.$$

- *There exist two positive constants C, c such that for all $z \in U$*

$$\forall j \geq k \geq 0, \quad \left|X_j(z) Q_U^+(z) X_k(z)^{-1}\right| \leq C e^{-c|j-k|}, \tag{3.10a}$$
$$\forall k \geq j \geq 0, \quad \left|X_j(z)(Id - Q_U^+(z)) X_k(z)^{-1}\right| \leq C e^{-c|j-k|}, \tag{3.10b}$$
$$\forall j \leq k \leq 0, \quad \left|X_j(z) Q_U^-(z) X_k(z)^{-1}\right| \leq C e^{-c|j-k|}, \tag{3.10c}$$
$$\forall k \leq j \leq 0, \quad \left|X_j(z)(Id - Q_U^-(z)) X_k(z)^{-1}\right| \leq C e^{-c|j-k|}. \tag{3.10d}$$

Lemma 3.3 has been developped in the thesis of Pauline Godillon [LG01, Section III.1.5] and is inspired by the exponential dichotomy discussed by Coppel in [Cop78]. Just as it is explained in [Cop78], to better understand the meaning of this lemma, it is interesting to see that the inequalities (3.10a) and (3.10b) imply that for all $\xi \in \mathbb{C}^{(p+q)d}$

$$\forall j \geq k \geq 0, \quad \left|X_j(z) Q_U^+(z) \xi\right| \leq C e^{-c(j-k)} \left|X_k(z)\xi\right|,$$
$$\forall k \geq j \geq 0, \quad \left|X_j(z)(Id - Q_U^+(z))\xi\right| \leq C e^{-c(k-j)} \left|X_k(z)\xi\right|.$$

The first inequality implies that there exists a dp-dimensional subspace of solutions $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (3.3) which converge exponentially fast toward 0 as j tends to $+\infty$. The second inequality translates to the fact that there exists a supplementary of the previous subspace of solutions for which the solutions explode exponentially as j tends towards $+\infty$. Thus, $Q_U^+(z)$ plays a similar role for the dynamical system (3.3) as any projector for which the range is $E_s^+(z)$ (for instance the spectral projector $P_s^+(z)$) for the dynamical system (3.6a). The same kind of conclusion can be achieved with Q_U^-.

Thus, the construction of those two projectors $Q_U^{\pm}$ is fundamental to study the solutions $(W_j)_{j\in\mathbb{Z}}$ of (3.3) that converge toward 0 as j tends to $\pm\infty$, i.e. the elements of the set $E^{\pm}(z)$ defined by (3.5a). We will see in Lemma 3.6 that the projectors $Q_U^{\pm}(z)$ allow to completely characterize the elements of the vectors spaces $E_0^{\pm}(z)$.

Let us point out that the construction of the projectors $Q_U^{\pm}$ is done on relatively compact subsets of $\mathcal{O}$. More importantly, for two sets U_1 and U_2 that satisfy the conditions of Lemma 3.3, even if $U_1 \subset U_2$, the construction of the proof of Lemma 3.3 does not imply that the projectors $Q_{U_1}^{\pm}$ and $Q_{U_2}^{\pm}$ are equal on $U_1 \cap U_2$. Therefore, we cannot immediately construct two functions $Q^{\pm}$ that are defined on $\mathcal{O}$ which would verify similar properties as $Q_U^{\pm}$. However, it turns out that $\Im Q_{U_1}^{\pm}(z) = \Im Q_{U_2}^{\pm}(z)$ for $z \in U_1 \cap U_2$. We will prove this fact later on and use it to extend uniformly the geometric dichotomy on a large part of $\mathcal{O}$ (see Lemma 3.8).

Proof The construction of both functions $Q_U^{\pm}$ is similar so we focus here on the construction of Q_U^+. The proof will be separated in four steps. In the first step, we will construct the function Q_U^+ using a fixed point argument. The second step will be dedicated to proving that for all $z \in U$, $Q_U^+(z)$ is a projector for which the kernel and the range are respectively of dimension dq and dp. The third and fourth steps concern the proof of the inequalities (3.10a) and (3.10b).

Step 1: Construction of Q_U^+.

We set for $z \in \mathcal{O}$ that

$$\eta_s^{\pm}(z) := \max\left\{\ln(|\zeta|), \quad \zeta \in \sigma(M^{\pm}(z)) \cap \mathbb{D}\right\}$$
$$\eta_u^{\pm}(z) := \min\left\{\ln(|\zeta|), \quad \zeta \in \sigma(M^{\pm}(z)) \cap \mathbb{U}\right\}.$$

The functions η_s^+ and η_u^+ are continuous on $\mathcal{O}$ and verify that

$$\forall z \in \mathcal{O}, \quad \eta_s^+ < 0 \quad \text{and} \quad \eta_u^+ > 0.$$

The set $\overline{U}$ being a compact included in $\mathcal{O}$, there exists a constant c_H such that

$$\max_{z\in\overline{U}} \eta_s^+(z) < -c_H < 0 \quad \text{and} \quad 0 < c_H < \min_{z\in\overline{U}} \eta_u^+(z).$$

We will also ask that $c_H < \alpha$ where α is the constant appearing in (3.2). By definition of η_s^+ and η_u^+, there exists a positive constant C_H such that

$$\forall z \in U, \forall j \in \mathbb{N}, \qquad \begin{array}{l} \left|M^+(z)^j P_s^+(z)\right| \leq C_H e^{-c_H j}, \\ \left|M^+(z)^{-j} P_u^+(z)\right| \leq C_H e^{-c_H j}. \end{array} \tag{3.11}$$

Furthermore, using (3.2), since U is bounded, there exists a positive constant $C_{\mathcal{E}}$ such that

$$\forall z \in U, \forall j \in \mathbb{N}, \quad |\mathcal{E}_j^+(z)| \leq C_{\mathcal{E}} e^{-\alpha j}. \tag{3.12}$$

We fix an integer $J \in \mathbb{N}$ and we will make a more precise choice later. We define the Banach space

$$\ell_J^\infty := \left\{ (Y_j)_{j\geq J} \in \mathcal{M}_{(p+q)d}(\mathbb{C})^{\{j\in\mathbb{N}, j\geq J\}}, \quad \sup_{j\geq J} |Y_j| < +\infty \right\}$$

with the norm

$$\|Y\|_{\infty,J} := \sup_{j\geq J} |Y_j|.$$

Furthermore, for $z \in U$, we define $\varphi(z) \in \mathcal{L}\left(\ell_J^\infty\right)$ and $T(z) : \ell_J^\infty \to \ell_J^\infty$ such that for $Y \in \ell_J^\infty$ and $j \geq J$, we have

$$\left(\varphi(z)Y\right)_j := \sum_{k=J}^{j-1} M^+(z)^{j-1-k} P_s^+(z) \mathcal{E}_k^+(z) Y_k - \sum_{k=j}^{+\infty} M^+(z)^{j-1-k} P_u^+(z) \mathcal{E}_k^+(z) Y_k$$

and

$$T(z)Y := \left(M^+(z)^{j-J} P_s^+(z)\right)_{j\geq J} + \varphi(z)Y.$$

We observe that

$$\forall Y \in \ell_J^\infty, \forall j \geq J, \quad \left(\varphi(z)Y\right)_{j+1} = M^+(z)\left(\varphi(z)Y\right)_j + \mathcal{E}_j^+(z)Y_j,$$

and thus

$$\forall Y \in \ell_J^\infty, \forall j \geq J, \quad \left(T(z)Y\right)_{j+1} = M^+(z)\left(T(z)Y\right)_j + \mathcal{E}_j^+(z)Y_j. \tag{3.13}$$

Our goal will be to find a fixed point of $T(z)$. To do so, we will have to prove that there exists J large enough so that

$$\|\varphi(z)\|_{\mathcal{L}\left(\ell_J^\infty\right)} < 1.$$

We begin by proving that the applications $\varphi(z)$ and $T(z)$ are well-defined. We consider $Y \in \ell_J^\infty$ and $j \geq J$. We have using (3.11) and (3.12)

$$\begin{aligned} \left|\left(\varphi(z)Y\right)_j\right| &\leq \sum_{k=J}^{j-1} C_H C_{\mathcal{E}} e^{-c_H|j-1-k|} e^{-\alpha k} |Y_k| + \sum_{k=j}^{+\infty} C_H C_{\mathcal{E}} e^{-c_H|j-1-k|} e^{-\alpha k} |Y_k| \\ &\leq \|Y\|_{\infty,U} C_H C_{\mathcal{E}} e^{-\alpha J} \sum_{k=J}^{+\infty} e^{-c_H|j-1-k|} \\ &\leq \|Y\|_{\infty,U} C_H C_{\mathcal{E}} e^{-\alpha J} \frac{1+e^{-c_H}}{1-e^{-c_H}}. \end{aligned}$$

If we set

$$\theta := C_H C_{\mathcal{E}} e^{-\alpha J} \frac{1+e^{-c_H}}{1-e^{-c_H}}, \tag{3.14}$$

then we just proved that the operator $\varphi(z)$ is defined, bounded and

$$\|\varphi(z)\|_{\mathcal{L}\left(\ell_J^\infty\right)} \leq \theta.$$

We also observe that (3.11) implies that $(M^+(z)^{j-J}P_s^+(z))_{j\geq J} \in \ell^\infty_J$. Therefore, $T(z)$ is well-defined.

We consider that we chose J large enough so that $\theta < 1$. For $z \in U$, we have that $Id - \varphi(z)$ is invertible. Thus, we can define

$$Y(z) := (Id - \varphi(z))^{-1} \left(M^+(z)^{j-J}P_s^+(z)\right)_{j\geq J}.$$

It is the only fixed point of $T(z)$ and it depends holomorphically on z. We observe that (3.13) implies that

$$\forall z \in U, \forall j \geq J, \quad (Y(z))_{j+1} = M_j(z)(Y(z))_j. \tag{3.15}$$

We define

$$\forall z \in U, \quad Q_U^+(z) := X_J(z)^{-1}(Y(z))_J X_J(z). \tag{3.16}$$

Since Y depends holomorphically on z and is bounded on U, $Q_U^+(z)$ also depends holomorphically on z for $z \in U$ and is bounded in U.

Step 2: Q_U^+ is a projector.

We are going to prove that for all $z \in U$ the matrix $Q_U^+(z)$ we just constructed is a projector such that

$$\ker Q_U^+(z) = X_J(z)^{-1}E^u(M^+(z)) \quad \text{and} \quad \dim \Im Q_U^+(z) = dp.$$

By observing that $P_s^+(z)^2 = P_s^+(z)$, we prove that $((Y(z))_j P_s^+(z))_{j\geq J}$ is a fixed point $T(z)$. Since $Y(z)$ is the only fixed point of $T(z)$ in ℓ^∞_J, we prove that

$$(Y(z))_J P_s^+(z) = (Y(z))_J. \tag{3.17}$$

Using that $Y(z)$ is a fixed point of $T(z)$, we have that

$$\begin{aligned} P_s^+(z)(Y(z))_J &= P_s^+(z)(T(z)Y(z))_J \\ &= P_s^+(z)\left(P_s^+(z) - \sum_{k=J}^{+\infty} M^+(z)^{J-1-k}P_u^+(z)\mathcal{E}_k^+(z)(Y(z))_k\right). \end{aligned}$$

Because $P_s^+(z)$ commutes with $M^+(z)$, $P_s^+(z)^2 = P_s^+(z)$ and $P_s^+(z)P_u^+(z) = 0$, we have proved

$$P_s^+(z)(Y(z))_J = P_s^+(z). \tag{3.18}$$

Using (3.18), we prove that $((Y(z))_j(Y(z))_J)_{j\geq J}$ is a fixed point $T(z)$. Since $Y(z)$ is the only fixed point of $T(z)$ in ℓ^∞_J, we prove that

$$(Y(z))_J{}^2 = (Y(z))_J.$$

Therefore, we have proved that $(Y(z))_J$ is a projector. The equalities (3.17) and (3.18) allow us to prove that $\ker(Y(z))_J = \ker P_s^+(z) = E^u(M^+(z))$. The definition of $Q_U^+(z)$ implies that $Q_U^+(z)$ is a projector and

$$\Im Q_U^+(z) = X_J(z)^{-1}\Im(Y(z))_J \quad \text{and} \quad \ker Q_U^+(z) = X_J(z)^{-1}\ker(Y(z))_J = X_J(z)^{-1}E^u(M^+(z)).$$

Step 3: Q_U^+ satisfies the inequalities (3.10a) and (3.10b) respectively for $j \geq k \geq J$ and $k \geq j \geq J$.

First, we are going to prove the inequality (3.10a) for $j \geq k \geq J$ and the inequality (3.10b) for $k \geq j \geq J$. We observe that (3.15) implies that

$$\forall z \in U, \forall j \geq J, \quad (Y(z))_{j+1} = M_j(z)(Y(z))_j.$$

and thus

$$\forall z \in U, \forall j \geq J, \quad (Y(z))_j = X_j(z)X_J(z)^{-1}(Y(z))_J = X_j(z)Q_U^+(z)X_J(z)^{-1}.$$

We introduce

$$\forall z \in U, \forall j \geq J, \quad (Z(z))_j := X_j(z)X_J(z)^{-1}(Id - (Y(z))_J) = X_j(z)\left(Id - Q_U^+(z)\right)X_J(z)^{-1}.$$

We have the following lemma.

Lemma 3.4. *We have that*

$$\forall j \geq k \geq J, \quad (Y(z))_j = M^+(z)^{j-k} P_s^+(z)(Y(z))_k + \sum_{l=k}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Y(z))_l - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l,$$

and

$$\forall k \geq j \geq J, \quad (Z(z))_j = M^+(z)^{j-k} P_u^+(z)(Z(z))_k + \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Z(z))_l - \sum_{l=j}^{k-1} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Z(z))_l.$$

Proof

- Since we have that

$$\forall j \geq J, \quad (Y(z))_{j+1} = M_j(z)(Y(z))_j = (M^+(z) + \mathcal{E}_j^+(z))(Y(z))_j,$$

using the variation of constants formula, we find that

$$\forall k \geq J, \quad (Y(z))_k = M^+(z)^{k-J}(Y(z))_J + \sum_{l=J}^{k-1} M^+(z)^{k-1-l}\mathcal{E}_l^+(z)(Y(z))_l.$$

Knowing that $Y(z)$ is a fixed point of $T(z)$ and that $P_s^+(z)(Y(z))_J = P_s^+(z)$, we have for $j \geq k \geq J$

$$\begin{aligned}
&(Y(z))_j \\
=&(T(z)Y(z))_j \\
=&M^+(z)^{j-J} P_s^+(z) + \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Y(z))_l - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l \\
=&M^+(z)^{j-k} P_s^+(z) \left(M^+(z)^{k-J}(Y(z))_J + \sum_{l=J}^{k-1} M^+(z)^{k-1-l}\mathcal{E}_l^+(z)(Y(z))_l \right) \\
&+ \sum_{l=k}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Y(z))_l - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l \\
=&M^+(z)^{j-k} P_s^+(z)(Y(z))_k + \sum_{l=k}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Y(z))_l - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l.
\end{aligned}$$

- Since we have that

$$\forall j \geq J, \quad (Z(z))_{j+1} = M_j(z)(Z(z))_j = (M^+(z) + \mathcal{E}_j^+(z))(Z(z))_j,$$

using the variation of constants formula, we find that for $k \geq j \geq J$

$$(Z(z))_j = M^+(z)^{j-J}(Z(z))_J + \sum_{l=J}^{j-1} M^+(z)^{j-1-l}\mathcal{E}_l^+(z)(Z(z))_l \tag{3.19}$$

$$(Z(z))_k = M^+(z)^{k-j}(Z(z))_j + \sum_{l=j}^{k-1} M^+(z)^{k-1-l}\mathcal{E}_l^+(z)(Z(z))_l. \tag{3.20}$$

Using (3.19) and knowing that $P_s^+(z)(Z(z))_J = P_s^+(z)(Id - (Y(z))_J) = 0$, we have that

$$P_s^+(z)(Z(z))_j = \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Z(z))_l.$$

Furthermore, (3.20) implies that

$$P_u^+(z)(Z(z))_j = M^+(z)^{j-k}P_u^+(z)(Z(z))_k - \sum_{l=j}^{k-1} M^+(z)^{j-1-l}P_u^+(z)\mathcal{E}_l^+(z)(Z(z))_l.$$

We end the proof of the lemma by observing that $(Z(z))_j = P_s^+(z)(Z(z))_j + P_u^+(z)(Z(z))_j$.

□

We introduce the constant

$$\Theta := \theta\frac{1-e^{-c_H}}{1+e^{-c_H}}. \tag{3.21}$$

Using Lemma 3.4 and (3.11), we obtain that for any $\xi \in \mathbb{C}^{(p+q)d}$

$$\forall j \geq k \geq J, \quad |(Y(z))_j\xi| \leq C_H e^{-c_H(j-k)}\,|(Y(z))_k\xi| + \Theta\sum_{l=k}^{+\infty} e^{-c_H|j-1-l|}\,|(Y(z))_l\xi|\,, \tag{3.22}$$

and

$$\forall k \geq j \geq J, \quad |(Z(z))_j\xi| \leq C_H e^{-c_H(k-j)}\,|(Z(z))_k\xi| + \Theta\sum_{l=J}^{k-1} e^{-c_H|j-1-l|}\,|(Z(z))_l\xi|\,. \tag{3.23}$$

The following lemma will allow us to obtain clearer bounds on $|(Y(z))_j\xi|$ and $|(Z(z))_j\xi|$.

Lemma 3.5. *For all $y \in \ell^\infty(\mathbb{N})$ with non negative coefficients that satisfies*

$$\forall j \in \mathbb{N}, \quad y_j \leq C_H e^{-c_H j} + \Theta\sum_{k=0}^{+\infty} e^{-c_H|j-1-k|}y_k, \tag{3.24}$$

then we have

$$\forall j \in \mathbb{N}, \quad y_j \leq \rho r^j,$$

where

$$r := \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right)\sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta} \in]e^{-c_H}, 1[\quad \text{and} \quad \rho := \frac{C_H}{\Theta}(r - e^{-c_H}) > 0. \tag{3.25}$$

The proof can be found in the Appendix (Section 6). We will now use Lemma 3.5 to prove that

$$\forall j \geq k \geq J, \quad |(Y(z))_j\xi| \leq \rho r^{j-k}|(Y(z))_k\xi|, \tag{3.26}$$

$$\forall k \geq j \geq J, \quad |(Z(z))_j\xi| \leq \rho r^{k-j}|(Z(z))_k\xi|. \tag{3.27}$$

We consider $k \geq J$. If $(Y(z))_k\xi \neq 0$, then by applying Lemma 3.5 to the bounded sequence $y = \left(\frac{|(Y(z))_{k+j}\xi|}{|(Y(z))_k\xi|}\right)_{j\in\mathbb{N}}$, we obtain (3.26). Else, if $(Y_U(z))_k = 0$, then for $j \geq k$, we have

$$(Y(z))_j\xi = X_j(z)X_k(z)^{-1}(Y(z))_k\xi = 0.$$

Thus, (3.26) is also verified in this case.

The proof of (3.27) is similar. If $(Z(z))_k\xi \neq 0$, then we apply the Lemma 3.5 to the sequence y defined by

$$\forall j \in \mathbb{N}, \quad y_j = \begin{cases} \frac{|(Z(z))_{k-j}\xi|}{|(Z(z))_k\xi|} & \text{if } j \in \{0,\dots,k\}\,, \\ 0 & \text{else.} \end{cases}$$

This proves (3.27) in this case. If $(Z(z))_k\xi = 0$, then since

$$\forall j \in \{J,\dots,k\}\,, \quad (Z(z))_j\xi = X_j(z)X_k(z)^{-1}(Z(z))_k\xi = 0,$$

(3.27) is also verified in this case.

Using (3.26) and (3.27), we proved that

$$\forall j \geq k \geq J, \quad |X_j(z)Q_U^+(z)X_k(z)^{-1}| \leq \rho r^{j-k}|X_k(z)Q_U^+(z)X_k(z)^{-1}|, \tag{3.28}$$
$$\forall k \geq j \geq J, \quad |X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}| \leq \rho r^{k-j}|X_k(z)(Id - Q_U^+(z))X_k(z)^{-1}|. \tag{3.29}$$

If we prove that the families $(X_k(z)Q_U^+(z)X_k(z)^{-1})_{k\geq J}$ and $(X_k(z)(Id - Q_U^+(z))X_k(z)^{-1})_{k\geq J}$ are uniformly bounded for $z \in U$, we would have proved (3.10a) and (3.10b) respectively for $j \geq k \geq J$ and $k \geq j \geq J$.

Using Lemma 3.4, we prove that for $j \geq J$

$$P_u^+(z)(Y(z))_j = -\sum_{l=j}^{+\infty} M^+(z)^{j-1-l}P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l,$$
$$P_s^+(z)(Z(z))_j = \sum_{l=J}^{j-1} M^+(z)^{j-1-l}P_s^+(z)\mathcal{E}_l^+(z)(Z(z))_l.$$

Thus,

$$P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1} = -\sum_{l=j}^{+\infty} M^+(z)^{j-1-l}P_u^+(z)\mathcal{E}_l^+(z)X_l(z)Q_U^+(z)X_j(z)^{-1},$$
$$P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} = \sum_{l=J}^{j-1} M^+(z)^{j-1-l}P_s^+(z)\mathcal{E}_l^+(z)X_l(z)(Id - Q_U^+(z))X_j(z)^{-1}.$$

Using (3.11), the definitions (3.14), (3.21) and (3.25) of the constants θ, Θ and ρ, as well as (3.28), we have

$$\begin{aligned}
|P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}| &\leq \Theta \sum_{l=j}^{+\infty} e^{-c_H(l-(j-1))}|X_l(z)Q_U^+(z)X_j(z)^{-1}| \\
&\leq \Theta e^{-c_H}\rho \sum_{l=j}^{+\infty}(re^{-c_H})^{l-j}|X_j(z)Q_U^+(z)X_j(z)^{-1}| \\
&= \Theta \frac{e^{-c_H}}{1 - re^{-c_H}}\rho|X_j(z)Q_U^+(z)X_j(z)^{-1}| \\
&= C_H \frac{r - e^{-c_H}}{e^{c_H} - r}|X_j(z)Q_U^+(z)X_j(z)^{-1}|.
\end{aligned}$$

Similarly, using (3.29), we have

$$\begin{aligned}
|P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| &\leq \Theta \sum_{l=J}^{j-1} e^{-c_H(j-1-l)}|X_l(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\
&\leq \Theta e^{c_H}\rho \sum_{l=J}^{j-1}(re^{-c_H})^{j-l}|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\
&\leq \Theta e^{c_H} \frac{1}{1 - re^{-c_H}}\rho|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\
&= C_H e^{2c_H} \frac{r - e^{-c_H}}{e^{c_H} - r}|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|.
\end{aligned}$$

Therefore, if we define $\eta := C_H \frac{e^{2c_H}}{e^{c_H}-1}(r - e^{-c_H})$, we have for all $j \geq J$

$$\begin{aligned}
|P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}| &\leq \eta|X_j(z)Q_U^+(z)X_j(z)^{-1}|, \\
|P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| &\leq \eta|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|.
\end{aligned} \tag{3.30}$$

Using the definition (3.25) of r, we observe that

$$\eta = C_H \frac{e^{2c_H}}{e^{c_H} - 1} 2\sinh\left(\frac{c_H}{2}\right)\left(\cosh\left(\frac{c_H}{2}\right) - \sqrt{\cosh\left(\frac{c_H}{2}\right)^2 - \theta}\right).$$

We already supposed that J was taken large enough so that $\theta < 1$. We will now also suppose that we took J large enough so that θ is close enough to 0 so that $\eta < \frac{1}{2}$. To conclude this step of the proof, we observe that

$$X_j(z)Q_U^+(z)X_j(z)^{-1} - P_s^+(z) = P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1} - P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}$$

and

$$X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} - P_u^+(z) = P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} - P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}.$$

Thus, using (3.30) and (3.11) to bound P_s^+ and P_u^+, we have

$$|X_j(z)Q_U^+(z)X_j(z)^{-1}| \leq C_H + \eta\left(|X_j(z)Q_U^+(z)X_j(z)^{-1}| + |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|\right)$$

and

$$|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \leq C_H + \eta\left(|X_j(z)Q_U^+(z)X_j(z)^{-1}| + |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|\right).$$

This implies that

$$\forall j \geq J, \quad \begin{array}{c} |X_j(z)Q_U^+(z)X_j(z)^{-1}| \leq \dfrac{2C_H}{1-2\eta}, \\ |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \leq \dfrac{2C_H}{1-2\eta}. \end{array}$$

Therefore, we have proved that for all $z \in U$, we have

$$\forall j \geq k \geq J, \quad |X_j(z)Q_U^+(z)X_k(z)^{-1}| \leq \rho\frac{2C_H}{1-2\eta}r^{j-k}, \tag{3.31}$$

$$\forall k \geq j \geq J, \quad |X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}| \leq \rho\frac{2C_H}{1-2\eta}r^{k-j}. \tag{3.32}$$

Step 4: Q_U^+ satisfies the inequalities (3.10a) and (3.10b) respectively for all $j \geq k \geq 0$ and $k \geq j \geq 0$

We will only finish the proof of (3.10a) since the proof for (3.10b) would be similar. We have proved (3.10a) for $j \geq k \geq J$. We consider a constant $C > 0$ such that

$$\forall z \in U, \quad \begin{array}{c} C > r^{-J}\max_{j\in\{0,\dots,J-1\}}|X_j(z)Q_U^+(z)X_J(z)^{-1}| \\ C > r^{-J}\max_{j\in\{0,\dots,J-1\}}|X_J(z)Q_U^+(z)X_j(z)^{-1}|. \end{array}$$

This can be done since the projector Q_U^+ defined by (3.16) is bounded on U.

- If $j \geq J > k \geq 0$, we have

$$\begin{aligned} |X_j(z)Q_U^+(z)X_k(z)^{-1}| &\leq |X_j(z)Q_U^+(z)X_J(z)^{-1}||X_J(z)Q_U^+(z)X_k(z)^{-1}| \\ &\leq \rho\frac{2C_H}{1-2\eta}r^{j-J}Cr^J \\ &\leq C\rho\frac{2C_H}{1-2\eta}r^{j-k}. \end{aligned}$$

- If $J > j \geq k \geq 0$, we have

$$\begin{aligned} |X_j(z)Q_U^+(z)X_k(z)^{-1}| &\leq |X_j(z)Q_U^+(z)X_J(z)^{-1}||X_J(z)Q_U^+(z)X_k(z)^{-1}| \\ &\leq C^2r^{2J} \\ &\leq C^2r^{j-k}. \end{aligned}$$

Therefore, there exists two constants $C, c > 0$ such that for all $z \in U$, (3.10a) is verified. □

3.4 Spectrum of $\mathscr{L}$ and extended geometric dichotomy

Now that we have proved the geometric dichotomy, let us go back on studying the vector spaces $E_0^{\pm}(z)$ which characterize the solutions of (3.3) converging towards 0 as j tends towards $\pm\infty$. The previous section about the geometric dichotomy allows us to prove the following lemma.

Lemma 3.6. *For all open bounded sets U such that $\overline{U} \subset \mathcal{O}$, we have*

$$\forall z \in U, \quad E_0^+(z) = \Im(Q_U^+(z)) \quad \text{and} \quad E_0^-(z) = \Im(Q_U^-(z)).$$

Therefore, for all $z \in \mathcal{O}$, $\dim E_0^+(z) = dp$ and $\dim E_0^-(z) = dq$. Also, for $W_0 \in E_0^+(z) \cap E_0^-(z)$, we have that $(X_j(z)W_0)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C}^{(p+q)d})$.

Proof We prove the first set equality.

- For $W_0 \in \Im(Q_U^+(z))$, we have for $j \in \mathbb{N}$ using (3.10a)

$$X_j(z)W_0 = X_j(z)Q_U^+(z)X_0(z)^{-1}W_0 \underset{j\to+\infty}{\to} 0.$$

  Thus, $W_0 \in E_0^+(z)$.

- For $W_0 \in E_0^+(z)$, we have for $j \in \mathbb{N}$ using (3.10b)

$$(Id - Q_U^+(z))W_0 = X_0(z)(Id - Q_U^+(z))X_j(z)^{-1}X_j(z)W_0 \underset{j\to+\infty}{\to} 0.$$

  Thus, $(Id - Q_U^+(z))W_0 = 0$, i.e. $W_0 \in \Im(Q_U^+(z))$.

Therefore, we have proved that

$$E_0^{\pm}(z) = \Im(Q_U^{\pm}(z)).$$

We now consider $W_0 \in E_0^+(z) \cap E_0^-(z) = \Im(Q_U^+(z)) \cap \Im(Q_U^-(z))$. Because of the inequalities (3.10a) and (3.10c), the equalities

$$\forall j \in \mathbb{Z}, \quad X_j(z)W_0 = X_j(z)Q_U^+(z)X_0(z)^{-1}W_0 = X_j(z)Q_U^-(z)X_0(z)^{-1}W_0$$

imply that $(X_j(z)W_0)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C}^{(p+q)d})$. □

Let us now come back to the heart of the matter: the study of the spectrum of the operator $\mathscr{L}$. We introduced the dynamical system (3.3) to study the solutions of the eigenvalue problem

$$(zId_{\ell^2} - \mathscr{L})u = 0.$$

The following lemma, for which the main part is proved in [Ser07, Theorem 4.1], is deduced using the geometric dichotomy.

Lemma 3.7. *For $z \in \mathcal{O}$, we have that*

$$\dim\ker(zId_{\ell^2} - \mathscr{L}) = \dim E_0^+(z) \cap E_0^-(z). \tag{3.33}$$

Furthermore, $zId_{\ell^2} - \mathscr{L}$ is a Fredholm operator of index 0, i.e.

$$\sigma_{ess}(\mathscr{L}) \cap \mathcal{O} = \emptyset.$$

Before proving the lemma, let us thus introduce the linear map which extracts the center values of a vector of size $d(p+q)$

$$\Pi : \begin{array}{ccc} \mathbb{C}^{d(p+q)} & \to & \mathbb{C}^d \\ (x_j)_{j\in\{1,\dots,d(p+q)\}} & \mapsto & (x_j)_{j\in\{d(q-1)+1,\dots,dq\}} \end{array}. \tag{3.34}$$

Proof We consider $z \in \mathcal{O}$ and start by proving (3.33).

- For $w \in \ker(zId_{\ell^2} - \mathscr{L})$, if we introduce

$$W_0 := \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix}$$

then we have that

$$\forall j \in \mathbb{Z}, \quad X_j(z)W_0 = \begin{pmatrix} w_{j+q-1} \\ \vdots \\ w_{j-p} \end{pmatrix}.$$

Since w belongs to $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $W_0 \in E_0^+(z) \cap E_0^-(z)$. This implies that the linear application

$$\begin{array}{rccc} \varphi : & \ker(zId_{\ell^2} - \mathscr{L}) & \to & E_0^+(z) \cap E_0^-(z) \\ & w & \mapsto & \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix} \end{array}$$

is well-defined.

- We consider $W_0 \in E_0^+(z) \cap E_0^-(z)$ and define for $j \in \mathbb{Z}$

$$w_j := \Pi(X_j(z)W_0) \in \mathbb{C}^d$$

where the operator Π is defined by (3.34). Lemma 3.6 implies that the sequence $w := (w_j)_{j\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z}, \mathbb{C}^d)$. Furthermore, since $(X_j(z)W_0)_{j\in\mathbb{Z}}$ is a solution of (3.3), we have that

$$(zId_{\ell^2} - \mathscr{L})w = 0.$$

Therefore, the linear application

$$\begin{array}{rccc} \psi : & E_0^+(z) \cap E_0^-(z) & \to & \ker(zId_{\ell^2} - \mathscr{L}) \\ & W_0 & \mapsto & (\Pi(X_j(z)W_0))_{j\in\mathbb{Z}} \end{array}$$

is well-defined.

We have that

$$\varphi \circ \psi = Id_{E_0^+(z)\cap E_0^-(z)} \quad \text{and} \quad \psi \circ \varphi = Id_{\ker(zId_{\ell^2}-\mathscr{L})}.$$

This concludes the proof of (3.33).

We now focus on the second part of the lemma. Our first goal is to prove that $zId_{\ell^2} - \mathscr{L}$ is a Fredholm operator. We have that

$$\dim \ker(zId_{\ell^2} - \mathscr{L}) = \dim E_0^+(z) \cap E_0^-(z) < +\infty.$$

There remains to prove that $\Im(zId_{\ell^2} - \mathscr{L})$ is closed and that

$$\text{codim}\Im(zId_{\ell^2} - \mathscr{L}) < +\infty.$$

We fix a bounded open neighborhood U of $z \in \mathcal{O}$ such that $\overline{U} \subset \mathcal{O}$. We consider $h \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$. The sequence h belongs to the range of $zId_{\ell^2} - \mathscr{L}$ if and only if there exists $v \in \ell^2(Z, \mathbb{C}^d)$ such that if we define

$$\forall j \in \mathbb{Z}, \quad W_j = \begin{pmatrix} v_{j+q-1} \\ \vdots \\ v_{j-p} \end{pmatrix}, \quad H_j = \begin{pmatrix} -\mathbb{A}_{j,p}(z)^{-1}h_j \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

then

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j + H_j.$$

For $j \geq 0$, we define

$$Z_j^+ := \sum_{k=0}^{j} X_j(z)Q_U^+(z)X_k(z)^{-1}H_{k-1} - \sum_{k=j+1}^{+\infty} X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}H_{k-1}.$$

Those elements are well-defined and verify that

$$\forall j \geq 0, \quad Z_{j+1}^+ = M_j(z)Z_j^+ + H_j$$

and

$$\left(\sum_{j\geq 0}|Z_j^+|^2\right)^{\frac{1}{2}}\leq C\left(\sum_{j\in\mathbb{Z}}|H_j|^2\right)^{\frac{1}{2}}$$

where $C>0$ is independent from h. We can then prove that the sequences $(\tilde{Z}_j)_{j\geq 0}$ which satisfy that

$$\sum_{j\geq 0}|\tilde{Z}_j|^2<+\infty$$

and

$$\forall j\geq 0,\quad \tilde{Z}_{j+1}=M_j(z)\tilde{Z}_j+H_j$$

are the sequences defined by

$$\forall j\geq 0,\quad \tilde{Z}_j=Z_j^++X_j(z)V^+$$

where $V^+\in E_0^+(z)$. We prove in a similar way that the sequences $(\tilde{Z}_j)_{j\leq 0}$ which satisfy that

$$\sum_{j\leq 0}|\tilde{Z}_j|^2<+\infty$$

and

$$\forall j\leq 0,\quad \tilde{Z}_{j+1}=M_j(z)\tilde{Z}_j+H_j$$

are the sequences defined by

$$\forall j\leq 0,\quad \tilde{Z}_j=Z_j^-+X_j(z)V^-$$

where $V^-\in E_0^-(z)$ and

$$\forall j\leq 0,\quad Z_j^-:=\sum_{k=-\infty}^{j}X_j(z)(Id-Q_U^-(z))X_k(z)^{-1}H_{k-1}-\sum_{k=j+1}^{1}X_j(z)Q_U^-(z)X_k(z)^{-1}H_{k-1}.$$

We also have that

$$\left(\sum_{j\leq 0}|Z_j^-|^2\right)^{\frac{1}{2}}\leq C\left(\sum_{j\in\mathbb{Z}}|H_j|^2\right)^{\frac{1}{2}}$$

where $C>0$ is independent from h. Using all those information, we conclude that h is in the range of $zId_{\ell^2}-\mathscr{L}$ if and only if there exists $(V^+,V^-)\in E_0^+(z)\times E_0^-(z)$ such that

$$Z_0^+-Z_0^-=V^--V^+.$$

If we define the operator

$$\nu:h\in\ell^2(\mathbb{Z},\mathbb{C}^d)\mapsto Z_0^+-Z_0^-\in\mathbb{C}^{d(p+q)}$$

which is bounded and

$$\varphi:(V^+,V^-)\in E_0^+(z)\times E_0^-(z)\mapsto V^--V^+\in\mathbb{C}^{d(p+q)}$$

which is an operator from a finite dimension vector space to a finite dimension vector space, then we proved that

$$\Im(zId_{\ell^2}-\mathscr{L})=\nu^{-1}(\Im\varphi).$$

Therefore, $\Im(zId_{\ell^2}-\mathscr{L})$ is closed. We now want to prove that $\text{codim}\Im(zId_{\ell^2}-\mathscr{L})<+\infty$. We consider $N\geq 1$ such that there exists $(h_1,\dots,h_N)$ a linearly independent family of $\ell^2(\mathbb{Z},\mathbb{C}^d)$ such that

$$\Im(zId_{\ell^2}-\mathscr{L})\cap Span(h_1,\dots,h_N)=\{0\}.$$

We are going to prove that the family $(\nu(h_1),\dots,\nu(h_N))$ is linearly independent in $\mathbb{C}^{d(p+q)}$ and therefore that $N\leq d(p+q)$. We consider $\lambda_1,\dots,\lambda_N\in\mathbb{C}$ such that

$$0=\sum_{i=1}^{N}\lambda_i\nu(h_i)=\nu\left(\sum_{i=1}^{N}\lambda_ih_i\right).$$

We therefore have that

$$\nu\left(\sum_{i=1}^{N}\lambda_i h_i\right)\in\Im\varphi,$$

thus

$$\sum_{i=1}^{N}\lambda_i h_i\in\Im(zId_{\ell^2}-\mathscr{L})\cap Span(h_1,\dots,h_N).$$

This implies that

$$\sum_{i=1}^{N}\lambda_i h_i=0$$

and the linear independency of $(h_1,\dots,h_N)$ allows us to conclude that $\lambda_1=\ldots=\lambda_N=0$. We have thus proved that $zId_{\ell^2}-\mathscr{L}$ is a Fredholm operator for all $z\in\mathcal{O}$. We also know that, since $\mathcal{O}$ is unbounded, there exists $z\in\mathcal{O}$ such that $zId_{\ell^2}-\mathscr{L}$ is an isomorphism. The set $\mathcal{O}$ being connected and by continuity of the Fredholm index, the statement of the lemma is true. □

We now introduce the sets

$$\mathcal{O}_\rho:=\mathcal{O}\cap\rho(\mathscr{L})\quad\text{ and }\quad\mathcal{O}_\sigma:=\mathcal{O}\cap\sigma(\mathscr{L}).$$

Lemma 3.7 implies that $\mathcal{O}_\sigma$ only contains eigenvalues of $\mathscr{L}$. Then, because of Hypothesis 6, we have that

$$\overline{\mathbb{U}}\setminus\{1\}\subset\mathcal{O}_\rho.$$

We also observe that Lemma 3.6 gives us the dimension of the subspaces $E_0^\pm(z)$. Then, (3.33) implies that, for $z\in\mathcal{O}_\rho$, we have

$$E_0^+(z)\oplus E_0^-(z)=\mathbb{C}^{(p+q)d}.$$

Thus, for $z\in\mathcal{O}_\rho$, we can define the projector $Q(z)$ such that

$$\Im Q(z)=E_0^+(z)\quad\text{ and }\quad\ker Q(z)=E_0^-(z).$$

The function $z\in\mathcal{O}_\rho\mapsto Q(z)$ is holomorphic. Indeed, for $z_0\in\mathcal{O}_\rho$, we have $r>0$ such that if we define $U:=B(z_0,r)$, then $\overline{U}\subset\mathcal{O}_\rho$. Lemma 3.6 then implies that

$$\forall z\in U,\quad E_0^+(z)=\Im Q_U^+(z)\quad\text{ and }\quad E_0^-(z)=\Im Q_U^-(z).$$

Knowing that $Q_U^\pm$ depend holomorphically on z, we have proved that E_0^+ and E_0^- are also depending holomorphically on z in a neighborhood of z_0. The holomorphy of Q ensues. We will now prove that the function Q is fundamental to the study of (3.3) by extending the geometric dichotomy. The following lemma is once again very much inspired by [LG01, Section III.1.5] and [Cop78].

Lemma 3.8 (Extended geometric dichotomy)**.** *For all bounded open set U such that $\overline{U}\subset\mathcal{O}_\rho$, there exist two positive constants $C,c>0$ such that for all $z\in U$*

$$\forall j\geq k,\quad\left|X_j(z)Q(z)X_k(z)^{-1}\right|\leq Ce^{-c|j-k|},\tag{3.35a}$$
$$\forall k\geq j,\quad\left|X_j(z)(Id-Q(z))X_k(z)^{-1}\right|\leq Ce^{-c|j-k|}.\tag{3.35b}$$

Proof We begin by assuming that we proved the existence of two constants $C,c>0$ such that for all $z\in U$

$$\forall j\geq k\geq 0,\quad\left|X_j(z)Q(z)X_k(z)^{-1}\right|\leq Ce^{-c|j-k|},\tag{3.36a}$$
$$\forall k\geq j\geq 0,\quad\left|X_j(z)(Id-Q(z))X_k(z)^{-1}\right|\leq Ce^{-c|j-k|},\tag{3.36b}$$
$$\forall k\leq j\leq 0,\quad\left|X_j(z)Q(z)X_k(z)^{-1}\right|\leq Ce^{-c|j-k|},\tag{3.36c}$$
$$\forall j\leq k\leq 0,\quad\left|X_j(z)(Id-Q(z))X_k(z)^{-1}\right|\leq Ce^{-c|j-k|}.\tag{3.36d}$$

Then, we observe that to prove the assertion (3.35a), there would only remain to look at the case where $j\geq 0\geq k$. Using (3.36a) and (3.36c), we have

$$\left|X_j(z)Q(z)X_k(z)^{-1}\right|\leq\left|X_j(z)Q(z)X_0(z)^{-1}\right|\left|X_0(z)Q(z)X_k(z)^{-1}\right|\leq C^2e^{-c(j-k)}.$$

We would have proved the existence of two constants $C, c > 0$ such that assertion (3.35a) would be true for all $z \in U$. We would use the same kind of proof to prove the existence of $C, c > 0$ such that assertion (3.35b) holds true for all $z \in U$.

Therefore, there only remains to prove the existence of $C, c > 0$ such that (3.36a)-(3.36d) are true for all $z \in U$. We will prove (3.36a) and (3.36c). The proof for (3.36b) and (3.36d) would be dealt with similarly. First, we need to consider a bounded open set V such that $\overline{V} \subset \mathcal{O}_\rho$ and $\overline{U} \subset V$. This will be useful later on to bound the difference $Q_V^+ - Q$. For $z \in U$, Lemma 3.6 implies that $E_0^+(z) = \Im Q_V^+(z) = \Im Q(z)$, i.e.

$$Q_V^+(z)Q(z) = Q(z) \quad \text{and} \quad Q(z)Q_V^+(z) = Q_V^+(z).$$

This allows us to prove that

$$Q_V^+(z) - Q(z) = Q_V^+(z)(Q_V^+(z) - Q(z))(Id - Q_V^+(z)).$$

Therefore, for $j, k \in \mathbb{N}$, we have

$$X_j(z)(Q_V^+(z) - Q(z))X_k(z)^{-1} = X_j(z)Q_V^+(z)X_0(z)^{-1}(Q_V^+(z) - Q(z))X_0(z)(Id - Q_V^+(z))X_k(z)^{-1}.$$

Thus, because of the inequalities (3.10a) and (3.10b)

$$\left|X_j(z)(Q_V^+(z) - Q(z))X_k(z)^{-1}\right| \leq C^2 e^{-c(j+k)}|Q_V^+(z) - Q(z)|. \tag{3.37}$$

Using the inequalities (3.10a), (3.10b) and (3.37), we can thus prove that

$$\forall j \geq k \geq 0, \quad |X_j(z)Q(z)X_k(z)^{-1}| \leq Ce^{-c(j-k)} + C^2 e^{-c(j+k)}|Q_V^+(z) - Q(z)|$$

and

$$\forall k \geq j \geq 0, \quad |X_j(z)(Id - Q(z))X_k(z)^{-1}| \leq Ce^{-c(k-j)} + C^2 e^{-c(j+k)}|Q_V^+(z) - Q(z)|.$$

Since $z \in V \mapsto |Q_V^+(z) - Q(z)|$ is continuous and $\overline{U} \subset V$, we can uniformly bound $|Q_V^+(z) - Q(z)|$ for $z \in U$. We can then deduce the existence of two positive constants C, c to verify the inequalities (3.36a) and (3.36c). □

3.5 Bounds on the spatial Green's function far from 1

For $z \in \mathcal{O}_\rho$ and $j_0 \in \mathbb{Z}$, since $z \in \rho(\mathscr{L})$, the spatial Green's function $G(z, j_0, \cdot)$ defined by (1.25) exists. We observe that the function $z \in \mathcal{O}_\rho \mapsto G(z, j_0, \cdot)$ is holomorphic.

We consider $\vec{e} \in \mathbb{C}^d$. We then observe that $G(z, j_0, \cdot)\vec{e} \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$ and

$$zG(z, j_0, \cdot)\vec{e} - \mathscr{L}G(z, j_0, \cdot)\vec{e} = \delta_{j_0}\vec{e},$$

i.e.

$$\forall j \in \mathbb{Z}, \quad \sum_{k=-p}^{q} \mathbb{A}_{j,k}(z)G(z, j_0, j+k)\vec{e} = \delta_{j_0,j}\vec{e}.$$

Thus, we have that

$$\forall j \in \mathbb{Z}, \quad W(z, j_0, j+1, \vec{e}) = M_j(z)W(z, j_0, j, \vec{e}) - \begin{pmatrix} A_{j,q}^{-1}\delta_{j_0,j}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.38}$$

where $W(z, j_0, j, \vec{e}) := \begin{pmatrix} G(z, j_0, j+q-1)\vec{e} \\ \vdots \\ G(z, j_0, j-p)\vec{e} \end{pmatrix}$. We will now prove the following proposition using the extended geometric dichotomy (Lemma 3.8).

Proposition 2 (Bounds far from 1)**.** *For U a bounded open set such that $\overline{U} \subset \mathcal{O}_\rho$, there exist two constants $C, c > 0$ such that*

$$\forall z \in U, \forall \vec{e} \in \mathbb{C}^d, \forall j, j_0 \in \mathbb{Z}, \quad |W(z, j_0, j, \vec{e})| \leq C|\vec{e}|e^{-c|j-j_0|}.$$

The result of Proposition 2 is in particular true in a neighborhood of any point $z \in \overline{\mathbb{U}} \setminus \{1\}$.

Proof We consider $z \in U$, $j, j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$ such that $|\vec{e}| \leq 1$. The equality (3.38) implies the following results:

- We have
$$\forall j \geq j_0 + 1, \quad W(z, j_0, j+1, \vec{e}) = M_j(z) W(z, j_0, j, \vec{e}),$$
i.e.
$$\forall j \geq j_0 + 1, \quad W(z, j_0, j, \vec{e}) = X_j(z) X_{j_0+1}(z)^{-1} W(z, j_0, j_0+1, \vec{e}). \tag{3.39}$$
Also, since $G(z, j_0, \cdot)\vec{e} \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $X_{j_0+1}(z)^{-1} W(z, j_0, j_0+1, \vec{e}) \in E_0^+(z)$.

- We have
$$\forall j \leq j_0 - 1, \quad W(z, j_0, j+1, \vec{e}) = M_j(z) W(z, j_0, j, \vec{e}),$$
i.e.
$$\forall j \leq j_0, \quad W(z, j_0, j, \vec{e}) = X_j(z) X_{j_0}(z)^{-1} W(z, j_0, j_0, \vec{e}). \tag{3.40}$$
Also, since $G(z, j_0, \cdot)\vec{e} \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $X_{j_0}(z)^{-1} W(z, j_0, j_0, \vec{e}) \in E_0^-(z)$.

- We have
$$W(z, j_0, j_0+1, \vec{e}) = M_{j_0}(z) W(z, j_0, j_0, \vec{e}) - \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
i.e.
$$X_{j_0+1}(z)^{-1} W(z, j_0, j_0+1, \vec{e}) - X_{j_0}(z)^{-1} W(z, j_0, j_0, \vec{e}) = -X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $Q(z)$ is the projection on $E_0^+(z)$ with respect to $E_0^-(z)$, we have that

$$X_{j_0+1}(z)^{-1} W(z, j_0, j_0+1, \vec{e}) = -Q(z) X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$X_{j_0}(z)^{-1} W(z, j_0, j_0, \vec{e}) = (Id - Q(z)) X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using (3.39) and (3.40), we have

$$\forall j \geq j_0 + 1, \quad W(z, j_0, j, \vec{e}) = -X_j(z) Q(z) X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.41a}$$

$$\forall j \leq j_0, \quad W(z, j_0, j, \vec{e}) = X_j(z)(Id - Q(z)) X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.41b}$$

Thus, the inequalities (3.35a) and (3.35b) imply

$$\forall z \in U, \forall j, j_0 \in \mathbb{Z}, \quad |W(z, j_0, j, \vec{e})| \leq C e^{-c|j-(j_0+1)|} \left| \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|.$$

We can then easily conclude Proposition 2. □

4 Extension of the spatial Green's function near 1

The analysis of the spatial Green's function done in the previous section does not hold near 1. The first reason is that we can prove that 1 is an eigenvalue of $\mathscr{L}$ and the curves describing the spectrum of the limit operators $\mathscr{L}^{\pm}$ in (1.16) should belong to the essential spectrum of the operator $\mathscr{L}$. Thus, the definition of the spatial Green's function breaks down near 1. The second reason is that the matrices $M^{\pm}(1)$ have central eigenvalues equal to 1 as explained in Lemma 3.2, thus the geometric dichotomy will not work near 1. The solution will be to refine the analysis of (3.3) near 1 by finding a particular basis of $E_0^{\pm}(z)$ and using this basis to express the spatial Green's function. In some sense, it amounts at using the projections on a basis of solutions of (3.3) rather than the projection associated with the geometric dichotomy.

4.1 Right and left eigenvectors of $M^{\pm}(z)$ for z near 1

To study the spatial Green's function for z near 1, we will need to study the solutions of the dynamical system (3.3) with more precision. The first step is to have a better understanding of the eigenvalues and eigenvectors of $M^{\pm}(z)$ when z is close to 1.

First, let us make some observations on the eigenvalues of $M_l^{\pm}(1)$ defined by (3.8) for $l \in \{1, \dots, d\}$. Using Lemma 3.2, we know that the eigenvalues $\kappa \in \mathbb{C} \setminus \{0\}$ of $M_l^{\pm}(1)$ are the solutions of

$$\mathcal{F}_l^{\pm}(\kappa) = 1.$$

Hypothesis 9 allows us to conclude that the matrix $M_l^{\pm}(1)$ only has simple eigenvalues. Furthermore, Lemma 3.2 implies that 1 is a simple eigenvalue of $M_l^{\pm}(1)$ and that the rest of the eigenvalues are in $\mathbb{D} \setminus \{0\}$ or $\mathbb{U}$ and we know the number of eigenvalues in each set depending on the sign of $\alpha_l^{\pm}$. Thus, we can define a family $(\underline{\zeta}_m^{\pm})_{m \in \{1, \dots, d(p+q)\}} \in \mathbb{C}^{d(p+q)}$ such that

$$\forall l \in \{1, \dots, d\}, \quad \sigma(M_l^{\pm}(1)) = \left\{ \underline{\zeta}_l^{\pm}, \underline{\zeta}_{l+d}^{\pm}, \dots, \underline{\zeta}_{l+(p+q-1)d}^{\pm} \right\}.$$

Furthermore, using Hypothesis 1 to determine the sign of $\alpha_l^{\pm}$ defined by (1.17) and Lemma 3.2, we can index them in order to have the following fact.

- For all $l \in \{1, \dots, I\}$, since $\alpha_l^{+} < 0$, we choose
$$\underline{\zeta}_l^{+}, \dots, \underline{\zeta}_{l+d(p-1)}^{+} \in \mathbb{D}, \quad \underline{\zeta}_{l+dp}^{+} = 1, \quad \underline{\zeta}_{l+d(p+1)}^{+}, \dots, \underline{\zeta}_{l+d(p+q-1)}^{+} \in \mathbb{U}.$$
- For all $l \in \{I+1, \dots, d\}$, since $\alpha_l^{+} > 0$, we choose
$$\underline{\zeta}_l^{+}, \dots, \underline{\zeta}_{l+d(p-2)}^{+} \in \mathbb{D}, \quad \underline{\zeta}_{l+d(p-1)}^{+} = 1, \quad \underline{\zeta}_{l+dp}^{+}, \dots, \underline{\zeta}_{l+d(p+q-1)}^{+} \in \mathbb{U}.$$
- For all $l \in \{1, \dots, I-1\}$, since $\alpha_l^{-} < 0$, we choose
$$\underline{\zeta}_l^{-}, \dots, \underline{\zeta}_{l+d(p-1)}^{-} \in \mathbb{D}, \quad \underline{\zeta}_{l+dp}^{-} = 1, \quad \underline{\zeta}_{l+d(p+1)}^{-}, \dots, \underline{\zeta}_{l+d(p+q-1)}^{-} \in \mathbb{U}.$$
- For all $l \in \{I, \dots, d\}$, since $\alpha_l^{-} > 0$, we choose
$$\underline{\zeta}_l^{-}, \dots, \underline{\zeta}_{l+d(p-2)}^{-} \in \mathbb{D}, \quad \underline{\zeta}_{l+d(p-1)}^{-} = 1, \quad \underline{\zeta}_{l+dp}^{-}, \dots, \underline{\zeta}_{l+d(p+q-1)}^{-} \in \mathbb{U}.$$

We indexed the eigenvalues to separate the stable, central and unstable eigenvalues of the matrices $M_l^{\pm}(1)$. More precisely, we observe that if we introduce the sets

$$\begin{aligned} I_{ss}^{+} &:= \{1, \dots, d(p-1)+I\}, & I_{ss}^{-} &:= \{1, \dots, d(p-1)+I-1\}, \\ I_{cs}^{+} &:= \{d(p-1)+I+1, \dots, dp\}, & I_{cs}^{-} &:= \{d(p-1)+I, \dots, dp\}, \\ I_{cu}^{+} &:= \{dp+1, \dots, dp+I\}, & I_{cu}^{-} &:= \{dp+1, \dots, dp+I-1\}, \\ I_{su}^{+} &:= \{dp+I+1, \dots, d(p+q)\}, & I_{su}^{-} &:= \{dp+I, \dots, d(p+q)\}, \end{aligned}$$

then we have that

$$\begin{aligned} \forall m \in I_{ss}^{\pm}, &\quad \underline{\zeta}_m^{\pm} \in \mathbb{D}, \\ \forall m \in I_{cs}^{\pm} \cup I_{cu}^{\pm}, &\quad \underline{\zeta}_m^{\pm} = 1, \\ \forall m \in I_{su}^{\pm}, &\quad \underline{\zeta}_m^{\pm} \in \mathbb{U}. \end{aligned}$$

Since those are simple eigenvalues of $M_l^{\pm}(1)$, we are able to extend them holomorphically in a neighborhood of 1. We consider $\delta_0 > 0$ a radius such that for each $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$, there exists a holomorphic function $\zeta_m^{\pm} : B(1, \delta_0) \to \mathbb{C}$ such that $\zeta_m^{\pm}(1) = \underline{\zeta}_m^{\pm}$ and for all $z \in B(1, \delta_0)$, $\zeta_m^{\pm}(z)$ is a simple eigenvalue of $M_l^{\pm}(z)$. We will also separate the different types of eigenvalues by assuming that we chose δ_0 small enough so that there exists a constant $c_* > 0$ such that for all $z \in B(1, \delta_0)$

$$\forall m \in I_{ss}^{\pm}, \quad |\zeta_m^{\pm}(z)| \leq \exp(-2c_*), \tag{4.1}$$

$$\forall m \in I_{cs}^{\pm} \cup I_{cu}^{\pm}, \quad \exp(-c_*) \leq |\zeta_m^{\pm}(z)| \leq \exp(c_*) \tag{4.2}$$

$$\forall m \in I_{su}^{\pm}, \quad \exp(2c_*) \leq |\zeta_m^{\pm}(z)|. \tag{4.3}$$

When we will study the temporal Green's function $\mathscr{G}(n, j_0, j)$ later on in Section 5, we will have to bound terms of the form

$$|\zeta_m^{\pm}(z)|^j |\zeta_{m'}^{\pm}(z)|^{-j_0}.$$

The inequalities (4.1)-(4.3) will allow us in a lot of cases to obtain directly exponential bounds for some of those terms.

Using Lemma 3.1, we thus have a complete description of the eigenvalues of $M^{\pm}(z)$ for z in a neighborhood of 1. The following lemma also allows us to introduce a basis of eigenvectors for the matrices $M^{\pm}(z)$.

Lemma 4.1. *For $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$, the vector*

$$R_m^{\pm}(z) := \begin{pmatrix} \zeta_m^{\pm}(z)^{q-1} \boldsymbol{r}_l^{\pm} \\ \vdots \\ \zeta_m^{\pm}(z)^{-p} \boldsymbol{r}_l^{\pm} \end{pmatrix} \in \mathbb{C}^{d(p+q)} \tag{4.4}$$

is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$. Furthermore, for all $z \in B(1, \delta_0)$, the family $(R_m^{\pm}(z))_{m \in \{1, \dots, d(p+q)\}}$ is a basis of $\mathbb{C}^{d(p+q)}$.

Proof Let us start by proving that the vector $R_m^{\pm}(z)$ defined by (4.4) is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$. We have that $\zeta_m^{\pm}(z)$ is an eigenvalue of $M_l^{\pm}(z)$ so Lemma 3.2 implies that

$$\mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z)) = z.$$

We use the definition (3.7) of the functions $\Lambda_{l,k}^{\pm}$ and the definition (1.15) of the function $\mathcal{F}_l^{\pm}$ to prove that

$$\begin{aligned} -\sum_{k=-p}^{q-1} \mathbb{A}_q^{\pm}(z)^{-1} \mathbb{A}_k^{\pm}(z) \zeta_m^{\pm}(z)^k \boldsymbol{r}_l^{\pm} &= -\sum_{k=-p}^{q-1} \Lambda_{l,q}^{\pm}(z)^{-1} \Lambda_{l,k}^{\pm}(z) \zeta_m^{\pm}(z)^k \boldsymbol{r}_l^{\pm} \\ &= \left(\zeta_m^{\pm}(z)^q + \Lambda_{l,q}^{\pm}(z)^{-1} \left(z - \mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z)) \right) \right) \boldsymbol{r}_l^{\pm} \\ &= \zeta_m^{\pm}(z)^q \boldsymbol{r}_l^{\pm}. \end{aligned}$$

This allows us to conclude that the vector $R_m^{\pm}(z)$ is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$.

We now consider $z \in B(1, \delta_0)$ and $(\lambda_m)_{m \in \{1, \dots, d(p+q)\}}$ a family of complex numbers such that

$$0 = \sum_{m=1}^{d(p+q)} \lambda_m R_m^{\pm}(z).$$

Separating the blocks of coefficients of size d in the previous equality and observing that the family $(\boldsymbol{r}_l^{\pm})_{l \in \{1, \dots, d\}}$ is linearly independent, we have for all $l \in \{1, \dots, d\}$,

$$\forall j \in \{-p, \dots, q-1\}, \quad 0 = \sum_{k=1}^{p+q} \lambda_{l+(k-1)d} \zeta_{l+(k-1)d}^{\pm}(z)^j.$$

We have that, for each integer $k \in \{1, \dots, p+q\}$, $\zeta_{l+(k-1)d}^{\pm}(z)$ is a simple eigenvalue of $M_l^{\pm}(z)$ for all $z \in B(1, \delta_0)$. Therefore, the complex values $(\zeta_{l+(k-1)d}^{\pm}(z))_{k \in \{1, \dots, p+q\}}$ are distinct and thus

$$\forall k \in \{1, \dots, p+q\}, \quad \lambda_{l+d(k-1)} = 0.$$

Since this is true for all $l \in \{1, \dots, d\}$, we proved that the family $(R_m^{\pm}(z))_{m \in \{1,\dots,d(p+q)\}}$ is linearly independent and is thus a basis of $\mathbb{C}^{d(p+q)}$. □

Thus, we have a characterization of the eigenvalues and eigenvectors of $M^{\pm}(z)$ for $z \in B(1, \delta_0)$. Lemma 3.2 implies that, for all $z \in \mathcal{O} \cap B(1, \delta_0)$, we have that $|\zeta_m^{\pm}(z)| < 1$ for $m \in \{1, \dots, dp\}$ and $|\zeta_m^{\pm}(z)| > 1$ for $m \in \{dp+1, \dots, d(p+q)\}$. Thus, for $z \in \mathcal{O} \cap B(1, \delta_0)$

$$E^s(M^{\pm}(z)) = \text{Span}\left\{R_m^{\pm}(z), \quad m \in \{1, \dots, dp\}\right\}$$

and

$$E^u(M^{\pm}(z)) = \text{Span}\left\{R_m^{\pm}(z), \quad m \in \{dp+1, \dots, d(p+q)\}\right\}.$$

This equality implies that we can extend holomorphically the definitions of $E^s(M^{\pm}(z))$ and $E^u(M^{\pm}(z))$ for $z \in B(1, \delta_0)$.

We now conclude this section by studying the dual basis associated with the basis $(R_m^{\pm}(z))_{m \in \{1,\dots,d(p+q)\}}$. We introduce the invertible matrix

$$\forall z \in B(1, \delta_0), \quad N^{\pm,\infty}(z) := \left(R_1^{\pm}(z) \quad |\dots| \quad R_{d(p+q)}^{\pm}(z)\right) \in \mathcal{M}_{d(p+q)}(\mathbb{C}) \tag{4.5}$$

and the vectors $L_1^{\pm}(z), \dots, L_{d(p+q)}^{\pm}(z) \in \mathbb{C}^{d(p+q)}$ defined by

$$\forall z \in B(1, \delta_0), \quad \left(L_1^{\pm}(z) \quad |\dots| \quad L_{d(p+q)}^{\pm}(z)\right)^T := N^{\pm,\infty}(z)^{-1}. \tag{4.6}$$

We observe that

$$\forall z \in B(1, \delta_0), \forall m, \widetilde{m} \in \{1, \dots, d(p+q)\}, \quad L_m^{\pm}(z)^T R_{\widetilde{m}}^{\pm}(z) = \delta_{m,\widetilde{m}} \tag{4.7}$$

and

$$\forall z \in B(1, \delta_0), \forall m \in \{1, \dots, d(p+q)\}, \quad M^{\pm}(z)^T L_m^{\pm}(z) = \zeta_m^{\pm}(z) L_m^{\pm}(z). \tag{4.8}$$

We will now prove the following lemma which gives a more precise description of the vectors $L_m^{\pm}(z)$ of the dual basis.

Lemma 4.2. *We consider $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$. For all $z \in \mathbb{C}$, there exist coefficients $x_1^{\pm}(z), \dots, x_{p+q}^{\pm}(z) \in \mathbb{C}$ such that*

$$L_m^{\pm}(z) := \begin{pmatrix} x_1^{\pm}(z) \boldsymbol{l}_l^{\pm} \\ \vdots \\ x_{p+q}^{\pm}(z) \boldsymbol{l}_l^{\pm} \end{pmatrix}. \tag{4.9}$$

Furthermore, we have

$$\forall z \in B(1, \delta_0), \quad x_1^{\pm}(z) = \lambda_{l,q}^{\pm} \zeta_m^{\pm\prime}(z). \tag{4.10}$$

In the proof of Lemma 4.2, we also find the expression of the coefficients $x_2^{\pm}(z), \dots, x_{p+q}^{\pm}(z)$ but, contrarily with $x_1^{\pm}(z)$, they will not be used later on in the paper.

Proof The proof Lemma 4.2 uses calculations similar to those done at the end of [Coe22, Lemma 2.4]. We fix $z \in B(1, \delta_0)$. We begin by introducing the vectors $\boldsymbol{x}_1^{\pm}(z), \dots, \boldsymbol{x}_{p+q}^{\pm}(z) \in \mathbb{C}^d$ defined by

$$\begin{pmatrix} \boldsymbol{x}_1^{\pm}(z) \\ \vdots \\ \boldsymbol{x}_{p+q}^{\pm}(z) \end{pmatrix} := L_m^{\pm}(z).$$

We consider $\widetilde{l} \in \{1, \dots, d\} \setminus \{l\}$. Using the definition (4.4) and the linear independence of the vectors $R_{\widetilde{m}}^{\pm}(z)$, we have that

$$\text{Span}\left\{R_{\widetilde{l}+(\widetilde{k}-1)d}^{\pm}(z), \quad \widetilde{k} \in \{1, \dots, p+q\}\right\} = \left\{\begin{pmatrix} y_1 \boldsymbol{r}_{\widetilde{l}}^{\pm} \\ \vdots \\ y_{p+q} \boldsymbol{r}_{\widetilde{l}}^{\pm} \end{pmatrix}, \quad y_1, \dots, y_{p+q} \in \mathbb{C}\right\}.$$

Using (4.7), we can thus prove that

$$\forall j \in \{1, \dots, p+q\}, \quad \boldsymbol{x}_j^{\pm}(z)^T \boldsymbol{r}_{\widetilde{l}}^{\pm} = 0$$

Since this is true for all $\widetilde{l} \in \{1, \dots, d\} \setminus \{l\}$, we have that $\boldsymbol{x}_j^{\pm}(z) \in \text{Span } \boldsymbol{l}_l^{\pm}$ for all $j \in \{1, \dots, p+q\}$.

Now that we know that we can express the vector $L_m^{\pm}(z)$ as (4.9), let us prove (4.10). Using (1.14) and the definitions (1.5) and (3.7) respectively of the vectors $\boldsymbol{l}_l^{\pm}$ and of the functions $\Lambda_{l,k}^{\pm}$, we have looking at the j-th block of size d of (4.8) that

$$\forall j \in \{1, \dots, p+q-1\}, \quad \zeta_m^{\pm}(z) x_j^{\pm}(z) = x_{j+1}^{\pm}(z) - \Lambda_{l,q}^{\pm}(z)^{-1} \Lambda_{l,q-j}^{\pm}(z) x_1^{\pm}(z)$$

and

$$\zeta_m^{\pm}(z) x_{p+q}^{\pm}(z) = -\Lambda_{l,q}^{\pm}(z)^{-1} \Lambda_{l,-p}^{\pm}(z) x_1^{\pm}(z).$$

Thus, we have

$$\forall j \in \{1, \dots, p+q\}, \quad x_j^{\pm}(z) = -\left(\sum_{k=-p}^{q-j} \frac{\Lambda_{l,k}^{\pm}(z)}{\zeta_m^{\pm}(z)^{q-j-k+1}} \right) \Lambda_{l,q}^{\pm}(z)^{-1} x_1^{\pm}(z).$$

We now have an expression of each $x_j^{\pm}(z)$ depending on $x_1^{\pm}(z)$. We also recall that

$$\boldsymbol{l}_l^{\pm T} \boldsymbol{r}_l^{\pm} = 1.$$

Using the expressions (4.4) and (4.9) respectively of the vectors $R_m^{\pm}(z)$ and $L_m^{\pm}(z)$ as well as (4.7), we have

$$1 = L_m^{\pm}(z)^T R_m^{\pm}(z) = \sum_{j=1}^{p+q} x_j^{\pm}(z) \zeta_m^{\pm}(z)^{q-j} = -\left(\sum_{j=1}^{p+q} \sum_{k=-p}^{q-j} \Lambda_{l,k}^{\pm}(z) \zeta_m^{\pm}(z)^{k-1} \right) \Lambda_{l,q}^{\pm}(z)^{-1} x_1^{\pm}(z).$$

Using the definitions (3.7) and (1.15) of the functions $\Lambda_{l,k}^{\pm}$ and $\mathcal{F}_l^{\pm}$, we have

$$\begin{aligned}
1 &= -\left(\sum_{k=-p}^{q} (q-k) \Lambda_{l,k}^{\pm}(z) \zeta_m^{\pm}(z)^{k-1} \right) \Lambda_{l,q}^{\pm}(z)^{-1} x_1^{\pm}(z) \\
&= -\left(q \zeta_m^{\pm}(z)^{-1} z - \sum_{k=-p}^{q} (q-k) \lambda_{l,k}^{\pm} \zeta_m^{\pm}(z)^{k-1} \right) \Lambda_{l,q}^{\pm}(z)^{-1} x_1^{\pm}(z) \\
&= -\left(q \zeta_m^{\pm}(z)^{-1} (z - \mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z))) + \mathcal{F}_l^{\pm\prime}(\zeta_m^{\pm}(z)) \right) \Lambda_{l,q}^{\pm}(z)^{-1} x_1^{\pm}(z).
\end{aligned}$$

We observe that since $\zeta_m^{\pm}(z)$ is an eigenvalue $M_l^{\pm}(z)$, Lemma 3.2 allows us to prove that

$$\mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z)) = z \quad \text{and} \quad \zeta_m^{\pm\prime}(z) \mathcal{F}_l^{\pm\prime}(\zeta_m^{\pm}(z)) = 1.$$

Thus, since $\Lambda_{l,q}^{\pm}(z) = -\lambda_{l,q}^{\pm}$, we have that

$$1 = \zeta_m^{\pm\prime}(z)^{-1} {\lambda_{l,q}^{\pm}}^{-1} x_1^{\pm}(z)$$

and we deduce (4.10). □

4.2 Choice of a precise basis of $E_0^{\pm}(z)$ for z near 1

Now that we have a better understanding of the spectrum of $M^{\pm}(z)$, we are going to prove a lemma that is quite similar to the geometric dichotomy. This lemma is a direct reference to [God03, Lemma 3.1], itself inspired by [ZH98, Proposition 3.1].

Lemma 4.3. *There exist a radius $\delta_1 \in]0, \delta_0]$ and two constants $C, c > 0$ such that for all $m \in \{1, \dots, d(p+q)\}$ and $z \in B(1, \delta_1)$, there exists a sequence $(V_m^{\pm}(z, j))_{j \in \mathbb{Z}} \in \mathbb{C}^{(p+q)d^{\mathbb{Z}}}$ such that :*

- *The functions* $z \in B(1,\delta_1) \mapsto (V_m^+(z,j))_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{C}^{(p+q)d})$ *and* $z \in B(1,\delta_1) \mapsto (V_m^-(z,j))_{j\in-\mathbb{N}} \in \ell^\infty(-\mathbb{N},\mathbb{C}^{(p+q)d})$ *are holomorphic. Furthermore, up to considering a smaller radius* δ_1*, those functions and their derivatives are bounded on* $B(1,\delta_1)$*.*
- *For* $z \in B(1,\delta_1)$*, if we define* $W_m^\pm(z,j) := \zeta_m^\pm(z)^j V_m^\pm(z,j)$ *for all* $j \in \mathbb{Z}$*, then* $W_m^\pm(z,\cdot)$ *is a solution of* (3.3)*, i.e.*
$$\forall j \in \mathbb{Z}, \quad W_m^\pm(z,j+1) = M_j(z) W_m^\pm(z,j).$$
- *We have*
$$\forall z \in B(1,\delta_1), \quad \begin{array}{ll} \forall j \in \mathbb{N}, & |V_m^+(z,j) - R_m^+(z)| \leq Ce^{-c|j|}, \\ \forall j \in -\mathbb{N}, & |V_m^-(z,j) - R_m^-(z)| \leq Ce^{-c|j|}. \end{array}$$

The proof of this lemma is fairly similar to the construction of $Q_U^\pm(z)$ in Lemma 3.3.

Proof We will focus on the construction of $(V_m^+(z,j))_{j\in\mathbb{Z}}$ for an integer $m \in \{1,\dots,d(p+q)\}$. Because of (3.2), we have a constant $C>0$ such that

$$\forall j \in \mathbb{N}, \forall z \in B(1,\delta_0), \quad |\mathcal{E}_j^+(z)| \leq Ce^{-\alpha j}.$$

We fix $\Delta := \frac{\alpha}{4}$ and define the sets

$$\begin{array}{l} I_m^s = \left\{\nu \in \{1,\dots,d(p+q)\}, \quad |\zeta_\nu^+(1)| < |\zeta_m^+(1)|e^{-\Delta}\right\} \\ I_m^u = \left\{\nu \in \{1,\dots,d(p+q)\}, \quad |\zeta_\nu^+(1)| \geq |\zeta_m^+(1)|e^{-\Delta}\right\}. \end{array}$$

Because the functions ζ_ν^+ depend holomorphicaly on z in $B(1,\delta_0)$, there exists $\delta_1 \in]0,\delta_0[$ such that

$$\forall z \in B(1,\delta_1), \quad \left\{ \begin{array}{ll} \forall \nu \in I_m^s, & |\zeta_\nu^+(z)| < |\zeta_m^+(z)|e^{-\Delta} \\ \forall \nu \in I_m^u, & |\zeta_\nu^+(z)| > |\zeta_m^+(z)|e^{-\frac{3}{2}\Delta}. \end{array} \right. \tag{4.11}$$

We define for $z \in B(1,\delta_1)$

$$\begin{array}{l} E_m^s(z) := Span\left(R_\nu^+(z), \quad \nu \in I_m^s\right) \\ E_m^u(z) := Span\left(R_\nu^+(z), \quad \nu \in I_m^u\right). \end{array}$$

We have that

$$\mathbb{C}^{(p+q)d} = E_m^s(z) \oplus E_m^u(z).$$

We define $P_m^s(z)$ and $P_m^u(z)$ the projectors defined by this decomposition of $\mathbb{C}^{(p+q)d}$. They depend holomorphically on z and commute with $M^+(z)$. Because of (4.11), there exists a constant $C>0$ such that

$$\forall z \in B(1,\delta_1), \forall j \in \mathbb{N}, \quad \begin{array}{l} \left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^j P_m^s(z)\right| \leq C\exp\left(-\frac{\Delta}{2}j\right) \\ \left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{-j} P_m^u(z)\right| \leq C\exp\left(2\Delta j\right) \end{array} \tag{4.12}$$

We fix $J \in \mathbb{N}$ and we will make a more precise choice later. For $z \in B(1,\delta_1)$, we define $\varphi(z) \in \mathcal{L}\left(\ell^\infty\left(\{j\in\mathbb{Z}, j\geq J\}, \mathbb{C}^{(p+q)d}\right)\right)$ such that for $Y \in \ell^\infty\left(\{j\in\mathbb{Z}, j\geq J\}, \mathbb{C}^{(p+q)d}\right)$ and $j \geq J$, we have

$$\begin{array}{rl} (\varphi(z)Y)_j := & \displaystyle\sum_{k=J}^{j-1} \left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k \\ & \displaystyle - \sum_{k=j}^{+\infty} \left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k. \end{array} \tag{4.13}$$

Using the inequalities (4.12), we have that

$$\begin{array}{rl} \displaystyle\sum_{k=J}^{j-1} \left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k\right| & \lesssim \|Y\|_\infty \displaystyle\sum_{k=J}^{j-1} e^{-\frac{\Delta}{2}(j-k)}e^{-\alpha k} \\ & \lesssim \|Y\|_\infty e^{-\frac{\Delta}{2}j} \displaystyle\sum_{k=J}^{+\infty} e^{\left(\frac{\Delta}{2}-\alpha\right)k} \\ & \lesssim \|Y\|_\infty e^{-\frac{\Delta}{2}(j-J)}e^{-\alpha J} \end{array} \tag{4.14}$$

and

$$\sum_{k=j}^{+\infty}\left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k}P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k\right| \lesssim \|Y\|_\infty e^{-\alpha j}\sum_{k=j}^{+\infty}e^{(2\Delta-\alpha)(k-j)} \lesssim \|Y\|_\infty e^{-\alpha j}. \tag{4.15}$$

We have thus proved that $\varphi(z)$ is in $\mathcal{L}\left(\ell^\infty\left(\{j\in\mathbb{Z}, j\geq J\},\mathbb{C}^{(p+q)d}\right)\right)$ and there exists a constant $C>0$ independent from J such that

$$\forall z\in B(1,\delta_1),\quad \|\varphi(z)\|_{\mathcal{L}(\ell^\infty)}\leq Ce^{-\alpha J}.$$

Furthermore, φ depends holomorphically on z. We can choose J large enough so that there exists a constant $C\in]0,1[$ such that

$$\forall z\in B(1,\delta_1),\quad \|\varphi(z)\|_{\mathcal{L}(\ell^\infty)}\leq C<1.$$

We can thus define for $z\in B(1,\delta_1)$

$$V(z):=(Id-\varphi(z))^{-1}\left(R_m^+(z)\right)_{j\geq J}\in\ell^\infty\left(\{j\in\mathbb{Z}, j\geq J\},\mathbb{C}^{(p+q)d}\right)$$

which depends holomorphically on z. We have that

$$V(z)=\left(R_m^+(z)\right)_{j\geq J}+\varphi(z)V(z).$$

Thus, for $j\geq J$, we find that

$$\begin{aligned}
(V(z))_{j+1} &= R_m^+(z)+\sum_{k=J}^{j}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-k}P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)\left(V(z)\right)_k\\
&\quad-\sum_{k=j+1}^{+\infty}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-k}P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)\left(V(z)\right)_k\\
&=\zeta_m^+(z)^{-1}M^+(z)\left(R_m^+(z)+\sum_{k=J}^{j}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k}P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)\left(V(z)\right)_k\right.\\
&\quad\left.-\sum_{k=j+1}^{+\infty}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k}P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)\left(V(z)\right)_k\right)\\
&=\zeta_m^+(z)^{-1}M^+(z)\left(\left(V(z)\right)_j+\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{-1}\zeta_m^+(z)^{-1}\mathcal{E}_j^+(z)\left(V(z)\right)_j\right)\\
&=\zeta_m^+(z)^{-1}M_j(z)\left(V(z)\right)_j.
\end{aligned}$$

Thus, for $j\geq J$, we have

$$(V(z))_j=\zeta_m^+(z)^{J-j}X_j(z)X_J(z)^{-1}(V(z))_J.$$

We define for $z\in B(1,\delta_1)$ and $j\in\mathbb{Z}$

$$V_m^+(z,j):=\zeta_m^+(z)^{J-j}X_j(z)X_J(z)^{-1}(V(z))_J$$

and

$$W_m^+(z,j):=\zeta_m^+(z)^jV_m^+(z,j).$$

The two first points are easily proved from the previous observations. There remains to prove the inequalities in the third point. For $z\in B(1,\delta_1)$ and $j\geq J$, we have using (4.13)-(4.15)

$$\left|V_m^+(z,j)-R_m^+(z)\right|=\left|(\varphi(z)V(z))_j\right|\lesssim e^{-\frac{\Delta}{2}j}+e^{-\alpha j}.$$

□

We recall that for $z\in\mathcal{O}_\rho\cap B(1,\delta_1)$, we have for $m\in\{1,\dots,dp\}$ that

$$|\zeta_m^+(z)|<1.$$

Therefore, $(W_m^+(z,0))_{m\in\{1,\dots,dp\}}$ is a family of elements of $E_0^+(z)=\Im Q(z)$ for $z\in\mathcal{O}_\rho\cap B(1,\delta_1)$. In the same way, we prove that for all $z\in\mathcal{O}_\rho\cap B(1,\delta_1)$, $(W_m^-(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$ is a family of elements of $E_0^-(z)=\ker Q(z)$. We are going to prove the following lemma.

Lemma 4.4. *For all $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$, $(W_m^+(z,j))_{m\in\{1\dots,d(p+q)\}}$ and $(W_m^-(z,j))_{m\in\{1\dots,d(p+q)\}}$ are bases of $\mathbb{C}^{d(p+q)}$. The same is then also true for the families $(V_m^+(z,j))_{m\in\{1\dots,d(p+q)\}}$ and $(V_m^-(z,j))_{m\in\{1\dots,d(p+q)\}}$.*

Proof We will write the proof for the family of vectors $(W_m^+(z,j))_{m\in\{1\dots,d(p+q)\}}$. We consider $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$ such that the family of vectors $(W_m^+(z,j))_{m\in\{1,\dots,d(p+q)\}}$ is not linearly independent. We can then introduce a family $(c_m)_{m\in\{1,\dots,d(p+q)\}} \in \mathbb{C}^{d(p+q)}\setminus\{0\}$ such that

$$0 = \sum_{m=1}^{d(p+q)} c_m W_m^+(z,j). \tag{4.16}$$

Since the sequences $(W_m^+(z,j))_{j\in\mathbb{Z}}$ are solutions of (3.3), (4.16) is verified for all $j \in \mathbb{Z}$. We define

$$\begin{aligned} I_n &:= \{m \in \{1,\dots,d(p+q)\}, \quad c_m \neq 0\} \neq \emptyset, \\ R &:= \max_{m\in I_n} |\zeta_m^+(z)| > 0, \\ I_R &:= \operatorname{argmax}_{m\in I_n} |\zeta_m^+(z)| \neq \emptyset. \end{aligned}$$

Using (4.16), we obtain

$$\begin{aligned} 0 &= \sum_{m\in I_n} c_m \frac{W_m^+(z,j)}{R^j} \\ &= \sum_{m\in I_n} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) + \sum_{m\in I_n} c_m \frac{W_m^+(z,j) - \zeta_m^+(z)^j R_m^+(z)}{R^j}. \end{aligned}$$

- Using Lemma 4.3, there exist two positive constants C, c such that we have for $m \in I_n$ and $j \in \mathbb{N}$

$$\left|\frac{W_m^+(z,j) - \zeta_m^+(z)^j R_m^+(z)}{R^j}\right| \leq Ce^{-cj}\left(\frac{|\zeta_m^+(z)|}{R}\right)^j \leq Ce^{-cj}.$$

  Thus,

$$\frac{W_m^+(z,j) - \zeta_m^+(z)^j R_m^+(z)}{R^j} \underset{j\to+\infty}{\to} 0.$$

- For $m \in I_n\setminus I_R$, we have

$$\left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) \underset{j\to+\infty}{\to} 0.$$

Thus, we have that

$$\sum_{m\in I_R} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) \underset{j\to+\infty}{\to} 0.$$

Since $I_R \neq \emptyset$, we fix $m_0 \in I_R$. Because of Lemma 4.1, the projection of the previous expression on $\operatorname{Span}(R_{m_0}^+(z))$ along $\operatorname{Span}(R_m^+(z), m \neq m_0)$ implies that

$$c_{m_0}\left(\frac{\zeta_{m_0}^+(z)}{R}\right)^j \underset{j\to+\infty}{\to} 0.$$

But, m_0 belongs to I_R so $|\zeta_{m_0}^+(z)| = R$. This implies that $c_{m_0} = 0$. However, $m_0 \in I_R \subset I_n$ so $c_{m_0} \neq 0$. This is a contradiction. □

For $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, we recall that

$$\dim E_0^+(z) = dp \quad \text{and} \quad \dim E_0^-(z) = dq.$$

Thus, Lemma 4.4 implies that, $(W_m^+(z,0))_{m\in\{1,\dots,dp\}}$ (resp. $(W_m^-(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$) is a basis of $E_0^+(z)$ (resp. $E_0^-(z)$). We can then extend holomorphically the subspaces $E_0^+(z)$ and $E_0^-(z)$ on the whole ball $B(1,\delta_1)$ as

$$\forall z \in B(1,\delta_1), \quad E_0^+(z) := \operatorname{Span}\left(W_m^+(z,0)\right)_{m\in\{1,\dots,dp\}} \quad \text{and} \quad E_0^-(z) := \operatorname{Span}\left(W_m^-(z,0)\right)_{m\in\{dp+1,\dots,d(p+q)\}}. \tag{4.17}$$

We will also define

$$\begin{aligned} I_{ss} &:= I_{ss}^+, & I_{cs} &:= I_{cs}^+, \\ I_{cu} &:= I_{cu}^-, & I_{su} &:= I_{su}^-. \end{aligned}$$

4.3 Definition of the Evans function

In this section, we are going to define an Evans function and prove that 1 is a simple eigenvalue of the operator $\mathscr{L}$ when it acts on $\ell^2(\mathbb{Z},\mathbb{C}^d)$.

An important part of the study of the spatial Green's function far from 1 was dedicated to introduce the projection Q of the geometric dichotomy. The main ingredient of the introduction of $Q(z)$ has been to understand when $E_0^+(z)$ and $E_0^-(z)$ are supplementary, as it allowed to conclude via Lemma 3.7 which elements of $\mathcal{O}$ where eigenvalues of the operator $\mathscr{L}$. For z near 1, because of (4.17), studying whether the vector subspace $E_0^+(z)$ and $E_0^-(z)$ are supplementary comes down to knowing when $\left(W_1^+(z,0),\dots,W_{dp}^+(z,0),W_{dp+1}^-(z,0),\dots,W_{d(p+q)}^-(z,0)\right)$ is a basis of $\mathbb{C}^{d(p+q)}$. We define the Evans function as

$$\forall z\in B(1,\delta_1),\quad \mathrm{Ev}(z):=\det(W_1^+(z,0),\dots,W_{dp}^+(z,0),W_{dp+1}^-(z,0),\dots,W_{d(p+q)}^-(z,0)). \tag{4.18}$$

The function Ev is holomorphic on $B(1,\delta_1)$. Furthermore, for $z\in\mathcal{O}\cap B(1,\delta_1)$, (3.33) and the set equality (4.17) imply that the function Ev vanishes when z is an eigenvalue of the operator $\mathscr{L}$. Thus, Hypothesis 6 implies that the function Ev is not uniformly equal to 0. We will now prove the following lemma which links the behavior at $z=1$ of the Evans function Ev, the eigenspace associated with the eigenvalue 1 for the operator $\mathscr{L}$ and the vector subspace

$$\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right).$$

Lemma 4.5. *We have that* $\mathrm{Ev}(1)=0$ *and*

$$\dim\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)=1. \tag{4.19}$$

Furthermore, if we introduce $V_0\in\mathbb{C}^{d(p+q)}\backslash\{0\}$ *such that*

$$\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)=\mathrm{Span}V_0,$$

then

$$\ker(Id_{\ell^2}-\mathscr{L})=\mathrm{Span}(\Pi(X_j(z)V_0))_{j\in\mathbb{Z}} \tag{4.20}$$

where the operator Π *defined by* (3.34) *is the linear map which extracts the center values of a vector of size* $d(p+q)$.

Proof The proof is separated in several steps.

- **Step 1:** We start by proving that

$$\dim\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)>0.$$

We fix $m\in\{1,\dots,d(p+q)\}$. For all $j\in\mathbb{Z}$, we define $W_m^{\pm -p}(1,j),\dots,W_m^{\pm q-1}(1,j)\in\mathbb{C}^d$ such that

$$W_m^\pm(1,j)=:\begin{pmatrix}W_m^{\pm q-1}(1,j)\\ \vdots\\ W_m^{\pm -p}(1,j)\end{pmatrix}$$

and we notate $u_m^\pm(j):=W_m^{\pm 0}(1,j)$. Since $W_m^\pm$ satisfies

$$\forall j\in\mathbb{Z},\quad W_m^\pm(1,j+1)=M_j(1)W_m^\pm(1,j),$$

we have

$$\forall j\in\mathbb{Z},\forall k\in\{-p,\dots,q-2\},\quad W_m^{\pm k}(1,j+1)=W_m^{\pm k+1}(1,j) \tag{4.21}$$

and

$$\forall j\in\mathbb{Z},W_m^{\pm q-1}(1,j+1)=-\mathbb{A}_{j,q}(1)^{-1}\left(\sum_{k=-p}^{q-1}\mathbb{A}_{j,k}(1)W_m^{\pm k}(1,j)\right). \tag{4.22}$$

The equality (4.21) implies that

$$\forall j\in\mathbb{Z},\forall k\in\{-p,\dots,q-1\},\quad W_m^{\pm k}(1,j)=u_m^\pm(j+k).$$

We then obtain using (4.22)

$$\sum_{k=-p}^{q} \mathbb{A}_{j,k}(1) u_m^{\pm}(j+k) = 0.$$

Using the definition (1.20) of $A_{j,k}$, we have that

$$\forall j \in \mathbb{Z}, \quad \sum_{k=-p}^{q-1} B_{j+1,k} u_m^{\pm}(j+1+k) = \sum_{k=-p}^{q-1} B_{j,k} u_m^{\pm}(j+k).$$

Therefore, the sequence $\left(\sum_{k=-p}^{q-1} B_{j,k} u_m^{\pm}(j+k)\right)_{j\in\mathbb{Z}}$ is constant.

- If $m \in I_{ss}^{\pm}$, since $\zeta_m^{\pm}(1) = \underline{\zeta}_m^{\pm} \in \mathbb{D}$, Lemma 4.3 implies that

$$W_m^{\pm}(1,j) \underset{j\to+\infty}{\to} 0.$$

  Therefore, $u_m^{\pm}(j)$ tends to 0 as j tends to $+\infty$.

- If $m \in I_{su}^{\pm}$, since $\zeta_m^{-}(1) = \underline{\zeta}_m^{-} \in \mathbb{U}$, Lemma 4.3 implies that

$$W_m^{\pm}(1,j) \underset{j\to-\infty}{\to} 0.$$

  Therefore, $u_m^{\pm}(j)$ tends to 0 as j tends to $-\infty$.

Therefore, the two previous points imply that for $m \in I_{ss}^{\pm} \cup I_{su}^{\pm}$

$$\forall j \in \mathbb{Z}, \quad \sum_{k=-p}^{q-1} B_{j,k} u_m^{\pm}(j+k) = 0.$$

Therefore,

$$\forall j \in \mathbb{Z}, \quad u_m^{\pm}(j-p) = D_j \begin{pmatrix} u_m^{\pm}(j+q-1) \\ \vdots \\ u_m^{\pm}(j-p+1) \end{pmatrix}$$

where

$$D_j := \begin{pmatrix} -B_{j,q}^{-1}B_{j,q-1} & \dots & -B_{j,q}^{-1}B_{j,-p+1} \end{pmatrix} \in \mathcal{M}_{d,(d-1)(p+q)}(\mathbb{C}).$$

This implies that

$$\forall m \in I_{ss}^{\pm} \cup I_{su}^{\pm}, \quad W_m^{\pm}(1,0) \in \left\{ \begin{pmatrix} V \\ D_0 V \end{pmatrix}, \quad V \in \mathbb{C}^{(d-1)(p+q)} \right\}$$

and thus

$$\mathrm{Span}\left(W_m^{+}(1,0), m \in I_{ss}\right) \cup \mathrm{Span}\left(W_m^{-}(1,0), m \in I_{su}\right) \subset \left\{ \begin{pmatrix} V \\ D_0 V \end{pmatrix}, \quad V \in \mathbb{C}^{(d-1)(p+q)} \right\}.$$

Also,

$$\dim \left\{ \begin{pmatrix} V \\ D_0 V \end{pmatrix}, \quad V \in \mathbb{C}^{(d-1)(p+q)} \right\} = (d-1)(p+q),$$

and Hypothesis 1 implies that

$$\# I_{ss} \cup I_{su} = (d-1)(p+q) + 1.$$

Therefore, since Lemma 4.4 implies that the families $(W_m^{+}(1,0))_{m\in I_{ss}}$ and $(W_m^{-}(1,0))_{m\in I_{su}}$ are both linearly independent, this allows us to conclude that

$$\dim \mathrm{Span}\left(W_m^{+}(1,0), m \in I_{ss}\right) \cap \mathrm{Span}\left(W_m^{-}(1,0), m \in I_{su}\right) \geq 1. \tag{4.23}$$

- **Step 2:** The inequality (4.23) allows us to conclude that $\mathrm{Ev}(1) = 0$. We recall that Hypothesis 7 implies that

$$\frac{\partial \mathrm{Ev}}{\partial z}(1) \neq 0.$$

Therefore, using (4.23) and the expression of the Evans function Ev, we easily deduce (4.19).

• **Step 3:** We introduce the vector $V_0 \in \mathbb{C}^{d(p+q)} \setminus \{0\}$ defined in the statement of Lemma 4.5. Since V_0 belongs to $\text{Span}\,(W_m^+(1,0), m \in I_{ss}) \cap \text{Span}\,(W_m^-(1,0), m \in I_{su})$, we obviously have that $\left(\Pi\left(X_j(1)V_0\right)\right)_{j\in\mathbb{Z}}$ has exponential decay as j tends towards $+\infty$ and $-\infty$ and thus belongs to $\ell^2(\mathbb{Z})$. Furthermore, since $(X_j(1)V_0)_{j\in\mathbb{Z}}$ is a solution of (3.3) for $z = 1$, we have that $\left(\Pi\left(X_j(1)V_0\right)\right)_{j\in\mathbb{Z}}$ belongs to $\ker(Id_{\ell^2} - \mathscr{L})$. Thus,

$$\text{Span}\left(\Pi\left(X_j(1)V_0\right)\right)_{j\in\mathbb{Z}} \subset \ker(Id_{\ell^2} - \mathscr{L}).$$

We now consider $w \in \ker\left(Id_{\ell^2} - \mathscr{L}\right)$ and define

$$W_0 := \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix} \in \mathbb{C}^{d(p+q)} \quad \text{and} \quad \forall j \in \mathbb{Z}, \quad W_j = X_j(1)W_0 = \begin{pmatrix} w_{j+q-1} \\ \vdots \\ w_{j-p} \end{pmatrix}.$$

If we decompose

$$W_0 = \sum_{m=1}^{d(p+q)} c_m W_m^+(1,0) \quad \text{and thus} \quad \forall j \in \mathbb{Z}, \quad W_j = \sum_{m=1}^{d(p+q)} c_m W_m^+(1,j)$$

with a family of complex scalars $(c_m)_{m\in\{1,\dots,d(p+q)\}} \in \mathbb{C}^{d(p+q)}$, since $W_j \underset{j\to+\infty}{\to} 0$, we have that

$$\forall m \in \{1,\dots,d(p+q)\} \setminus I_{ss}, \quad c_m = 0.$$

This can be proved using a similar idea as in Lemma 4.4 by proving that the scalar

$$R := \max\left\{\left|\underline{\zeta}_m^+\right|, \quad m \in \{1,\dots,d(p+q)\} \text{ such that } c_m \neq 0\right\}$$

cannot be larger or equal than 1. Thus, using a similar proof with the family $(W_m^-(1,0))_{m\in\{1,\dots,d(p+q)\}}$, we have that W_0 belongs to $\text{Span}(W_m^+(1,0), m \in I_{ss}) \cap \text{Span}(W_m^-(1,0), m \in I_{su}) = \text{Span}V_0$. Therefore, since for all $j \in \mathbb{Z}$ we have $w_j = \Pi(W_j) = \Pi(X_j(1)W_0)$, we conclude that the sequence w belongs to $\text{Span}\left(\Pi\left(X_j(1)V_0\right)\right)_{j\in\mathbb{Z}}$. We can thus finally verify (4.20). □

First, as a consequence of Lemma 4.5, since the Evans function Ev is holomorphic on $B(1,\delta_1)$ and not uniformly equal to 0, the equality $\text{Ev}(1) = 0$ implies that we can consider δ_1 small enough so that the Evans function Ev only vanishes at $z = 1$.

Our new goal for the rest of this section will be to use Lemma 4.5 to introduce in (4.26) below two new bases $(\Phi_m(z,0))_{m\in\{1,\dots,dp\}}$ and $(\Phi_m(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$ for the vector spaces $E_0^+(z)$ and $E_0^-(z)$ more suitable for the study of the spatial Green's function when z is close to 1. We will also define in (4.30c) a new Evans function D^Φ associated with this new choice of bases which will share the same properties as Ev.

The equality (4.19) of Lemma 4.5 implies that there exist two non zero families of complex numbers $(\theta_{s,m})_{m\in I_{ss}}$ and $(\theta_{u,m})_{m\in I_{su}}$ such that

$$\sum_{m\in I_{ss}} \theta_{s,m} W_m^+(1,0) = \sum_{m\in I_{su}} \theta_{u,m} W_m^-(1,0). \tag{4.24}$$

In the rest of the paper, we fix the choice of families of coefficients $\theta_{s,m}$ and $\theta_{u,m}$. Even if we have to reindex the eigenvalues $\underline{\zeta}_m^\pm$, we will suppose that $\theta_{s,1}, \theta_{u,d(p+q)} \neq 0$. Furthermore, we also define

$$\theta_{s,m} := 0 \text{ for } m \in \{1,\dots,d(p+q)\} \setminus I_{ss} \quad \text{and} \quad \theta_{u,m} := 0 \text{ for } m \in \{1,\dots,d(p+q)\} \setminus I_{su}. \tag{4.25}$$

We define for $m \in \{1,\dots,d(p+q)\}$

$$\forall z \in B(1,\delta_1), \forall j \in \mathbb{Z}, \quad \Phi_m(z,j) := \begin{cases} \sum_{m\in I_{ss}} \theta_{s,m} W_m^+(z,j), & \text{if } m = 1, \\ W_m^+(z,j), & \text{if } m \in \{2,\dots,dp\}, \\ W_m^-(z,j), & \text{if } m \in \{dp+1,\dots,d(p+q)-1\}, \\ \sum_{m\in I_{su}} \theta_{u,m} W_m^-(z,j), & \text{if } m = d(p+q). \end{cases} \tag{4.26}$$

Since $\theta_{s,1}, \theta_{u,d(p+q)} \neq 0$, we have that $(\Phi_m(z,0))_{m\in\{1,\dots,dp\}}$ and $(\Phi_m(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$ are respectively bases of $E_0^+(z)$ and $E_0^-(z)$. (4.24) implies that $\Phi_1(1,0) = \Phi_{d(p+q)}(1,0)$ and since $\Phi_1(1,\cdot)$ and $\Phi_{d(p+q)}(1,\cdot)$ are solutions of (3.3) for $z=1$, we have

$$\forall j \in \mathbb{Z}, \quad \Phi_1(1,j) = \Phi_{d(p+q)}(1,j). \tag{4.27}$$

Furthermore, using the expression of $\Phi_1(z,j)$ and $\Phi_{d(p+q)}(z,j)$ as well as Lemma 4.3 and inequalities (4.1) and (4.3), we prove that there exists a positive constant C such that

$$\begin{aligned} \forall z \in B(1,\delta_1), \forall j \in \mathbb{N}, \quad & |\Phi_1(z,j)| \leq Ce^{-2c_*|j|}, \\ \forall z \in B(1,\delta_1), \forall j \in -\mathbb{N}, \quad & \left|\Phi_{d(p+q)}(z,j)\right| \leq Ce^{-2c_*|j|}. \end{aligned} \tag{4.28}$$

Using (4.20), we have that if we define

$$\forall j \in \mathbb{Z}, \quad V(j) = \Pi(\Phi_1(1,j)) = \Pi(\Phi_{d(p+q)}(1,j)) \tag{4.29}$$

then V is a sequence in $\ell^2(\mathbb{Z},\mathbb{C}^d)\setminus\{0\}$ such that (1.21) and (1.22) are verified. It will correspond to the sequence V in Theorem 1.

If we summarize, for $z \in B(1,\delta_1)$, we have five families to describe the solutions of the dynamical system (3.3):

- The bases $\left(\Phi_1(z,j), W_2^+(z,j), \dots, W_{d(p+q)}^+(z,j)\right)$ and $\left(W_1^+(z,j), W_2^+(z,j), \dots, W_{d(p+q)}^+(z,j)\right)$ for which we know the asymptotic behavior when j tends to $+\infty$.
- The bases $\left(W_1^-(z,j), \dots, W_{d(p+q)-1}^-(z,j), \Phi_{d(p+q)}(z,j)\right)$ and $\left(W_1^-(z,j), \dots, W_{d(p+q)-1}^-(z,j), W_{d(p+q)}^-(z,j)\right)$ for which we know the asymptotic behavior when j tends to $-\infty$.
- The family $(\Phi_m(z,j))_{m\in\{1,\dots,d(p+q)\}}$ which is linked to the solution which tend towards 0 as j tends towards $+\infty$ or $-\infty$, at least when $z \in \mathcal{O}$. It is a basis of $\mathbb{C}^{d(p+q)}$ if and only if $E_0^+(z) \oplus E_0^-(z) = \mathbb{C}^{d(p+q)}$.

We introduce a few more notations. For $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$, we define

$$\mathcal{G}^\Phi(z,j) := \left(\Phi_1(z,j)|\dots|\Phi_{d(p+q)}(z,j)\right), \qquad D^\Phi(z) := \det(\mathcal{G}^\Phi(z,0)), \tag{4.30a}$$

$$\mathcal{G}^+(z,j) := \left(\Phi_1(z,j)|W_2^+(z,j)|\dots|W_{d(p+q)}^+(z,j)\right), \qquad D^+(z) := \det(\mathcal{G}^+(z,0)), \tag{4.30b}$$

$$\mathcal{G}^-(z,j) := \left(W_1^-(z,j)|\dots|W_{d(p+q)-1}^-(z,j)|\Phi_{d(p+q)}(z,j)\right), \qquad D^-(z) := \det(\mathcal{G}^-(z,0)), \tag{4.30c}$$

$$\widetilde{\mathcal{G}}^+(z,j) := \left(W_1^+(z,j)|W_2^+(z,j)|\dots|W_{d(p+q)}^+(z,j)\right), \tag{4.30d}$$

$$\widetilde{\mathcal{G}}^-(z,j) := \left(W_1^-(z,j)|W_2^-(z,j)|\dots|W_{d(p+q)}^-(z,j)\right). \tag{4.30e}$$

Those functions are holomorphic on $B(1,\delta_1)$. Let us now conclude the section with a few observations:

- We observe using the definition (4.26) of Φ_1 and $\Phi_{d(p+q)}$ that for $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$

$$\mathcal{G}^+(z,j) = \widetilde{\mathcal{G}}^+(z,j)\begin{pmatrix} \theta_{s,1} & & \\ \theta_{s,2} & 1 & \\ \vdots & & \ddots \\ \theta_{s,d(p+q)} & & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{G}^-(z,j) = \widetilde{\mathcal{G}}^-(z,j)\begin{pmatrix} 1 & & & \theta_{u,1} \\ & \ddots & & \vdots \\ & & 1 & \theta_{u,d(p+q)-1} \\ & & & \theta_{u,d(p+q)} \end{pmatrix}. \tag{4.31}$$

- Using (4.26) and (4.25), we have for $z \in B(1,\delta_1)$

$$\mathcal{G}^\Phi(z,0) = \left(W_1^+(z,0)|\dots|W_{dp}^+(z,0)|W_{dp+1}^-(z,0)|\dots|W_{d(p+q)}^-(z,0)\right)\begin{pmatrix} \theta_{s,1} & & & \theta_{u,1} \\ \theta_{s,2} & 1 & & \theta_{u,2} \\ \vdots & & \ddots & \vdots \\ \theta_{s,d(p+q)-1} & & 1 & \theta_{u,d(p+q)-1} \\ \theta_{s,d(p+q)} & & & \theta_{u,d(p+q)} \end{pmatrix}$$

and thus

$$D^\Phi(z) = \theta_{s,1}\theta_{u,d(p+q)}\mathrm{Ev}(z).$$

The function D^Φ thus shares the same properties as the Evans function Ev, i.e. the function D^Φ is holomorphic on $B(1,\delta_1)$, vanishes only at $z=1$ and 1 is a simple zero of D^Φ. We will thus also call D^Φ Evans function.

• For $z \in B(1,\delta_1)\backslash\{1\}$, the function D^Φ does not vanish at z and thus $(\Phi_m(z,0))_{m\in\{1,\dots,d(p+q)\}}$ is a basis. We can define for $m \in \{1,\dots,(p+q)d\}$ the projector $\Pi_m(z)$ on $\mathrm{Span}(\Phi_m(z,0))$ along $\mathrm{Span}(\Phi_\nu(z,0))_{\nu\in\{1,\dots,(p+q)d\}\backslash\{m\}}$. We observe that

$$\Pi_m(z) = \mathcal{G}^\Phi(z,0) P_m \mathcal{G}^\Phi(z,0)^{-1} \tag{4.32}$$

where $P_m = (\delta_{i,m}\delta_{j,m})_{i,j\in\{1,\dots,(p+q)d\}} \in \mathcal{M}_{(p+q)d}(\mathbb{C})$. The function Π_m is holomorphic on $B(1,\delta_1)\backslash\{1\}$.

4.4 Behavior of the spatial Green's function near 1

We have now introduced all the tools necessary to study the spatial Green's function near 1. In Section 4.4.1, we will use the families of vectors previously indicated to decompose the expressions (3.41a) and (3.41b) proved on the spatial Green's function in Section 3 and obtain the expressions (4.35) of the spatial Green's function on $B(1,\delta_1)\cap\mathcal{O}_\rho$. Since the sequence $W_m^\pm$ are holomorphic on the whole ball $B(1,\delta_1)$, this will allow us to to extend the spatial Green's function meromorphically on $B(1,\delta_1)$ with a pole of order 1 at 1. The calculations performed in this section will be fairly inspired by [God03, Section 3] and the expression (4.35) corresponds to the result of [God03, Proposition 3.1]. However, we will need a more accurate description of the spatial Green's function to improve on the result of [God03]. Those calculations will be performed in Sections 4.4.2 and 4.4.3.

4.4.1 Meromorphic extension of the spatial Green's function near 1

We observe that for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, we have that $Q(z)$ is the projector on $E_0^+(z)$ along $E_0^-(z)$, so

$$Q(z) = \sum_{m=1}^{dp} \Pi_m(z) \quad \text{and} \quad Id - Q(z) = \sum_{m=dp+1}^{(p+q)d} \Pi_m(z).$$

Since the equalities (3.41a) and (3.41b) are still verified for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, if we define for $m \in \{1,\dots,d(p+q)\}$ the functions

$$\forall z \in B(1,\delta_1)\backslash\{1\}, \forall j_0, j \in \mathbb{Z}, \forall \vec{e} \in \mathbb{C}^d, \quad \nu_m(z,j_0,j,\vec{e}) := X_j(z)\Pi_m(z)X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then, we have that for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$

$$\forall j \geq j_0+1, \quad W(z,j_0,j,\vec{e}) = -\sum_{m=1}^{dp} \nu_m(z,j_0,j,\vec{e}), \tag{4.33a}$$

$$\forall j \leq j_0, \quad W(z,j_0,j,\vec{e}) = \sum_{m=dp+1}^{(p+q)d} \nu_m(z,j_0,j,\vec{e}). \tag{4.33b}$$

Since the right hand terms of (4.33) are holomorphic on $B(1,\delta_1)\backslash\{1\}$, we can extend holomorphically the spatial Green's function on $B(1,\delta_1)\backslash\{1\}$.

We observe that (4.32) implies for $z \in B(1,\delta_1)\backslash\{1\}$ and $m \in \{1,\dots,(p+q)d\}$,

$$\nu_m(z,j_0,j,\vec{e}) = \frac{\mathcal{G}^\Phi(z,j)}{D^\Phi(z)} P_m \mathrm{com}(\mathcal{G}^\Phi(z,0))^T X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we define

$$\begin{pmatrix} \widehat{D}_1(z,j_0,\vec{e}) \\ \vdots \\ \widehat{D}_{d(p+q)}(z,j_0,\vec{e}) \end{pmatrix} := \mathrm{com}(\mathcal{G}^\Phi(z,0))^T X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.34}$$

then the functions $\widehat{D}_m(\cdot, j_0, \vec{e})$ is holomorphic on $B(1,\delta_1)$ and we have

$$\nu_m(z, j_0, j, \vec{e}) = \frac{\widehat{D}_m(z, j_0, \vec{e})}{D^\Phi(z)} \Phi_m(z, j).$$

Therefore, (4.33a) and (4.33b) can be rewritten for $z \in B(1,\delta_1)\backslash\{1\}$ as

$$\forall j \geq j_0 + 1, \quad W(z, j_0, j, \vec{e}) = -\sum_{m=1}^{dp} \frac{\widehat{D}_m(z, j_0, \vec{e})}{D^\Phi(z)} \Phi_m(z, j), \tag{4.35a}$$

$$\forall j \leq j_0, \quad W(z, j_0, j, \vec{e}) = \sum_{m=dp+1}^{(p+q)d} \frac{\widehat{D}_m(z, j_0, \vec{e})}{D^\Phi(z)} \Phi_m(z, j). \tag{4.35b}$$

Thus, recalling that 1 is a simple zero of the Evans function D^Φ, the expressions (4.35a) and (4.35b) allow us to conclude that the spatial Green's function has been meromorphically extended on $B(1,\delta_1)\backslash\{1\}$ with a pole of order 1 at 1.

Now that we have found the meromorphic extension of the spatial Green's function near 1 using the family $(\Phi_m(z,j))_{m\in\{1,\dots,d(p+q)\}}$, we will use the other families $\left(W_1^\pm(z,j),\dots,W_{d(p+q)}^\pm(z,j)\right)$ for which we know precisely the behavior as j tends towards $\pm\infty$ to find a new improved expression of the spatial Green's function with terms that we can estimate more easily.

4.4.2 Decomposing the function $\widehat{D}_m$

We begin this section by introducing the vectors

$$\forall z \in B(1,\delta_1), \forall j_0 \in \mathbb{Z}, \forall \vec{e} \in \mathbb{C}^d, \quad \mathscr{C}^\pm(z, j_0, \vec{e}) = \begin{pmatrix} \mathscr{C}_1^\pm(z, j_0, \vec{e}) \\ \vdots \\ \mathscr{C}_{d(p+q)}^\pm(z, j_0, \vec{e}) \end{pmatrix} := \mathcal{G}^\pm(z, j_0+1)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.36a}$$

$$\forall z \in B(1,\delta_1), \forall j_0 \in \mathbb{Z}, \forall \vec{e} \in \mathbb{C}^d, \quad \widetilde{\mathscr{C}}^\pm(z, j_0, \vec{e}) = \begin{pmatrix} \widetilde{\mathscr{C}}_1^\pm(z, j_0, \vec{e}) \\ \vdots \\ \widetilde{\mathscr{C}}_{d(p+q)}^\pm(z, j_0, \vec{e}) \end{pmatrix} := \widetilde{\mathcal{G}}^\pm(z, j_0+1)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.36b}$$

Using (4.31) and (4.25), we obtain that for $z \in B(1,\delta_1)$, $j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$

$$\widetilde{\mathscr{C}}_1^+(z, j_0, \vec{e}) = \theta_{s,1} \mathscr{C}_1^+(z, j_0, \vec{e}), \tag{4.37a}$$

$$\forall m \in I_{ss}\backslash\{1\}, \quad \widetilde{\mathscr{C}}_m^+(z, j_0, \vec{e}) = \mathscr{C}_m^+(z, j_0, \vec{e}) + \theta_{s,m} \mathscr{C}_1^+(z, j_0, \vec{e}), \tag{4.37b}$$

$$\forall m \in \{1,\dots,d(p+q)\}\backslash I_{ss}, \quad \widetilde{\mathscr{C}}_m^+(z, j_0, \vec{e}) = \mathscr{C}_m^+(z, j_0, \vec{e}), \tag{4.37c}$$

$$\forall m \in \{1,\dots,d(p+q)\}\backslash I_{su}, \quad \widetilde{\mathscr{C}}_m^-(z, j_0, \vec{e}) = \mathscr{C}_m^-(z, j_0, \vec{e}), \tag{4.37d}$$

$$\forall m \in I_{su}\backslash\{d(p+q)\}, \quad \widetilde{\mathscr{C}}_m^-(z, j_0, \vec{e}) = \mathscr{C}_m^-(z, j_0, \vec{e}) + \theta_{u,m} \mathscr{C}_{d(p+q)}^-(z, j_0, \vec{e}), \tag{4.37e}$$

$$\widetilde{\mathscr{C}}_{d(p+q)}^-(z, j_0, \vec{e}) = \theta_{u,d(p+q)} \mathscr{C}_{d(p+q)}^-(z, j_0, \vec{e}). \tag{4.37f}$$

In Section 4.4.4, we will prove estimates to bound $\widetilde{\mathscr{C}}_m^\pm$. However, the functions $\widetilde{\mathscr{C}}_m^\pm$ will be put on the side for now and will naturally reappear later on in Section 4.4.3 using (4.37). For now, we will mainly focus on properties linked to the functions $\mathscr{C}_m^\pm$.

We introduce the matrices

$$\forall z \in B(1,\delta_1), \quad \mathcal{M}^\pm(z) := \mathcal{G}^\pm(z,0)^{-1}\mathcal{G}^\Phi(z,0) \quad \text{and} \quad \left(g_{m',m}^\pm(z)\right)_{(m',m)\in\{1,\dots,d(p+q)\}} := \operatorname{com}\left(\mathcal{M}^\pm(z)\right). \tag{4.38}$$

Using the definition (4.34) of $\widehat{D}_m$ and a Laplace expansion of the determinant, we prove that

$$\forall m \in \{1,\dots,d(p+q)\}, \forall z \in B(1,\delta_1), \forall j_0 \in \mathbb{Z}, \forall \vec{e} \in \mathbb{C}^d, \quad \widehat{D}_m(z, j_0, \vec{e}) = D^\pm(z) \sum_{m'=1}^{d(p+q)} g_{m',m}^\pm(z) \mathscr{C}_{m'}^\pm(z, j_0, \vec{e}). \tag{4.39}$$

Looking at (4.35), we are interested in studying the quotient of $\widehat{D}_m(\cdot, j_0, \vec{e})$ and D^Φ. In order to have less crowded expressions later on, we also introduce the functions

$$\forall m, m' \in \{1, \dots, d(p+q)\}, \forall z \in B(1, \delta_1) \backslash \{1\}, \quad \widetilde{g}^{\pm}_{m',m}(z) := D^{\pm}(z) \frac{g^{\pm}_{m',m}(z)}{D^\Phi(z)}. \tag{4.40}$$

The function $\widetilde{g}^{\pm}_{m',m}$ is meromorphic on $B(1, \delta_1) \backslash \{1\}$ with a pole of order less than 1 at 1 since 1 is a simple zero of the Evans function D^Φ. If $g^{\pm}_{m',m}(1) = 0$, it can thus be extended holomorphically on the whole ball $B(1, \delta_1)$.

This decomposition of the functions $\widehat{D}_m$ will be used in Section 4.4.3 with (4.35) to obtain a better expression of the spatial Green's function. We end this section by proving the following lemma using (4.27).

Lemma 4.6. *1. For $m' \in \{1, \dots, dp\}$ and $m \in \{1, \dots, d(p+q)\}$, we have*

$$\forall z \in B(1, \delta_1), \quad g^{+}_{m',m}(z) = \delta_{m',m} \frac{D^\Phi(z)}{D^+(z)}.$$

2. For $m \notin \{1, d(p+q)\}$ and $m' \in \{1, \dots, d(p+q)\}$, we have

$$g^{+}_{m',m}(1) = 0.$$

3. For $m' \in \{dp+1, \dots, d(p+q)\}$ and $m \in \{1, \dots, d(p+q)\}$, we have

$$\forall z \in B(1, \delta_1), \quad g^{-}_{m',m}(z) = \delta_{m',m} \frac{D^\Phi(z)}{D^-(z)}.$$

4. For $m \notin \{1, d(p+q)\}$ and $m' \in \{1, \dots, d(p+q)\}$, we have

$$g^{-}_{m',m}(1) = 0.$$

5. We have

$$g^{+}_{1,1}(1) = g^{+}_{1,d(p+q)}(1) = 0 \quad \text{and} \quad g^{-}_{d(p+q),d(p+q)}(1) = g^{-}_{d(p+q),1}(1) = 0.$$

6. For $m' \in \{1, \dots, d(p+q)\}$, we have

$$g^{+}_{m',1}(1) = -g^{+}_{m',d(p+q)}(1) \quad \text{and} \quad g^{-}_{m',1}(1) = -g^{-}_{m',d(p+q)}(1). \tag{4.41}$$

Lemma 4.6 determines the couple of indexes $m, m' \in \{1, \dots, d(p+q)\}$ such that $g^{\pm}_{m',m}$ is equal to 0. This allows us to remove many terms in (4.39). It also determines the couple of indexes $m, m' \in \{1, \dots, d(p+q)\}$ such that $g^{\pm}_{m',m}$ vanishes at $z = 1$ which implies that the function $\widetilde{g}_{m',m}$ defined by (4.40) can be holomorphically extended on the whole ball $B(1, \delta_1)$.

Proof We will focus on proving the statements involving $g^{+}_{m',m}$ since every statement involving $g^{-}_{m',m}$ will have similar proofs. We observe that the definition (4.26) of $\Phi_m(z, 0)$ implies that if we define for $z \in B(1, \delta_1)$ and $m \in \{1, \dots, d(p+q)\}$

$$C^{+}_m(z) := \mathcal{G}^{+}(z, 0)^{-1} \Phi_m(z, 0)$$

then, for $m \in \{1, \dots, dp\}$, we have

$$C^{+}_m(z) = e_m$$

where $(e_j)_{j \in \{1, \dots, d(p+q)\}}$ is the canonical basis of $\mathbb{C}^{d(p+q)}$. The equality (4.27) also implies that

$$C^{+}_{d(p+q)}(1) = e_1 = C^{+}_1(1). \tag{4.42}$$

Thus,

$$\mathcal{M}^{+}(z) = \left(\begin{array}{c|c|c|c} \dfrac{I_{dp}}{0} & C^{+}_{dp+1}(z) & \dots & C^{+}_{d(p+q)}(z) \end{array} \right). \tag{4.43}$$

For $m, m' \in \{1, \dots, d(p+q)\}$, we recall that $g^{+}_{m',m}(z)$ is the (m', m)-cofactor of the matrix above.

We observe that

$$\mathrm{com}(\mathcal{M}^{+}(z))^T \mathcal{M}^{+}(z) = \frac{D^\Phi(z)}{D^+(z)} I_{d(p+q)}. \tag{4.44}$$

Looking at the first dp columns of $\mathcal{M}^+(z)$ in (4.43), we then conclude that the first dp lines of $\mathrm{com}(\mathcal{M}^+(z))$ are equal to

$$\left(\frac{D^\Phi(z)}{D^+(z)} I_{dp} \;\middle|\; 0 \right)$$

which implies Point 1.

We observe that the equality (4.42) implies

$$\mathcal{M}^+(1) = \left(\begin{array}{c|c|c|c|c} \dfrac{I_{dp}}{0} & C^+_{dp+1}(1) & \dots & C^+_{d(p+q)-1}(1) & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right).$$

Points 2 and 5 are then easily deduced by equality between the first and last columns of the matrix above.

There just remains to prove Point 6. We observe that

$$\mathcal{M}^+(1)\mathrm{com}(\mathcal{M}^+(1))^T = 0.$$

Looking at the coefficient at the first line and m'-th column, we have that

$$g^+_{m',1}(1) + \sum_{m=dp+1}^{d(p+q)-1} (C^+_m(1))_1 g^+_{m',m}(1) + g^+_{m',d(p+q)}(1) = 0.$$

Using Point 2, we easily conclude the proof of Point 6. □

4.4.3 Final expression of the spatial Green's function

Depending on the sign of j_0 and on the localization of j depending on j_0 and 0, we will exhibit an expression of the spatial Green's function which will be useful to study the temporal Green's function. We will only present the expressions for $j_0 \geq 0$. The case where $j_0 < 0$ would be handled similarly and give another expression of the spatial Green's function on $B(1, \delta_1)$ that would be necessary to prove a decomposition of the temporal Green's function similar to (1.30) when $j_0 < 0$.

Case where $j_0 \geq 0$ and $j \geq j_0 + 1$:

Using (4.35a) and (4.39), we have

$$W(z, j_0, j, \vec{e}) = -\sum_{m=1}^{dp} \sum_{m'=1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \mathscr{C}^+_{m'}(z, j_0, \vec{e}) \Phi_m(z, j).$$

Point 1 of Lemma 4.6 then implies that

$$W(z, j_0, j, \vec{e}) = -\sum_{m=1}^{dp} \mathscr{C}^+_m(z, j_0, \vec{e}) \Phi_m(z, j) - \sum_{m=1}^{dp} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \mathscr{C}^+_{m'}(z, j_0, \vec{e}) \Phi_m(z, j).$$

Using the definition (4.26) of Φ_1, we then have that

$$\begin{aligned} W(z, j_0, j, \vec{e}) = -\theta_{s,1} \mathscr{C}^+_1(z, j_0, \vec{e}) W^+_1(z, j) - \sum_{m \in I_{ss} \setminus \{1\}} (\mathscr{C}^+_m(z, j_0, \vec{e}) + \theta_{s,m} \mathscr{C}^+_1(z, j_0, \vec{e})) W^+_m(z, j) \\ - \sum_{m \in I_{cs}} \mathscr{C}^+_m(z, j_0, \vec{e}) W^+_m(z, j) - \sum_{m=1}^{dp} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \mathscr{C}^+_{m'}(z, j_0, \vec{e}) \Phi_m(z, j). \end{aligned}$$

Using (4.37a)-(4.37c) which links the functions $\mathscr{C}^\pm_m$ and $\widetilde{\mathscr{C}}^\pm_m$ and the definition (4.26) of Φ_m for $m \in \{2, \dots, dp\}$, we obtain that

$$\begin{aligned} W(z, j_0, j, \vec{e}) = -\sum_{m=1}^{dp} \widetilde{\mathscr{C}}^+_m(z, j_0, \vec{e}) W^+_m(z, j) - \sum_{m=2}^{dp} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \vec{e}) W^+_m(z, j) \\ - \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',1}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \vec{e}) \Phi_1(z, j). \quad (4.45) \end{aligned}$$

Remark 2. If $m \in \{2, \dots, dp\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$, we have that $g^+_{m',m}(1) = 0$ because of Lemma 4.6. Thus, the function

$$z \in B(1,\delta_1)\backslash\{1\} \mapsto \widetilde{g}^+_{m',m}(z)\widetilde{\mathscr{C}}^+_{m'}(z, j_0, \vec{e})\Phi_m(z,j)$$

can be holomorphically extended on the whole ball $B(1,\delta_1)$.

Case where $j_0 \geq 0$ and $j \in \{0, \dots, j_0\}$:

For $m \in \{dp+1, \dots, d(p+q)\}$, using the respective definitions (4.30a), (4.30b) and (4.38) of the matrices $\mathcal{G}^\Phi$, $\mathcal{G}^+$ and $\mathcal{M}^+$, we have the following expression of the vector $\Phi_m(z,j)$ depending on the family $(W^+_k(z,j))_{k\in\{1,\dots,d(p+q)\}}$:

$$\Phi_m(z,j) = \mathcal{G}^+(z,j)(\mathcal{M}^+(z)_{k,m})_{k\in\{1,\dots,d(p+q)\}} = \mathcal{M}^+(z)_{1,m}\Phi_1(z,j) + \sum_{k=2}^{d(p+q)} \mathcal{M}^+(z)_{k,m}W^+_k(z,j).$$

Thus, using (4.39) and the fact that Lemma 4.6 implies $g^+_{m',m} = 0$ for $m' \in \{1, \dots, dp\}$, we have

$$\frac{\widehat{D}_m(z,j_0,\vec{e})}{D^\Phi(z)}\Phi_m(z,j) = \frac{D^+(z)}{D^\Phi(z)} \sum_{m'=dp+1}^{d(p+q)} g^+_{m',m}(z)\mathscr{C}^+_{m'}(z,j_0,\vec{e}) \left(\mathcal{M}^+(z)_{1,m}\Phi_1(z,j) + \sum_{k=2}^{d(p+q)} \mathcal{M}^+(z)_{k,m}W^+_k(z,j) \right).$$

Using (4.35b), we then have that

$$\begin{aligned} &W(z,j_0,j,\vec{e}) \\ &= \sum_{m'=dp+1}^{d(p+q)} \left[\left(\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{1,m}g^+_{m',m}(z) \right) \Phi_1(z,j) + \sum_{k=2}^{d(p+q)} \left(\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g^+_{m',m}(z) \right) W^+_k(z,j) \right] \\ &\qquad \frac{D^+(z)}{D^\Phi(z)}\mathscr{C}^+_{m'}(z,j_0,\vec{e}). \end{aligned}$$

Let us find expressions for the sums $\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g^+_{m',m}(z)$ when $m' \in \{dp+1, \dots, d(p+q)\}$ and $k \in \{1, \dots, d(p+q)\}$. We recall that $g^+_{m',m}(z)$ is the (m',m)-cofactor of the matrix $\mathcal{M}^+(z)$. Furthermore, by definition (4.38) of the matrix $\mathcal{M}^+$, we have that

$$\mathcal{M}^+(z)\mathrm{com}(\mathcal{M}^+(z))^T = \frac{D^\Phi(z)}{D^+(z)}Id. \tag{4.46}$$

Thus, by observing (4.43) implies that for $k \in \{dp+1, \dots, d(p+q)\}$ and $m \in \{1, \dots, dp\}$ we have $\mathcal{M}^+(z)_{k,m} = 0$, we conclude looking at the k-th line and m'-th column of (4.46) that

$$\forall k \in \{dp+1, \dots, d(p+q)\}, \forall m' \in \{dp+1, \dots, d(p+q)\}, \quad \sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g^+_{m',m}(z) = \frac{D^\Phi(z)}{D^+(z)}\delta_{k,m'}.$$

Furthermore, (4.43) implies that

$$\forall k \in \{1, \dots, dp\}, \forall m \in \{1, \dots, dp\}, \quad \mathcal{M}^+(z)_{k,m} = \delta_{k,m}.$$

Thus, looking once again at the k-th line and m'-th column of (4.46), we have

$$\forall k \in \{1, \dots, dp\}, \forall m' \in \{dp+1, \dots, d(p+q)\}, \quad \sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g^+_{m',m}(z) = -g^+_{m',k}(z).$$

We finally conclude using (4.37a)-(4.37c) which links the functions $\mathscr{C}^\pm_m$ and $\widetilde{\mathscr{C}}^\pm_m$ that

$$\begin{aligned} W(z,j_0,j,\vec{e}) = &\sum_{m=dp+1}^{d(p+q)} \widetilde{\mathscr{C}}^+_m(z,j_0,\vec{e})W^+_m(z,j) - \sum_{m=2}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z)\widetilde{\mathscr{C}}^+_{m'}(z,j_0,\vec{e})W^+_m(z,j) \\ &- \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',1}(z)\widetilde{\mathscr{C}}^+_{m'}(z,j_0,\vec{e})\Phi_1(z,j). \end{aligned} \tag{4.47}$$

Remark 3. We observe that some terms in (4.47) are equal to terms of (4.45). We will see later on that they contribute to the reflected waves in the decomposition of Theorem 1.

Case where $j_0 \geq 0$ and $j < 0$:
Using (4.35b) and (4.39),

$$W(z,j_0,j,\vec{e}) = \sum_{m=dp+1}^{d(p+q)} \sum_{m'=1}^{d(p+q)} \widetilde{g}^{+}_{m',m}(z)\mathscr{C}^{+}_{m'}(z,j_0,\vec{e})\Phi_m(z,j).$$

Lemma 4.6 implies that

$$\forall m \in \{dp+1,\dots,d(p+q)\}, \forall m' \in \{1,\dots,dp\}, \quad g^{+}_{m',m} = 0.$$

Thus, using (4.37a)-(4.37c) which links the functions $\mathscr{C}^{\pm}_m$ and $\widetilde{\mathscr{C}}^{\pm}_m$,

$$W(z,j_0,j,\vec{e}) = \sum_{m=dp+1}^{d(p+q)-1} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^{+}_{m',m}(z)\widetilde{\mathscr{C}}^{+}_{m'}(z,j_0,\vec{e})W^{-}_m(z,j) + \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^{+}_{m',d(p+q)}(z)\widetilde{\mathscr{C}}^{+}_{m'}(z,j_0,\vec{e})\Phi_{d(p+q)}(z,j). \quad (4.48)$$

Remark 4. If $m \in \{dp+1,\dots,d(p+q)-1\}$ and $m' \in \{dp+1,\dots,d(p+q)\}$, we have that $g^{+}_{m',m}(1) = 0$. Thus, the function

$$z \in B(1,\delta_1)\backslash\{1\} \mapsto \widetilde{g}^{+}_{m',m}(z)\widetilde{\mathscr{C}}^{+}_{m'}(z,j_0,\vec{e})\Phi_m(z,j)$$

can be holomorphically extended on the whole ball $B(1,\delta_1)$.

4.4.4 Useful estimates

In this section, we will introduce the necessary observations to properly bound the terms appearing in the decomposition of the spatial Green's function of Section 4.4.3. We will in particular introduce a new expression of the functions $\widetilde{\mathscr{C}}^{\pm}_m(z,j_0,\vec{e})$, prove that they roughly act like $\zeta^{\pm}_m(z)^{-j_0}$ and determine their behavior as j_0 tends towards $\pm\infty$.

For $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$, we recall that Lemma 4.4 implies that $(V^{\pm}_m(z,j))_{m\in\{1,\dots,d(p+q)\}}$ is a basis of $\mathbb{C}^{d(p+q)}$. Thus, we can define for $z \in B(1,\delta_1)$, $j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$

$$N^{\pm}(z,j_0) := \left(V^{\pm}_1(z,j_0) \,\middle|\, \dots \,\middle|\, V^{\pm}_{d(p+q)}(z,j_0) \right) \quad \text{and} \quad \begin{pmatrix} \Delta^{\pm}_1(z,j_0,\vec{e}) \\ \vdots \\ \Delta^{\pm}_{d(p+q)}(z,j_0,\vec{e}) \end{pmatrix} := N^{\pm}(z,j_0)^{-1} \begin{pmatrix} A^{-1}_{j_0,q}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (4.49)$$

We observe that (4.36b) implies that for all $z \in B(1,\delta_1)$, $j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$, we have

$$\sum_{m=1}^{d(p+q)} \zeta^{\pm}_m(z)^{j_0+1}\widetilde{\mathscr{C}}^{\pm}_m(z,j_0,\vec{e})V^{\pm}_m(z,j_0+1) = \begin{pmatrix} A^{-1}_{j_0,q}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, we have that for $m \in \{1,\dots,d(p+q)\}$, $z \in B(1,\delta_1)$, $j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$

$$\widetilde{\mathscr{C}}^{\pm}_m(z,j_0,\vec{e}) = \zeta^{\pm}_m(z)^{-j_0-1}\Delta^{\pm}_m(z,j_0,\vec{e}). \quad (4.50)$$

We now prove the following lemma which gives us the asymptotic behavior of $\Delta^{\pm}_m(z,j_0,\vec{e})$.

Lemma 4.7. *There exist a radius $\delta_2 \in]0,\delta_1]$ and two constants $C,c>0$ such that for all $z \in B(1,\delta_2)$, $m = l+(k-1)d \in \{1,\dots,d(p+q)\}$ with $k \in \{1,\dots,p+q\}$ and $l \in \{1,\dots,d\}$ and $\vec{e} \in \mathbb{C}^d$, we have*

$$\forall j \in \mathbb{N}, \quad \left|V_m^+(z,j) - V_m^+(1,j)\right| \leq C|z-1|, \tag{4.51a}$$
$$\forall j \in -\mathbb{N}, \quad \left|V_m^-(z,j) - V_m^-(1,j)\right| \leq C|z-1|, \tag{4.51b}$$
$$\forall j \in \mathbb{N}, \quad |\Phi_1(z,j) - \Phi_1(1,j)| \leq C|z-1|e^{-\frac{3c_*}{2}|j|}, \tag{4.51c}$$
$$\forall j \in -\mathbb{N}, \quad \left|\Phi_{d(p+q)}(z,j) - \Phi_{d(p+q)}(1,j)\right| \leq C|z-1|e^{-\frac{3c_*}{2}|j|}, \tag{4.51d}$$
$$\forall j_0 \in \mathbb{N}, \quad \left|\Delta_m^+(z,j_0,\vec{e}) - {\zeta_m^+}'(z){\boldsymbol{l}_l^+}^T\vec{e}\right| \leq C|\vec{e}|e^{-c|j_0|}, \tag{4.51e}$$
$$\forall j_0 \in -\mathbb{N}, \quad \left|\Delta_m^-(z,j_0,\vec{e}) - {\zeta_m^-}'(z){\boldsymbol{l}_l^-}^T\vec{e}\right| \leq C|\vec{e}|e^{-c|j_0|}, \tag{4.51f}$$
$$\forall j_0 \in \mathbb{N}, \quad \left|\Delta_m^+(z,j_0,\vec{e})\right| \leq C|\vec{e}|, \tag{4.51g}$$
$$\forall j_0 \in -\mathbb{N}, \quad \left|\Delta_m^-(z,j_0,\vec{e})\right| \leq C|\vec{e}|. \tag{4.51h}$$
$$\forall j_0 \in \mathbb{N}, \quad \left|\Delta_m^+(z,j_0,\vec{e}) - \Delta_m^+(1,j_0,\vec{e})\right| \leq C|z-1||\vec{e}|, \tag{4.51i}$$
$$\forall j_0 \in -\mathbb{N}, \quad \left|\Delta_m^-(z,j_0,\vec{e}) - \Delta_m^-(1,j_0,\vec{e})\right| \leq C|z-1||\vec{e}|. \tag{4.51j}$$

Proof The first two inequalities are direct consequences from the fact that $z \in B(1,\delta_1) \mapsto \frac{\partial V_m^\pm}{\partial z}(z,\cdot) \in \ell^\infty(\pm\mathbb{N},\mathbb{C}^{d(p+q)})$ is bounded (see Lemma 4.3).

We will prove (4.51c). The proof of (4.51d) would be similar. Using the definition (4.26) of Φ_1, we conclude that we only have to prove that for all $m \in I_{ss}$ that there exists a constant $C>0$ and a radius $\delta_2 \in]0,\delta_1]$ such that

$$\forall z \in B(1,\delta_2), \forall j \in \mathbb{N}, \quad \left|W_m^+(z,j) - W_m^+(1,j)\right| \leq C|z-1|e^{-\frac{3c_*}{2}|j|}.$$

We observe that for $z \in B(1,\delta_1)$ and $j \in \mathbb{N}$, we have

$$\left|W_m^+(z,j) - W_m^+(1,j)\right| \leq \left|\zeta_m^+(z)\right|^j \left|V_m^+(z,j) - V_m^+(1,j)\right| + \left|V_m^+(1,j)\right| \left|\zeta_m^+(z)^j - \zeta_m^+(1)^j\right|.$$

Using (4.51a), (4.1) and the fact that $\frac{d\zeta_m^+}{dz}$ is bounded on $B(1,\delta_2)$ for $\delta_2 \in]0,\delta_2[$, we easily conclude.

We observe that (4.51g) and (4.51h) are direct consequences of (4.51e) and (4.51f). We will focus on (4.51e) as (4.51f) would be proved in a similar way. We observe that Lemma 4.9, (1.14) and (1.5) imply that

$$L_m^+(z)^T \begin{pmatrix} {A_q^+}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_{l,q}^+ {\zeta_m^+}'(z){\boldsymbol{l}_l^+}^T {A_q^+}^{-1}\vec{e} = {\zeta_m^+}'(z){\boldsymbol{l}_l^+}^T\vec{e}.$$

Thus, $\Delta_m^+(z,j_0,\vec{e}) - {\zeta_m^+}'(z){\boldsymbol{l}_l^+}^T\vec{e}$ is the m-th coefficient of the vector

$$N^+(z,j_0)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - N^{+,\infty}(z)^{-1} \begin{pmatrix} {A_q^+}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We then just have to find bounds for this difference of vectors. We have

$$\begin{aligned} &N^+(z,j_0)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - N^{+,\infty}(z)^{-1} \begin{pmatrix} {A_q^\pm}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ =&N^+(z,j_0)^{-1}\left(N^{+,\infty}(z) - N^+(z,j_0)\right)N^{+,\infty}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + N^{+,\infty}(z)^{-1} \begin{pmatrix} (A_{j_0,q}^{-1} - {A_q^+}^{-1})\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{4.52}$$

We wish to bound each term in the right-hand side of the equality (4.52). Let us start by looking at the first term. Lemma 4.3 implies that the functions $N^+(\cdot, j_0)^{-1}$ are bounded on $B(1,\delta_1)$ and that the bound can considered to be uniform for $j_0 \in \mathbb{N}$. The function $N^{+,\infty}(\cdot)^{-1}$ is also bounded on $B(1,\delta_1)$. Since $A_{j_0,q}^{-1}$ converges towards A_q^{+-1} as j_0 converges towards $+\infty$, we also have that the family of matrices $(A_{j_0,q}^{-1})_{j_0\in\mathbb{N}}$ is bounded. Finally, using Lemma 4.3, we have that there exist two constants $C, c > 0$ such that

$$\forall z \in B(1,\delta_1), \forall j_0 \in \mathbb{N}, \quad \left|N^{+,\infty}(z) - N^+(z,j_0)\right| \leq Ce^{-cj_0}.$$

Thus, we have that there exists another constant $C > 0$ such that

$$\forall z \in B(1,\delta_1), \forall \vec{e} \in \mathbb{C}^d, \forall j_0 \in \mathbb{N}, \quad \left| N^+(z,j_0)^{-1}\left(N^{+,\infty}(z) - N^+(z,j_0)\right) N^{+,\infty}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \leq C|\vec{e}|e^{-cj_0}.$$

We now focus on the second term. The function $N^{\pm,\infty}(\cdot)^{-1}$ is bounded on $B(1,\delta_1)$. Furthermore, Hypothesis 3 allows us to determine that there exist two constants $C, c > 0$ such that

$$\forall j_0 \in \mathbb{N}, \quad \left|A_{j_0,q}^{-1} - A_q^{+-1}\right| \leq Ce^{-cj_0}.$$

Therefore, there exists a new constant C such that

$$\forall z \in B(1,\delta_1), \forall \vec{e} \in \mathbb{C}^d, \forall j_0 \in \mathbb{N}, \quad \left| N^{+,\infty}(z)^{-1} \begin{pmatrix} (A_{j_0,q}^{-1} - A_q^{+-1})\vec{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \leq C|\vec{e}|e^{-cj_0}.$$

There remains to prove (4.51i) as (4.51j) would be proved similarly. We observe that for $z \in B(1,\delta_2)$ and $j_0 \in \mathbb{N}$, we have

$$N^+(z,j_0)^{-1} - N^+(1,j_0)^{-1} = N^+(z,j_0)^{-1}\left(N^+(1,j_0) - N^+(z,j_0)\right) N^+(1,j_0)^{-1}.$$

Using (4.51a) and the observations above which claimed that $N^+(z,j_0)^{-1}$ is bounded uniformly for $z \in B(1,\delta_2)$ and $j_0 \in \mathbb{N}$, we have that there exists a positive constant C such that

$$\forall z \in B(1,\delta_2), \forall j_0 \in \mathbb{N}, \quad |N^+(z,j_0)^{-1} - N^+(1,j_0)^{-1}| \leq C|z-1|.$$

The definition (4.49) and the fact that the family of matrices $(A_{j_0,q}^{-1})_{j_0\in\mathbb{N}}$ is bounded imply that (4.51i) is verified for some constant $C > 0$. □

5 Temporal Green's function and proof of Theorem 1

The previous Sections 3 and 4 served respectively to describe the spatial Green's function far from 1 and near 1. Our objective is now to focus on the core of the article: the study of temporal Green's function and the proof of Theorem 1. In the present section, we will express the temporal Green's function with the spatial Green's function using functional analysis. We will then use the different results of the previous sections (mainly Proposition 2 and the decompositions (4.45), (4.47) and (4.48) of the spatial Green's function near 1) to obtain the result of Theorem 1. Just as when proved the decompositions (4.45), (4.47) and (4.48) of the spatial Green's function near 1, the proof of Theorem 1 will be done whilst assuming that j_0 is larger than 0 to obtain (1.30). The case where $j_0 < 0$ would be handled similarly and would necessitate to prove expressions of the spatial Green's function on $B(1,\delta_1)$ similar to (4.45)-(4.48) when $j_0 < 0$.

5.1 Link between the spatial and temporal Green's function

First, we recall that in Sections 3 and 4, we studied the vectors $W(z,j_0,j,\vec{e})$ defined in Section 3.5 which are composed of several components of the spatial Green's function. The inverse Laplace transform implies that if we introduce a path $\widetilde{\Gamma}$ that surrounds the spectrum $\sigma(\mathscr{L})$, for instance $\widetilde{\Gamma}_r := r\mathbb{S}^1$ where $r > 1$, then we have

$$\mathscr{G}(n,j_0,j)\vec{e} = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_r} z^n G(z,j_0,j)\vec{e}\,dz = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_r} z^n \Pi(W(z,j_0,j,\vec{e}))dz$$

where Π is the linear application defined by (3.34) which extracts the center values of a large vector. We consider the change of variables $z = \exp(\tau)$. If we define $\Gamma_r = \{r + it, t \in [-\pi, \pi]\}$, then we have

$$\mathscr{G}(n, j_0, j)\vec{e} = \frac{1}{2i\pi} \int_{\Gamma_r} e^{n\tau} e^{\tau} \Pi(W(e^{\tau}, j_0, j, \vec{e}))d\tau. \tag{5.1}$$

The goal will now be to use Cauchy's formula and/or the residue theorem to modify our choice of path Γ and to use at best the properties we proved on the spatial Green's function.

In Section 4, we have found a decomposition and meromorphically extended the spatial Green's function on a ball $B(1, \delta_1)$ with a pole of order 1 at 1 and introduced an even smaller ball $B(1, \delta_2)$ on which we have more precise bounds (Lemma 4.7) that will help us later on in the proof. We consider a radius $\varepsilon_0^\star \in]0, \pi[$ such that

$$\forall \tau \in B(0, \varepsilon_0^\star), \quad e^{\tau} \in B(1, \delta_2).$$

Lemma 5.1. *For all radii $\varepsilon \in]0, \varepsilon_0^\star[$, there exists a width $\eta_\varepsilon > 0$ such that if we define*

$$\Omega_\varepsilon := \{\tau \in \mathbb{C}, \quad \Re\tau \in]-\eta_\varepsilon, \pi], \Im\tau \in [-\pi, \pi]\} \cup B(0, \varepsilon),$$

then for all $j, j_0 \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$, the function $\tau \mapsto W(e^{\tau}, j_0, j, \vec{e})$ is meromorphically defined on $\Omega_\varepsilon \setminus \{0\}$ with a pole of order 1 at 0 and there exist two positive constants $C_\varepsilon, c_\varepsilon$ such that

$$\forall \tau \in \Omega_\varepsilon \setminus B(0, \varepsilon), \forall j, j_0 \in \mathbb{Z}, \forall \vec{e} \in \mathbb{C}^d, \quad |W(e^{\tau}, j_0, j, \vec{e})| \leq C_\varepsilon |\vec{e}| e^{-c_\varepsilon |j - j_0|}. \tag{5.2}$$

Defining this width η_ε is important for the following calculations since we have defined a set Ω_ε on which we can change the path of integration of (5.1) using the residue theorem. Furthermore, for $\tau \in \Omega_\varepsilon$, either $\tau \in B(0, \varepsilon)$ which implies that $e^{\tau} \in B(1, \delta_2)$ and that we can thus use the decomposition of the spatial Green's function we obtained in Section 4.4, or $\tau \notin B(0, \varepsilon)$ and we can use (5.2) to obtain exponential bounds on the spatial Green's function.

Proof The proof is identical as [Coe23, Lemma 5.2] and will thus not be detailled. It just relies on observing that for any radius $\varepsilon \in]0, \varepsilon_0^\star[$, the results of Section 4 imply that for all $j_0, j \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$, the function $\tau \mapsto W(e^{\tau}, j_0, j, \vec{e})$ is meromorphically extended on $B(0, \varepsilon) \setminus \{0\}$ with a pole of order 1 at 0. We then use Proposition 2 on a neighborhood of each point of the set

$$U_\varepsilon := \{\tau \in \mathbb{C}, \quad \Re\tau \in [0, \pi], \Im\tau \in [-\pi, \pi]\} \setminus B(0, \varepsilon)$$

and conclude via a compactness argument on the existence of a width $\eta_\varepsilon \in]0, \varepsilon[$ and of two positive constants $C_\varepsilon, c_\varepsilon$ such that (5.2) is verified. □

Let us observe that for all $m \in I_{cs}^{\pm} \cup I_{cu}^{\pm}$, we have that the function $\zeta_m^{\pm}$ (which we recall are defined in Section 4.1 and are eigenvalues of the matrices $M^{\pm}(z)$) is holomorphic and $\zeta_m^{\pm}(1) = 1$. Therefore, there exists a radius $\varepsilon_1^\star \in]0, \varepsilon_0^\star[$ so that for all $m \in I_{cs}^{\pm} \cup I_{cu}^{\pm}$ that we write as $m = l + (k-1)d$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$, there exists an holomorphic function $\varpi_l^{\pm} : B(0, \varepsilon_1^\star) \to \mathbb{C}$ such that $\varpi_l^{\pm}(0) = 0$ and

$$\forall \tau \in B(0, \varepsilon_1^\star), \quad \zeta_m^{\pm}(e^{\tau}) = \exp(\varpi_l^{\pm}(\tau)). \tag{5.3}$$

Since $\zeta_m^{\pm}(e^{\tau})$ is an eigenvalue of $M_l^{\pm}(e^{\tau})$, Lemma 3.2 implies that

$$\forall \tau \in B(0, \varepsilon_1^\star), \quad \mathcal{F}_l^{\pm}(e^{\varpi_l^{\pm}(\tau)}) = e^{\tau}.$$

If we define the holomorphic function

$$\begin{array}{rccc} \varphi_l^{\pm} : & \mathbb{C} & \to & \mathbb{C} \\ & \tau & \mapsto & -\frac{\tau}{\alpha_l^{\pm}} + (-1)^{\mu+1} \frac{\beta_l^{\pm}}{\alpha_l^{\pm 2\mu+1}} \tau^{2\mu} \end{array}, \tag{5.4}$$

then, up to considering $\varepsilon_1^\star$ to be slightly smaller, the asymptotic expansion (1.18) implies that there exists a bounded holomorphic function $\xi_l^{\pm} : B(0, \varepsilon_1^\star) \to \mathbb{C}$ such that

$$\forall \tau \in B(0, \varepsilon_1^\star), \quad \varpi_l^{\pm}(\tau) = \varphi_l^{\pm}(\tau) + \tau^{2\mu+1} \xi_l^{\pm}(\tau). \tag{5.5}$$

Lemma 5.2. *There exists a radius $\varepsilon_2^\star \in]0, \varepsilon_1^\star[$ and two positive constants A_R, A_I such that for all $l \in \{1, \dots, d\}$*

$$\forall \tau \in \mathbb{C}, \qquad \alpha_l^{\pm} \Re(\varphi_l^{\pm}(\tau)) \leq -\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \Im(\tau)^{2\mu}, \tag{5.6}$$

$$\forall \tau \in B(0, \varepsilon_2^\star), \qquad \alpha_l^{\pm} \Re(\varpi_l^{\pm}(\tau)) + |\alpha_l^{\pm} \xi_l^{\pm}(\tau) \tau^{2\mu+1}| \leq -\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \Im(\tau)^{2\mu}. \tag{5.7}$$

The proof is identical as [Coe23, Lemma 5.3] and will thus not be detailed here.

Choice of the radius ε and of the width η

We will now fix choices for a radius $\varepsilon > 0$ and a width $\eta > 0$ which will satisfy a list of conditions. Those conditions will be are centralized here in order to fix the notations and are especially important to prove some technical lemmas in Section 5.2.1. We will try to indicate at best where those conditions are used.

First, we fix a choice of radius $\varepsilon \in \left]0, \min\left(\varepsilon_2^\star, \left(\frac{1}{2\mu A_R}\right)^{\frac{1}{2\mu-1}}\right)\right[$ where the radius $\varepsilon_2^\star$ is defined in Lemma 5.2. This choice for ε will allow us to use the results of Lemmas 5.1 and 5.2. Furthermore, if we introduce the function

$$\begin{array}{rccc} \Psi: & \mathbb{R} & \to & \mathbb{R} \\ & \tau_p & \mapsto & \tau_p - A_R {\tau_p}^{2\mu} \end{array} \tag{5.8}$$

which we will use to define a family of parameterized curve later on in Lemma 5.4, then the function Ψ is continuous and strictly increasing on $]-\infty, \varepsilon]$. This conclusion on the function Ψ will be essential in the proof of Lemma 5.4 to construct the path Γ appearing in (5.17c).

We now introduce the function

$$\begin{array}{rccc} r_\varepsilon: &]0, \varepsilon[& \to & \mathbb{R} \\ & \eta & \mapsto & \sqrt{\varepsilon^2 - \eta^2} \end{array} \tag{5.9}$$

which serves to define the extremities of the curve $-\eta + i\mathbb{R} \cap B(0, \varepsilon)$. We recall that we defined a width η_ε in Lemma 5.1. We claim that there exists a width $\eta \in]0, \eta_\varepsilon[$ that we fix for the rest of the paper such that:

- The following inequality is satisfied:
$$\frac{\eta}{2} > A_R \eta^{2\mu}. \tag{5.10}$$
It is used for instance in the proof of Lemma 5.4.

- We have
$$\eta + A_R \eta^{2\mu} - \frac{A_I}{2} r_\varepsilon(\eta)^{2\mu} < 0. \tag{5.11}$$
It is quite clear that we can choose η small enough to satisfy this condition since, when η tends towards 0, the first two terms on the left hand side converge towards 0 and the third converges towards $-\frac{A_I}{2}\varepsilon$. The condition (5.11) is used in Lemma 5.4 to prove (5.17b). A consequence of (5.11) is that
$$\forall n \in \mathbb{N}, \forall x \in \left[\frac{n}{2}, 2n\right], \forall t \in [-\eta, \eta], \quad (n-x)t + x A_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq 0. \tag{5.12}$$
Indeed, using the convexity with regards to t of the left hand side of (5.12), we have that
$$(n-x)t + x A_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq |n-x|\eta + x A_R \eta^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2}.$$
We observe that $n \in \left[\frac{x}{2}, 2x\right]$ and thus, using (5.11), we have
$$(n-x)t + x A_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq x\left(\eta + A_R \eta^{2\mu} - \frac{A_I r_\varepsilon(\eta)^{2\mu}}{2}\right) \leq 0.$$
The consequence (5.12) of (5.11) will be used in the proof of Lemma 5.6.

- There exists a radius $\varepsilon_\# \in]0, \varepsilon[$ such that if we define
$$l_{extr} := \left(\frac{\Psi(\varepsilon_\#) - \Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}, \tag{5.13}$$
then $-\eta + i l_{extr} \in B(0, \varepsilon)$. It is used in the proof of Lemma 5.4.

We introduce the paths $\Gamma_{out}(\eta)$, $\Gamma_{in}^{\pm}(\eta)$, $\Gamma_{in}^{0}(\eta)$, $\Gamma_{in}(\eta)$, $\Gamma(\eta)$, $\Gamma_d(\eta)$ represented on Figure 2 and defined as

$$\begin{aligned} \Gamma_{out}(\eta) &:= [-\eta - i\pi, -\eta - i r_\varepsilon(\eta)] \cup [-\eta + i r_\varepsilon(\eta), -\eta + i\pi], \\ \Gamma_{in}^{\pm}(\eta) &:= [-\eta \pm i r_\varepsilon(\eta), \eta \pm i r_\varepsilon(\eta)], \\ \Gamma_{in}^{0}(\eta) &:= [\eta - i r_\varepsilon(\eta), \eta + i r_\varepsilon(\eta)], \\ \Gamma_{in}(\eta) &:= \Gamma_{in}^{-}(\eta) \cup \Gamma_{in}^{0}(\eta) \cup \Gamma_{in}^{+}(\eta) \\ \Gamma(\eta) &:= \Gamma_{in}(\eta) \cup \Gamma_{out}(\eta), \\ \Gamma_d(\eta) &:= [-\eta - i r_\varepsilon(\eta), -\eta + i r_\varepsilon(\eta)]. \end{aligned} \tag{5.14}$$

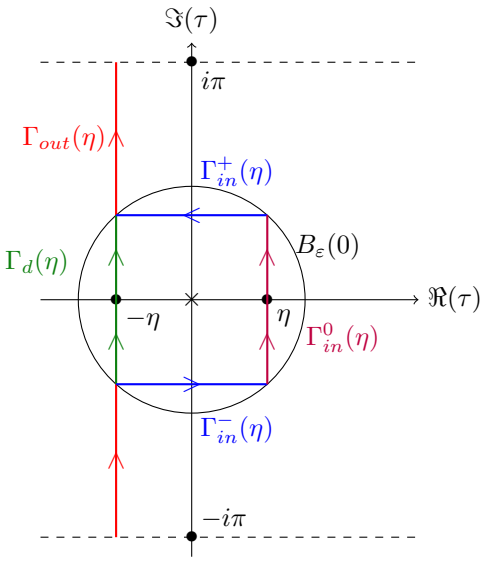

Figure 2: A representation of the path described in (5.14): $\Gamma_{out}(\eta)$ (in red), $\Gamma^{\pm}_{in}(\eta)$ (in blue), $\Gamma^{0}_{in}(\eta)$ (in purple) and $\Gamma_d(\eta)$ (in green)

We observe that those paths lie in Ω_ε. Using the Cauchy formula and acknowledging the "$2i\pi$"-periodicity of $\tau \mapsto W(e^\tau, j_0, j, \vec{e})$, we can prove via the equality (5.1) that

$$\begin{aligned}\mathscr{G}(n, j_0, j)\vec{e} &= \frac{1}{2i\pi}\int_{\Gamma(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau \\ &= \frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau + \frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau.\end{aligned} \tag{5.15}$$

Lemma 5.3. *There exist two positive constants C, c such that for all $n \in \mathbb{N}$, $j_0, j \in \mathbb{Z}$ and $\vec{e} \in \mathbb{C}^d$ we have that*

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau\right| \leq C|\vec{e}|e^{-n\eta}.$$

Proof The conclusion of the lemma directly follows from (5.2) and the definition of $\Gamma_{out}(\eta)$ which implies that

$$\forall \tau \in \Gamma_{out}(\eta), \quad |e^{n\tau}| = e^{-n\eta}.$$

□

The equality (5.15) and the sharp exponential bounds on

$$\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau,$$

we just proved imply that this term will belong to the residual term $\mathcal{R}(n, j_0, j)$ in (1.30). There just remains to handle the term

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau}\Pi(W(e^\tau, j_0, j, \vec{e}))d\tau \tag{5.16}$$

to have the description (1.30) of the temporal Green's function expected in Theorem 1. We recall that $\Gamma_{in}(\eta)$ is a path that lies inside the set $B(0, \varepsilon)$ by construction and that we chose the radius $\varepsilon \in]0, \varepsilon_2^*[$ to be small enough so that

$$\forall \tau \in B(0, \varepsilon), \quad e^\tau \in B(1, \delta_2).$$

Thus, recalling that we consider $j_0 \geq 0$, we can use the expressions (4.45), (4.47) and (4.48) to decompose the integral (5.16) into different terms depending on the position of j with respect to 0 and j_0. Our goal is to associate those terms with the different behaviors of the temporal Green's function presented in (1.30) of

Theorem 1 using Lemmas 5.7-5.11 that we will proven below. To clarify, let us give details on one of the cases. If we consider $j \geq j_0 + 1$, then (4.45) implies that

$$\begin{aligned}\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\Pi\left(W(e^{\tau},j_0,j,\vec{e})\right)d\tau = &-\sum_{m=1}^{dp}\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}^{+}_{m}(e^{\tau},j_0,\vec{e})\Pi\left(W^{+}_{m}(e^{\tau},j)\right)d\tau\\ &-\sum_{m=2}^{dp}\sum_{m'=dp+1}^{d(p+q)}\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',m}(e^{\tau})\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau},j_0,\vec{e})\Pi\left(W^{+}_{m}(e^{\tau},j)\right)d\tau\\ &-\sum_{m'=dp+1}^{d(p+q)}\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',1}(e^{\tau})\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau},j_0,\vec{e})\Pi\left(\Phi_1(e^{\tau},j)\right)d\tau.\end{aligned}$$

Using Lemma 5.7 proved below, we can prove that the first terms are either belonging to the residual terms $\mathcal{R}(n,j_0,j,\vec{e})$ when $m \in I_{ss}$ or, when $m \in I_{cs}$, they are a combination of a generalized Gaussian waves $S^{+}_{l}(n,j_0,j)\vec{e}$ for some $l \in \{1,\dots,I\}$ and residual terms $\mathcal{R}(n,j_0,j,\vec{e})$. In a similar manner, using Lemmas 5.8, 5.10 and 5.11 proven below, the second and third terms can be shown to relate to the reflected waves $R^{+}_{l',l}(n,j_0,j)\vec{e}$, the excited eigenvector $E^{+}_{l'}(n,j_0)\vec{e}V_j$ and residual terms $\mathcal{R}(n,j_0,j,\vec{e})$. We observe that the transmitted waves $T^{+}_{l',l}(n,j_0,j)\vec{e})$ and the generalized Gaussian waves $S^{+}_{l}(n,j_0,j)\vec{e}$ for $l \in \{I+1,\dots,d\}$ are equal to 0 in this setting. Thus, we obtain the decomposition (1.30).

Thus, once we will have proved Lemmas 5.7-5.11, the proof of Theorem 1 will be concluded.

5.2 Decomposition of the integral within $B(0,\varepsilon)$

This section will be mainly devoted to the proof of Lemmas 5.7-5.11 that will allow to study each term that can appear in the decomposition of the integral (5.16) using the expressions (4.45), (4.47) and (4.48). However, we are first going to need to introduce a few more technical lemmas that will be used relentlessly throughout the rest of the paper.

5.2.1 Gaussian estimates

First and foremost, we define the set X of the paths going from $-\eta - ir_{\varepsilon}(\eta)$ to $-\eta + ir_{\varepsilon}(\eta)$ whilst remaining in $B(0,\varepsilon)$. We observe in particular that $\Gamma_d(\eta), \Gamma_{in}(\eta) \in X$.

Lemma 5.4. *We consider an integer $k \in \mathbb{N}$.*

- *There exist two positive constants C, c such that for all $n \in \mathbb{N}\setminus\{0\}$ and $x \in \left[0, \frac{n}{2}\right]$*

$$\int_{\Gamma_d(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq Ce^{-cn}. \tag{5.17a}$$

- *There exist two positive constants C, c such that for all $n \in \mathbb{N}\setminus\{0\}$ and $x \in [2n, +\infty[$*

$$\int_{\Gamma_{in}(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq Ce^{-cn}. \tag{5.17b}$$

- *There exist two positive constants C, c such that for all $n \in \mathbb{N}\setminus\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$, there exists a path $\Gamma \in X$ such that*

$$\begin{aligned}\int_{\Gamma} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|&\\ \leq \frac{C}{n^{\frac{k+1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).&\end{aligned} \tag{5.17c}$$

Lemma 5.4 will allow us to obtain generalized Gaussian bounds for several terms throughout the proof of Theorem 1. The inequalities (5.17a)-(5.17c) separate different cases depending on x. An important point to observe is that the path Γ appearing in (5.17c) depends on n and x whereas the constants C, c are uniform.

The way Lemma 5.4 will be used is to first observe that the integral of some holomorphic function over some path of X is equal by Cauchy's formula to the integral of the same function over any path of X. We then prove that the integrand can be well bounded and use the result of the lemma. The proof of Lemma 5.4 can be adapted from [CF22, CF21, Coe22, Coe23] and will be done in the Appendix (Section 6).

Lemma 5.5. *There exists a constant $C>0$ such that for all $l\in\{1,\dots,d\}$, $n\in\mathbb{N}\setminus\{0\}$ and $x\in[0,2n]$*

$$\left|e^{x\alpha_l^{\pm}\varpi_l^{\pm}(\tau)}-e^{x\alpha_l^{\pm}\varphi_l^{\pm}(\tau)}\right|\leq Cn|\tau|^{2\mu+1}\exp\left(x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right).$$

The proof of Lemma 5.5 is similar to the proof of [Coe22, Lemma 16] and will not be detailed here. We recall that the functions $\varpi_l^{\pm}$ and $\varphi_l^{\pm}$ respectively defined by (5.3) and (5.4) are linked by the equality (5.5). Lemma 5.5 will allow us to "extract" the principal part $\varphi_l^{\pm}$ of the function $\varpi_l^{\pm}$. This principal part will then appear in terms that can be studied using the following lemma.

Lemma 5.6. *We consider $l,l'\in\{1,\dots,d\}$ and $?,?'\in\{-,+\}$. There exist two constants $C,c>0$ such that for all $n\in\mathbb{N}\setminus\{0\}$, we have:*

- *For $x,y\in[0,+\infty[$ such that $x+y\in\left[\frac{n}{2},2n\right]$ and $\Gamma\in X$*

$$\left|\frac{1}{2i\pi}\int_\Gamma\exp\left(n\tau+x\alpha_l^?\varphi_l^?(\tau)+y\alpha_{l'}^{?'}\varphi_{l'}^{?'}(\tau)\right)d\tau-\frac{|\alpha_l^?|}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{x}{n}\beta_l^?+\frac{y}{n}\beta_{l'}^{?'}\left(\frac{\alpha_l^?}{\alpha_{l'}^{?'}}\right)^{2\mu};\frac{\alpha_l^?(n-(x+y))}{n^{\frac{1}{2\mu}}}\right)\right|\leq Ce^{-cn}. \tag{5.18a}$$

- *For $x\in\left[\frac{n}{2},2n\right]$ and $\Gamma\in X$*

$$\left|\frac{1}{2i\pi}\int_\Gamma\exp\left(n\tau+x\alpha_l^?\varphi_l^?(\tau)\right)d\tau-\frac{|\alpha_l^?|}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\beta_l^?;\frac{\alpha_l^?(n-x)}{n^{\frac{1}{2\mu}}}\right)\right|\leq\frac{C}{n^{\frac{1}{\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{5.18b}$$

- *For $x\in\left[\frac{n}{2},2n\right]$,*

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^?\varphi_l^?(\tau)\right)}{\tau}d\tau-E_{2\mu}\left(\beta_l^?;\frac{-|\alpha_l^?|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right|\leq\frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{5.18c}$$

The proof of Lemma 5.6 is a summary of calculations performed in [Coe22, Coe23] and will be done in the Appendix (Section 6). Let us consider that there is no condition on the paths on integration in (5.18a) and (5.18b). However, since the integrand is only meromoprhic in (5.18a), we only consider the path $\Gamma_{in}(\eta)$.

5.2.2 Outgoing and entering waves

We will start by looking at the outgoing and entering waves by proving the following lemma.

Lemma 5.7. *We consider $m\in\{1,\dots,d(p+q)\}$ and write it as $m=l+(k-1)d$ with $k\in\{1,\dots,p+q\}$ and $l\in\{1,\dots,d\}$. There exists a constant $c>0$ such that for all $n\in\mathbb{N}\setminus\{0\}$, $j_0,j\in\mathbb{N}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\vec{e}\in\mathbb{C}^d$ we have:*

- *If $m\in I_{cs}^+\cup I_{cu}^+$ and $\frac{j-j_0}{\alpha_l^+}\in\left[\frac{n}{2},2n\right]$, we have*

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau-S_l^+(n,j_0,j)\vec{e}$$
$$=\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\vec{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_s\left(\frac{|\vec{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^+ +O_s\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{l}_l^{+T}\vec{e}\boldsymbol{r}_l^+\right). \tag{5.19a}$$

- *If $m\in I_{cs}^+\cup I_{cu}^+$, $\frac{j-j_0}{\alpha_l^+}\notin\left[\frac{n}{2},2n\right]$ and $\frac{j-j_0}{\alpha_l^+}\geq 0$, we have*

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau-S_l^+(n,j_0,j)\vec{e}=O\left(|\vec{e}|e^{-cn}\right). \tag{5.19b}$$

- *If $m\in I_{ss}^+$ and $j\geq j_0+1$ or if $m\in I_{su}^+$ and $j\in\{0,\dots,j_0\}$*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau=O\left(|\vec{e}|e^{-cn}\right). \tag{5.19c}$$

Proof • We start by proving (5.19a). The proofs of (5.19b) and (5.19c) will be done afterwards as they are fairly less complicated. We consider $m \in I_{cs}^+ \cup I_{cu}^+$ such that $\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$.

Using the expression of W_m^+ given by Lemma 4.3 and (4.50), we have using Cauchy's formula that for any $\Gamma \in X$

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^+(e^\tau, j)) d\tau = \int_{\Gamma} e^{n\tau} \zeta_m^+(e^\tau)^{j-j_0-1} e^\tau \Delta_m^+(e^\tau, j_0, \vec{e}) \Pi(V_m^+(e^\tau, j)) d\tau. \tag{5.20}$$

Using Cauchy's formula once again, (5.3) since the eigenvalue we consider is central and the definition (1.29a) of the function S_l^+, we then have that

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi} \int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^+(e^\tau, j)) d\tau - S_l^+(n, j_0, j)\vec{e} = E_1 + E_2 \boldsymbol{r}_l^+ + (E_3 + E_4 + E_5)\, \boldsymbol{l}_l^{+T} \vec{e} \boldsymbol{r}_l^+ \tag{5.21}$$

where E_1 is a vector and $E_2, \dots, E_5$ are complex scalars defined by

$$\begin{aligned}
E_1 &= -\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi} \int_{\Gamma_1} e^{n\tau} e^{(j-j_0-1)\varpi_l^+(\tau)} e^\tau \Delta_m^+(e^\tau, j_0, \vec{e}) \Pi(V_m^+(e^\tau, j) - R_m^+(e^\tau)) d\tau \\
E_2 &= -\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi} \int_{\Gamma_2} e^{n\tau} e^{(j-j_0-1)\varpi_l^+(\tau)} e^\tau \left(\Delta_m^+(e^\tau, j_0, \vec{e}) - \zeta_m^{+\prime}(e^\tau) \boldsymbol{l}_l^{+T} \vec{e} \right) d\tau \\
E_3 &= -\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi} \int_{\Gamma_3} e^{n\tau} e^{(j-j_0)\varpi_l^+(\tau)} \left(e^\tau \zeta_m^+(e^\tau)^{-1} \zeta_m^{+\prime}(e^\tau) + \frac{1}{\alpha_l^+} \right) d\tau \\
E_4 &= \frac{1}{2i\pi|\alpha_l^+|} \int_{\Gamma_4} e^{n\tau} \left(e^{(j-j_0)\varpi_l^+(\tau)} - e^{(j-j_0)\varphi_l^+(\tau)} \right) d\tau \\
E_5 &= \frac{1}{2i\pi|\alpha_l^+|} \int_{\Gamma_5} e^{n\tau} e^{(j-j_0)\varphi_l^+(\tau)} d\tau - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{n\alpha_l^+ + j_0 - j}{n^{\frac{1}{2\mu}}} \right)
\end{aligned}$$

and $\Gamma_1, \dots, \Gamma_5$ are paths belonging to X. We just have to prove correct bounds on the terms $E_1, \dots, E_5$ appearing in (5.21) to obtain (5.19a). In particular, we will use good choices of paths $\Gamma_1, \dots, \Gamma_5$ to optimize the bounds using Lemma 5.4.

► Using (4.51g), (5.7) and Lemma 4.3 which claims that the vectors $V_m^+(z, j)$ converge exponentially fast towards $R_m^+(e^\tau)$, we have that there exist two positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq Ce^{-c|j|}|\vec{e}| \int_{\Gamma_1} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using (4.51e) and (5.7), we have that there exist two positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$|E_2| \leq Ce^{-c|j_0|}|\vec{e}| \int_{\Gamma_2} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

► We notice that $\zeta_m^+(1) = 1$ and $\zeta_m^{+\prime}(1) = -\frac{1}{\alpha_l^+}$. Using a Taylor expansion and (5.7), we have that there exists a positive constant C independent from n, j_0 and j such that

$$|E_3| \leq C \int_{\Gamma_3} |\tau| \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Since $\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$, we can use Lemma 5.5 and prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_4| \leq Cn \int_{\Gamma_4} |\tau|^{2\mu+1} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 5.4 which gives a good choices of path $\Gamma_1, \dots, \Gamma_4 \in X$ depending on n, j_0 and j to handle the integrals in the terms above as well as Lemma 5.6 to take care of the term E_5, there exist new constants

$C, c > 0$ independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq \frac{Ce^{-c|j|}|\vec{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad |E_2| \leq \frac{Ce^{-c|j_0|}|\vec{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad |E_4| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_5| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

We have thus obtained (5.19a).

• We now focus on (5.19b). We thus consider $m \in I_{cs}^+$ and $j \geq j_0 + 1$ or $m \in I_{cu}^+$ and $j \in \{0, \dots, j_0\}$ such that $\frac{j-j_0}{\alpha_l^+} \notin \left[\frac{n}{2}, 2n\right]$. We observe that in particular, $S_l^+(n, j_0, j) = 0$. Using (5.20), (5.3) since the eigenvalue we consider is central, Lemma 4.3 which claims that the vectors $V_m^+(z, j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, (4.51g) and (5.7), there exists a positive constant C independent from n, j_0, j and $\vec{e}$ such that for all $\Gamma \in X$

$$\left|-\frac{\mathrm{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}_m^+}(e^\tau, j_0, \vec{e})\Pi(W_m^+(e^\tau, j))d\tau - S_l^+(n, j_0, j)\vec{e}\right|$$
$$\leq C|\vec{e}|\int_\Gamma \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

We then use Lemma 5.4 to prove that there exist two new positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$\left|-\frac{\mathrm{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}_m^+}(e^\tau, j_0, \vec{e})\Pi(W_m^+(e^\tau, j))d\tau - S_l^+(n, j_0, j)\vec{e}\right| \leq C|\vec{e}|e^{-cn}.$$

• We now focus on (5.19c). We will consider the case where the integer m belongs to I_{cs}^+ and $j \geq j_0 + 1$. The second case considered in (5.19c) would be handled similarly. Using (5.20), Lemma 4.3 which claims that the vectors $V_m^+(z, j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, (4.51g) and (4.1) whilst noticing that $j \geq j_0 + 1$, there exists a positive constant C independent from n, j_0, j and $\vec{e}$ such that

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}_m^+}(e^\tau, j_0, \vec{e})\Pi(W_m^+(e^\tau, j))d\tau\right| \leq C|\vec{e}|e^{-2c_*|j-j_0|}\int_{\Gamma_d(\eta)} e^{n\Re(\tau)}|d\tau|$$
$$\leq 2r_\varepsilon(\eta)C|\vec{e}|e^{-n\eta-2c_*|j-j_0|}.$$

□

5.2.3 Reflected waves

We now look at the reflected waves.

Lemma 5.8. *We consider $m \in \{2, \dots, dp\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$ and write them as $m = l + (k-1)d$ and $m' = l' + (k'-1)d$ with $k, k' \in \{1, \dots, p+q\}$ and $l, l' \in \{1, \dots, d\}$. There exists a positive constant c such that for all $n \in \mathbb{N}\setminus\{0\}$, $j_0, j \in \mathbb{N}$ such that $j - j_0 \in \{-nq, \dots, np\}$ and $\vec{e} \in \mathbb{C}^d$, we have:*

- *If $m \in I^+_{cs}$, $m' \in I^+_{cu}$ and $\frac{j}{\alpha^+_l} - \frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau - \frac{\alpha^+_l}{\alpha^+_{l'}} \widetilde{g}^+_{m',m}(1) R^+_{l',l}(n, j_0, j)\vec{e}$$
$$= \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha^+_l} - \frac{j_0}{\alpha^+_{l'}}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\vec{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right) + O_s\left(\frac{|\vec{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}^+_l + O_s\left(\frac{1}{n^{\frac{1}{\mu}}}\right){\boldsymbol{l}^+_{l'}}^T \vec{e}\boldsymbol{r}^+_l\right). \tag{5.22a}$$

- *If $m \in I^+_{cs}$, $m' \in I^+_{cu}$ and $\frac{j}{\alpha^+_l} - \frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau - \frac{\alpha^+_l}{\alpha^+_{l'}} \widetilde{g}^+_{m',m}(1) R^+_{l',l}(n, j_0, j)\vec{e} = O(|\vec{e}|e^{-cn}). \tag{5.22b}$$

- *If $m \in I^+_{ss}$, $m' \in I^+_{cu}$ and $-\frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau = O\left(\frac{|\vec{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n + \frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right). \tag{5.22c}$$

- *If $m \in I^+_{ss}$, $m' \in I^+_{cu}$ and $-\frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau = O\left(|\vec{e}|e^{-cn}\right). \tag{5.22d}$$

- *If $m \in I^+_{cs}$, $m' \in I^+_{su}$ and $\frac{j}{\alpha^+_l} \in \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau$$
$$= O(|\vec{e}|e^{-cn}) + O_s\left(\frac{|\vec{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j}{\alpha^+_l}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)\boldsymbol{r}^+_l. \tag{5.22e}$$

- *If $m \in I^+_{cs}$, $m' \in I^+_{su}$ and $\frac{j}{\alpha^+_l} \notin \left[\frac{n}{2}, 2n\right]$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau = O\left(|\vec{e}|e^{-cn}\right). \tag{5.22f}$$

- *If $m \in I^+_{ss}$, $m' \in I^+_{su}$, we have*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau = O\left(|\vec{e}|e^{-cn}\right). \tag{5.22g}$$

We observe that since we consider $m \in \{2, \dots, dp\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$, Lemma 4.6 implies that $\widetilde{g}^+_{m',m}$ can be holomorphically extended on the whole ball $B(1, \delta_1)$ and thus the term $\widetilde{g}^+_{m',m}(1)$ is well defined.

Proof • We start by proving (5.22a). We consider that $m \in I^+_{cs}$, $m' \in I^+_{cu}$ and $\frac{j}{\alpha^+_l} - \frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$.

Since the function $\widetilde{g}^+_{m',m}$ can be holomorphically extended on the whole ball $B(1, \varepsilon)$, using the expression of W^+_m given by Lemma 4.3 and (4.50), we have using Cauchy's formula that for any $\Gamma \in X$

$$\int_{\Gamma_{in}(\eta)} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau$$
$$= \int_{\Gamma} e^{n\tau} \widetilde{g}^+_{m',m}(e^\tau) \zeta^+_m(e^\tau)^j \zeta^+_{m'}(e^\tau)^{-j_0-1} e^{\tau} \Delta^+_{m'}(e^\tau, j_0, \vec{e}) \Pi(V^+_m(e^\tau, j)) d\tau. \tag{5.23}$$

Using Cauchy's formula once again, (5.3) since the eigenvalues we consider are central and (1.29b), we then have that

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',m}(e^{\tau})\widetilde{\mathscr{C}}^{+}_{m}(e^{\tau},j_0,\vec{e})\Pi(W^{+}_{m}(e^{\tau},j))d\tau - \frac{\alpha^{+}_{l}}{\alpha^{+}_{l'}}\widetilde{g}^{+}_{m',m}(1)R^{+}_{l',l}(n,j_0,j)\vec{e}$$
$$= E_1 + E_2\boldsymbol{r}^{+}_{l} + (E_3+E_4+E_5+E_6)\,\boldsymbol{l}^{+\,T}_{l'}\vec{e}\boldsymbol{r}^{+}_{l} \quad (5.24)$$

where E_1 is a vector and $E_2,\dots,E_6$ are complex scalars defined by

$$E_1 = -\frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}\widetilde{g}^{+}_{m',m}(e^{\tau})e^{j\varpi^{+}_{l}(\tau)}e^{-(j_0+1)\varpi^{+}_{l'}(\tau)}e^{\tau}\Delta^{+}_{m'}(e^{\tau},j_0,\vec{e})\Pi(V^{+}_{m}(e^{\tau},j)-R^{+}_{m}(e^{\tau}))d\tau$$

$$E_2 = -\frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}\widetilde{g}^{+}_{m',m}(e^{\tau})e^{j\varpi^{+}_{l}(\tau)}e^{-(j_0+1)\varpi^{+}_{l'}(\tau)}e^{\tau}\left(\Delta^{+}_{m'}(e^{\tau},j_0,\vec{e}) - \left(\zeta^{+}_{m'}\right)'(e^{\tau})\boldsymbol{l}^{+\,T}_{l'}\vec{e}\right)d\tau$$

$$E_3 = -\frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{j\varpi^{+}_{l}(\tau)}e^{-j_0\varpi^{+}_{l'}(\tau)}\left(\widetilde{g}^{+}_{m',m}(e^{\tau})e^{\tau}\zeta^{+}_{m'}(e^{\tau})^{-1}\left(\zeta^{+}_{m'}\right)'(e^{\tau}) + \frac{\widetilde{g}^{+}_{m',m}(1)}{\alpha^{+}_{l'}}\right)d\tau$$

$$E_4 = \frac{\widetilde{g}^{+}_{m',m}(1)}{2i\pi\alpha^{+}_{l'}}\int_{\Gamma_4} e^{n\tau}\left(e^{j\varpi^{+}_{l}(\tau)} - e^{j\varphi^{+}_{l}(\tau)}\right)e^{-j_0\varpi^{+}_{l'}(\tau)}d\tau$$

$$E_5 = \frac{\widetilde{g}^{+}_{m',m}(1)}{2i\pi\alpha^{+}_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi^{+}_{l}(\tau)}\left(e^{-j_0\varpi^{+}_{l'}(\tau)} - e^{-j_0\varphi^{+}_{l'}(\tau)}\right)d\tau$$

$$E_6 = \frac{\widetilde{g}^{+}_{m',m}(1)}{2i\pi\alpha^{+}_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi^{+}_{l}(\tau)}e^{-j_0\varphi^{+}_{l'}(\tau)}d\tau - \frac{\alpha^{+}_{l}}{\alpha^{+}_{l'}}\widetilde{g}^{+}_{m',m}(1)\frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{j}{n\alpha^{+}_{l}}\beta^{+}_{l} - \frac{j_0}{n\alpha^{+}_{l'}}\beta^{+}_{l'}\left(\frac{\alpha^{+}_{l}}{\alpha^{+}_{l'}}\right)^{2\mu}, \frac{n\alpha^{+}_{l} + j_0\frac{\alpha^{+}_{l}}{\alpha^{+}_{l'}} - j}{n^{\frac{1}{2\mu}}}\right)$$

and $\Gamma_1,\dots,\Gamma_6$ are paths belonging to X. We just have to prove bounds on the terms $E_1,\dots,E_6$. In particular, we will use good choices of paths $\Gamma_1,\dots,\Gamma_6$ to optimize the bounds using Lemma 5.4.

▶ Using (4.51g), (5.7) and Lemma 4.3 which claims that the vectors $V^{+}_{m}(z,j)$ converge exponentially fast towards $R^{+}_{m}(e^{\tau})$, we have that there exist two positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq Ce^{-c|j|}|\vec{e}|\int_{\Gamma_1}\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Using (4.51e) and (5.7), we have that there exist two positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$|E_2| \leq Ce^{-c|j_0|}|\vec{e}|\int_{\Gamma_2}\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ We notice that $\zeta^{+}_{m'}(1) = 1$ and $\zeta^{+}_{m'}{}'(1) = -\frac{1}{\alpha^{+}_{l'}}$. Using a Taylor expansion and (5.7), we have that there exists a positive constant C independent from n, j_0 and j such that

$$|E_3| \leq C\int_{\Gamma_3}|\tau|\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ We observe that $\frac{j}{\alpha^{+}_{l}}$ and $-\frac{j_0}{\alpha^{+}_{l'}}$ are positive and $\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}} \in \left[\frac{n}{2}, 2n\right]$. Thus, we have that $\frac{j}{\alpha^{+}_{l}} \in [0, 2n]$. We can then use Lemma 5.5 and (5.7) to prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_4| \leq Cn\int_{\Gamma_4}|\tau|^{2\mu+1}\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Furthermore, we also have that and $-\frac{j_0}{\alpha^{+}_{l'}} \in [0, 2n]$. We can also use Lemma 5.5 and (5.6) to prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_5| \leq Cn\int_{\Gamma_5}|\tau|^{2\mu+1}\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^{+}_{l}} - \frac{j_0}{\alpha^{+}_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 5.4 which gives a good choices of path $\Gamma_1,\dots,\Gamma_5 \in X$ depending on n, j_0 and j to handle the integrals in the terms above as well as Lemma 5.6 to take care of the term E_6, there exist new constants $C, c > 0$ independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq \frac{Ce^{-c|j|}|\vec{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad |E_2| \leq \frac{Ce^{-c|j_0|}|\vec{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad |E_4| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_5| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad |E_6| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

We have thus obtained (5.22a).

• We now focus on (5.22b). We consider that $m \in I_{cs}^+$, $m' \in I_{cu}^+$ and $\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$. We observe that in particular, $R_{l',l}^+(n,j_0,j) = 0$. Using (5.23), (5.3) since the eigenvalue we consider is central, Lemma 4.3 which claims that the vectors $V_m^+(z,j)$ are uniformly bounded for $z \in B(1,\delta_1)$ and $j \in \mathbb{N}$, (4.51g) and (5.7), there exists a positive constant C independent from n, j_0, j and $\vec{e}$ such that for all $\Gamma \in X$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau - \frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1)R_{l',l}^+(n,j_0,j)\vec{e}\right|$$
$$\leq C|\vec{e}|\int_\Gamma \exp\left(n\Re(\tau)+\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

We then use Lemma 5.4 to prove that there exist two new positive constants C, c independent from n, j_0, j and $\vec{e}$ such that

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau - \frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1)R_{l',l}^+(n,j_0,j)\vec{e}\right| \leq C|\vec{e}|e^{-cn}.$$

• We now focus on (5.22c) and (5.22d). We consider that $m \in I_{ss}^+$, $m' \in I_{cu}^+$. Using (5.23), (4.1) to bound ζ_m^+, (5.3) since the index m' belongs to I_{cu}^+, Lemma 4.3 which claims that the vectors $V_m^+(z,j)$ are uniformly bounded for $z \in B(1,\delta_1)$ and $j \in \mathbb{N}$, (4.51g) and (5.7), there exists a positive constant C independent from n, j_0, j and $\vec{e}$ such that for all $\Gamma \in X$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau - \frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1)R_{l',l}^+(n,j_0,j)\vec{e}\right|$$
$$\leq Ce^{-2c_\star|j|}|\vec{e}|\int_\Gamma \exp\left(n\Re(\tau)+\left(-\frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

We observe that $-\frac{j_0}{\alpha_{l'}^+}$ is positive since m' belongs to I_{cu}^+. Using (5.17c) when $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$ and (5.17a) and (5.17b) else, we end up proving (5.22c) and (5.22d).

• We now focus on (5.22e) and (5.22f). We consider that $m \in I_{cs}^+$, $m' \in I_{su}^+$. Using (5.23), (5.3) since the index m belongs to I_{cs}^+ and Cauchy's formula, we have that

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\vec{e})\Pi(W_m^+(e^\tau,j))d\tau = E_1 + E_2\boldsymbol{r}_l^+$$

where the vector E_1 and the complex scalar E_2 are defined by

$$E_1 := \int_{\Gamma_1} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) e^{j\varpi_l^+(\tau)} \zeta^+_{m'}(e^\tau)^{-j_0-1} \Delta^+_{m'}(e^\tau, j_0, \vec{e}) \Pi(V^+_m(e^\tau, j) - R^+_m(e^\tau)) d\tau$$
$$E_2 := \int_{\Gamma_2} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) e^{j\varpi_l^+(\tau)} \zeta^+_{m'}(e^\tau)^{-j_0-1} \Delta^+_{m'}(e^\tau, j_0, \vec{e}) d\tau$$

where Γ_1, Γ_2 are paths belonging to X. Using (4.3) to bound $\zeta^+_{m'}$, Lemma 4.3 which claims that the vectors $V^+_m(z,j)$ converge exponentially fast towards $R^+_m(e^\tau)$, (4.51g) and (5.7), we prove that there exist two positive constants C, c independent from n, j_0, j, $\vec{e}$, Γ_1 and Γ_2 such that

$$|E_1| \leq C|\vec{e}| e^{-c|j|} e^{-c|j_0|} \int_{\Gamma_1} \exp\left(n\Re(\tau) + \frac{j}{\alpha_l^+} \left(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \Im(\tau)^{2\mu} \right) \right) |d\tau|,$$
$$|E_2| \leq C|\vec{e}| e^{-c|j_0|} \int_{\Gamma_2} \exp\left(n\Re(\tau) + \frac{j}{\alpha_l^+} \left(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \Im(\tau)^{2\mu} \right) \right) |d\tau|.$$

Whilst observing that $\frac{j}{\alpha_l^+}$ is positive since m belongs to I^+_{cs}, when $\frac{j}{\alpha_l^+} \notin \left[\frac{n}{2}, 2n\right]$, Lemma 5.4 allows us to prove exponential bounds with regard to n on the terms E_1 and E_2 and to thus immediately conclude the proof of (5.22f). When $\frac{j}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$, Lemma 5.4 allows us to choose Γ_1 and Γ_2 depending on n, j_0, j and $\vec{e}$ so that there exist new constants $C, c > 0$ independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq \frac{C|\vec{e}| e^{-c|j|} e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n - \frac{j}{\alpha_l^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \quad |E_2| \leq \frac{C|\vec{e}| e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n - \frac{j}{\alpha_l^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right).$$

Since $j \in \left[\frac{n}{2}, 2n\right]$, we have that there exist two other constants $C, c > 0$ independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \leq C|\vec{e}| e^{-cn}.$$

This allows us to conclude (5.22e).

• There only remains to prove (5.22g). We observe using (5.23) with $\Gamma = \Gamma_d(\eta) \in X$, (4.1) and (4.3) to bound ζ^+_m and $\zeta^+_{m'}$, (4.51g) and Lemma 4.3 which claims that the vectors $V^+_m(z,j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, we have that there exist a positive constant C independent from n, j_0, j and $\vec{e}$ such that

$$\left| -\frac{1}{2i\pi} \int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^+_m(e^\tau, j)) d\tau \right| \leq C|\vec{e}| e^{-2c_*(|j|+|j_0|)-n\eta}.$$

□

5.2.4 Transmitted waves

We now look at the transmitted waves.

Lemma 5.9. *We consider $m \in \{dp+1, \dots, d(p+q)-1\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$ and write them as $m = l + (k-1)d$ and $m' = l' + (k'-1)d$ with $k, k' \in \{1, \dots, p+q\}$ and $l, l' \in \{1, \dots, d\}$. There exists a positive constant c such that for all $n \in \mathbb{N}\setminus\{0\}$, $j_0 \in \mathbb{N}$, $j \in -\mathbb{N}$ such that $j - j_0 \in \{-nq, \dots, np\}$ and $\vec{e} \in \mathbb{C}^d$, we have:*

• *If $m \in I^-_{cu}$, $m' \in I^+_{cu}$ and $\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$, we have*

$$\frac{1}{2i\pi} \int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^+_{m',m}(e^\tau) \widetilde{\mathscr{C}}^+_m(e^\tau, j_0, \vec{e}) \Pi(W^-_m(e^\tau, j)) d\tau - \frac{\alpha_l^-}{\alpha_{l'}^+} \widetilde{g}^+_{m',m}(1) T^+_{l',l}(n, j_0, j) \vec{e}$$
$$= \exp\left(-c \left(\frac{\left| n - \left(\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^+} \right) \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \left(O\left(\frac{|\vec{e}| e^{-c|j|}}{n^{\frac{1}{2\mu}}} \right) + O_s\left(\frac{|\vec{e}| e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \right) \boldsymbol{r}_l^- + O_s\left(\frac{1}{n^{\frac{1}{\mu}}} \right) {\boldsymbol{l}_{l'}^+}^T \vec{e} \boldsymbol{r}_l^- \right). \tag{5.25a}$$

- *If* $m \in I_{cu}^-$, $m' \in I_{cu}^+$ *and* $\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau - \frac{\alpha_l^-}{\alpha_{l'}^+} \widetilde{g}_{m',m}^+(1) T_{l',l}^+(n, j_0, j)\vec{e} = O\left(|\vec{e}| e^{-cn}\right). \quad (5.25b)$$

- *If* $m \in I_{su}^-$, $m' \in I_{cu}^+$ *and* $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau = O\left(\frac{e^{-c|j|}|\vec{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n + \frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right). \quad (5.25c)$$

- *If* $m \in I_{su}^-$, $m' \in I_{cu}^+$ *and* $-\frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau = O\left(|\vec{e}| e^{-cn}\right). \quad (5.25d)$$

- *If* $m \in I_{cu}^-$, $m' \in I_{su}^+$ *and* $\frac{j}{\alpha_l^-} \in \left[\frac{n}{2}, 2n\right]$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau$$
$$= O(|\vec{e}| e^{-cn}) + O_s\left(\frac{|\vec{e}| e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j}{\alpha_l^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) \boldsymbol{r}_l^-. \quad (5.25e)$$

- *If* $m \in I_{cu}^-$, $m' \in I_{su}^+$ *and* $\frac{j}{\alpha_l^-} \notin \left[\frac{n}{2}, 2n\right]$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau = O\left(|\vec{e}| e^{-cn}\right). \quad (5.25f)$$

- *If* $m \in I_{su}^-$, $m' \in I_{su}^+$, *we have*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau = O\left(|\vec{e}| e^{-cn}\right). \quad (5.25g)$$

Just like in the case of the reflected waves, since we consider $m \in \{dp+1, \dots, d(p+q)-1\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$, Lemma 4.6 implies that $\widetilde{g}_{m',m}^+$ can be holomorphically extended on the whole ball $B(1, \delta_1)$ and thus the term $\widetilde{g}_{m',m}^+(1)$ is well defined.

Proof The proof of Lemma 5.9 is sensibly the same one as for Lemma 5.8 so the proof is left to the reader. Let us just point out that in order to prove (5.25a), we have using Cauchy's formula that

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \vec{e}) \Pi(W_m^-(e^\tau, j)) d\tau - \frac{\alpha_l^-}{\alpha_{l'}^+} \widetilde{g}_{m',m}^+(1) T_{l',l}^+(n, j_0, j)\vec{e}$$
$$= E_1 + E_2 \boldsymbol{r}_l^- + (E_3 + E_4 + E_5 + E_6)\, \boldsymbol{l}_{l'}^{+T} \vec{e} \boldsymbol{r}_l^-$$

where E_1 is a vector and $E_2, \dots, E_6$ are complex scalars defined by

$$E_1 = \frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi_l^-(\tau)}e^{-(j_0+1)\varpi_{l'}^+(\tau)}e^\tau\Delta^+_{m'}(e^\tau,j_0,\vec{e})\Pi(V_m^-(e^\tau,j)-R_m^-(e^\tau))d\tau$$

$$E_2 = \frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi_l^-(\tau)}e^{-(j_0+1)\varpi_{l'}^+(\tau)}e^\tau\left(\Delta^+_{m'}(e^\tau,j_0,\vec{e})-\left(\zeta^+_{m'}\right)'(e^\tau)\boldsymbol{l}_{l'}^{+\,T}\vec{e}\right)d\tau$$

$$E_3 = \frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{j\varpi_l^-(\tau)}e^{-j_0\varpi_{l'}^+(\tau)}\left(\widetilde{g}^+_{m',m}(e^\tau)e^\tau\zeta^+_{m'}(e^\tau)^{-1}\left(\zeta^+_{m'}\right)'(e^\tau)+\frac{\widetilde{g}^+_{m',m}(1)}{\alpha^+_{l'}}\right)d\tau$$

$$E_4 = -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_4} e^{n\tau}\left(e^{j\varpi_l^-(\tau)}-e^{j\varphi_l^-(\tau)}\right)e^{-j_0\varpi_{l'}^+(\tau)}d\tau$$

$$E_5 = -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi_l^-(\tau)}\left(e^{-j_0\varpi_{l'}^+(\tau)}-e^{-j_0\varphi_{l'}^+(\tau)}\right)d\tau$$

$$E_6 = -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi_l^-(\tau)}e^{-j_0\varphi_{l'}^+(\tau)}d\tau - \frac{\alpha^-_l}{\alpha^+_{l'}}\widetilde{g}^+_{m',m}(1)\frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{j}{n\alpha^-_l}\beta^-_l-\frac{j_0}{n\alpha^+_{l'}}\beta^+_{l'}\left(\frac{\alpha^-_l}{\alpha^+_{l'}}\right)^{2\mu},\frac{n\alpha^-_l+j_0\frac{\alpha^-_l}{\alpha^+_{l'}}-j}{n^{\frac{1}{2\mu}}}\right)$$

and $\Gamma_1, \dots, \Gamma_6$ are paths belonging to X. We just have to prove bounds on the terms $E_1, \dots, E_6$ just like in the proof of (5.22a). □

5.2.5 Unstable excited mode

Lemma 5.10. *There exists a positive constant c such that for all $m' \in I^+_{su}$, $n \in \mathbb{N}\setminus\{0\}$, $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$ such that $j - j_0 \in \{-nq, \dots, np\}$ and $\vec{e} \in \mathbb{C}^d$, we have*

- *For $j \in \mathbb{N}$,*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau = O_s\left(|\vec{e}|e^{-c|j_0|}\right)V(j)+O\left(|\vec{e}|e^{-cn}\right). \tag{5.26}$$

- *For $j \in -\mathbb{N}$,*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',d(p+q)}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau = O_s\left(|\vec{e}|e^{-c|j_0|}\right)V(j)+O\left(|\vec{e}|e^{-cn}\right). \tag{5.27}$$

We recall that the sequence V is defined by (4.29).

Proof We are going to prove (5.26). We consider $m' \in I^+_{su}$, $n \in \mathbb{N}\setminus\{0\}$, $j_0 \in \mathbb{N}$. For $j \in \mathbb{N}$, using the residue theorem and the equality (4.50), we have

$$\begin{aligned}-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau&\\ = -\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}\zeta^+_{m'}(e^\tau)^{-j_0-1}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\Delta^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau&\\ -\zeta^+_{m'}(1)^{-j_0-1}\Delta^+_{m'}(1,j_0,\vec{e})\mathrm{Res}(\widetilde{g}^+_{m',1},1)V(j).&\end{aligned}$$

We observe using (4.3) and (4.51g) that there exists a positive constant C independent from n, j_0, j and $\vec{e}$ such that

$$\left|\zeta^+_{m'}(1)^{-j_0-1}\Delta^+_{m'}(1,j_0,\vec{e})\mathrm{Res}(\widetilde{g}^+_{m',1},1)\right| \le C|\vec{e}|e^{-2c_*|j_0|}.$$

Furthermore, (4.3) and (4.28) imply that there exists another positive constant C independent from n, j_0, j and $\vec{e}$ such that

$$\begin{aligned}\left|-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}\zeta^+_{m'}(e^\tau)^{-j_0-1}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\Delta^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau\right|&\\ \le C|\vec{e}|\int_{\Gamma_d(\eta)} e^{n\Re(\tau)}|d\tau| \le 2r_\varepsilon(\eta)C|\vec{e}|e^{-n\eta}.&\end{aligned}$$

We thus obtain (5.26). The proof of (5.27) is fairly similar and is left to the reader. The only point to observe is that we have to use the equality (4.27). □

5.2.6 Central excited mode

Lemma 5.11. *We consider $m' \in I_{cu}^+$ and write it as $m' = l' + (k'-1)d$ with $k' \in \{1,\dots,p+q\}$ and $l' \in \{1,\dots,d\}$. There exists a positive constant c such that for all $n \in \mathbb{N}\setminus\{0\}$, $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$ such that $j - j_0 \in \{-nq,\dots,np\}$ and $\vec{e} \in \mathbb{C}^d$, we have*

- *For $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$ and $j \geq 0$,*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',1}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+,1)}{\alpha_{l'}^+}E_{l'}^+(n,j_0)\vec{e}V(j) = O\left(\frac{|\vec{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) + O_s\left(|\vec{e}|e^{-c|j_0|}\right)V(j). \quad (5.28a)$$

- *For $-\frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$ and $j \geq 0$,*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',1}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+,1)}{\alpha_{l'}^+}E_{l'}^+(n,j_0)\vec{e}V(j) = O(|\vec{e}|e^{-cn}). \quad (5.28b)$$

- *For $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$ and $j \geq 0$,*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',d(p+q)}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\vec{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+,1)}{\alpha_{l'}^+}E_{l'}^+(n,j_0)\vec{e}V(j) = O\left(\frac{|\vec{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) + O_s\left(|\vec{e}|e^{-c|j_0|}\right)V(j). \quad (5.28c)$$

- *For $-\frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$ and $j < 0$,*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',d(p+q)}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\vec{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+,1)}{\alpha_{l'}^+}E_{l'}^+(n,j_0)\vec{e}V(j) = O(|\vec{e}|e^{-cn}). \quad (5.28d)$$

We recall once again that the sequence V is defined by (4.29).

Proof We will focus on proving (5.28a) and (5.28b) as the proof of (5.28c) and (5.28d) would be similar whilst observing that (4.27) implies that

$$\forall j \in \mathbb{Z}, \quad \Phi_1(1,j) = \Phi_{d(p+q)}(1,j)$$

and the definition (4.40) of $\widetilde{g}_{m',m}^{\pm}$ and (4.41) which imply that

$$\mathrm{Res}(\widetilde{g}_{m',d(p+q)}^+,1) = -\mathrm{Res}(\widetilde{g}_{m',1}^+,1).$$

- **Proof of (5.28a):**

Using Cauchy's formula, (4.50) and (5.3) since the eigenvalue $\zeta_{m'}^+$ is central, we have

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',1}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+,1)}{\alpha_{l'}^+}E_{l'}^+(n,j_0)\vec{e}\Pi\left(\Phi_1(1,j)\right) = E_1 + E_2\Pi(\Phi(1,j)) + (E_3+E_4+E_5)\Delta_{m'}^+(1,j_0,\vec{e})\Pi(\Phi(1,j)) + E_6\Pi(\Phi(1,j)) \quad (5.29)$$

where

$$E_1 := -\frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)e^{-(j_0+1)\varpi^+_{l'}(\tau)}\Delta^+_{m'}(e^\tau,j_0,\vec{e})\Pi(\Phi_1(e^\tau,j)-\Phi_1(1,j))d\tau$$

$$E_2 := -\frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)e^{-(j_0+1)\varpi^+_{l'}(\tau)}\left(\Delta^+_{m'}(e^\tau,j_0,\vec{e})-\Delta^+_{m'}(1,j_0,\vec{e})\right)d\tau$$

$$E_3 := -\frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{-j_0\varpi^+_{l'}(\tau)}\left(e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\zeta^+_{m'}(e^\tau-1)-\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\tau}\right)d\tau$$

$$E_4 := -\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{2i\pi}\int_{\Gamma_4} e^{n\tau}\frac{e^{-j_0\varpi^+_{l'}(\tau)}-e^{-j_0\varphi^+_{l'}(\tau)}}{\tau}d\tau$$

$$E_5 := -\mathrm{Res}(\widetilde{g}^+_{m',1},1)\left(\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}\frac{e^{-j_0\varphi^+_{l'}(\tau)}}{\tau}d\tau - E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\right)$$

$$E_6 := -\mathrm{Res}(\widetilde{g}^+_{m',1},1)E_{2\mu}\left(\beta^+_{l'},\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\left(\Delta^+_{m'}(1,j_0,\vec{e})+\frac{\boldsymbol{l}^{+T}_{l'}\vec{e}}{\alpha^+_{l'}}\right)$$

and $\Gamma_1,\dots,\Gamma_4 \in X$. Let us observe that, since the function $\widetilde{g}^+_{m',1}$ has a simple pole of order 1 at 1, we have the right to use the Cauchy's formula for the first 4 terms as the functions inside the integrals can be holomorphically extended on the whole ball $B(1,\varepsilon)$.

▶ Using (4.51g) to bound $\Delta^+_{m'}(e^\tau,j_0,\vec{e})$, (4.51c) to bound $\widetilde{g}^+_{m',1}(e^\tau)\Pi\left(\Phi_1(e^\tau,j)-\Phi_1(1,j)\right)$ and (5.7), there exists a constant $C>0$ independent from n, j_0, j, $\vec{e}$ and Γ_1 such that

$$|E_1| \le C|\vec{e}|e^{-\frac{3c_*}{2}|j|}\int_{\Gamma_1}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Using (4.51i) to bound $\widetilde{g}^+_{m',1}(e^\tau)\left(\Delta^+_{m'}(e^\tau,j_0,\vec{e})-\Delta^+_{m'}(1,j_0,\vec{e})\right)$ and (5.7), there exists a constant $C>0$ independent from n, j_0, j, $\vec{e}$ and Γ_2 such that

$$|E_2| \le C|\vec{e}|\int_{\Gamma_2}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Using (5.7), there exists a constant $C>0$ independent from n, j_0, j, $\vec{e}$ and Γ_3 such that

$$|E_3| \le C\int_{\Gamma_3}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Using Lemma 5.2, there exists a constant $C>0$ independent from n, j_0, j, $\vec{e}$ and Γ_4 such that

$$|E_4| \le Cn\int_{\Gamma_4}|\tau|^{2\mu}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 5.4 which gives a good choices of path $\Gamma_1,\dots,\Gamma_4 \in X$ depending on n, j_0 and j to handle the integrals in the terms above, there exist new constants $C,c>0$ independent from n, j_0, j and $\vec{e}$ such that

$$|E_1| \le \frac{Ce^{-c|j|}|\vec{e}|}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \qquad |E_2| \le \frac{C|\vec{e}|}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_3| \le \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \qquad |E_4| \le \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Using (5.18c) of Lemma 5.6 to handle E_5 and using (4.51i) and the fact that the function $E_{2\mu}(\beta^+_{l'};\cdot)$ is bounded to handle E_6, we can also consider that we chose the constants C,c so that

$$|E_5| \le \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_{l'}}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \qquad |E_6| \le C|\vec{e}|e^{-c|j_0|}.$$

Using (4.28) and (4.51g) to bound $\Phi_1(1,j)$ and $\Delta^+_{m'}$ in (5.29), we can conclude the proof of (5.28a).

• **Proof of** (5.28b)**:**

We will separate this proof into two parts.

► Let us assume that $-\frac{j_0}{\alpha^+_{l'}} \in \left[0, \frac{n}{2}\right]$. We recall that $\widetilde{g}^+_{m',1}$ is a meromorphic function with a pole of order 1 at 1. Using the Residue Theorem, we have

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^{\tau})\widetilde{\mathscr{C}}^+_{m'}(e^{\tau},j_0,\vec{e})\Pi(\Phi_1(e^{\tau},j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\vec{e}\Pi\left(\Phi_1(1,j)\right) =$$
$$-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^{\tau})e^{-(j_0+1)\varpi^+_{l'}(\tau)}\Delta^+_{m'}(e^{\tau},j_0,\vec{e})\Pi(\Phi_1(e^{\tau},j))d\tau$$
$$-\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}\left(1-E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\right){\boldsymbol{l}^+_{l'}}^T\vec{e}\Pi\left(\Phi_1(1,j)\right).$$

We need to obtain exponential bounds on both terms on the right hand side of the equality above. First, we observe that since $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $\left[0,\frac{n}{2}\right]$, we have

$$\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}} \leq \frac{\alpha^+_{l'}}{2}n^{\frac{2\mu}{2\mu-1}}.$$

We have that $\alpha^+_{l'}<0$ since m' belongs to I^+_{cu} and thus, using (1.28c) and (4.28), we have that there exist two positive constants C,c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0,j\in\mathbb{N}$, $\vec{e}\in\mathbb{C}^d$

$$\left|\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}\left(1-E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\right){\boldsymbol{l}^+_{l'}}^T\vec{e}\Pi\left(\Phi_1(1,j)\right)\right| \leq C|\vec{e}|e^{-c|j|}e^{-cn}.$$

We now observe that using (4.51g), (4.28) and (5.7), we can prove that there exist two positive constants C,c such that such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0,j\in\mathbb{N}$, $\vec{e}\in\mathbb{C}^d$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^{\tau})e^{-(j_0+1)\varpi^+_{l'}(\tau)}\Delta^+_{m'}(e^{\tau},j_0,\vec{e})\Pi(\Phi_1(e^{\tau},j))d\tau\right|$$
$$\leq C|\vec{e}|e^{-c|j|}\int_{\Gamma_d(\eta)}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using (5.17a), we can find exponential bounds for the integral in the right hand term above. This allows us to conclude the proof of (5.28b) when $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $\left[0,\frac{n}{2}\right]$.

► Let us assume that $-\frac{j_0}{\alpha^+_{l'}}\in[2n,+\infty[$. Since $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $[2n,+\infty[$, we have

$$\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}} \geq -\alpha^+_{l'}n^{\frac{2\mu}{2\mu-1}}.$$

We have that $\alpha^+_{l'}<0$ since m' belongs to I^+_{cu} and thus, using (1.28b) and (4.28), we have that there exist two positive constants C,c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0,j\in\mathbb{N}$, $\vec{e}\in\mathbb{C}^d$

$$\left|\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right){\boldsymbol{l}^+_{l'}}^T\vec{e}\Pi\left(\Phi_1(1,j)\right)\right| \leq C|\vec{e}|e^{-c|j|}e^{-cn}.$$

Furthermore, using (4.51g), (4.28) and (5.7), we can prove that there exist two positive constants C,c such that such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0,j\in\mathbb{N}$, $\vec{e}\in\mathbb{C}^d$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^{\tau})\widetilde{\mathscr{C}}^+_{m'}(e^{\tau},j_0,\vec{e})\Pi(\Phi_1(e^{\tau},j))d\tau\right|$$
$$\leq C|\vec{e}|e^{-c|j|}\int_{\Gamma_{in}(\eta)}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using (5.17b), we can find exponential bounds for the integral in the right hand term above. This allows us to conclude the proof of (5.28b) when $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $[2n,+\infty[$. □

Acknowledgements: A faire

6 Appendix

Proof of Lemma 3.5

Proof We define recursively

$$\forall j \in \mathbb{N}, \quad z_j^0 := y_j$$

and

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{N}, \quad z_j^{n+1} := C_H e^{-c_H j} + \Theta \sum_{k=0}^{+\infty} e^{-c_H |j-1-k|} z_k^n. \tag{6.1}$$

We then prove recursively that for all $n \in \mathbb{N}$, the sequence z^n is bounded, has non negative coefficients and

$$\|z^n\|_\infty \leq \frac{C_H}{1-\theta} \quad \text{and} \quad \forall j \in \mathbb{N}, \quad z_j^n \leq z_j^{n+1}.$$

Indeed, this property is obviously true for $n = 0$ using the inequality (3.24) since $z^0 = y$. We now consider $n \in \mathbb{N}$ for which the property is verified we will prove that the property for $n+1$ is satisfied. We first observe using the equality (6.1) that for all $j \in \mathbb{N}$ the coefficient z_j^{n+1} is non negative and

$$z_j^{n+1} \leq C_H + \theta \|z^n\|_\infty .$$

Thus, we have that $z^{n+1} \in \ell^\infty(\mathbb{N})$ and

$$\left\|z^{n+1}\right\|_\infty \leq \frac{C_H}{1-\theta}.$$

Finally, we observe that using the equality (6.1) for n and $n+1$, we have

$$z_j^{n+2} - z_j^{n+1} = \Theta \sum_{k=0}^{+\infty} e^{-c_H |j-1-k|} \underbrace{(z_k^{n+1} - z_k^n)}_{\geq 0} \geq 0.$$

This concludes the recurrence. We now observe that, for $p, q \in \mathbb{N} \setminus \{0\}$, using the equality (6.1), we have

$$z_j^p - z_j^q = \Theta \sum_{k=0}^{+\infty} e^{-c_H |j-1-k|} (z_k^{p-1} - z_k^{q-1}).$$

This implies that

$$\|z^p - z^q\|_\infty \leq \theta \left\|z^{p-1} - z^{q-1}\right\|_\infty .$$

Thus, we have that

$$\forall p \geq q \geq 0, \quad \|z^p - z^q\|_\infty \leq \theta^q \left\|z^{p-q} - y\right\|_\infty \leq \theta^q \frac{2C_H}{1-\theta}.$$

Since $\theta < 1$, the sequence $(z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\ell^\infty(\mathbb{N})$, thus it converges towards a sequence $z^\infty \in \ell^\infty(\mathbb{N})$. Since we have $y_j \leq z_j^n$ for all $n, j \in \mathbb{N}$, we obviously have

$$\forall j \in \mathbb{N}, \quad y_j \leq z_j^\infty.$$

Also, the equality (6.1) implies that

$$\forall j \in \mathbb{N}, \quad z_j^\infty = C_H e^{-c_H j} + \Theta \sum_{k=0}^{+\infty} e^{-c_H |j-1-k|} z_k^\infty. \tag{6.2}$$

Thus, there just remain to prove that there exists only one bounded sequence that satisfies (6.2) and that it has the form $z^\infty = (\rho r^j)_{j\in\mathbb{N}}$ where ρ and r satisfies the properties we expected.

We write (6.2) for j, $j+1$ and $j+2$ and reassemble the terms z_j^∞, z_{j+1}^∞ and z_{j+2}^∞ on the left side. We then have

$$\begin{aligned}\left(1 - e^{-c_H}\Theta\right) z_j^\infty - e^{-2c_H}\Theta z_{j+1}^\infty - e^{-3c_H}\Theta z_{j+2}^\infty \\ = C_H e^{-c_H j} + \Theta \left(\sum_{k=0}^{j-1} e^{-c_H |j-1-k|} z_k^\infty + \sum_{k=j+3}^{+\infty} e^{-c_H |j-1-k|} z_k^\infty \right)\end{aligned} \tag{6.3a}$$

$$
\begin{aligned}
&-\Theta z_j^\infty + \left(1 - e^{-c_H}\Theta\right) z_{j+1}^\infty - e^{-2c_H}\Theta z_{j+2}^\infty \\
&\qquad = e^{-c_H} C_H e^{-c_H j} + \Theta \left(e^{-c_H} \sum_{k=0}^{j-1} e^{-c_H|j-1-k|} z_k^\infty + e^{c_H} \sum_{k=j+3}^{+\infty} e^{-c_H|j-1-k|} z_k^\infty \right) \qquad (6.3b)
\end{aligned}
$$

$$
\begin{aligned}
&-e^{-c_H}\Theta z_j^\infty - \Theta z_{j+1}^\infty + \left(1 - e^{-c_H}\Theta\right) z_{j+2}^\infty \\
&\qquad = e^{-2c_H} C_H e^{-c_H j} + \Theta \left(e^{-2c_H} \sum_{k=0}^{j-1} e^{-c_H|j-1-k|} z_k^\infty + e^{2c_H} \sum_{k=j+3}^{+\infty} e^{-c_H|j-1-k|} z_k^\infty \right). \qquad (6.3c)
\end{aligned}
$$

We consider $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$
\begin{cases}
\alpha + \beta e^{-c_H} + \gamma e^{-2c_H} &= 0 \\
\alpha + \beta e^{c_H} + \gamma e^{2c_H} &= 0.
\end{cases}
$$

A solution is $\alpha = \gamma = 1$ and $\beta = -\frac{e^{2c_H} - e^{-2c_H}}{e^{c_H} - e^{-c_H}} = -\frac{\sinh(2c_H)}{\sinh(c_H)} = -2\cosh(c_H)$. We then have that

$$
\begin{aligned}
\alpha - \Theta\left(\alpha e^{-c_H} + \beta + \gamma e^{-c_H}\right) &= 1 - \theta \frac{e^{\frac{c_H}{2}} - e^{-\frac{c_H}{2}}}{e^{\frac{c_H}{2}} + e^{-\frac{c_H}{2}}} \left(2e^{-c_H} - 2\cosh(c_H)\right) \\
&= 1 + 2\theta \sinh(c_H) \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \\
&= 1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right).
\end{aligned}
$$

Multiplying the equalities (6.3) respectively by α, β, γ and summing them, we obtain that

$$
\forall j \in \mathbb{N}, \quad z_{j+2}^\infty - 2\cosh(c_H) z_{j+1}^\infty + \left(1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right)\right) z_j^\infty = 0.
$$

We are thus led to study the polynomial $P := X^2 - 2\cosh(c_H)X + \left(1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right)\right)$. Its discriminant is

$$
\Delta = 4\sinh^2(c_H) - 16\theta \sinh^2\left(\frac{c_H}{2}\right) = 16 \sinh^2\left(\frac{c_H}{2}\right)\left(\cosh^2\left(\frac{c_H}{2}\right) - \theta\right) > 0.
$$

Its roots are

$$
r_\pm := \cosh(c_H) \pm 2\sinh\left(\frac{c_H}{2}\right) \sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta}.
$$

We observe that evaluating the polynomial P at 0 and 1 gives us

$$
P(0) = 1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right) > 0
$$

and

$$
P(1) = 2 - 2\cosh(c_H) + 4\theta \sinh^2\left(\frac{c_H}{2}\right) = -4(1-\theta)\sinh^2\left(\frac{c_H}{2}\right) < 0.
$$

Thus, $r_+ > 1$ and $r := r_- \in]0, 1[$. We have that $\rho, \tilde{\rho} \in \mathbb{R}$ such that

$$
\forall j \in \mathbb{N}, z_j^\infty = \rho r^j + \tilde{\rho} r_+^j.
$$

Since the sequence z^∞ is bounded, we have that $\tilde{\rho} = 0$, i.e.

$$
\forall j \in \mathbb{N}, z_j^\infty = \rho r^j.
$$

We are now going to be using the equality (6.2) to determine ρ. We first observe that $r > e^{-c_H}$. Indeed, we have

$$
\begin{aligned}
r &= \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right) \sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta} \\
&> \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right) \cosh\left(\frac{c_H}{2}\right) \\
&= \cosh(c_H) - \sinh(c_H) \\
&= e^{-c_H}.
\end{aligned}
$$

Now, using the equality (6.2), we obtain

$$\begin{aligned}\rho r^j &= C_H e^{-c_H j} + \Theta\rho \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|} r^k \\ &= C_H e^{-c_H j} + \Theta\rho \left(e^{-c_H(j-1)} \frac{(re^{c_H})^j - 1}{re^{c_H} - 1} + e^{c_H(j-1)} \frac{(re^{-c_H})^j}{1 - re^{-c_H}} \right) \\ &= e^{-c_H j} \left(C_H - \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1} \right) + r^j \Theta\rho \left(\frac{e^{c_H}}{re^{c_H} - 1} + \frac{e^{-c_H}}{1 - r^{-c_H}} \right).\end{aligned}$$

Using the definition (3.21) of Θ, we have that

$$\begin{aligned}\Theta \left(\frac{e^{c_H}}{re^{c_H} - 1} + \frac{e^{-c_H}}{1 - r^{-c_H}} \right) &= \theta \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \frac{2\sinh(c_H)}{-r^2 + 2\cosh(c_H) - 1} \\ &= \theta \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \frac{2\sinh(c_H)}{4\theta \sinh^2\left(\frac{c_H}{2}\right)} \\ &= 1.\end{aligned}$$

Thus,

$$\rho r^j = e^{-c_H j} \left(C_H - \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1} \right) + \rho r^j$$

i.e.

$$C_H = \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1}.$$

Therefore,

$$\rho = \frac{C_H}{\Theta}(r - e^{-c_H}).$$

□

Proof of Lemma 5.4

Proof • **Proof of** (5.17a)

We consider $n \in \mathbb{N}\setminus\{0\}$ and $x \in \left[0, \frac{n}{2}\right]$. Noticing that $\Gamma_d(\eta) \subset B(0,\varepsilon)$ and using (5.10), we have

$$\begin{aligned}\int_{\Gamma_d(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu})\right) |d\tau| &\le \varepsilon^k \int_{-r_\varepsilon(\eta)}^{r_\varepsilon(\eta)} \exp\left(-(n-x)\eta + xA_R\eta^{2\mu} - xA_I t^{2\mu}\right) dt \\ &\le 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-\frac{n}{2}\left(\eta - A_R\eta^{2\mu}\right)\right) \\ &\le 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-\frac{n\eta}{4}\right).\end{aligned}$$

• **Proof of** (5.17b)**:**

We consider $n \in \mathbb{N}\setminus\{0\}$ and $x \in [2n, +\infty[$. We will separate the integral on the path $\Gamma_{in}(\eta)$ using the paths $\Gamma_{in}^0(\eta)$ and $\Gamma_{in}^{\pm}(\eta)$ introduced in (5.14).

▶ Noticing that $\Gamma_d(\eta) \subset B(0,\varepsilon)$ and that $n - x \le -\frac{x}{2}$, we have using condition (5.10)

$$\begin{aligned}\int_{\Gamma_{in}^0(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu})\right) |d\tau| &\le \varepsilon^k \int_{-r_\varepsilon(\eta)}^{r_\varepsilon(\eta)} \exp\left((n-x)\eta + xA_R\eta^{2\mu} - xA_I t^{2\mu}\right) dt \\ &\le 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-x\left(\frac{\eta}{2} - A_R\eta^{2\mu}\right)\right) \\ &\le 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-2n\left(\frac{\eta}{2} - A_R\eta^{2\mu}\right)\right).\end{aligned}$$

We have proved exponential bounds on this first term.

▶ We have that using (5.11) and that $x \geq 2n$

$$\begin{aligned}\int_{\Gamma^{\pm}_{in}(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu})\right)|d\tau| &\leq \varepsilon^k \int_{-\eta}^{\eta} \exp\left((n-x)t + xA_R t^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu}\right) dt \\ &\leq 2\eta\varepsilon^k \exp\left((x-n)\eta + xA_R\eta^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu}\right) \\ &\leq 2\eta\varepsilon^k \exp\left(x\left(\eta + A_R\eta^{2\mu} - A_I r_\varepsilon(\eta)^{2\mu}\right)\right) \\ &\leq 2\eta\varepsilon^k \exp\left(-A_I r_\varepsilon(\eta)^{2\mu} n\right).\end{aligned}$$

We have proved exponential bounds on this second term.

We can then easily conclude the proof of (5.17b)

• **Proof of** (5.17c)**:**

We consider $n \in \mathbb{N}\setminus\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$. We start by observing that

$$\forall x \in \left[\frac{n}{2}, 2n\right], \quad \left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}} \leq n. \tag{6.4}$$

Thus, obtaining exponential bounds on certain terms when $x \in [\frac{n}{2}, 2n]$ would also allow to conclude on the proof of (5.17c).

We will now follow a strategy developed in [ZH98] in a continuous setting, which has also been used in [God03, CF22, CF21, Coe22, Coe23] in the discrete case, and introduce a family of parameterized curves.

We recall that we introduced in (5.8) the function Ψ defined by

$$\forall \tau_p \in \mathbb{R}, \quad \Psi(\tau_p) := \tau_p - A_R {\tau_p}^{2\mu}.$$

and that we chose ε small enough so that the function Ψ is continuous and strictly increasing on $]-\infty, \varepsilon]$. We can therefore introduce for $\tau_p \in [-\eta, \varepsilon]$ the curve Γ_p defined by

$$\Gamma_p := \left\{\tau \in \mathbb{C}, -\eta \leq \Re(\tau) \leq \tau_p, \quad \Re(\tau) - A_R\Re(\tau)^{2\mu} + A_I\Im(\tau)^{2\mu} = \Psi(\tau_p)\right\}.$$

It is a symmetric curve with respect to the axis $\mathbb{R}$ which intersects this axis on the point τ_p. If we introduce $\ell_p = \left(\frac{\Psi(\tau_p)-\Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}$, then $-\eta + i\ell_p$ and $-\eta - i\ell_p$ are the end points of Γ_p. We can also introduce a parametrization of this curve by defining $\gamma_p : [-\ell_p, \ell_p] \to \mathbb{C}$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad \Im(\gamma_p(t)) = t, \quad \Re(\gamma_p(t)) = h_p(t) := \Psi^{-1}\left(\Psi(\tau_p) - A_I t^{2\mu}\right). \tag{6.5}$$

The above parametrization immediately yields that there exists a constant $C > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad |h_p'(t)| \leq C. \tag{6.6}$$

Also, there exists a constant $c_p > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall \tau \in \Gamma_p, \quad \Re(\tau) - \tau_p \leq -c_p\Im(\tau)^{2\mu}. \tag{6.7}$$

For $\tau \in \Gamma_p$, it follows from (6.7) that

$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right) \leq -nc_p\Im(\tau)^{2\mu} + (n-x)\tau_p + xA_R\tau_p^{2\mu}. \tag{6.8}$$

There remains to make an appropriate choice of τ_p depending on n and x that minimizes the right-hand side of the inequality (6.8) whilst the paths Γ_p have to remain within the ball $B(0,\varepsilon)$. We recall that when we fixed our choice of width η, we defined a radius $\varepsilon_\# \in]0, \varepsilon[$ such that $-\eta + il_{extr} \in B(0,\varepsilon)$ where the real number l_{extr} is defined by (5.13). This implies that the curve Γ_p associated with $\tau_p = \varepsilon_\#$ intersects the axis $-\eta + i\mathbb{R}$ within $B(0,\varepsilon)$. We let

$$\zeta = \frac{x-n}{2\mu n}, \quad \gamma = \frac{xA_R}{n}, \quad \rho\left(\frac{\zeta}{\gamma}\right) = \operatorname{sgn}(\zeta)\left(\frac{|\zeta|}{\gamma}\right)^{\frac{1}{2\mu-1}}.$$

Inequality (6.8) thus becomes

$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right) \leq -nc_p\Im(\tau)^{2\mu} + n(\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p). \tag{6.9}$$

Our limiting estimates will come from the case where ζ is close to 0. We observe that the condition $x \geq \frac{n}{2}$ implies

$$\gamma \geq \frac{A_R}{2}. \tag{6.10}$$

Then, we take

$$\tau_p := \begin{cases} \rho\left(\frac{\zeta}{\gamma}\right), & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) \in [-\frac{\eta}{2}, \varepsilon_\#], \quad \text{(Case } \mathbf{A}\text{)} \\ \varepsilon_\#, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_\#, \quad \text{(Case } \mathbf{B}\text{)} \\ -\frac{\eta}{2}, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}. \quad \text{(Case } \mathbf{C}\text{)} \end{cases}$$

The case **A** corresponds to the choice to minimize the right-hand side of (6.9) since $\rho\left(\frac{\zeta}{\gamma}\right)$ is the unique real root of the polynomial

$$\gamma X^{2\mu-1} = \zeta.$$

The cases **B** and **C** allow the path Γ_p to stay within $B(0,\varepsilon)$.

We now define the paths:

$$\begin{aligned} \Gamma_{p,res} &:= \{-\eta + it, \quad t \in [-r_\varepsilon(\eta), -\ell_p] \cup [\ell_p, r_\varepsilon(\eta)]\}, \\ \Gamma_{p,in} &:= \Gamma_p \cup \Gamma_{p,res}, \end{aligned}$$

where the function r_ε is defined by (5.9). We observe that $\Gamma_{p,in}$ belongs to the set of paths X. We will decompose the integral

$$\int_{\Gamma_{p,in}} |\tau|^k \exp(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right))|d\tau|$$

using the paths Γ_p and $\Gamma_{p,res}$ and we will then bound each term.

▶ Let us assume that x and n are such that we are in Case **A**. Since $\tau_p = \rho\left(\frac{\zeta}{\gamma}\right)$ is the unique root of $\gamma X^{2\mu-1} - \zeta$, we have

$$\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p = -(2\mu-1)\gamma\tau_p^{2\mu} \leq 0. \tag{6.11}$$

Thus, the inequality (6.9) becomes for $\tau \in \Gamma_p$

$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right) \leq -nc_p\Im(\tau)^{2\mu} - (2\mu-1)\gamma n\tau_p^{2\mu}.$$

Therefore, we have

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq \int_{\Gamma_p} |\tau|^k \exp\left(-nc_p\Im(\tau)^{2\mu}\right)|d\tau| \exp\left(-(2\mu-1)\gamma n\tau_p^{2\mu}\right).$$

Using the parametrization (6.5) and the inequality (6.6), we have that

$$\int_{\Gamma_p} |\tau|^k \exp\left(-nc_p\Im(\tau)^{2\mu}\right)|d\tau| \lesssim \int_{-\ell_p}^{-\ell_p} (|\tau_p|^k + t^k)e^{-nc_pt^{2\mu}}dt.$$

The change of variables $u = n^{\frac{1}{2\mu}}t$ and the fact that the functions $y \geq 0 \mapsto y^k \exp\left(-\frac{2\mu-1}{2}\gamma y^{2\mu}\right)$ are uniformly bounded with respect to $\gamma \geq \frac{A_R}{2}$ imply

$$\begin{aligned} \int_{-\ell_p}^{-\ell_p} |t|^k e^{-nc_pt^{2\mu}}dt &\lesssim \frac{1}{n^{\frac{k+1}{2\mu}}} \\ \int_{-\ell_p}^{-\ell_p} |\tau_p|^k e^{-nc_pt^{2\mu}}dt &\lesssim \frac{1}{n^{\frac{k+1}{2\mu}}} \exp\left(\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu}\right). \end{aligned}$$

Thus,

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \lesssim \frac{1}{n^{\frac{k+1}{2\mu}}} \exp\left(-\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu}\right).$$

Furthermore, since we are in the Case **A**

$$-\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu} = -\frac{2\mu-1}{2(2\mu A_R)^{\frac{2\mu}{2\mu-1}}}A_R\left(\frac{|n-x|}{x^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}.$$

Therefore, there exist two positive constants C, c independent from n and x such that if we are in Case **A**,

$$\int_{\Gamma_p}|\tau|^k\exp\left(n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq \frac{C}{n^{\frac{k+1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{x^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $x\in[\frac{n}{2},2n]$, this gives us two new constants C, c independent from n and x such that if we are in Case **A** and $x\in[\frac{n}{2},2n]$, then as expected

$$\int_{\Gamma_p}|\tau|^k\exp\left(n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq \frac{C}{n^{\frac{k+1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

► Let us assume that x and n are such that we are in Case **B**. Since $\tau_p=\varepsilon_\#<\rho\left(\frac{\zeta}{\gamma}\right)$, we have

$$-\zeta\leq-\gamma\varepsilon_\#^{2\mu-1}$$

and thus using (6.10)

$$\gamma\tau_p^{2\mu}-2\mu\zeta\tau_p\leq-(2\mu-1)\gamma\varepsilon_\#^{2\mu}\leq-\frac{2\mu-1}{2}A_R\varepsilon_\#^{2\mu}. \tag{6.12}$$

Therefore, the inequality (6.9) becomes for $\tau\in\Gamma_p$

$$\begin{aligned}n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)&\leq-nc_p\Im(\tau)^{2\mu}-\frac{2\mu-1}{2}A_Rn\varepsilon_\#^{2\mu}\\&\leq-\frac{2\mu-1}{2}\varepsilon_\#^{2\mu}A_Rn.\end{aligned}$$

We conclude that there exist two positive constants C, c independent from n and x such that if we are in Case **B**,

$$\int_{\Gamma_p}|\tau|^k\exp\left(n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|\leq Ce^{-cn}.$$

Using (6.4) if necessary, we obtain the bound expected in the statement of the lemma.

► Let us assume that x and n are such that we are in Case **C**. Since $\tau_p=-\frac{\eta}{2}>\rho\left(\frac{\zeta}{\gamma}\right)$, we have

$$\zeta\leq-\gamma\left(\frac{\eta}{2}\right)^{2\mu-1}$$

and thus using (6.10)

$$\gamma\tau_p^{2\mu}-2\mu\zeta\tau_p=\gamma\left(\frac{\eta}{2}\right)^{2\mu}+2\mu\zeta\frac{\eta}{2}\leq-(2\mu-1)\gamma\left(\frac{\eta}{2}\right)^{2\mu}\leq-\frac{2\mu-1}{2}A_R\left(\frac{\eta}{2}\right)^{2\mu}. \tag{6.13}$$

Therefore, the inequality (6.9) becomes for $\tau\in\Gamma_p$

$$\begin{aligned}n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)&\leq-nc_p\Im(\tau)^{2\mu}-\frac{2\mu-1}{2}A_Rn\left(\frac{\eta}{2}\right)^{2\mu}\\&\leq-\frac{2\mu-1}{2}\left(\frac{\eta}{2}\right)^{2\mu}A_Rn.\end{aligned}$$

We then conclude that there exist two positive constants C, c independent from n and x such that if we are in Case **C**,

$$\int_{\Gamma_p}|\tau|^k\exp\left(n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\Im(\tau)^{2\mu}\right)\right)|d\tau|\leq Ce^{-cn}.$$

Using (6.4) if necessary, we obtain the bound expected in the statement of the lemma.

► We recall that $-\eta \pm \ell_p$ belongs to Γ_p. For $\tau \in \Gamma_{p,res}$, we have that

$$\Re(\tau) = -\eta \quad \text{and} \quad |\Im(\tau)| \geq \ell_p.$$

Thus,

$$\begin{aligned} n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}) &\leq -n\eta + x(\eta + A_R\eta^{2\mu} - A_I\ell_p^{2\mu}) \\ &\leq -n\eta + x(-\tau_p + A_R\tau_p^{2\mu}) \\ &\leq -n(\eta + \tau_p) + n(\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p). \end{aligned}$$

In each cases **A**, **B** and **C**, we have that

$$\eta + \tau_p \geq \frac{\eta}{2} \quad \text{and} \quad \gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p \leq 0.$$

Therefore, for all $\tau \in \Gamma_{p,res}$,

$$n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}) \leq -n\frac{\eta}{2}.$$

We then conclude that

$$\int_{\Gamma_{p,res}} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\Im(\tau)^{2\mu}\right)\right)|d\tau| \leq 2\pi\varepsilon^k e^{-n\frac{\eta}{2}}.$$

Using (6.4) if necessary, we obtain the bound expected in the statement of the lemma.

Combining all the results we encountered, we easily conclude the proof of (5.17c). □

Proof of Lemma 5.6

Proof We will prove the statement of Lemma 5.6 with $? = ?' = +$ in order to alleviate the notations.

• **Proof of (5.18a):**

► We start by defining the paths

$$\Gamma_0 := \{it, t \in [-r_\varepsilon(\eta), r_\varepsilon(\eta)]\} \quad \text{and} \quad \Gamma^{\pm}_{comp} := \{t \pm ir_\varepsilon(\eta), t \in [-\eta, 0]\}.$$

For all $\Gamma \in X$, Cauchy's formula implies that

$$\begin{aligned} &\left|\int_\Gamma \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau - \int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau\right| \\ \leq &\left|\int_{\Gamma^+_{comp}} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau\right| + \left|\int_{\Gamma^-_{comp}} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau\right|. \end{aligned}$$

We observe that (5.6) implies that

$$\begin{aligned} &\left|\int_{\Gamma^\pm_{comp}} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau\right| \\ &\qquad \leq \int_{-\eta}^0 \exp\left((n - (x+y))t + (x+y)A_Rt^{2\mu} - (x+y)A_Ir_\varepsilon(\eta)^{2\mu}\right)dt. \end{aligned}$$

Using (5.12) since $t \in [-\eta, 0]$ and $x + y \in \left[\frac{n}{2}, 2n\right]$, we have that

$$(n - (x+y))t + (x+y)A_Rt^{2\mu} - (x+y)A_Ir_\varepsilon(\eta)^{2\mu} \leq -\frac{A_Ir_\varepsilon(\eta)^{2\mu}}{4}n.$$

Combining the observations above, we have thus proved that for all path $\Gamma \in X$, $n \in \mathbb{N}\setminus\{0\}$, $x, y \in [0, +\infty[$ such that $x + y \in \left[\frac{n}{2}, 2n\right]$

$$\begin{aligned} &\left|\int_\Gamma \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau - \int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right)d\tau\right| \\ &\qquad \leq 2\eta \exp\left(-\frac{A_Ir_\varepsilon(\eta)^{2\mu}}{4}n\right). \end{aligned} \tag{6.14}$$

► Since $x+y \geq \frac{n}{2} \geq \frac{1}{2}$, we observe that

$$\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-(x+y)A_I t^{2\mu}\right) dt \leq \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4} n\right) \int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{4}\right) dt.$$

Therefore, if we introduce the path

$$\Gamma_0^\infty := \{it, t \in \mathbb{R}\}$$

then, using (5.6), the integral

$$\int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau) + y\alpha_{l'}^+ \varphi_{l'}^+(\tau)\right) d\tau$$

is defined and we have that

$$\left| \int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau) + y\alpha_{l'}^+ \varphi_{l'}^+(\tau)\right) d\tau - \int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau) + y\alpha_{l'}^+ \varphi_{l'}^+(\tau)\right) d\tau \right| \leq 2 \int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{4}\right) dt \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4} n\right). \quad (6.15)$$

► Using the change of variables $u = \frac{n^{\frac{1}{2\mu}}}{\alpha_l^+} t$, we obtain

$$\begin{aligned} \frac{1}{2i\pi} \int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau) + y\alpha_{l'}^+ \varphi_{l'}^+(\tau)\right) d\tau &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(it(n-(x+y)) - \left(\frac{x\beta_l^+}{{\alpha_l^+}^{2\mu}} + \frac{y\beta_{l'}^+}{{\alpha_{l'}^+}^{2\mu}}\right) t^{2\mu}\right) dt \\ &= \frac{|\alpha_l^+|}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{x}{n}\beta_l^+ + \frac{y}{n}\beta_{l'}^+ \left(\frac{\alpha_l^+}{\alpha_{l'}^+}\right)^{2\mu}; \frac{\alpha_l^+(n-(x+y))}{n^{\frac{1}{2\mu}}}\right). \end{aligned}$$

Combining (6.14), (6.15) and the observation above, we obtain the inequality (5.18a).

• **Proof of** (5.18b)**:**

We observe using the change of variables $u = \frac{x^{\frac{1}{2\mu}}}{\alpha_l^+} t$ that

$$\begin{aligned} \frac{1}{2i\pi} \int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right) d\tau &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(it(n-x) - \frac{x\beta_l^+}{{\alpha_l^+}^{2\mu}} t^{2\mu}\right) dt \\ &= \frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right). \end{aligned}$$

Combining (6.14), (6.15) and the observation above, we have proved that there exist two constants $C, c > 0$ such that

$$\forall x \in \left[\frac{n}{2}, 2n\right], \quad \left| \frac{1}{2i\pi} \int_\Gamma \exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right) d\tau - \frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) \right| \leq Ce^{-cn}. \quad (6.16)$$

Using (6.4), we can obtain the same generalized Gaussian bound as the one expected in (5.18b).

► We observe that

$$\begin{aligned} &\frac{1}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right) \\ &= \frac{1}{x^{\frac{1}{2\mu}}} \left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right) + H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right) \left(\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right). \quad (6.17) \end{aligned}$$

We want to prove generalized Gaussian bounds for the two terms on the right hand side of (6.17). Applying the mean value inequality and (1.28a), we have that there exist two constants $C, c > 0$ such that for all $x \in [\frac{n}{2}, 2n]$

$$\left| \frac{1}{x^{\frac{1}{2\mu}}} \left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right) \right| \leq \frac{C}{n^{\frac{1}{2\mu}}} |n-x| \left| \frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}} \right| \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $x \in \left[\frac{n}{2}, 2n\right]$, we also have using the mean value inequality that

$$\left| \frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}} \right| \leq \frac{|n-x|}{2\mu} \sup_{t \in [x,n]} \frac{1}{|t|^{1+\frac{1}{2\mu}}} \leq \frac{2^{\frac{1}{2\mu}}}{\mu} \frac{|n-x|}{n^{1+\frac{1}{2\mu}}}. \tag{6.18}$$

Therefore, since the function $y \mapsto y^2 \exp\left(-\frac{c}{2} y^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, there exist two new constants $C, c > 0$ such that for all $x \in [\frac{n}{2}, 2n]$

$$\left| \frac{1}{x^{\frac{1}{2\mu}}} \left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right)\right| \leq \frac{C}{n} \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{6.19}$$

We have thus proved generalized Gaussian bounds for the first term of the right hand side in (6.17). We now focus on the second term. Using (1.28a), (6.18) and the fact that, for any constant $c > 0$, the function $y \mapsto y \exp\left(-c y^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, we have that there exist two constants $C, c > 0$ such that

$$\left| H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right) \left(\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}} \right)\right| \leq \frac{C}{n} \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{6.20}$$

Combining (6.17), (6.19) and (6.20), we have proved generalized Gaussian bounds for the difference

$$\frac{1}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right).$$

With (6.16), we easily conclude the proof of (5.18b).

• **Proof of** (5.18c)**:**

► We observe that (5.6) implies that

$$\left| \int_{\Gamma_{in}^{\pm}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right)}{\tau} d\tau \right| \leq \frac{1}{r_\varepsilon(\eta)} \int_{-\eta}^{\eta} \exp\left((n-x)t + xA_R t^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu}\right) dt.$$

Using (5.12) since $t \in [-\eta, \eta]$ and $x \in \left[\frac{n}{2}, 2n\right]$, we have that

$$(n-x)t + xA_R t^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu} \leq -\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4} n.$$

Using the observations above and Cauchy's formula, we have thus proved that for all $n \in \mathbb{N}\setminus\{0\}$, $x \in \left[\frac{n}{2}, 2n\right]$ and paths $\Gamma \in X$

$$\left| \int_{\Gamma_{in}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right)}{\tau} d\tau - \int_{\Gamma_{in}^0(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right)}{\tau} d\tau \right| \leq \frac{4\eta}{r_\varepsilon(\eta)} \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4} n\right). \tag{6.21}$$

► Since $x \geq \frac{n}{2} \geq \frac{1}{2}$, we observe that

$$\int_{r_\varepsilon(\eta)}^{+\infty} \frac{\exp\left((n-x)\eta + xA_R\eta^{2\mu} - xA_I t^{2\mu}\right)}{|\eta + it|} dt \leq \frac{1}{\eta} \underbrace{\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{8}\right) dt}_{<+\infty} \exp\left((n-x)\eta + xA_R\eta^{2\mu} - x\frac{3}{4}A_I r_\varepsilon(\eta)^{2\mu}\right).$$

Furthermore, using (5.12) since $x \in \left[\frac{n}{2}, 2n\right]$, we have that

$$\exp\left((n-x)\eta + xA_R\eta^{2\mu} - x\frac{3}{4}A_I r_\varepsilon(\eta)^{2\mu}\right) \leq \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{8} n\right).$$

Therefore, if we introduce the path

$$\Gamma_{in}^{\infty}(\eta) := \{\eta + it, t \in \mathbb{R}\}$$

then, using (5.6), the integral

$$\int_{\Gamma_{in}^{\infty}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+ \varphi_l^+(\tau)\right)}{\tau} d\tau$$

is defined and we have that

$$\left|\int_{\Gamma^0_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau-\int_{\Gamma^\infty_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau\right| \leq \frac{2}{\eta}\int_{r_\varepsilon(\eta)}^{+\infty}\exp\left(-\frac{A_I t^{2\mu}}{8}\right)dt\exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{8}n\right). \quad (6.22)$$

▶ We observe that using the change of variables $t=-\frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}}u$, we have

$$\begin{aligned}\frac{1}{2i\pi}\int_{\Gamma^\infty_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau &= \frac{1}{2i\pi}\int_{-\infty}^{+\infty}\frac{\exp\left(i(n-x)(t-i\eta)-x\frac{\beta_l^+}{\alpha_l^{+2\mu}}(\eta+it)^{2\mu}\right)}{t-i\eta}dt\\ &= -\frac{1}{2i\pi}\int_{-\infty}^{+\infty}\frac{\exp\left(i\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\left(u+i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}\right)-\beta_l^+\left(u+i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}\right)^{2\mu}\right)}{u+i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}}du.\end{aligned}$$

Furthermore, we can prove that

$$\forall s\in]0,+\infty[,\forall x\in\mathbb{R},\quad -\frac{1}{2i\pi}\int_{-\infty}^{+\infty}\frac{\exp\left(ix\left(u+is\right)-\beta_l^+\left(u+is\right)^{2\mu}\right)}{u+is}du=E_{2\mu}\left(\beta_l^+;x\right).$$

The proof is done in [Coe23, (5.65)]. Therefore,

$$\frac{1}{2i\pi}\int_{\Gamma^\infty_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau=E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right).$$

Combining this observation with (6.21), (6.22) and (6.4), we have that there exist two positive constants C,c such that for all $n\in\mathbb{N}\backslash\{0\}$ and $x\in\left[\frac{n}{2},2n\right]$

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)}\frac{\exp\left(n\tau+x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau-E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right)\right|\leq\frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

▶ We notice that $\partial_x E_{2\mu}\left(\beta_l^+;\cdot\right)=-H_{2\mu}\left(\beta_l^+;\cdot\right)$. Therefore, we have using the mean value inequality and (1.28a) that there exist two positive constants C,c such that for all $n\in\mathbb{N}\backslash\{0\}$ and $x\in\left[\frac{n}{2},2n\right]$

$$\left|E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right)-E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right|\leq C|n-x|\left|\frac{1}{x^{\frac{1}{2\mu}}}-\frac{1}{n^{\frac{1}{2\mu}}}\right|\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Using (6.18) and the fact that $y\mapsto y^2\exp\left(-\frac{c}{2}y^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, we have that there exist two new positive constants C,c such that for all $n\in\mathbb{N}\backslash\{0\}$ and $x\in\left[\frac{n}{2},2n\right]$

$$\left|E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right)-E_{2\mu}\left(\beta_l^+;\frac{-|\alpha_l^+|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right|\leq\frac{C}{n^{1-\frac{1}{2\mu}}}\exp\left(-\frac{c}{2}\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

This allows us to conclude the proof of (5.18c). □

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