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# Hypercomplex representation of finite-dimensional unital Archimedean $f$ -algebras

Sayed Kossentini\*

## Abstract

In this paper, we characterize all  $N$ -dimensional hypercomplex numbers having unital Archimedean  $f$ -algebra structure. We use matrix representation of hypercomplex numbers to define an order structure on the matrix spectra. We prove that the unique (up to isomorphism) unital Archimedean  $f$ -algebra of hypercomplex numbers of dimension  $N \geq 1$  is that with real and simple spectrum. We also show that these number systems can be made into unital Banach lattice algebras and we establish some of their properties. Furthermore, we prove that every  $2N$ -dimensional unital Archimedean  $f$ -algebra is the *hyperbolization* of that of dimension  $N$ . Finally, we consider hypercomplex number systems of dimension  $N = 2, 3, 4, 6$  and give their explicit matrix representation and eigenvalue operators. This work is a multidimensional generalization of the results obtained in [11] and [4] for, respectively, the two and four-dimensional systems.

**Key words:** Hypercomplex numbers, Hyperbolic numbers, Canonical hyperbolic quaternions,  $f$ - algebra, Riesz homomorphism, Banach lattices

## 1 Introduction

Hypercomplex numbers are finite-dimensional extensions of real and complex numbers. They have many applications in geometry and physics (see e.g. [12]). The first example of hypercomplex numbers, and probably the most studied, is the system  $\mathbb{H}$  of non-commutative Hamilton quaternions [13].

The algebraic theory of hypercomplex numbers was developed by Wedderburn [20] and Cartan [7]. Lie [16] generalized all extensions of complex numbers in relation with their group properties. In this context geometries generated by hypercomplex numbers were considered by Cattoni et al [8]. The study of hypercomplex numbers has been further advanced through their classification by Peirce [18].

One important aspect of hypercomplex numbers is that as the dimension of the numbers increases, certain algebraic properties are lost. For example, as noted above, quaternions are non-commutative, while complex numbers commute. This loss of properties has

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\*Université de Tunis El Manar, Faculté des Sciences de Tunis, Department of Mathematics, 2092, Tunis, Tunisia. E-mail address: sayed.kossentini@fst.utm.tn

been observed, a half century before Hamilton, by Gauss. In particular, one must either renounce the commutative property of the product or acknowledge that the product between some non-zero numbers is zero. In this vein, Frobenius theorem states that  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$  are the only real associative division algebras.

In this paper, we deviate from Hamilton's choice and consider commutative (and a fortiori) non-division  $N$ -dimensional algebras of hypercomplex numbers ( $N \geq 2$ ).

According to [8] commutative algebras of hypercomplex numbers are of interest for their own sake. In particular, a theorem due to Sheffers [19] states that *for distributive systems with unity the differential and integral calculus does exist only if the systems are commutative*. Catoni et al. suggest that functions of hypercomplex variables can be useful for studying certain partial differential equations, specifically the Generalized Cauchy-Riemann equations, as well as non-flat spaces associated with the geometries introduced by multidimensional commutative hypercomplex numbers. This property can be particularly relevant in general relativity and field theories.

In this article, we present a new approach to the study of hypercomplex numbers, focusing on the order structure. Specifically, we examine the ordering of hypercomplex numbers and the relationship between the order structure and the algebraic properties of the numbers. This new approach provides a deeper understanding of hypercomplex numbers and has the potential to lead to new applications. The aim of this work is to investigate  $N$ -dimensional hypercomplex numbers in terms of Riesz algebras (also known as lattice algebras or  $\ell$ -algebras). More precisely, we will establish a necessary and sufficient condition for which  $N$ -dimensional algebra of hypercomplex numbers can be made into unital Archimedean  $f$ -algebra. This is, indeed, not possible for many  $\ell$ -algebras including finite-dimensional ones (see e.g. [15]).

The first example of such system of numbers are the 2-dimensional hyperbolic numbers  $\mathbb{D} = \{x + \mathbf{j}y : x, y \in \mathbb{R}; \mathbf{j} \notin \mathbb{R}; \mathbf{j}^2 = 1\}$ . Note that complex numbers and hyperbolic numbers are the only real commutative Clifford algebras:  $\mathbb{D} \cong \text{Cl}_{\mathbb{R}}(1, 0)$  and  $\mathbb{C} \cong \text{Cl}_{\mathbb{R}}(0, 1)$ .

As complex numbers are related to the euclidean geometry, hyperbolic numbers are related to the *Minkowski geometry* in the plane also called *Lorentzian* or *space-time geometry* [8, 22].

It has been shown in [11] that  $\mathbb{D}$  is the unique, up to isomorphism, Archimedean  $f$ -algebra of dimension two.  $f$ -algebra structure is an order structure. More precisely, an  $f$ -algebra is simultaneously a Riesz space (or vector lattice) and an associative real algebra that fulfills certain "positivity" conditions. A typical example of  $f$ -algebras is the linear space of real valued continuous functions on a topological space. Moreover, Archimedean  $f$ -algebras are known to be commutative (see e.g. [14]) and are even automatically associative ([3]). Of course, the fundamental example of Archimedean  $f$ -algebras is the field of real numbers.

Another example of order structure on hypercomplex numbers is given in a recent work due to Bilgin and Ersoy [4]: the algebra of bihyperbolic numbers is equipped by a partial order compatible with its algebraic structure. We prove in section 6.3 that this order structure is the "natural" Archimedean  $f$ -algebra structure on bihyperbolic numbers.

This paper is organized in the following way: in section 2 we recall the basic notions and properties of matrix representation of hypercomplex numbers, Riesz spaces and  $f$ -algebras we will use throughout this article. In section 3 we prove, for a given  $N$ , that under a well defined order there is a unique, up to isomorphism,  $N$ -dimensional Archimedean  $f$ -algebra of hypercomplex numbers (denoted  $\mathbf{H}_N$ ). As a consequence, fundamental order properties including Dedekind completeness, group of signs, Banach lattice algebra norm are given in section 4 as well as eigenvalue and conjugation operators. Section 5 is devoted to the new notion of hypercomplex interval. Section 6 is devoted to the study of  $\mathbf{H}_N$  for  $N = 2, 3$  and 4. We give their explicit matrix representation and eigenvalue operators. We propose, in section 7, an hyperbolic analogue of the Cayley-Dixon construction called *hyperbolization*. we prove that every  $2N$ -dimensional unital Archimedean  $f$ -algebra is the hyperbolization of that of dimension  $N$ . As an example, we consider the properties of  $\mathbf{H}_6$  as a hyperbolization of  $\mathbf{H}_3$ .

## 2 Basic concepts

We recall some notions about hypercomplex numbers (see [8]) and Riesz space theory ([17], [23]) that we will use in our presentation.

### 2.1 $N$ -dimensional commutative hypercomplex numbers

An  $N$ -Dimensional commutative hypercomplex number system is the set

$$\left\{ \mathbf{x} = \sum_{k=0}^{N-1} x_k \mathbf{e}_k : x_k \in \mathbb{R}, \mathbf{e}_k \notin \mathbb{R} \right\}, \quad (2.1)$$

where the multiplication is given by

$$\mathbf{e}_i \mathbf{e}_j = \sum_{k=0}^{N-1} C_{ij}^k \mathbf{e}_k, \quad C_{ij}^k = C_{ji}^k, \quad (i, j, k = 0, \dots, N-1).$$

The constants  $C_{ij}^k \in \mathbb{R}$  are called *structural constants*. We consider a unital system with a multiplicative identity  $e_0$  denoted by 1 and then omitted. The multiplication between two numbers  $\mathbf{x}, \mathbf{y}$  can be expressed as

$$\mathbf{x} \mathbf{y} = \mathcal{M}(\mathbf{x}) \mathbf{y}, \quad (2.2)$$

where the *characteristic matrix*  $\mathcal{M}(\mathbf{x})$  depend on the structure constants and the components of  $\mathbf{x}$  and determined by the column vectors  $\mathbf{e}_i \mathbf{x}$ ,  $i = 0, \dots, N-1$ .

Throughout the paper,  $\mathbf{H}_N$  will denote the algebra given by the system (2.1) and satisfying the eigenvalue problem

$$\text{For each } \mathbf{x}, \mathcal{M}(\mathbf{x}) \text{ admits } N \text{ distinct real eigenvalues } \lambda_0(\mathbf{x}), \dots, \lambda_{N-1}(\mathbf{x}). \quad (2.3)$$

Therefore,  $\mathbf{H}_N$  has a basis  $(\mathbf{v}_0, \dots, \mathbf{v}_{N-1})$  of eigenvectors such that

$$\sum_{k=0}^{N-1} \mathbf{v}_k = 1, \mathbf{v}_k \mathbf{v}_n = 0 \text{ for } k \neq n \Rightarrow \mathbf{v}_k^2 = \mathbf{v}_k \text{ for each } k \quad (2.4)$$

In this basis, called *idempotent basis*, every  $\mathbf{x} \in \mathbf{H}_N$  can be expressed uniquely as

$$\mathbf{x} = \sum_{k=0}^{N-1} \lambda_k(\mathbf{x}) \mathbf{v}_k. \quad (2.5)$$

Using formula (2.5), called *spectral representation*, algebraic calculations are given by component-wise operations. Therefore, the group  $\mathbf{H}_N^*$  of units (i.e, invertible elements) of  $\mathbf{H}_N$  is characterized by all numbers  $\mathbf{x}$  such that  $\lambda_k(\mathbf{x}) \neq 0$  for each  $k$ . Moreover, one can define the  $N - 1$  conjugations of  $\mathbf{x}$  as

$$\bar{\mathbf{x}}^s = \sum_{k=0}^{N-1} \lambda_{[s+k]_N}(\mathbf{x}) \mathbf{v}_k \quad (s = 1, 2, \dots, N - 1), \quad (2.6)$$

where  $[s + k]_N$  denotes the remainder of the division of  $s + k$  by  $N$ .

Let

$$\Lambda_N = \left\{ \sigma_s : \mathbf{x} \mapsto \bar{\mathbf{x}}^s, s = 0, 1, \dots, N - 1 \right\}, \quad (2.7)$$

be the set all conjugation operators where  $\sigma_0$  is the identity operator. Therefore,  $\Lambda_N$  is a commutative group of automorphisms of  $\mathbf{H}_N$  with  $\sigma_s \circ \sigma_t = \sigma_{[s+t]_N}$ ,  $\sigma_s^{-1} = \sigma_{N-s}$ . It follows that  $\Lambda_N$  contains a non-trivial symmetry operator if and only if  $N = \dim(\mathbf{H}_N)$  is even, and in this case there is only one given by  $\sigma_{N/2}$ .

Having in mind that the set of real numbers  $\mathbb{R}$  is viewed now as a subalgebra of  $\mathbf{H}_N$ , so an element  $\mathbf{x} \in \mathbf{H}_N$  is a real number if and only if  $\lambda_k(\mathbf{x}) = \mathbf{x}$  for each  $k$ , which means that  $\sigma(\mathbf{x}) = \mathbf{x}$  for every  $\sigma \in \Lambda_N$ . It follows that the product of  $\mathbf{x}$  by its conjugates  $\bar{\mathbf{x}}^s$  ( $s = 1, \dots, N - 1$ ) belongs to  $\mathbb{R}$  and coincides with the *characteristic determinant*  $\|\mathbf{x}\|_{\mathbf{H}_N} := \det \mathcal{M}(\mathbf{x})$ . That is

$$\|\mathbf{x}\|_{\mathbf{H}_N} = \prod_{\sigma \in \Lambda_N} \sigma(\mathbf{x}) = \prod_{k=0}^{N-1} \lambda_k(\mathbf{x}). \quad (2.8)$$

The modulus of an element  $\mathbf{x} \in \mathbf{H}_N$  is denoted by  $|\mathbf{x}|_{\mathbf{H}_N}$  and defined by the formula

$$|\mathbf{x}|_{\mathbf{H}_N} := \left| \|\mathbf{x}\|_{\mathbf{H}_N} \right|^{\frac{1}{N}}. \quad (2.9)$$

Geometrically speaking, the number  $|\mathbf{x} - \mathbf{y}|_{\mathbf{H}_N}$  is interpreted as the pseudo-euclidean distance between  $\mathbf{x}$  and  $\mathbf{y}$  viewed as two points in  $\mathbb{R}^N$ . Therefore, the *nul cone* of  $\mathbf{H}_N$  (the set of non-invertible elements) is the set  $\mathcal{NC}_N$  of all points that coincide with the origin via the pseudo-euclidean distance.

## 2.2 Riesz spaces

An ordered vector space  $L$  is a real vector space that is simultaneously an ordered set connected by the implication:  $u, v, w \in L$ ,  $\alpha \in \mathbb{R}^+$ ;  $u \leq v$  implies  $u + w \leq v + w$ ,  $\alpha u \leq \alpha v$ . The set  $L^+$  of all positive elements ( i.e,  $u \in L$ ;  $u \geq 0$ ) is called positive cone of  $L$ . A positive operator is a linear map  $T : L \longrightarrow W$  such that  $T(L^+) \subset W^+$ , i.e.,  $T(u) \leq T(v)$

whenever  $u \leq v$ .

An ordered vector space  $L$  is said to be Riesz space or (vector lattice) if the supremum  $u \vee v$ ; equivalently, the infimum  $u \wedge v$  of two elements  $u$  and  $v$  exist in  $L$ . In this case the absolute value of  $u \in L$  is defined by  $|u| = u \vee (-u)$ . Two elements  $u, v \in L$  are called disjoint, and denoted by  $u \perp v$ , if  $|u| \wedge |v| = 0$ . The set of all elements disjoint from  $u$  is denoted by  $u^\perp$ , that is,  $u^\perp = \{v \in L : |u| \wedge |v| = 0\}$ .

A Riesz space  $L$  is said to be Archimedean if,

$$u, v \in L^+ \text{ then, } nu \leq v \text{ for all } n = 1, 2, \dots \text{ implies } u = 0.$$

A Riesz homomorphism is an operator  $T$  from a Riesz space  $L$  to a Riesz space  $W$  such that  $T(u \vee v) = T(u) \vee T(v)$  and  $T(u \wedge v) = T(u) \wedge T(v)$ , i.e.,  $T(|u|) = |T(u)|$ .

A real algebra  $A$  (associative algebra with usual algebraic operations) is said to be an  $f$ -algebra if its underlying group is a Riesz space with a positive cone  $A^+$  satisfying the properties:  $a, b, c \in A^+$  then,  $ab \in A^+$  and  $a \wedge b = 0$  implies  $ac \wedge b = a \wedge cb = 0$ . In any  $f$ -algebra  $A$  the squares are positive and if in addition  $A$  is Archimedean with a multiplicative unit, then  $a \perp b$  holds if and only if  $ab = 0$ .

Finally, we recall a basic fact involving bases in a finite-dimensional unital Archimedean  $f$ -algebras. This property is a direct consequence of a fundamental theorem [17, Theorem 26.10] that states that every finite-dimensional Archimedean Riesz space has a basis  $\beta$  of mutually disjoint elements (atoms). However, since in any  $f$ -algebra,  $|a| \wedge |b| = 0$  implies that  $ab = 0$ . Thus, multiplying the elements of  $\beta$  by the components of the identity element, we obtain the following result.

**Corollary 2.1.** *Every  $N$ -dimensional Archimedean  $f$ -algebra with identity element  $e$  has a basis  $(c_0, \dots, c_{N-1})$  satisfying*

$$e = c_0 + \dots + c_{N-1}, \quad c_i c_j = 0 \text{ for } i \neq j \Rightarrow c_i^2 = c_i \text{ for all } i.$$

### 3 Characterization of Archimedean $f$ -algebras of hypercomplex numbers

The next theorem characterize Finite-dimensional unital Archimedean  $f$ -algebras by means of hypercomplex numbers system.

**Theorem 3.1** (Structure theorem). *The algebra  $\mathbf{H}_N$  is, up to Riesz and algebra isomorphism, the unique  $N$ -dimensional unital Archimedean  $f$ -algebra with the set  $\{\mathbf{x} : \lambda_k(\mathbf{x}) \geq 0, \text{ for all } k\}$  as positive cone.*

*Proof.* According to the ordering in  $\mathbf{H}_N$  defined above we have that

$$\mathbf{x}, \mathbf{y} \in \mathbf{H}_N, \quad \mathbf{x} \leq \mathbf{y} \text{ if and only if } \lambda_k(\mathbf{x}) \leq \lambda_k(\mathbf{y}) \text{ for all } k.$$

Thus in view of the Archimedean  $f$ -algebra structure of  $\mathbb{R}$  and by the fact that the  $\lambda_k$  are ring homomorphisms from  $\mathbf{H}_N$  to  $\mathbb{R}$ , we deduce that  $\mathbf{H}_N$  is an Archimedean Riesz space

with the positive cone  $\mathbf{H}_N^+$  closed under multiplication. Moreover,

$$\mathbf{x} \vee \mathbf{y} = \sum_{k=0}^{N-1} \max\{\lambda_k(\mathbf{x}), \lambda_k(\mathbf{y})\} \mathbf{v}_k \quad (3.1)$$

and

$$\mathbf{x} \wedge \mathbf{y} = \sum_{k=0}^{N-1} \min\{\lambda_k(\mathbf{x}), \lambda_k(\mathbf{y})\} \mathbf{v}_k. \quad (3.2)$$

Let  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{H}_N^+$ , and  $m(\mathbf{z}) = \min_k \{1, \lambda_k(\mathbf{z})\}$ ,  $M(\mathbf{z}) = \max_k \{1, \lambda_k(\mathbf{z})\}$ , we get

$$m(\mathbf{z})(\mathbf{x} \wedge \mathbf{y}) \leq \mathbf{z} \mathbf{x} \wedge \mathbf{y} \leq M(\mathbf{z})(\mathbf{x} \wedge \mathbf{y}).$$

The above inequalities show that  $\mathbf{z} \mathbf{x} \wedge \mathbf{y} = 0$  whenever  $\mathbf{x} \wedge \mathbf{y} = 0$ , and so  $\mathbf{H}_N$  is an  $f$ -algebra. It remains to prove uniqueness (up to Riesz and algebra isomorphism). To do this, let  $\mathcal{A}_N$  be an  $N$ -dimensional Archimedean  $f$ -algebra with multiplicative identity  $e$ . By corollary 2.1, we get a basis  $(\mathbf{c}_0, \dots, \mathbf{c}_{N-1})$  in  $\mathcal{A}_N$  satisfying

$$e = \mathbf{c}_0 + \dots + \mathbf{c}_{N-1}, \quad \mathbf{c}_i \mathbf{c}_j = 0 \text{ for } i \neq j, \quad \mathbf{c}_i^2 = \mathbf{c}_i \text{ for all } i. \quad (3.3)$$

Therefore, up to an order isomorphism, the  $f$ -algebra ordering in  $\mathcal{A}_N$  is given by the positive cone  $\{\alpha_0 \mathbf{c}_0 + \dots + \alpha_{N-1} \mathbf{c}_{N-1} : \alpha_k \in \mathbb{R}^+\}$ . It follows from (3.3) and (2.4), that the mapping  $\varphi : \lambda_0 \mathbf{v}_0 + \dots + \lambda_{N-1} \mathbf{v}_{N-1} \mapsto \lambda_0 \mathbf{c}_0 + \dots + \lambda_{N-1} \mathbf{c}_{N-1}$  is an algebra isomorphism with  $\varphi(|\mathbf{x}|) = |\varphi(\mathbf{x})|$ , where from (3.1) the absolute value of  $\mathbf{x}$  is given by

$$|\mathbf{x}| := \mathbf{x} \vee (-\mathbf{x}) = \sum_{k=0}^{N-1} |\lambda_k(\mathbf{x})| \mathbf{v}_k. \quad (3.4)$$

Hence  $\varphi$  is a Riesz isomorphism. □

It is well established that for every integer  $N \geq 1$ , there exists an  $N$ -dimensional unital commutative hypercomplex number system satisfying the condition (2.3). Namely,  $\mathbb{R}^N$  with coordinate-wise multiplication. Consequently, according to Theorem (3.1), we obtain the following characterization Theorem

**Theorem 3.2** (Characterization theorem). *An  $N$ -dimensional unital commutative hypercomplex numbers system is an Archimedean  $f$ -algebra if and only if it satisfies the condition (2.3), and in this case the positive cone is given by all elements  $\mathbf{x}$  with a positive spectrum of the associate matrix  $\mathcal{M}(\mathbf{x})$ .*

## 4 Order properties of $\mathbf{H}_N$

In this section we establish some properties of  $\mathbf{H}_N$  related to its order structure. These results are multidimensional generalizations of those obtained in [11]. We start by order properties of two useful operators.

## 4.1 Eigenvalue and conjugation operators

**Proposition 4.1.** *One has*

- (i) *the eigenvalue operators  $\lambda_k$  are surjective Riesz homomorphisms with  $\ker(\lambda_k) = \mathbf{v}_k^\perp$ ;*
- (ii) *the conjugation operators are Riesz isomorphisms of  $\mathbf{H}_N$ .*

*Proof.* (i) Let  $k = 0, 1, \dots, N-1$ . From (3.4), we deduce that  $\lambda_k(|\mathbf{x}|) = |\lambda_k(\mathbf{x})|$  for every  $\mathbf{x} \in \mathbf{H}_N$  which means that  $\lambda_k$  is a Riesz homomorphism from  $\mathbf{H}_N$  to  $\mathbb{R}$ . It is also surjective, since for every real  $x$  we have  $\lambda_k(x.1) = x$ . Now, as  $\ker(\lambda_k) = \text{Vect}\{\mathbf{v}_n : n \neq k\}$  and  $\mathbf{v}_k^\perp$  is the set of all  $\mathbf{x}$  disjoint from  $\mathbf{v}_k$ . One has

$$\ker(\lambda_k) = \{\mathbf{x} : \mathbf{v}_k \mathbf{x} = 0\} = \mathbf{v}_k^\perp.$$

(ii) Let  $\sigma \in \Lambda_N$ . Then (3.4) and (2.6) shows that  $\sigma$  is a positive operator, and so is  $\sigma^{-1}$  since  $\Lambda_N$  is a group. This proves that  $\sigma$  is a Riesz isomorphism of  $\mathbf{H}_N$ . □

## 4.2 Dedekind completeness

Recall that a Riesz space  $L$  is said *Dedekind complete* if every nonempty subset of  $L$  bounded from above (resp. from below) has a supremum in  $L$  (resp. an infimum) in  $L$ . In the case of  $\mathbf{H}_N$  the result below is a direct consequence of the Dedekind completeness of  $\mathbb{R}$  and the spectral representation (2.4).

**Theorem 4.1.** *The Riesz space  $\mathbf{H}_N$  is Dedekind complete. Moreover, for every nonempty subset  $S$ , we have*

- (i) *if  $S$  is bounded from above, then*

$$\sup(S) = \sum_{k=0}^{N-1} \sup(\lambda_k(S)) \mathbf{v}_k;$$

- (ii) *if  $S$  is bounded from below, then*

$$\inf(S) = \sum_{k=0}^{N-1} \inf(\lambda_k(S)) \mathbf{v}_k.$$

**Proposition 4.2.** *Let  $S$  be a nonempty bounded subset of  $\mathbf{H}_N$  and  $\mathbf{a} \in \mathbf{H}_N$ . Then the set  $\mathbf{a}S := \{\mathbf{a}s : s \in S\}$  is bounded and*

$$\begin{aligned} \sup(\mathbf{a}S) &= \mathbf{a}^+ \sup(S) - \mathbf{a}^- \inf(S), \\ \inf(\mathbf{a}S) &= \mathbf{a}^+ \inf(S) - \mathbf{a}^- \sup(S). \end{aligned}$$



*Proof.* From Theorem 4.1 one has

$$\sup(\mathbf{a}S) = \sum_{k=0}^{N-1} \sup(\lambda_k(\mathbf{a})\lambda_k(S)) \mathbf{v}_k.$$

As for every  $k$ ,  $\lambda_k(\mathbf{a})\lambda_k(S)$  is a bounded set in the Dedekind complete  $\mathbb{R}$ , then

$$\sup(\lambda_k(\mathbf{a})\lambda_k(S)) = (\lambda_k(\mathbf{a}))^+ \sup \lambda_k(S) - (\lambda_k(\mathbf{a}))^- \inf \lambda_k(S).$$

Since  $\lambda_k$  is a Riesz homomorphism (see proposition 4.1), we get  $(\lambda_k(\mathbf{a}))^+ = \lambda_k(\mathbf{a}^+)$  and  $(\lambda_k(\mathbf{a}))^- = \lambda_k(\mathbf{a}^-)$ . Therefore, by observing that  $\lambda_k(\mathbf{x})\mathbf{v}_k = \mathbf{x}\mathbf{v}_k$  holds for every  $\mathbf{x}$  we obtain

$$\begin{aligned} \sup(\mathbf{a}S) &= \mathbf{a}^+ \sum_{k=0}^{N-1} \sup \lambda_k(S) \mathbf{v}_k - \mathbf{a}^- \sum_{k=0}^{N-1} \inf \lambda_k(S) \mathbf{v}_k \\ &= \mathbf{a}^+ \sup(S) - \mathbf{a}^- \inf(S). \end{aligned}$$

Using duality formula  $\inf(\mathbf{a}S) = -\sup(-\mathbf{a}S)$  and the identities  $(-\mathbf{a})^+ = \mathbf{a}^-$ ,  $(-\mathbf{a})^- = \mathbf{a}^+$  one obtains the second formula.  $\square$

### 4.3 Group of signs

The multiplicity of the absolute value in  $\mathbf{H}_N$  leads to introduce the following group.

**Definition 4.1.** The kernel of the group homomorphism  $|\cdot|$  from  $\mathbf{H}_N^*$  to  $(\mathbf{H}_N^*)^+$  is called *group of signs*, denoted by  $\mathfrak{S}_N$ .

From the definition above,  $\mathfrak{S}_N$  is the set of all elements  $\mathbf{x}$  such that  $|\mathbf{x}| = 1$ , and so it coincides with the set of the square roots of the unity 1, given by

$$\mathfrak{S}_N = \{\epsilon_0 \mathbf{v}_0 + \cdots \epsilon_{N-1} \mathbf{v}_{N-1} : (\epsilon_0, \cdots, \epsilon_{N-1}) \in \{-1, 1\}^N\} \cong (\mathbb{Z}/2\mathbb{Z})^N. \quad (4.1)$$

Thus, from ordering in  $\mathbf{H}_N$  the unity 1 is the unique positive element in  $\mathfrak{S}_N$ . The concept of group of signs is the natural generalization of that in real numbers  $\mathbb{R}$  with the two signs  $-1, +1$ . Therefore, a multidimensional formulation of the property that for every real number  $x$  there exists  $\epsilon \in \{-1, 1\}$  such that  $\epsilon x \geq 0$  gives the following result.

**Theorem 4.2** (Theorem of signs). *For every  $\mathbf{x} \in \mathbf{H}_N$  there exists  $\varepsilon \in \mathfrak{S}_N$  such that*

$$\varepsilon \mathbf{x} \geq \mathbf{0}.$$

*If  $\mathbf{x}$  is invertible then  $\varepsilon$  is unique, called sign of  $\mathbf{x}$ , noted  $\text{sign}(\mathbf{x})$  and given by*

$$\text{sign}(\mathbf{x}) = \frac{|\mathbf{x}|}{\mathbf{x}}.$$

*Proof.* Let  $\mathbf{x} \in \mathbf{H}_N$ . Then, as noted above, for every  $k = 0, 1, \dots, N-1$  there exists  $\epsilon_k \in \{-1, 1\}$  such that  $\epsilon_k \lambda_k(\mathbf{x}) \geq 0$ . Write  $\varepsilon = \epsilon_0 \mathbf{v}_0 + \dots + \epsilon_{N-1} \mathbf{v}_{N-1}$ . We get from (4.1)  $\varepsilon \in \mathfrak{S}$  and  $\lambda_k(\varepsilon \mathbf{x}) \geq 0$  for each  $k$ , that is  $\varepsilon \mathbf{x} \geq 0$ . Suppose now that  $\mathbf{x}$  is invertible and let  $\varepsilon_1, \varepsilon_2 \in \mathfrak{S}_N$  be such that  $\varepsilon_1 \mathbf{x} \geq 0$ ,  $\varepsilon_2 \mathbf{x} \geq 0$ . So  $\varepsilon_1 \varepsilon_2 \mathbf{x}^2 \geq 0$ , which implies that  $\varepsilon_1 \varepsilon_2 \geq 0$ . Since the unity 1 is the unique positive element of the set  $\mathfrak{S}_N$  of its square roots we must have  $\varepsilon_1 \varepsilon_2 = 1$ . Hence  $\varepsilon_1 = \varepsilon_2 = \frac{|\mathbf{x}|}{\mathbf{x}}$ . □

**Remark 4.1.** Geometrically speaking, Theorem 4.2 states that for every  $\mathbf{x} \in \mathbf{H}_N$  there exists a symmetry operator  $\psi \in \{\psi_\varepsilon : \mathbf{x} \mapsto \varepsilon \mathbf{x} : \varepsilon \in \mathfrak{S}_N\}$  that maps  $\mathbf{x}$  into its absolute value  $|\mathbf{x}|$ . Moreover,  $\psi$  is uniquely determined by  $\mathbf{x}$  if and only if  $\mathbf{x}$  is invertible, and in this case  $\psi = \psi_\varepsilon$  with  $\varepsilon = \text{sign}(\mathbf{x})$ . Further,  $\mathcal{H}_1(\varepsilon) = \ker(\psi_\varepsilon - Id)$  and  $\mathcal{H}_2(\varepsilon) = \ker(\psi_\varepsilon + Id)$  are the hyperplanes

$$\begin{aligned}\mathcal{H}_1(\varepsilon) &= \{\mathbf{x} : (\varepsilon - 1)\mathbf{x} = 0\} = (\varepsilon - 1)^\perp; \\ \mathcal{H}_2(\varepsilon) &= \{\mathbf{x} : (\varepsilon + 1)\mathbf{x} = 0\} = (\varepsilon + 1)^\perp.\end{aligned}$$

We will now give a decomposition of  $\mathbf{x} \wedge \mathbf{y}$  and  $\mathbf{x} \vee \mathbf{y}$  using the set  $I(\mathbf{H}_N)$  of all idempotent elements of  $\mathbf{H}_N$ .  $I(\mathbf{H}_N)$  is a distributive lattice with 0 and the multiplicative identity 1 as smallest and greatest elements, respectively. For every  $\mathbf{e} \in I(\mathbf{H}_N)$  there exists a unique  $\tilde{\mathbf{e}} \in I(\mathbf{H}_N)$ , called the *complement* of  $\mathbf{e}$ , such that

$$\mathbf{e}\tilde{\mathbf{e}} = \mathbf{e} \wedge \tilde{\mathbf{e}} = \mathbf{0} \text{ and } \mathbf{e} + \tilde{\mathbf{e}} = \mathbf{e} \vee \tilde{\mathbf{e}} = 1.$$

This means that  $(I(\mathbf{H}_N), \wedge, \vee)$  is a Boolean algebra.

**Proposition 4.3.** *For every  $(\mathbf{x}, \mathbf{y}) \in \mathbf{H}_N^2$  there exists  $\mathbf{e} = \mathbf{e}(\mathbf{x}, \mathbf{y}) \in I(\mathbf{H}_N)$  such that*

$$\mathbf{x} \vee \mathbf{y} = \mathbf{e}\mathbf{x} + \tilde{\mathbf{e}}\mathbf{y} \quad \text{and} \quad \mathbf{x} \wedge \mathbf{y} = \tilde{\mathbf{e}}\mathbf{x} + \mathbf{e}\mathbf{y}.$$

*Proof.* Let  $(\mathbf{x}, \mathbf{y}) \in \mathbf{H}_N^2$ . From Theorem 4.2, there exists  $\varepsilon \in \mathfrak{S}_N$  such that  $|\mathbf{x} - \mathbf{y}| = \varepsilon(\mathbf{x} - \mathbf{y})$ . Put  $\mathbf{e} = \frac{1}{2}(1 + \varepsilon)$ , one then has  $\mathbf{e} \in I(\mathbf{H}_N)$  and  $\tilde{\mathbf{e}} = \frac{1}{2}(1 - \varepsilon)$ . Using the following identities that holds in any Riesz space

$$\mathbf{x} \vee \mathbf{y} = \frac{1}{2}(\mathbf{x} + \mathbf{y} + |\mathbf{x} - \mathbf{y}|) \text{ and } \mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{x} + \mathbf{y} - |\mathbf{x} - \mathbf{y}|),$$

we obtain

$$\mathbf{x} \vee \mathbf{y} = \mathbf{e}\mathbf{x} + \tilde{\mathbf{e}}\mathbf{y} \quad \text{and} \quad \mathbf{x} \wedge \mathbf{y} = \tilde{\mathbf{e}}\mathbf{x} + \mathbf{e}\mathbf{y}.$$

□

## 4.4 Banach lattice algebra norm

One of the major problems arising in the study of algebras is that they cannot be normed in a sufficiently "nice" manner. It is well known, by Hurwitz Theorem, that  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$  (octonions) are the only finite-dimensional real algebras with a multiplicative norm. Therefore, apart from these algebras, multiplicity of the norm must fail and then, in the best case, one has a sub-multiplicative norm (i.e.  $\|uv\| \leq \|u\|\|v\|$ ). A similar norm-

problem in Riesz spaces leads to the notion of *lattice norm* (i.e.  $\|u\| \leq \|v\|$  whenever  $|u| \leq |v|$ ). Therefore, working in a "good" structure provides the following definition: a *Banach lattice algebra norm* in a Riesz algebra  $A$  is a sub-multiplicative lattice norm. If in addition,  $A$  has a multiplicative identity with norm one, then  $A$  is said to be *unital Banach lattice algebra*.

In our case, we will prove that the structure of  $\mathbf{H}_N$  allows us to make it into unital Banach lattice algebra by means of conjugation operators. This is due to the important algebraic and order properties claimed in Proposition 4.1.

Let  $\bigvee_{\sigma \in \Lambda_N}$ ,  $\bigwedge_{\sigma \in \Lambda_N}$  be the supremum and the infimum taken over  $\Lambda_N$ , respectively.

**Theorem 4.3.** *The algebra  $\mathbf{H}_N$  is a unital Banach lattice algebra under the norm*

$$\|\mathbf{x}\|_R := \min \left\{ \alpha \in \mathbb{R}^+ : \alpha \cdot 1 \geq |\mathbf{x}| \right\} = \bigvee_{\sigma \in \Lambda_N} \sigma(|\mathbf{x}|) \quad \text{for all } \mathbf{x} \in \mathbf{H}_N.$$

*Proof.* It is clear, by definition and by the fact that conjugations are positive operators, that  $\|\cdot\|_R$  is a lattice norm and  $\|1\|_R = 1$ . So, we have to prove that  $\|\cdot\|_R$  is a sub-multiplicative norm. Let  $\mathbf{x}, \mathbf{y} \in \mathbf{H}_N$ ,  $\sigma \in \Lambda_N$ . One then has  $0 \leq \sigma(|\mathbf{x}|) \leq \|\mathbf{x}\|_R$  and  $0 \leq \sigma(|\mathbf{y}|) \leq \|\mathbf{y}\|_R$ . Therefore,

$$\sigma(|\mathbf{x}\mathbf{y}|) = \sigma(|\mathbf{x}|)\sigma(|\mathbf{y}|) \leq \|\mathbf{x}\|_R \|\mathbf{y}\|_R.$$

Hence

$$\bigvee_{\sigma \in \Lambda_N} \sigma(|\mathbf{x}\mathbf{y}|) = \|\mathbf{x}\mathbf{y}\|_R \leq \|\mathbf{x}\|_R \|\mathbf{y}\|_R.$$

□

the importance of the norm  $\|\cdot\|_R$  lies not only in its Banach lattice algebra norm property but also in its lattice-conjugation connection. This approach allows us to obtain interesting properties, as we will explain in the next section. Let  $\mathbf{i}(\cdot)$  and  $\mathbf{s}(\cdot)$  be the real functions defined for every  $\mathbf{x} \in \mathbf{H}_N$  by

$$\mathbf{i}(\mathbf{x}) = \bigwedge_{\sigma \in \Lambda_N} \sigma(\mathbf{x}) \quad \text{and} \quad \mathbf{s}(\mathbf{x}) = \bigvee_{\sigma \in \Lambda_N} \sigma(\mathbf{x}). \quad (4.2)$$

Using the identity  $uv = (u \wedge v)(u \vee v)$  that holds in any  $f$ -algebra we obtain from (2.8),

$$\|\mathbf{x}\|_{\mathbf{H}_N} = \mathbf{i}(\mathbf{x})\mathbf{s}(\mathbf{x}). \quad (4.3)$$

Eq (4.2) shows that  $\|\mathbf{x}\|_R = \mathbf{s}(|\mathbf{x}|)$ . So, (4.3) and (2.9) give the formula

$$|\mathbf{x}|_{\mathbf{H}_N} = i_N(\mathbf{x})\|\mathbf{x}\|_R^{\frac{1}{N}}, \quad i_N(\mathbf{x}) = (\mathbf{i}(|\mathbf{x}|))^{\frac{1}{N}}$$

As norms in any finite-dimensional vector space are equivalent, the above identity implies that for every norm  $\|\cdot\|$ , there exist positive real constants  $c_1$  and  $c_2$ , which depend on the dimension  $N$  and the norm  $\|\cdot\|$ , such that for every  $\mathbf{x} \in \mathbf{H}_N$ , we have...

$$c_1 i_N(\mathbf{x})\|\mathbf{x}\|^{\frac{1}{N}} \leq |\mathbf{x}|_{\mathbf{H}_N} \leq c_2 i_N(\mathbf{x})\|\mathbf{x}\|^{\frac{1}{N}}.$$

## 5 Intervals of $\mathbf{H}_N$

The notion of interval in real numbers can be extended to  $\mathbf{H}_N$  in the following way.

**Definition 5.1.** A subset  $\mathcal{I} \subset \mathbf{H}_N$  is said to be an interval of  $\mathbf{H}_N$  if it satisfies the following property

$$\mathbf{a} \wedge \mathbf{b} \leq \mathbf{x} \leq \mathbf{a} \vee \mathbf{b} \text{ and } \mathbf{a}, \mathbf{b} \in \mathcal{I} \text{ implies } \mathbf{x} \in \mathcal{I}.$$

If moreover  $\|\mathbf{x} - \mathbf{y}\|_{\mathbf{H}_N} = 0$  for all  $\mathbf{x}, \mathbf{y} \in \mathcal{I}$ , then  $\mathcal{I}$  is said to be a degenerate interval.

**Example 5.1.** Let  $\mathbf{a}, \mathbf{b} \in \mathbf{H}_N$  be such that  $\mathbf{a} \leq \mathbf{b}$  then the set

$$[\mathbf{a}, \mathbf{b}]_{\mathbf{H}_N} := \{\mathbf{x} \in \mathbf{H}_N : \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\} \quad (5.1)$$

is a closed (for usual topology) interval of  $\mathbf{H}_N$  with endpoints  $\mathbf{a}$  and  $\mathbf{b}$ . It is a degenerate interval if and only if  $\|\mathbf{a} - \mathbf{b}\|_{\mathbf{H}_N} = 0$ . Write  $\mathbf{x} \prec \mathbf{y}$  when  $(\mathbf{y} - \mathbf{x}) \in (\mathbf{H}_*)^+$ . Then, similarly, one can define the open interval  $(\mathbf{a}, \mathbf{b})_{\mathbf{H}_N}$  and the half-open intervals  $(\mathbf{a}, \mathbf{b}]_{\mathbf{H}_N}$  and  $[\mathbf{a}, \mathbf{b})_{\mathbf{H}_N}$ , replacing  $\leq$  in (5.1) by  $\prec$ . In particular, the closed ball  $B_R(\mathbf{x}_0, r)$  and the open ball  $B_R^o(\mathbf{x}_0, r)$  in the space  $(\mathbf{H}_N, \|\cdot\|_R)$  are

$$\begin{aligned} B_R(\mathbf{x}_0, r) &= [\mathbf{x}_0 - r, \mathbf{x}_0 + r]_{\mathbf{H}_N}, \\ B_R^o(\mathbf{x}_0, r) &= (\mathbf{x}_0 - r, \mathbf{x}_0 + r)_{\mathbf{H}_N}. \end{aligned}$$

As is well known, intervals  $I$  on the real line are convex sets, meaning that if  $\alpha \in [0, 1]$  and  $a, b \in I$ , then  $a + \alpha(b - a) \in I$ . Our purpose in the following is to formulate a similar result in hypercomplex setting. This suggests introducing the concept of  $\mathbf{H}_N$ -convex set as follows :

**Definition 5.2.** A subset  $S \subset \mathbf{H}_N$  is said to be  $\mathbf{H}_N$ -convex if it satisfies the property

$$\mathbf{a}, \mathbf{b} \in S \text{ implies } [\mathbf{a}, \mathbf{b}]_{cov_{\mathbf{H}_N}} \subset S,$$

where  $[\mathbf{a}, \mathbf{b}]_{cov_{\mathbf{H}_N}}$  is the closed  $\mathbf{H}_N$ -segment with extremities  $\mathbf{a}$  and  $\mathbf{b}$ , given by

$$[\mathbf{a}, \mathbf{b}]_{cov_{\mathbf{H}_N}} = \{\mathbf{a} + \mathbf{u}(\mathbf{b} - \mathbf{a}) : \mathbf{u} \in [0, 1]_{\mathbf{H}_N}\}.$$

Now, in order to characterize intervals of  $\mathbf{H}_N$  as  $\mathbf{H}_N$ -convex sets we need the following lemma

**Lemma 5.1.** *For every  $\mathbf{a}, \mathbf{b} \in \mathbf{H}_N$ , we have*

$$[\mathbf{a}, \mathbf{b}]_{cov_{\mathbf{H}_N}} = [\mathbf{a} \wedge \mathbf{b}, \mathbf{a} \vee \mathbf{b}]_{\mathbf{H}_N}.$$

*Proof.* From proposition 4.3 one gets,

$$[\mathbf{a} \wedge \mathbf{b}, \mathbf{a} \vee \mathbf{b}]_{\mathbf{H}_N} = [\mathbf{a} \wedge \mathbf{b}, \mathbf{a} \vee \mathbf{b}]_{cov_{\mathbf{H}_N}} = [\mathbf{a}, \mathbf{b}]_{cov_{\mathbf{H}_N}}.$$

Therefore, it suffices to prove that the desired equality holds in the case  $\mathbf{a} \leq \mathbf{b}$ . In this case one has  $[\mathbf{a}, \mathbf{b}]_{\text{cov}_{\mathbf{H}_N}} \subset [\mathbf{a}, \mathbf{b}]_{\mathbf{H}_N}$ . For the second inclusion, let  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]_{\mathbf{H}_N}$  and let  $\theta_0, \dots, \theta_{N-1}$  be such that  $\theta_k = \lambda_k(\mathbf{b} - \mathbf{a})$  if  $\mathbf{b} - \mathbf{a} \notin \ker(\lambda_k)$  and  $\theta_k = 1$ , otherwise. Set

$$\mathbf{u} = \sum_{k=0}^{N-1} \frac{\lambda_k(\mathbf{b} - \mathbf{a})}{\theta_k} \mathbf{v}_k.$$

Thus, we have  $\mathbf{u} \in [0, 1]_{\mathbf{H}_N}$  and

$$\mathbf{a} + \mathbf{u}(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \sum_{k=0}^{N-1} \lambda_k(\mathbf{x} - \mathbf{a}) \mathbf{v}_k = \mathbf{x}.$$

Hence,

$$[\mathbf{a}, \mathbf{b}]_{\mathbf{H}_N} = [\mathbf{a}, \mathbf{b}]_{\text{cov}_{\mathbf{H}_N}}.$$

□

Finally, Definition 5.2 together with Definition 5.1 and Lemma 5.1 show that.

**Theorem 5.1.** *Intervals of  $\mathbf{H}_N$  are  $\mathbf{H}_N$ -convex sets.*

In the sequel, we will give another description of degenerate intervals which are (by Definition (5.1)) sets  $\mathcal{I}$  such that the pseudo-euclidean distance between each pair of points in  $\mathcal{I}$  is equal to 0. We show, modulo some point, that such intervals can be decomposed as a  $\mathcal{W}$ -combination of intervals in  $\mathbb{R}$ , where  $\mathcal{W} \subset \{\mathbf{v}_0, \dots, \mathbf{v}_{N-1}\}$  (the idempotent basis given by 2.4). We will first prove the following simple result.

**Lemma 5.2.** *For every eigenvalue operator  $\lambda_k$ , the direct image  $\lambda_k(\mathcal{I})$  of an interval  $\mathcal{I}$  of  $\mathbf{H}_N$  is an interval of  $\mathbb{R}$ .*

*Proof.* Let  $\mathcal{I}$  be a nonempty interval of  $\mathbf{H}_N$ ;  $\eta, \mu \in \mathbb{R}^+$ ,  $\eta + \mu = 1$  and  $a, b \in \lambda_k(\mathcal{I})$ . Thus,  $\eta\mathbf{a} + \mu\mathbf{b} \in \mathcal{I}$  and  $\eta a + \mu b = \lambda_k(\eta\mathbf{a} + \mu\mathbf{b})$  where  $a = \lambda_k(\mathbf{a})$  and  $b = \lambda_k(\mathbf{b})$ . This proves that  $\lambda_k(\mathcal{I})$  is an interval of  $\mathbb{R}$ . □

We prove now the following Theorem.

**Theorem 5.2.** *Degenerate intervals of  $\mathbf{H}_N$  are the sets*

$$\mathcal{I} = \mathbf{x}_0 + \sum_{i=1}^{N-1} \mathbf{w}_i I_i,$$

where  $\mathbf{x}_0 \in \mathbf{H}_N$ ,  $\{\mathbf{w}_1, \dots, \mathbf{w}_{N-1}\} \subset \{\mathbf{v}_0, \dots, \mathbf{v}_{N-1}\}$  and  $I_i$  are intervals of  $\mathbb{R}$ .

*Proof.* Suppose that  $\mathcal{I} = \mathbf{x}_0 + \sum_{i=1}^{N-1} \mathbf{w}_i I_i$  for some  $\mathbf{x}_0 \in \mathbf{H}_N$ ;  $\{\mathbf{w}_1, \dots, \mathbf{w}_{N-1}\} \subset \{\mathbf{v}_0, \dots, \mathbf{v}_{N-1}\}$  and intervals  $I_i$  of  $\mathbb{R}$ . So, we have  $\|\mathbf{x} - \mathbf{y}\|_{\mathbf{H}_N} = 0$  for every  $\mathbf{x}, \mathbf{y} \in \mathcal{I}$ . We will to prove  $\mathcal{I}$  is an interval. Let  $\mathbf{u}, \mathbf{v} \in \mathbf{H}_N^+$ ,  $\mathbf{u} + \mathbf{v} = 1$  and  $\mathbf{a}, \mathbf{b} \in \mathcal{I}$ . Then

$$\mathbf{u}\mathbf{a} + \mathbf{v}\mathbf{b} = \mathbf{x}_0 + \sum_{i=1}^{N-1} (\mathbf{u}a_i + \mathbf{v}b_i)\mathbf{w}_i, \quad a_i, b_i \in I_i, \quad (i = 1, \dots, N-1).$$

Since, for every  $i = 1, \dots, N-1$ , we can write  $\mathbf{u}\mathbf{w}_i = \alpha_i\mathbf{w}_i$  and  $\mathbf{v}\mathbf{w}_i = \beta_i\mathbf{w}_i$  for some  $\alpha_i, \beta_i \in \mathbb{R}^+$  with  $\alpha_i + \beta_i = 1$ , we obtain

$$\begin{aligned}\mathbf{u}\mathbf{a} + \mathbf{v}\mathbf{b} &= \mathbf{x}_0 + \sum_{i=1}^{N-1} (\alpha_i a_i + \beta_i b_i) \mathbf{w}_i \\ &= \mathbf{x}_0 + \sum_{i=1}^{N-1} s_i \mathbf{w}_i, \quad s_i \in I_i.\end{aligned}$$

This proves that  $\mathcal{I}$  is a degenerate interval of  $\mathbf{H}_N$ . Conversely, suppose that  $\mathcal{I}$  is a degenerate interval of  $\mathbf{H}_N$ . Fix  $\mathbf{x}_0 \in \mathcal{I}$ , so  $\mathcal{I} - \mathbf{x}_0$  is an interval of  $\mathbf{H}_N$ . Write

$$\mathcal{I} - \mathbf{x}_0 = \sum_{k=0}^{N-1} \lambda_k (\mathcal{I} - \mathbf{x}_0) \mathbf{v}_k.$$

Lemma 5.2 shows that the sets  $\lambda_k (\mathcal{I} - \mathbf{x}_0)$  are intervals of  $\mathbb{R}$  such that at least one of them is reduced to the single interval  $\{0\}$ , since  $\mathcal{I} - \mathbf{x}_0 \subset \mathcal{NC}_N$ . Then, from the above identity, one can conclude that

$$\mathcal{I} = \mathbf{x}_0 + \sum_{i=1}^{N-1} \mathbf{w}_i I_i,$$

where  $\{\mathbf{w}_1, \dots, \mathbf{w}_{N-1}\} \subset \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{N-1}\}$  and  $I_i$  are intervals of  $\mathbb{R}$ . □

## 6 Explicit description of $N$ - dimensional Archimedean $f$ -algebras, $N = 2, 3, 4$ .

### 6.1 Two-dimensional Archimedean $f$ -algebra

According to [8] the general commutative two-dimensional hypercomplex numbers are

$$\mathbb{C}_{\alpha, \beta} = \{z = x + \mathbf{u}y : \mathbf{u}^2 = \beta\mathbf{u} + \alpha; x, y, \alpha, \beta \in \mathbb{R}; \mathbf{u} \notin \mathbb{R}\}. \quad (6.1)$$

Then, the structure constants are  $C_{00}^1 = C_{01}^0 = 0$ ;  $C_{00}^0 = C_{01}^1 = 1$ ;  $C_{11}^0 = \alpha$ ;  $C_{11}^1 = \beta$  and the characteristic matrix is

$$\mathcal{M}(x + \mathbf{u}y) = \begin{pmatrix} x & \alpha y \\ y & x + \beta y \end{pmatrix}.$$

Thus, (6.1) satisfies (2.3) if and only if  $\beta^2 + 4\alpha > 0$ . In this case,  $\mathbb{C}_{\alpha, \beta}$  is isomorphic to the canonical system  $\mathbb{C}_{1,0}$  of hyperbolic numbers

$$\mathbb{D} = \{x + \mathbf{j}y : x, y \in \mathbb{R}; \mathbf{j} \notin \mathbb{R}; \mathbf{j}^2 = 1\}.$$

Thus (by Theorem 3.1), hyperbolic numbers  $\mathbb{D}$  (also called duplex, perplex or split complex numbers) is the unique (up to Riesz and algebra isomorphism) unital two-dimensional Archimedean  $f$ -algebra with the set

$$\mathbb{D}^+ = \{z \in \mathbb{D} : \lambda_0(z), \lambda_1(z) \geq 0\} = \{\alpha\mathbf{v}_0 + \beta\mathbf{v}_1 : \alpha, \beta \in \mathbb{R}^+\}, \quad (6.2)$$

as positive cone, where

$$\lambda_0(z) = x + y, \quad \lambda_1(z) = x - y,$$

and the idempotent basis  $(\mathbf{v}_0, \mathbf{v}_1)$  is

$$\mathbf{v}_0 = \frac{1 + \mathbf{j}}{2}, \quad \mathbf{v}_1 = \frac{1 - \mathbf{j}}{2}.$$

Note that the notion of partial order on  $\mathbb{D}$  under the positive cone (6.2) was mentioned at first time by [10, Section 2] and [1, Section 1.4] who established some of its elementary properties in order to study Hilbert spaces over bicomplex modulus. Independently, a lattice-approach [11] studied the properties of  $\mathbb{D}$  as the unique extension of real numbers into a two-dimensional Archimedean  $f$ -algebra, viewed now as  $\mathbf{H}_2$ .

The group of signs  $\mathfrak{S}_2$  in  $\mathbb{D}$  is the four-element Klein group  $\{1, -1, \mathbf{j}, -\mathbf{j}\}$ . From (2.6), a hyperbolic number  $z = x + \mathbf{j}y$  has one conjugation  $\bar{z}$  given by exchanging  $\lambda_0(z) \leftrightarrow \lambda_1(z)$ , i.e.,  $\bar{z} = x - \mathbf{j}y$ , and so  $\sigma(z) = \bar{z}$  is the symmetry with respect to the  $x$ -axis. The modulus  $|z|_{\mathbb{D}} = \sqrt{|z\bar{z}|} = \sqrt{|x^2 - y^2|}$  represents the space-time distance of the point  $z$  from the origin in the Minkowski plane.

It is worth noticing that the notion of a closed interval  $[\alpha, \beta]_{\mathbb{D}}$  was introduced by [2] and considered in [6] with the aim of extending the usual concept of Cantor sets in the real line to the hyperbolic numbers plane  $\mathbb{D}$ . Our contribution to this fact provides a general definition yielding further classes of intervals, including unbounded ones. In fact, there is no more to be added for a closed bounded interval that is necessarily of the form  $[\alpha, \beta]_{\mathbb{D}}$ , and so (by Theorem 5.2), it is a degenerate interval if and only if it is a line segment parallel to one of the two bisector axis  $\mathbb{R}\mathbf{v}_0 : y = x$  and  $\mathbb{R}\mathbf{v}_1 : y = -x$ .

## 6.2 Three-dimensional Archimedean $f$ -algebra

Recall that Hamilton was searching for a real, normed, three-dimensional associative, division algebra that does not exist. The three-dimensional Archimedean  $f$ -algebra

$$\mathbf{H}_3 := \left\{ \tau = x + \mathbf{i}y + \mathbf{h}z; x, y, z \in \mathbb{R}; \mathbf{h}, \mathbf{k} \notin \mathbb{R} \right\},$$

can be seen, in some sense, as an attempt to look for a different solution to Hamilton's problem. The multiplication in  $\mathbf{H}_3$  is given by the rules

$$\mathbf{i}^2 = \frac{1}{2}(1 + \mathbf{h}), \quad \mathbf{h}^2 = 1, \quad \mathbf{i}\mathbf{h} = \mathbf{h}\mathbf{i} = \mathbf{i}.$$

$\mathbf{H}_3$  contains one two-dimensional subalgebra given by hyperbolic numbers  $\mathbb{D}$ . The associate matrix is given by

$$\mathcal{M}(x + \mathbf{i}y + \mathbf{h}z) = \begin{pmatrix} x & \frac{y}{2} & z \\ y & x + z & y \\ z & \frac{y}{2} & x \end{pmatrix}. \quad (6.3)$$

Thus, the eigenvalue operators and the idempotent basis  $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$  are given by

$$\lambda_0(\tau) = x - z, \quad \lambda_1(\tau) = x + y + z, \quad \lambda_2(\tau) = x - y + z.$$

$$\mathbf{v}_0 = \frac{1}{2}(1 - \mathbf{h}), \quad \mathbf{v}_1 = \frac{1}{4}(1 + 2\mathbf{i} + \mathbf{h}), \quad \mathbf{v}_2 = \frac{1}{4}(1 - 2\mathbf{i} + \mathbf{h}).$$

By Theorem 3.1, the  $f$ -algebra ordering is given by the positive cone

$$\mathbf{H}_3^+ = \{\tau : \lambda_i(\tau) \geq 0; i = 0, 1, 2.\}.$$

which coincides, in the  $\mathbb{R}^3$ -representation, with

$$\mathbf{H}_3^+ = \{x + \mathbf{i}y + \mathbf{h}z : x \geq 0, z \leq x, |y| \leq x + z\}.$$

The two conjugations of  $\tau = \lambda_0(\tau)\mathbf{v}_0 + \lambda_1(\tau)\mathbf{v}_1 + \lambda_2(\tau)\mathbf{v}_2$  are

$$\begin{aligned} \bar{\tau}^1 &= \lambda_1(\tau)\mathbf{v}_0 + \lambda_2(\tau)\mathbf{v}_1 + \lambda_0(\tau)\mathbf{v}_2, \\ \bar{\tau}^2 &= \lambda_2(\tau)\mathbf{v}_0 + \lambda_0(\tau)\mathbf{v}_1 + \lambda_1(\tau)\mathbf{v}_2. \end{aligned}$$

### 6.3 Four-dimensional Archimedean $f$ -algebra

The four-dimensional unital Archimedean  $f$ -algebra  $\mathbf{H}_4$  is the set of hyperbolic quaternions (also called bihyperbolic numbers), given by

$$\mathbf{H}_4 =: \left\{ q = x + \mathbf{j}_1 y + \mathbf{j}_2 z + \mathbf{j}_3 t; x, y, z, t \in \mathbb{R}; \mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3 \notin \mathbb{R} \right\},$$

where the multiplication is given by the following rules

$$\mathbf{j}_1^2 = \mathbf{j}_2^2 = \mathbf{j}_3^2 = 1, \mathbf{j}_1 \mathbf{j}_2 = \mathbf{j}_3, \mathbf{j}_1 \mathbf{j}_3 = \mathbf{j}_2, \mathbf{j}_2 \mathbf{j}_3 = \mathbf{j}_1.$$

The algebra  $\mathbf{H}_4$  contains three copies of the algebra of hyperbolic numbers  $\mathbb{D}$ . The associate matrix  $\mathcal{M}(q)$  is

$$\mathcal{M}(q) = \begin{pmatrix} x & y & z & t \\ y & x & t & z \\ z & t & x & y \\ t & z & y & x \end{pmatrix}.$$

Thus the eigenvalues of  $\mathcal{M}(q)$  are

$$\begin{aligned} \lambda_0(q) &= x + y + z + t, & \lambda_1(q) &= x + y - z - t \\ \lambda_2(q) &= x - y + z - t, & \lambda_3(q) &= x - y - z + t; \end{aligned}$$

and the idempotent basis  $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is given by

$$\begin{aligned} \mathbf{v}_0 &= \frac{1}{4}(1 + \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3) & \mathbf{v}_1 &= \frac{1}{4}(1 + \mathbf{j}_1 - \mathbf{j}_2 - \mathbf{j}_3) \\ \mathbf{v}_2 &= \frac{1}{4}(1 - \mathbf{j}_1 + \mathbf{j}_2 - \mathbf{j}_3) & \mathbf{v}_3 &= \frac{1}{4}(1 - \mathbf{j}_1 - \mathbf{j}_2 + \mathbf{j}_3). \end{aligned}$$

The Archimedean  $f$ -algebra ordering is given by the positive cone

$$\mathbf{H}_4^+ = \{q : \lambda_k(q) \geq 0; k = 0, 1, 2, 3.\} = \{\alpha_0 \mathbf{v}_0 + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 : \alpha_i \in \mathbb{R}^+\}.$$

For a deeper study on the algebraic properties of  $\mathbf{H}_4$  as an ordered vector space under the above positive cone see [4].



## 7 Hyperbolization of $\mathbf{H}_N$

In this section we propose an hyperbolic analogue of the Cayley-Dixon construction called *hyperbolization*. Recall that  $\mathbf{H}_N$  denotes the  $N$ -dimensional hypercomplex numbers system (2.1) with a natural basis  $(\mathbf{e}_0, \dots, \mathbf{e}_{N-1})$  endowed by its Archimedean  $f$ -algebra structure under the positive cone

$$\mathbf{H}_N^+ = \{\lambda_0 \mathbf{v}_0 + \dots + \mathbf{v}_{N-1} : \lambda_i \in \mathbb{R}^+ \text{ for } i = 0, \dots, N-1\},$$

where  $(\mathbf{v}_0, \dots, \mathbf{v}_{N-1})$  is the idempotent basis of  $\mathbf{H}_N$ .

Define the hyperbolization of  $\mathbf{H}_N$  as

$$\mathbb{D}_N := \mathbf{H}_N + \mathbf{j}\mathbf{H}_N, \quad \mathbf{j}^2 = 1.$$

Thus,  $\mathbb{D}_N$  is a  $2N$ -dimensional system of hypercomplex numbers with the natural basis  $(\mathbf{f}_0, \dots, \mathbf{f}_{2N-1})$  given by

$$\mathbf{f}_i = \mathbf{e}_i \text{ if } 0 \leq i \leq N-1, \text{ and } \mathbf{f}_i = \mathbf{j}\mathbf{e}_{i-N} \text{ if } N \leq i \leq 2N-1. \quad (7.1)$$

Let us consider the idempotent elements  $e_1$  and  $e_2$  where

$$e_1 = \frac{1+\mathbf{j}}{2} \text{ and } e_2 = \frac{1-\mathbf{j}}{2} \Rightarrow e_1 + e_2 = 1, e_1 e_2 = 0. \quad (7.2)$$

Then, each  $\mathbf{z} = \mathbf{x} + \mathbf{j}\mathbf{y} \in \mathbf{H}_N + \mathbf{j}\mathbf{H}_N$  can be written as

$$\mathbf{z} = (\mathbf{x} + \mathbf{y})e_1 + (\mathbf{x} - \mathbf{y})e_2. \quad (7.3)$$

The matrix representation of  $\mathbf{z} = \mathbf{x} + \mathbf{j}\mathbf{y} \in \mathbf{H}_N + \mathbf{j}\mathbf{H}_N$  is given by

$$\mathcal{M}(\mathbf{z}) = \begin{pmatrix} \mathcal{M}_N(\mathbf{x}) & \mathcal{M}_N(\mathbf{y}) \\ \mathcal{M}_N(\mathbf{y}) & \mathcal{M}_N(\mathbf{x}) \end{pmatrix}, \quad (7.4)$$

where  $\mathcal{M}_N(\mathbf{x})$  and  $\mathcal{M}_N(\mathbf{y})$  are the associate matrices of  $\mathbf{x}$  and  $\mathbf{y}$ , respectively.

**Theorem 7.1.** *The hypercomplex numbers system  $\mathbb{D}_N$  satisfies the condition (2.3) where the eigenvalues of the associate matrix  $\mathcal{M}(\mathbf{z})$  of  $\mathbf{z} = \mathbf{x} + \mathbf{j}\mathbf{y} \in \mathbb{D}_N$  are*

$$\gamma_0(\mathbf{z}) = \lambda_0(\mathbf{x} + \mathbf{y}), \quad \gamma_1(\mathbf{z}) = \lambda_0(\mathbf{x} - \mathbf{y}), \quad \dots, \quad \gamma_{2N-2}(\mathbf{z}) = \lambda_{N-1}(\mathbf{x} + \mathbf{y}), \quad \gamma_{2N-1}(\mathbf{z}) = \lambda_{N-1}(\mathbf{x} - \mathbf{y}),$$

*and the idempotent basis  $(\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{2N-1})$  is*

$$\mathbf{w}_0 = e_1 \mathbf{v}_0, \quad \mathbf{w}_1 = e_2 \mathbf{v}_0, \quad \dots, \quad \mathbf{w}_{2N-2} = e_1 \mathbf{v}_{N-1}, \quad \mathbf{w}_{2N-1} = e_2 \mathbf{v}_{N-1}.$$

*Proof.* We have  $\mathcal{M}(\mathbf{z}) = PDP^{-1}$  with

$$P = \frac{1}{2} \begin{pmatrix} I_N & I_N \\ I_N & -I_N \end{pmatrix} \text{ and } D = \begin{pmatrix} \mathcal{M}_N(\mathbf{x} + \mathbf{y}) & 0 \\ 0 & \mathcal{M}_N(\mathbf{x} - \mathbf{y}) \end{pmatrix},$$

where  $I_N$  is the identity matrix. Therefore, the spectrum of  $\mathcal{M}(\mathfrak{z})$  is the set

$$\{\gamma_l(\mathfrak{z}) : l = 0, \dots, 2N-1\} = \{\lambda_k(\mathbf{x}+\mathbf{y}) : k = 0, \dots, N-1\} \cup \{\lambda_k(\mathbf{x}-\mathbf{y}) : k = 0, \dots, N-1\}.$$

We prove now that  $\{\mathbf{w}_l : l = 0, \dots, 2N-1\}$  is the corresponding set of orthonormal eigenvectors. From (2.2) and (7.3) one has

$$\begin{aligned} \mathcal{M}(\mathfrak{z}).\mathbf{w}_0 &= ((\mathbf{x} + \mathbf{y})e_1 + (\mathbf{x} - \mathbf{y})e_2) e_1 \mathbf{v}_0 \\ &= (\mathbf{x} + \mathbf{y})e_1 \mathbf{v}_0 \\ &= \left( \sum_{k=0}^{N-1} \lambda_k(\mathbf{x} + \mathbf{y}) \mathbf{v}_k \right) e_1 \mathbf{v}_0 \\ &= \lambda_0(\mathbf{x} + \mathbf{y}) \mathbf{w}_0 \\ &= \gamma_0(\mathfrak{z}) \mathbf{w}_0. \end{aligned}$$

Similarly, one can obtain  $\mathcal{M}(\mathfrak{z}).\mathbf{w}_l = \gamma_l(\mathfrak{z}) \mathbf{w}_l$  for  $l = 1, \dots, 2N-1$ . Finally, it follows from (7.2) and (2.4) that  $(\mathbf{w}_0, \dots, \mathbf{w}_{2N-1})$  is the idempotent basis of  $\mathbb{D}_N$ ; that is,

$$1 = \sum_{l=0}^{2N-1} \mathbf{w}_l, \quad \mathbf{w}_i \mathbf{w}_j = 0 \text{ for } i \neq j \Rightarrow \mathbf{w}_i^2 = \mathbf{w}_i \text{ for all } i.$$

□

From the fact that every multidimensional hypercomplex numbers system satisfying (2.3) has a spectral decomposition with respect to its idempotent basis, we deduce (from Theorem 7.1) that every  $\mathfrak{z} \in \mathbb{D}_N$  has a unique representation of the form

$$\mathfrak{z} = \sum_{l=0}^{2N-1} \gamma_l(\mathfrak{z}) \mathbf{w}_l. \tag{7.5}$$

Additionally, from (7.3) we get the following idempotent representation

$$\mathfrak{z} = \mathbf{z}_1 e_1 + \mathbf{z}_2 e_2, \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathbf{H}_N. \tag{7.6}$$

And then, from (7.6) and (7.3) we can write

$$\mathfrak{z} = \sum_{k=0}^{N-1} z_k \mathbf{v}_k, \quad z_k \in \mathbb{D}. \tag{7.7}$$

The next result is a direct consequence of Theorems 3.1 and 7.1.

**Theorem 7.2.** *Every  $2N$ -dimensional unital Archimedean  $f$ -algebra is, up to Riesz and algebra isomorphism, the hyperbolization  $\mathbb{D}_N$  of the  $N$ -dimensional unital Archimedean  $f$ -algebras  $\mathbf{H}_N$ .*

Representations (7.5) , (7.6) and (7.7) gives the various representations of  $\mathbb{D}_N^+$ ,

$$\begin{aligned}\mathbb{D}_N^+ &= \left\{ \mathfrak{z} = \sum_{k=0}^{2N-1} \gamma_k \mathbf{w}_k : \gamma_k \in \mathbb{R}^+ \right\}, \\ \mathbb{D}_N^+ &= \left\{ \mathfrak{z} = \mathbf{z}_1 e_1 + \mathbf{z}_2 e_2 : \mathbf{z}_1, \mathbf{z}_2 \in \mathbf{H}_N^+ \right\}, \\ \mathbb{D}_N^+ &= \left\{ \mathfrak{z} = \sum_{k=0}^{N-1} z_k \mathbf{v}_k, z_k \in \mathbb{D}^+ \right\}.\end{aligned}$$

**Example 7.1.** From (7.1) the hyperbolization of hyperbolic numbers  $\mathbb{D} = \{x + \mathbf{h}y : x, y \in \mathbb{R}, \mathbf{h}^2 = 1\}$  is the four Archimedean  $f$ -algebra

$$\mathbb{D} + \mathbf{j}\mathbb{D} = \left\{ \mathfrak{z} = x + \mathbf{h}y + \mathbf{j}z + \mathbf{k}t; x, y, z, t \in \mathbb{R}; \mathbf{j}, \mathbf{h}, \mathbf{k} \notin \mathbb{R} \right\},$$

where the multiplication is given by the following rules

$$\mathbf{h}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{h}\mathbf{j} = \mathbf{k}, \quad \mathbf{h}\mathbf{k} = \mathbf{j}, \quad \mathbf{k}\mathbf{j} = \mathbf{h}.$$

It follows that  $\mathbb{D} + \mathbf{j}\mathbb{D}$  is the set of bihyperbolic numbers  $\mathbf{H}_4$ (see section 6.3). Moreover, the three principal conjugations of  $\mathfrak{z} = x + \mathbf{h}y + \mathbf{j}z + \mathbf{k}t = z_1 + z_2\mathbf{j}$  are

$$\begin{aligned}\bar{\mathfrak{z}}^{\mathbf{h}} &= x + \mathbf{h}y - \mathbf{j}z - \mathbf{k}t = z_1 - \mathbf{j}z_2, \\ \bar{\mathfrak{z}}^{\mathbf{j}} &= x - \mathbf{h}y + \mathbf{j}z - \mathbf{k}t = \bar{z}_1 + \mathbf{j}\bar{z}_2, \\ \bar{\mathfrak{z}}^{\mathbf{k}} &= x - \mathbf{h}y - \mathbf{j}z + \mathbf{k}t = \bar{z}_1 - \mathbf{j}\bar{z}_2.\end{aligned}$$

**Example 7.2.** The six-dimentional unital Archimedean  $f$ -algebra is the hyperbolization  $\mathbf{H}_6 = \mathbf{H}_3 + \mathbf{j}\mathbf{H}_3$  of the three-dimensional unital Archimedean  $f$ -algebra  $\mathbf{H}_3$  (see section 7.2). By (7.1) one has

$$\mathbf{H}_6 = \left\{ x_0 + \mathbf{i}x_1 + \mathbf{h}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 + \mathbf{l}x_5 : x_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\},$$

with multiplication given by the rules

$$\begin{aligned}\mathbf{i}^2 = \mathbf{k}^2 = \mathbf{h}\mathbf{k} &= \frac{1 + \mathbf{h}}{2}, \quad \mathbf{j}^2 = \mathbf{l}^2 = \mathbf{h}^2 = 1 \\ \mathbf{i}\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\mathbf{h} = \mathbf{l}, \quad \mathbf{j}\mathbf{l} = \mathbf{h}, \quad \mathbf{i}\mathbf{k} &= \frac{\mathbf{j} + \mathbf{l}}{2}, \quad \mathbf{i}\mathbf{h} = \mathbf{j}\mathbf{k} = \mathbf{k}\mathbf{l} = \mathbf{i}.\end{aligned}$$

From (7.4) and (6.3) the associate matrix of  $\mathfrak{z} = x_0 + \mathbf{i}x_1 + \mathbf{h}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 + \mathbf{l}x_5$  is

$$\begin{aligned}\mathcal{M}(\mathfrak{z}) &= \begin{pmatrix} \mathcal{M}_3(x_0 + \mathbf{i}x_1 + \mathbf{h}x_2) & \mathcal{M}_3(x_3 + \mathbf{i}x_4 + \mathbf{h}x_5) \\ \mathcal{M}_3(x_3 + \mathbf{i}x_4 + \mathbf{h}x_5) & \mathcal{M}_3(x_0 + \mathbf{i}x_1 + \mathbf{h}x_2) \end{pmatrix} \\ &= \begin{pmatrix} x_0 & \frac{x_1}{2} & x_2 & x_3 & \frac{x_4}{2} & x_5 \\ x_1 & x_0 + x_2 & x_1 & x_4 & x_3 + x_5 & x_4 \\ x_2 & \frac{x_1}{2} & x_0 & x_5 & \frac{x_4}{2} & x_3 \\ x_3 & \frac{x_4}{2} & x_5 & x_0 & \frac{x_1}{2} & x_2 \\ x_4 & x_3 + x_5 & x_4 & x_1 & x_0 + x_2 & x_1 \\ x_5 & \frac{x_4}{2} & x_3 & x_2 & \frac{x_1}{2} & x_0 \end{pmatrix}.\end{aligned}$$

By Proposition 7.1, the eigenvalues  $\gamma_l(\mathfrak{z})$  of  $\mathcal{M}(\mathfrak{z})$  are

$$\begin{aligned}
\gamma_0(\mathfrak{z}) &= \lambda_0(x_0 + x_3 + \mathbf{i}(x_1 + x_4) + \mathbf{h}(x_2 + x_5)) = x_0 + x_3 - x_2 - x_5; \\
\gamma_1(\mathfrak{z}) &= \lambda_0(x_0 - x_3 + \mathbf{i}(x_1 - x_4) + \mathbf{h}(x_2 - x_5)) = x_0 - x_3 - x_2 + x_5; \\
\gamma_2(\mathfrak{z}) &= \lambda_1(x_0 + x_3 + \mathbf{i}(x_1 + x_4) + \mathbf{h}(x_2 + x_5)) = x_0 + x_1 + x_2 + x_3 + x_4 + x_5; \\
\gamma_3(\mathfrak{z}) &= \lambda_1(x_0 - x_3 + \mathbf{i}(x_1 - x_4) + \mathbf{h}(x_2 - x_5)) = x_0 + x_1 + x_2 - x_3 - x_4 - x_5; \\
\gamma_4(\mathfrak{z}) &= \lambda_2(x_0 + x_3 + \mathbf{i}(x_1 + x_4) + \mathbf{h}(x_2 + x_5)) = x_0 - x_1 + x_2 + x_3 - x_4 + x_5; \\
\gamma_5(\mathfrak{z}) &= \lambda_2(x_0 - x_3 + \mathbf{i}(x_1 - x_4) + \mathbf{h}(x_2 - x_5)) = x_0 - x_1 + x_2 - x_3 + x_4 - x_5,
\end{aligned}$$

and the orthonormal eigenvectors are

$$\begin{aligned}
\mathbf{w}_0 &= e_1 \mathbf{v}_0 = \frac{1}{4}(1 - \mathbf{h} + \mathbf{j} - \mathbf{l}); \\
\mathbf{w}_1 &= e_2 \mathbf{v}_0 = \frac{1}{4}(1 - \mathbf{h} - \mathbf{j} + \mathbf{l}); \\
\mathbf{w}_2 &= e_1 \mathbf{v}_1 = \frac{1}{8}(1 + 2\mathbf{i} + \mathbf{h} + \mathbf{j} + 2\mathbf{k} + \mathbf{l}); \\
\mathbf{w}_3 &= e_2 \mathbf{v}_1 = \frac{1}{8}(1 + 2\mathbf{i} + \mathbf{h} - \mathbf{j} - 2\mathbf{k} - \mathbf{l}); \\
\mathbf{w}_4 &= e_1 \mathbf{v}_2 = \frac{1}{8}(1 - 2\mathbf{i} + \mathbf{h} + \mathbf{j} - 2\mathbf{k} + \mathbf{l}); \\
\mathbf{w}_5 &= e_2 \mathbf{v}_2 = \frac{1}{8}(1 - 2\mathbf{i} + \mathbf{h} - \mathbf{j} + 2\mathbf{k} - \mathbf{l}).
\end{aligned}$$

We conclude our exposition with the following remark, which offers a new perspective on this topic.

**Remark 7.1.** Let us define the canonical complexifications of  $\mathbf{H}_N$  as follows. Consider the  $2N$ -dimensional systems

$$\mathbf{H}_N + \iota \mathbf{H}_N; \quad \iota^2 = 0, \pm 1. \quad (7.8)$$

In particular, for  $N = 2$  there are three canonical commutative Segre's quaternions: bicomplex numbers ( $\iota^2 = -1$ ), bihyperbolic numbers ( $\iota^2 = 1$ ), and parabolic quaternion numbers ( $\iota^2 = 0$ ). Examining the order structure of the systems (7.8), Theorem 7.2 shows that the hyperbolic system ( $\iota^2 = 1$ ) is the only one that can be made into an Archimedean  $f$ -algebra. In fact, even if we omit the Archimedean property, the elliptic system ( $\iota^2 = -1$ ) cannot be an  $f$ -algebra, and is probably not an Archimedean  $\ell$ -algebra either. However, an ordering may exist in the parabolic system ( $\iota^2 = 0$ ). A more detailed description of all multidimensional hypercomplex  $\ell$ -algebras must provide an affirmative answer.

Regarding the two-dimensional systems that are the hyperbolization, complexification, and dualization of the Archimedean  $f$ -algebra  $\mathbf{H}_1 = \mathbb{R}$ , Wickstead proved in [21, Proposition 4.1] that there is no Archimedean  $\ell$ -algebra ordering in complex numbers  $\mathbb{C}$ . He also characterizes all possible such orderings in hyperbolic and dual numbers. The question of  $\ell$ -ring ordering in complex numbers  $\mathbb{C}$  dates back to Birkhoff and Pierce [5] and remains an open problem that can be now extended to the system (7.8) with  $\iota^2 = -1$ .

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## References

- [1] Alpay, D., Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D.C.: Basics of Functional Analysis with Bicomplex scalars and Bicomplex Schur Analysis. Springer Breifs in Mathematics, 2014.
- [2] Balankin, A.S., Bory-Reys, J., Luna-Elizarrara's, M.E., and Shapiro, M.: Cantor-type sets in hyperbolic numbers,. *Factals* **24**(4) (2016) 1650051.
- [3] Bernau S.J.: On semi-normal lattice rings. *Proc. Cambridge Phil. Soc.* 61,613-616 (1965),
- [4] Bilgin, M and Ersoy, S: Algebraic Properties of Bihyperbolic Numbers. *Adv. Appl. Clifford Algebras* (2020).
- [5] Birkhoff, G., Pierce, R.S.: Lattice-ordered rings. *An.Acad.Bras.Cienc.* **28**, 41-69 (1956).
- [6] Bory-Reyes, J., Téllez-Sánchez, G.Y. : More about Cantor like sets in hyperbolic numbers. *Fractals* **25** (5) (2017) 1750046
- [7] Cartan, E.: *Oeuvres Complètes*, Gauthier Villars, Paris, (1953)
- [8] Catoni, F., Boccaletti, D., Cannata, R., Catoni, V., Nichelatti, E. and Zampatti, P.: *The Mathematics of Minkowski Space-Time: with an Introduction to Commutative Hypercomplex Numbers*. Springer Science & Business Media, 2008.
- [9] Drmota, M., Mauduit, R., and Rivat, J.: The sum of digits of primes in  $\mathbb{Z}(i)$ : *Monatsh. Math.* 155 (3-4): 317-347 (2008).
- [10] Colombo F., Sabadin I., Struppa D.C.: Bicomplex holomorphic functional calculus. *Math. Nachr.* **287** (13), 1093-1105( 2013) **49**, 277-294 (2011)
- [11] Gargoubi, H., Kossentini, S.: *f*-Algebra Srtucture On Hyperbolic Numbers. *Adv. Appl. Clifford Algebras* **26**(4),1211-1233 (2016)
- [12] Gu, Y.: Clifford Algebra and Hypercomplex Number as well as Their Applications in Physics. *Journal of Applied Mathematics and Physics*, 10(4), 1375-1393 (2022).
- [13] Hamilton W. R.: On quaternions or on a new system of imaginaries in algebra. *Philosophical Magazine.*, 1844, 25(3),489–495.
- [14] Huijsmans, C.B. and De Pagter, B.: Ideal theory in f-algebras. *Trans. Amer. Math. Soc.* 269 (1982) 225-245.
- [15] Jingjing, Ma: *Lecture Note On Algebraic Structure of Lattice-Ordered Rings*. World Scientific
- [16] Lie S. and Scheffers M. G.: *Vorlesungen über continuerliche Gruppen*. Kap. 21, Taubner, Leipzig (1893).

- [17] Luxemburg, W.A.J., Zaanen, A.C.: Riesz Spaces I, North-Holland, Amsterdam (1971).
- [18] Peirce, B.: Linear Associative Algebra. American Journal of Mathematics, 4 (1): 221–6 (1881).
- [19] Scheffers M. G., Sur la généralisation des fonctions analytiques, C. R. Acad. Sc., 116, 1114 (1893); and Berichte Verhandl. Sächsischen Akad. Wiss. Leipzig Mat.-Phis. Kl 45 (1893) and 46 (1894)
- [20] Wedderburn, J.H.: On Hypercomplex Numbers. Proc. London Math. Soc. 6, 77 (1908).
- [21] Wickstead, A.W., : Two dimensional unital Riesz algebras, their representations and norms. Positivity. Springer Basel (2015). DOI 10.1007/s 11117-015-0349-1.
- [22] Yaglom, I.M.: A Simple Non-Euclidean Geometry and it's Physical Basis, Springer-Verlag, New York (1979).
- [23] Zaanen, A.C: Riesz spaces II. North-Holland, Amsterdam (1983).