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# Invariant kernels on Riemannian symmetric spaces : a harmonic-analytic approach

Nathaël Da Costa\*, Cyrus Mostajeran\*, Juan-Pablo Ortega\*, and Salem Said†

**Abstract.** This work aims to prove that the classical Gaussian kernel, when defined on a non-Euclidean symmetric space, is never positive-definite for any choice of parameter. To achieve this goal, the paper develops new geometric and analytical arguments. These provide a rigorous characterization of the positive-definiteness of the Gaussian kernel, which is complete but for a limited number of scenarios in low dimensions that are treated by numerical computations. Chief among these results are the  $L^p$ -Godement theorems (where  $p = 1, 2$ ), which provide verifiable necessary and sufficient conditions for a kernel defined on a symmetric space of non-compact type to be positive-definite. A celebrated theorem, sometimes called the Bochner-Godement theorem, already gives such conditions and is far more general in its scope but is especially hard to apply. Beyond the connection with the Gaussian kernel, the new results in this work lay out a blueprint for the study of invariant kernels on symmetric spaces, bringing forth specific harmonic analysis tools that suggest many future applications.

**Key words.** Riemannian manifold, symmetric space, harmonic analysis, Bochner's theorem, Godement theorem, positive-definite kernel

**MSC codes.** 43A35, 43A85, 43A90, 46E22, 53C35, 53Z50

**1. Introduction.** Positive-definite functions play a fundamental role in harmonic analysis and the theory of group representations, as well as in probability and statistical inference [30, 36, 53, 54]. In machine learning, these functions are of paramount importance in the context of kernel-based methods. Indeed, kernels are most often required to be positive-definite for Reproducing Kernel Hilbert Space (RKHS) methods and associated linear geometry and classification algorithms to apply [7, 48]. While many positive-definite kernels are known in Euclidean spaces, this is not yet true for non-Euclidean metric spaces, such as non-trivial Riemannian manifolds. Even when such kernels are known, they are far from easy to work with (in the first place, to evaluate). For example, just one evaluation of the heat kernel on a Riemannian symmetric space may require an elaborate Monte Carlo scheme [1, 2].

This being the case, one hopes to develop a tractable means of constructing closed-form positive-definite kernels on Riemannian manifolds or at least to develop an explicit criterion to verify whether a given closed-form expression yields a positive-definite kernel or not. This should apply, in particular, to the so-called distance kernels, which are of the form  $k(x, y) = g(d(x, y))$ , with  $g$  a suitable function and  $d(x, y)$  the Riemannian distance.

The results developed in this work serve exactly this purpose. They are motivated by the study of the Gaussian kernel

$$(1.1) \quad k(x, y) = \exp(-\lambda d^2(x, y)) \quad \text{where } \lambda > 0.$$

In particular, they aim to show that this kernel is never positive-definite (meaning it is not

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38 positive-definite for any value of  $\lambda$ ) when defined on a non-Euclidean Riemannian symmetric  
 39 space  $M$ . In general, when  $M$  is a non-Euclidean Riemannian manifold, it is known that  
 40 there always exists some  $\lambda$  such that the Gaussian kernel is not positive-definite [34, 24, 23].  
 41 However, whether there exists any subset of values of the parameter  $\lambda$  for which the Gaussian  
 42 kernel is positive-definite is an open problem of theoretical and practical significance as noted  
 43 in the literature [41]. For instance, [50] shows that the Gaussian kernel may be positive-  
 44 definite for some non-trivial collection of parameters in certain non-Euclidean geometries.  
 45 Recently, it has been shown that if  $M$  is compact and not simply connected (*e.g.* a torus or  
 46 a real projective space), then the Gaussian kernel is never positive-definite [37]. While these  
 47 general results rely on somewhat complicated geometric arguments, they will be considerably  
 48 extended here, in the case of symmetric spaces, thanks to the introduction of new analytical  
 49 tools.

50 A typical result of choice in this circle of ideas is Bochner’s theorem, one of the most  
 51 famous theorems in harmonic analysis [25]. It states that a function defined on a locally  
 52 compact Abelian group is positive-definite if and only if it is the inverse Fourier transform  
 53 of some positive measure. For example, this theorem guarantees the existence of a spectral  
 54 power measure for any wide-sense stationary signal (Wiener-Khinchin theorem).

55 The generalization of Bochner’s theorem to symmetric spaces is due to Godement [27]. It  
 56 will be referred to as Godement’s theorem in this work, although it is sometimes called the  
 57 Bochner-Godement theorem elsewhere. This theorem is recalled in Section 2, where we explain  
 58 why it can be challenging to apply in practice. Focusing on non-compact symmetric spaces,  
 59 the  $L^2$ -Godement theorem (Section 3) and the  $L^1$ -Godement theorem (Appendix A), both  
 60 state that in the presence of additional integrability assumptions, the difficulties inherent in  
 61 its application are greatly alleviated: it turns out that a function is positive-definite if and only  
 62 if its spherical transform is positive and integrable (the definition of the spherical transform  
 63 is recalled in Section 3).

64 Basic examples of applications of the  $L^2$ -Godement theorem are provided in Section 4.  
 65 Sections 5 and 6 then return to the Gaussian kernel (1.1). Section 5 proves that the Gaussian  
 66 kernel is never positive-definite on a compact symmetric space, expanding on [19] and [37],  
 67 which prove the corresponding result on the circle and non-simply connected closed Riemann-  
 68 ian manifolds, respectively. Section 6 takes up the case of non-compact symmetric spaces  
 69 (under the general assumptions of Sections 2 and 3, compact or non-compact have the same  
 70 meaning as “of compact or non-compact type” — no Euclidean spaces are allowed).

71 Section 6 provides a rigorous proof that deals with the case of non-compact symmetric  
 72 spaces in all but a few limited scenarios in low dimensions, which are investigated compu-  
 73 tationally instead. First, in the case of the hyperbolic plane, a numerical evaluation shows  
 74 that the positivity condition in the  $L^2$ -Godement theorem does not hold, for any  $\lambda$  within  
 75 a certain continuous range, starting near  $\lambda = 0$ . However (Lemma 6.5), any non-compact  
 76 symmetric space contains an isometrically embedded hyperbolic plane, and this implies the  
 77 Gaussian kernel fails to be positive-definite on any non-compact symmetric space, for that  
 78 same range of values of  $\lambda$ . Section 6 also proves (with no recourse to numerical work) that  
 79 the Gaussian kernel is never positive-definite when defined on a three-dimensional hyperbolic  
 80 space. Again, this extends to any symmetric space that contains an isometrically embed-  
 81 ded hyperbolic space (*e.g.* the spaces of  $n \times n$  real, complex, or quaternion, positive-definite

82 matrices, with the requirement that  $n \geq 4$  for the real case).

83 Section 7 introduces an alternative to the Gaussian kernel (1.1) for hyperbolic spaces,  
84 giving it the name of hyperbolic secant or Herschel-Maxwell kernel. Once more, this is a  
85 distance kernel

$$86 \quad (1.2) \quad k(x, y) = (\cosh(d(x, y)))^{-a} \quad \text{where } a > 0.$$

87 This kernel is derived from first principles as a different generalization of the Gaussian kernel  
88 on hyperbolic spaces, similar to the famous Herschel-Maxwell derivation [35]. In contrast with  
89 the Gaussian kernel, it is shown to always be positive-definite on the hyperbolic plane and  
90 positive-definite on the hyperbolic space for integer values of  $a$ .

91 The  $L^p$ -Godement theorems (where  $p = 1, 2$ ) are most easily applied to distance kernels  
92 (for example, to the Gaussian or Herschel-Maxwell kernels). A detailed examination of their  
93 application to more general invariant kernels, which are not merely functions of distance, is  
94 one of the main tasks to be carried out in ongoing and future work (see in particular [42]).

95 The present work focuses on symmetric spaces and does not discuss other, more general  
96 classes of Riemannian manifolds. This restriction is rather essential if one wishes to pursue an  
97 approach based on harmonic analysis, but it is still a restriction (Stiefel manifolds are relevant  
98 to many applications, but are not symmetric spaces in general [13]). Fortunately, symmetric  
99 spaces include various spaces of covariance matrices, in addition to Grassmannians, compact  
100 Lie groups, and other kinds of manifolds, which arise across brain-computer interface analysis,  
101 radar signal processing, computer vision, quantum information, *etc.* [17, 11, 6, 16]. Moreover,  
102 symmetric spaces are also quite useful for graph-embedding applications [38]. Therefore, the  
103 ongoing efforts directed at constructing positive-definite (as in [1, 2, 10, 42]) or positively  
104 decomposable ([20]) kernels on symmetric spaces should lead to the emergence of new kernel-  
105 based methods (for classification, hypothesis testing, *etc.*) applicable to the above-mentioned  
106 fields.

107 The present work deals solely with kernels and does not make any mention of Gaussian  
108 processes (Gaussian processes are considered in detail in [1, 2, 10]). This should come as no  
109 surprise since the present approach is purely analytical rather than probabilistic. However, this  
110 is not a limitation of the present work as positive-definite kernels and Gaussian processes are  
111 equivalent. Given a Gaussian process, its covariance is a positive-definite kernel. The reverse  
112 construction can be obtained by Kolmogorov's extension theorem [55]. Let us finally note that  
113 the connection between invariant Gaussian processes and harmonic analysis is quite strong,  
114 since these processes are typically obtained by solving invariant stochastic partial differential  
115 equations [10, 43].

116 There is an additional direction where methods related to Godement's theorem can be  
117 applied. The general setting of this theorem is that of Gelfand pairs, which goes beyond  
118 symmetric spaces. In particular, Gelfand pairs include certain discrete spaces, such as spaces  
119 of phylogenetic trees, very important in bio-informatics, linguistics, *etc.* [12]. A tentative  
120 direction for future work is to use Godement's theorem to improve understanding of kernel  
121 methods for phylogenetic trees.

122 **2. Godement's theorem.** Let  $M$  be a set and let  $k : M \times M \rightarrow \mathbb{C}$ . Assume that  $k$  satisfies  
 123

$$124 \quad (2.1) \quad \sum_{i,j=1}^N k(x_i, x_j) c_i \bar{c}_j \geq 0$$

125 for all  $x_i \in M$ ,  $c_i \in \mathbb{C}$  and all  $N \in \mathbb{N}$ , where the bar denotes complex conjugation. Then,  $k$  is  
 126 said to be a positive-definite kernel on  $M$ . Without loss of generality, it is enough to consider  
 127 Hermitian kernels, that is,  $k(x, y)$  is the conjugate of  $k(y, x)$ . In addition, assume that a group  
 128  $G$  acts transitively on  $M$ ; this action is denoted by  $g \cdot x$ , for  $g \in G$  and  $x \in M$ . If  $k$  satisfies

$$129 \quad (2.2) \quad k(g \cdot x, g \cdot y) = k(x, y),$$

130 for all  $g \in G$  and  $x, y \in M$ , then  $k$  is said to be a  $G$ -invariant (or just an invariant) positive-  
 131 definite kernel. Let  $o \in M$  and denote by  $H$  the stabiliser of  $o$  in  $G$ . A function  $f : M \rightarrow \mathbb{C}$   
 132 called  $H$ -invariant if  $f(h \cdot x) = f(x)$  for all  $h \in H$  and  $x \in M$ . Using such a function, we can  
 133 define  $k_f : M \times M \rightarrow \mathbb{C}$ ,

$$134 \quad (2.3) \quad k_f(x, y) = f(g_1^{-1} g_2 \cdot o) \text{ for } x = g_1 \cdot o \text{ and } y = g_2 \cdot o.$$

135 Indeed,  $H$ -invariance of  $f$  implies the right-hand side of (2.3) depends only on  $x$  and  $y$  (and not  
 136 on the choice of  $g_1$  and  $g_2$ ). Now,  $k_f$  is  $G$ -invariant (it satisfies (2.2)). If  $k_f$  happens to be  
 137 positive-definite (to satisfy (2.1)), then  $f$  is said to be a positive-definite function.

138 All invariant positive-definite kernels can be represented in this way, since (2.1) and (2.2)  
 139 imply that  $k = k_f$ , where  $f(x) = k(o, x)$  and  $f$  is  $H$ -invariant and positive-definite. In  
 140 particular, if  $k = k_f$ , then  $k$  is positive-definite if and only if  $f$  is positive-definite.

141 Note that the setup that we just presented implies that the set  $M$  is necessarily isomorphic  
 142 to the homogeneous manifold  $G/H$ , where  $G$  is the group that acts transitively on  $M$  and  $H$   
 143 the isotropy group of some element  $o \in M$ . Conversely, given any group  $G$  and an arbitrary  
 144 subgroup  $H$ , we can cast the homogeneous space  $G/H$  in this framework by setting  $M = G/H$ ,  
 145  $o = eH$ , and considering the transitive left action of  $G$  on  $G/H$ . Given these observations,  
 146 in the sequel,  $M$  will be a Riemannian symmetric space and  $(G, H)$  a symmetric pair.  $G$  is  
 147 assumed to be semisimple, of either compact or non-compact type, and with finite center [32].  
 148 Godement's theorem generalizes Bochner's classical theorem to this setting [27]. Roughly  
 149 speaking, it states that an  $H$ -invariant continuous function  $f$  is positive-definite if and only if  
 150 it is a positive combination of positive-definite spherical functions. In [8], Godement's theorem  
 151 is called the Bochner-Godement theorem.

152 Recall that a spherical function  $\varphi : M \rightarrow \mathbb{C}$  is an  $H$ -invariant function, such that  $\varphi(o) = 1$   
 153 and  $\varphi$  is moreover a joint eigenfunction of all the  $G$ -invariant differential operators on  $M$  [30].  
 154 The set  $\Phi$  of positive-definite spherical functions on  $M$  is locally compact for the topology  
 155 of uniform convergence on compact subsets of  $M$  [27]. Godement's theorem states that an  
 156  $H$ -invariant continuous  $f$  is positive-definite if and only if

$$157 \quad (2.4) \quad f(x) = \int_{\Phi} \varphi(x) \mu_f(d\varphi)$$

158 where  $\mu_f$  is a unique finite positive measure on  $\Phi$ .

159 It is complicated to use Godement's theorem to verify whether a given  $f$  is positive-  
 160 definite. When  $M$  is non-compact, the set  $\Phi$  may be challenging to determine [30] (Page 484),  
 161 and there is no straightforward way of determining  $\mu_f$  from  $f$ . The following section identifies  
 162 a special case of Godement's theorem where these issues do not arise.

163 **3.  $L^2$ -Godement theorem.** The idea is to combine Godement's theorem with the so-called  
 164 spherical transform. Roughly speaking, the spherical transform can be seen as a radial Fourier  
 165 transform for  $H$ -invariant functions [51, 52]. When  $M = \mathbb{R}^2$ , it boils down to a zero-order  
 166 Hankel transform.

167 Assume that  $M$  is a non-compact symmetric space. Pick an Iwasawa decomposition of  
 168 the Lie algebra  $\mathfrak{g}$  of  $G$ , say  $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{h}$  (where  $\mathfrak{n}$  is nilpotent,  $\mathfrak{a}$  is Abelian, and  $\mathfrak{h}$  is the Lie  
 169 algebra of  $H$  [32]). For each  $g \in G$ ,  $g = n(g)\exp(a(g))h(g)$ , where  $n(g) \in N$ ,  $a(g) \in \mathfrak{a}$  and  
 170  $h(g) \in H$  are unique ( $N$  is the Lie subgroup of  $G$  with Lie algebra  $\mathfrak{n}$ ).

171 For each  $t \in \mathfrak{a}^*$  (the dual of  $\mathfrak{a}$  as a real vector space), there is a spherical function  
 172  $\varphi_t : M \rightarrow \mathbb{C}$ , given by the following integral formula [30],

$$173 \quad (3.1) \quad \varphi_t(g \cdot o) = \int_H \exp[(it + \rho)(a(hg))] dh \quad \text{for } g \in G.$$

174 Here,  $\rho$  is half the sum of the positive roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ , and  $dh$  denotes the  
 175 normalized Haar measure on  $H$  (a gentle introduction to these concepts is provided in [51, 52]).  
 176 Define the spherical transform of an  $H$ -invariant  $f : M \rightarrow \mathbb{C}$  as  $\hat{f} : \mathfrak{a}^* \rightarrow \mathbb{C}$ ,

$$177 \quad (3.2) \quad \hat{f}(t) = \int_M f(x)\varphi_{-t}(x)\text{vol}(dx)$$

178 where  $\text{vol}$  denotes the volume measure of  $M$ . This is well-defined when  $f$  is continuous and  
 179 compactly supported, in which case it admits an inversion formula,

$$180 \quad (3.3) \quad f(x) = c_M \int_{\mathfrak{a}^*} \hat{f}(t)\varphi_t(x)|c(t)|^{-2} dt$$

181 where  $c(t)$  is a certain function, called the Harish-Chandra function (for an overview, see [31]),  
 182  $dt$  denotes the Lebesgue measure, and  $c_M$  is a constant that is independent of the function  $f$ .

183 Defined as above, the spherical transform extends to the space  $L^2_H(M)$  of  $H$ -invariant  
 184 square-integrable functions (with respect to  $\text{vol}$ ) and satisfies the Plancherel formula

$$185 \quad (3.4) \quad \int_M |f(x)|^2 \text{vol}(dx) = c_M \int_{\mathfrak{a}^*} |\hat{f}(t)|^2 |c(t)|^{-2} dt$$

186 for any  $f \in L^2_H(M)$ . Here, the following proposition will be called the  $L^2$ -Godement theorem.  
 187 It is a special case of Godement's theorem, which holds for square-integrable functions.

188 **Theorem 3.1 ( $L^2$ -Godement Theorem).** *A continuous function  $f \in L^2_H(M)$  is positive-  
 189 definite if and only if  $\hat{f}(t) \geq 0$  for almost all  $t \in \mathfrak{a}^*$  and  $\hat{f}$  is integrable with respect to the  
 190 measure  $|c(t)|^{-2} dt$ .*

191 The connection with Godement's theorem, from the previous Section 2, comes from the fact  
 192 that all of the functions  $\varphi_t$  in (3.1) are positive-definite [30] (Page 484). The set of  $\varphi_t$  where  
 193  $t \in \mathfrak{a}^*$  is therefore a subset of  $\Phi$  (in fact, it is a proper subset). The proof of Theorem 3.1 is  
 194 given in Section 8.1, where a second alternative proof is also discussed.

195 The following section spells out the spherical transform pair (3.2)–(3.3) in two basic cases,  
 196 where  $M$  is a hyperbolic plane or a hyperbolic space of dimension equal to three, and provides  
 197 two examples of applications of Theorem 3.1.

#### 198 4. Basic examples.

199 **4.1. Hyperbolic plane.**  $M$  is a two-dimensional simply connected space with constant  
 200 negative curvature equal to  $-1$ . One may take  $G = SL(2, \mathbb{R})$  and  $H = SO(2)$  (in the usual  
 201 notation,  $SL(2, \mathbb{R}) = 2 \times 2$  real unit-determinant matrices,  $SO(2) = 2 \times 2$  rotation matrices).  
 202 A detailed exposition may be found in [51]. For any  $H$ -invariant function  $f : M \rightarrow \mathbb{C}$  there  
 203 exists  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ , with  $f(x) = g(r)$  where  $r = d(o, x)$ , the distance in  $M$  between  $x$  and  
 204 the “point of origin”  $o$  (the choice of this point is arbitrary). In particular, the spherical  
 205 functions (3.1) are given in terms of Legendre functions [30, 51],

$$206 \quad (4.1) \quad \varphi_t(x) = P_{-\frac{1}{2}+it}(\cosh(r))$$

207 for  $t \in \mathbb{R}$  (with regard to (3.1),  $\mathfrak{a}$  is one-dimensional, and therefore  $\mathfrak{a} \simeq \mathfrak{a}^* \simeq \mathbb{R}$ . Accordingly,  
 208  $t \in \mathfrak{a}^*$  becomes  $t \in \mathbb{R}$ ), and it should be noted that  $\varphi_t = \varphi_{-t}$  (see [51] (Page 141)).

209 The spherical transform pair (3.2)–(3.3) can be expressed as follows (see [51] (Page 149))

$$210 \quad (4.2) \quad \hat{f}(t) = 2\pi \int_0^\infty g(r) P_{-\frac{1}{2}+it}(\cosh(r)) \sinh(r) dr,$$

$$211 \quad (4.3) \quad g(r) = \frac{1}{2\pi} \int_0^\infty \hat{f}(t) P_{-\frac{1}{2}+it}(\cosh(r)) \alpha(t) dt,$$

212 where  $f(x) = g(r)$  as above, and  $\alpha(t) = t \tanh(\pi t)$ . As pointed out in [51], (4.2)–(4.3) is  
 213 known as a Mehler-Fock transform pair in the mathematical physics literature. Note that  
 214 (4.2) follows directly from (3.2) by using (4.1) and then expressing the volume integral in  
 215 geodesic spherical coordinates, which leads to the appearance of  $\sinh(r)$  under the integral  
 216 (roughly,  $\text{vol}(dx) = \sinh(r) dr d\theta$  where  $\theta$  is a polar angle, which yields the pre-factor  $2\pi$ ).

217 **4.2. Hyperbolic space.**  $M$  is now of dimension 3 instead of 2. One may take  $G = SL(2, \mathbb{C})$   
 218 and  $H = SU(2)$  (in other words,  $SL(2, \mathbb{C}) = 2 \times 2$  complex unit determinant matrices, and  
 219  $SU(2) =$  unitary matrices in  $SL(2, \mathbb{C})$ ). As before, any  $H$ -invariant function  $f$  is of the form  
 220  $f(x) = g(r)$ . The spherical functions (3.1) are given by [30] (Page 432)

$$221 \quad (4.4) \quad \varphi_t(x) = \frac{\sin(tr)}{t \sinh(r)}.$$

222 Again,  $\varphi_t = \varphi_{-t}$ . The spherical transform pair (3.2)–(3.3) is essentially a sine transform pair,

$$223 \quad (4.5) \quad \hat{f}(t) = \frac{4\pi}{t} \int_0^\infty g(r) \sinh(r) \sin(tr) dr,$$

$$224 \quad (4.6) \quad g(r) \sinh(r) = \frac{1}{2\pi^2} \int_0^\infty t \hat{f}(t) \sin(tr) dt,$$

225 as can be seen using (4.4) and computing the integral (3.2) in geodesic spherical coordinates.

226 **4.3. Distance kernels.** For the hyperbolic plane and space that we studied in the previous  
 227 examples, any  $H$ -invariant function  $f$  is of the form  $f(x) = g(d(o, x))$ . It follows from (2.3)  
 228 that  $k_f(x, y) = g(d(x, y))$ . But, as stated in Section 2, any invariant kernel  $k$  is of the form  
 229  $k = k_f$  and it is hence a distance kernel

$$230 \quad (4.7) \quad k(x, y) = g(d(x, y)),$$

231 for some  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ . The positive-definiteness of  $k$  implies that  $g$  is necessarily real-valued.  
 232 In other words, any positive-definite kernel is a real-valued distance kernel. This statement  
 233 holds whenever  $M$  is a rank-one symmetric space. This class of symmetric spaces includes  
 234 spheres, hyperbolic spaces, projective and hyperbolic projective spaces of any dimension [32].

235 Intuitively, the rank of a symmetric space  $M$  is the (minimum) number of geometric  
 236 invariants required to determine a pair of points  $(x, y)$  up to isometry. For a rank-one space,  
 237 there is exactly one such invariant, the distance  $d(x, y)$ . Thus, if  $k$  is an invariant kernel,  
 238  $k(x, y)$  depends only on  $d(x, y)$ .

**4.3.1. Wishart kernel.** Let  $M$  be the hyperbolic plane, and let  $g(r) = \exp(-2a \cosh(r))$ ,  
 where  $a > 0$ , in (4.7). It is well known that  $M$  can be identified with the space of  $2 \times 2$   
 unit-determinant positive-definite matrices [51]. Thinking of  $x \in M$  as a matrix of this kind,  
 and choosing  $o$  the identity matrix,  $d(o, x) = r$  implies the trace of  $x$  is  $\text{tr}(x) = 2 \cosh(r)$  [51]  
 (Page 149. Exercice 20). Thus, by (4.7)

$$k(o, x) = \exp(-a \text{tr}(x)).$$

Recalling the action  $g \cdot x = gxg^\dagger$  (a matrix product, with  $\dagger$  the transpose [51]) and  $o = \text{identity}$ ,  
 and setting  $x = g_1 \cdot o$  and  $y = g_2 \cdot o$ , it follows from (2.2) that

$$k(x, y) = k(o, g_1^{-1} g_2 \cdot o) = \exp(-a \text{tr}(x^{-1}y)).$$

239 This is the unnormalized density of a Wishart distribution, and  $k$  is the Wishart kernel. Now,  
 240 the  $L^2$ -Godement theorem can be used to show that this kernel is not positive-definite, for any  
 241 value of  $a$ . The spherical transform  $\hat{f}(t)$  is given in [51] (Page 151),

$$242 \quad (4.8) \quad g(r) = \exp(-2a \cosh(r)) \text{ in (4.2)} \implies \hat{f}(t) = \left(\frac{4\pi}{a}\right)^{\frac{1}{2}} K_{it}(2a)$$

243 where  $K_{it}$  is a modified Bessel function of the second kind with imaginary order. For fixed  
 244  $a$ ,  $K_{it}(2a)$  has infinitely many simple zeros in  $t$  [3]. In particular,  $\hat{f}(t)$  does not satisfy the  
 245 positivity condition in the  $L^2$ -Godement theorem (Theorem 3.1), so  $k$  is not positive-definite  
 246 (the theorem can be applied because  $f(x) = k(o, x)$  is continuous and in  $L^2_H(M)$  for any  $a$ ).

247 Furthermore, since the Wishart kernel is never positive-definite on the hyperbolic plane,  
 248 and since a hyperbolic space contains infinitely many isometric copies of the hyperbolic plane,  
 249 the Wishart kernel is never positive-definite on the hyperbolic space. This statement can be  
 250 made rigorous using the embedding lemma below (Lemma 5.2).



**4.3.2. Hyperbolic secant kernel.** Let  $M$  be the hyperbolic plane and  $g(r) = (\cosh(r))^{-a}$  where  $a > 1/2$ . Then,  $f(x) = g(d(o, x))$  is clearly continuous, and  $f \in L^2_H(M)$ , because

$$\int_M |f(x)|^2 \text{vol}(dx) = 2\pi \int_0^\infty (\cosh(r))^{-2a} \sinh(r) dr = 2\pi \int_1^\infty u^{-2a} du < \infty$$

251 as follows by integrating in geodesic spherical coordinates. From [51] (Page 151)

$$252 \quad (4.9) \quad g(r) = (\cosh(r))^{-a} \text{ in (4.2)} \implies \hat{f}(t) = 2^{a-1} \frac{\sqrt{\pi}}{\Gamma(a)} |\Gamma((a - 1/2 + it)/2)|^2$$

which is positive for all  $t$  ( $\Gamma$  is the Gamma function). Now, by the  $L^2$ -Godement theorem, the hyperbolic secant kernel

$$k(x, y) = (\cosh(d(x, y)))^{-a}$$

253 is positive-definite on the hyperbolic plane for any  $a > 1/2$ . This example of a positive-definite  
254 kernel on the hyperbolic plane does not seem to be known in the literature. We believe it is  
255 reasonable to name it the hyperbolic secant kernel.

256 The case where  $M$  is the hyperbolic space, rather than the hyperbolic plane, is investigated  
257 in Appendix B. Positive-definiteness seems to be, once more, the general rule.

258 **5. Compact symmetric spaces.** This section considers the case in which  $M$  is a compact  
259 symmetric space (for example, a sphere, a projective space, or a Grassmannian). The following  
260 proposition will now be proved.

261 **Proposition 5.1.** *Let  $M$  be a compact symmetric space. The Gaussian kernel (1.1) is not*  
262 *positive-definite on  $M$ , for any value of  $\lambda$ .*

263 In other words, the Gaussian kernel is never positive-definite on a compact symmetric space.  
264 The case where  $M$  is not simply connected (real projective space or real Grassmannian, etc.)  
265 is already taken care of in [37]. The case where  $M$  is simply connected will now be proved.  
266 First, we prove the following result, referred to as the Embedding Lemma.

267 **Lemma 5.2 (Embedding Lemma).** *Assume  $M_o$  and  $M$  are metric spaces, with  $M_o$  isomet-*  
268 *rically embedded in  $M$ . Let  $k$  be a distance kernel (i.e. a kernel of the form (4.7)). If  $k$  is*  
269 *positive-definite on  $M$ , then it is positive-definite on  $M_o$ .*

270 Here, isometric embedding means that the distance function in  $M_o$  is the restriction of the  
271 distance function in  $M$  (this is the isometric embedding of metric spaces, a stronger property  
272 than the isometric embedding of Riemannian manifolds). Due to (4.7), this implies the kernel  
273  $k$  on  $M_o$  is just the restriction of the same kernel  $k$  from  $M$ . The fact that positive-definiteness  
274 is hereditary from  $M$  to  $M_o$  is immediate from (2.1).

275 A corollary of the embedding lemma is that the Gaussian kernel (1.1) cannot be positive-  
276 definite on  $M$  if it is not positive-definite on  $M_o$ .

277 **Lemma 5.3.** *Any compact simply connected symmetric space  $M$  contains an isometrically*  
278 *embedded circle  $\sigma$  (the embedding being an isometric embedding of metric spaces).*

279 *Proof.* Assume that  $M$  is an irreducible symmetric space [32, Chapter VIII]. Denote the  
280 maximum of the sectional curvatures of  $M$  by  $\kappa^2$ . Then,  $M$  admits a simple periodic geodesic

281  $\sigma$  of length  $2\pi/\kappa$  [32] (Page 334). Let  $x$  be a point on  $\sigma$  and choose a parameterisation  
 282  $\sigma : [0, 2\pi] \rightarrow M$  with  $\sigma(0) = x$ . Then, let  $\sigma(\pi) = x'$ .

283 It can be shown that  $\sigma$  is length-minimising from  $x$  to  $x'$  (when restricted to  $[0, \pi]$ ) and  
 284 also from  $x'$  to  $x$  (when restricted to  $[\pi, 2\pi]$ ). Recall that the cut-locus of  $x$  in  $M$  is identical  
 285 to its first conjugate locus [18]. By definition of  $\kappa$ ,  $x$  has no conjugate points along  $\sigma$  before  
 286  $\sigma(\pi) = x'$  [14, Morse-Schönberg Theorem, Page 86]. Thus  $\sigma$  is length-minimising from  $x$  to  
 287  $x'$ . An identical argument shows  $\sigma$  is also length-minimising from  $x'$  to  $x$ .

288 Denote  $\sigma = \sigma([0, 2\pi])$  (the image of the geodesic curve  $\sigma$  in  $M$ ). This is the isometrically  
 289 embedded circle announced in Lemma 5.3. To see this, choose any  $y \in \sigma$ ,  $y = \sigma(t)$ . If  
 290  $t \in [0, \pi]$  the distance in  $M$  between  $x$  and  $y$  is  $d(x, y) = t$ . On the other hand, if  $t \in [\pi, 2\pi]$ ,  
 291  $d(x, y) = 2\pi - t$  (note both of these claims result from the length-minimising property of  $\sigma$ ,  
 292 just obtained). In other words, the restriction of the distance function from  $M$  to  $\sigma$  is the same  
 293 as the distance function of a circle (identified with the interval  $[0, 2\pi]$  with glued endpoints).

294 The assumption that  $M$  is irreducible can now be removed since any simply connected  
 295 symmetric space is a direct product of irreducible symmetric spaces [32, Page 381]. ■

296 We recall now (see [19]) that the Gaussian kernel is never positive-definite on the circle.  
 297 With this in mind, Lemmas 5.2 and 5.3 yield Proposition 5.1.

298 Even though it should be possible to prove Proposition 5.1 using an analytic approach  
 299 via Godement's theorem, the geometric approach used above is more elementary and easier  
 300 to visualise.

301 **6. Non-compact symmetric spaces.** In this section,  $M$  will be a non-compact symmetric  
 302 space, as in Section 3 (for example, a hyperbolic space, a space of positive-definite matrices,  
 303 *etc.*). Using a mixture of numerical and analytical results, we show that the Gaussian kernel  
 304 systematically fails to be positive-definite on  $M$ . First, the  $L^2$ -Godement theorem is applied  
 305 to the two basic cases where  $M$  is a hyperbolic plane or a three-dimensional hyperbolic space.  
 306 Specifically, from (1.1), the Gaussian kernel corresponds to  $g(r) = e^{-\lambda r^2}$ , and upon replacing  
 307 this into (4.2) and (4.5), the following observations are made:

- 308 • In both cases, the spherical transform  $\hat{f}(t)$  is positive when  $t$  is small in comparison  
 309 to  $\lambda$  (Proposition 6.1, and the remark after Formula (6.1)).
- 310 • In (4.2), a numerical evaluation shows that  $\hat{f}(t)$  oscillates between positive and negative  
 311 values, for any  $\lambda$  within a certain continuous range, starting near  $\lambda = 0$  (Figure 1).
- 312 • In (4.5),  $\hat{f}(t)$  has the same sign as  $\sin(t/2\lambda)$ . The Gaussian kernel is, therefore, never  
 313 positive-definite on the hyperbolic space (Proposition 6.3).

314 Second, the general case uses the Embedding Lemma 5.2.

315 **6.1. Hyperbolic plane.** For the Gaussian kernel on the hyperbolic plane, the behavior of  
 316 the spherical transform  $\hat{f}(t)$ , when  $t$  is small in comparison to  $\lambda$ , is the following.

317 **Proposition 6.1.** *Let  $g(r) = e^{-\lambda r^2}$  in (4.2). For any  $T > 0$ , there exists  $\lambda_T > 0$  such that*  
 318  *$\lambda > \lambda_T$  implies  $\hat{f}(t) \geq 0$  for all  $t \leq T$ .*

319 The proof of Proposition 6.1 is given in Section 8.2. The behavior of  $\hat{f}(t)$  for larger values of  
 320  $t$  was investigated numerically, with the results reported in Figure 1. Note that, at the time  
 321 of writing, we were not able to provide a full proof or counterexample to the negativity of the  
 322 function  $\hat{f}(t_\lambda)$  at some  $t_\lambda$  for all  $\lambda$ .

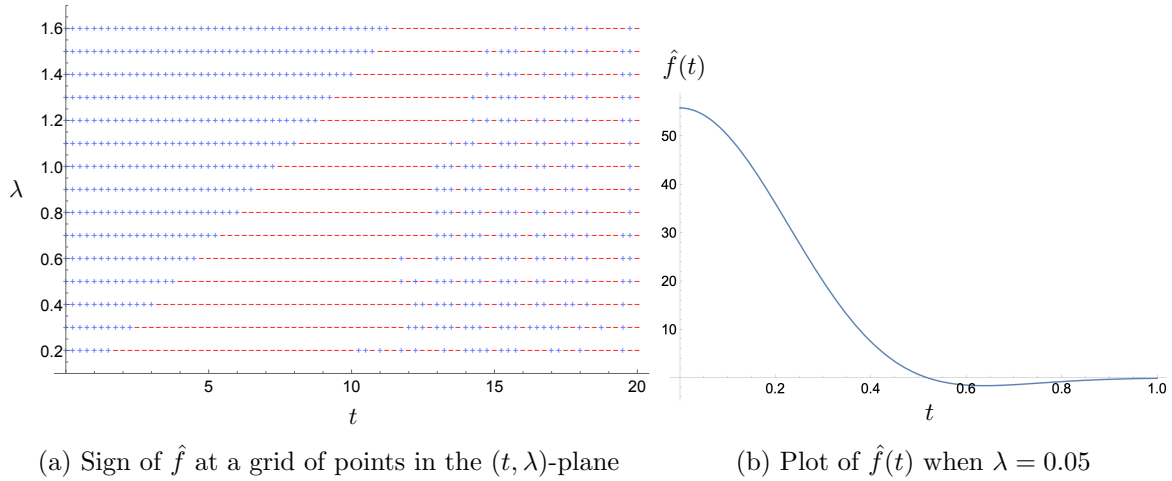


Figure 1: Spherical transform for the Gaussian kernel on the hyperbolic plane

323 Figure 1a shows the sign of  $\hat{f}$  in the  $(t, \lambda)$ -plane. The blue (positive) area on the left  
 324 confirms Proposition 6.1. It is followed to the right by an alternation of red and blue areas,  
 325 which shows that  $\hat{f}(t)$  oscillates between positive and negative values (note that the  $\lambda$  axis  
 326 begins at  $\lambda = 0.2$ , since the numerical evaluation of the integral (4.2) becomes unstable for  
 327 much smaller values of  $\lambda$ ). Figure 1b shows an individual plot of  $\hat{f}(t)$ , obtained for a fixed  
 328  $\lambda = 0.05$ .

329 It is clear from Figure 1a that the  $L^2$ -Godement theorem implies the Gaussian kernel is  
 330 not positive-definite on the hyperbolic plane, for any  $\lambda$  in the range of the  $\lambda$ -axis of this figure.

331 *Remark 6.2.* Proposition 6.1 continues to hold for a general non-compact  $M$ , as above.  
 332 The proof of this general statement is not provided in order to maintain a reasonable length.

333 **6.2. Hyperbolic space.** In this case,  $\hat{f}(t)$  can be found in closed form.

$$334 \quad (6.1) \quad g(r) = e^{-\lambda r^2} \text{ in (4.5)} \implies \hat{f}(t) = \left(\frac{2\pi}{t}\right) \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}} e^{(1-t^2)/4\lambda} \sin\left(\frac{t}{2\lambda}\right).$$

335 This shows that the sign of  $\hat{f}(t)$  is determined by  $\sin(t/2\lambda)$ . This remains positive for  $t \leq 2\pi\lambda$ ,  
 336 and then oscillates indefinitely. From the  $L^2$ -Godement theorem, the following is immediate.

337 **Proposition 6.3.** *The Gaussian kernel (1.1) is not positive-definite on the hyperbolic space*  
 338 *for any value of  $\lambda$ .*

339 The expression (6.1) can be obtained from the following general lemma, proved in Section 8.3.

340 **Lemma 6.4.** *In (4.5), extend  $g(r)$  to an even function of  $r \in \mathbb{R}$ . Then,*

$$341 \quad (6.2) \quad \hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{G(it + 1) - G(it - 1)\}$$

where  $\operatorname{Im}$  denotes the imaginary part and  $G$  is the moment-generating function

$$G(s) = \int_{-\infty}^{+\infty} g(r) e^{sr} dr.$$

Indeed, (6.1) follows from (6.2) using the expression of the moment-generating function of a univariate normal distribution,

$$G(s) = \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}} e^{s^2/4\lambda}.$$

342 **6.3. The general case.** Let  $M$  be any non-compact symmetric space, as in Section 3  
 343 (so the isometry group  $G$  of  $M$  is semisimple, of non-compact type, and has finite center [32]).  
 344 The following lemma will be proved in Section 8.4.

345 **Lemma 6.5.** *Under the assumptions just stated,  $M$  contains an isometrically embedded*  
 346 *hyperbolic plane.*

347 The numerical results of Paragraph 6.1 show that the Gaussian kernel is not positive-definite  
 348 on the hyperbolic plane, at least for any  $\lambda$  in the range appearing in Figure 1a. The Embedding  
 349 Lemma 5.2 and Lemma 6.5 therefore imply that the Gaussian kernel is not positive-definite  
 350 on any non-compact Riemannian symmetric space, for  $\lambda$  in this same range (the additional  
 351 assumptions of Lemma 6.5 merely serve to prevent  $M$  from being Euclidean).

352 On the other hand, Proposition 6.3 shows the Gaussian kernel is never positive-definite  
 353 on the hyperbolic space. By the Embedding Lemma, it is therefore never positive-definite on  
 354 symmetric spaces that contain an isometrically embedded hyperbolic space. Examples of such  
 355 symmetric spaces include all spaces of  $n \times n$  real, complex, or quaternion, positive-definite  
 356 matrices (with the requirement that  $n \geq 4$  for the real case) [38].

357 Hopefully, the numerical treatment of the hyperbolic plane case in Paragraph 6.1 will soon  
 358 be replaced with a complete analytical treatment, similar to the one that we conducted for  
 359 the hyperbolic space case in Paragraph 6.2. This is the only step still missing to obtain a  
 360 fully analytical proof of the statement that the Gaussian kernel is never positive-definite on  
 361 any non-Euclidean symmetric space.

362 **7. The hyperbolic secant revisited: non-Euclidean Herschel-Maxwell distributions.** Re-  
 363 call the hyperbolic secant kernel on the hyperbolic plane from Section 4.3.2, which was shown  
 364 to be positive-definite in contrast to the Gaussian kernel. Here, we will present a new inter-  
 365 pretation of the hyperbolic secant kernel as the Herschel-Maxwell kernel on the hyperbolic  
 366 plane. In Euclidean space, the Herschel-Maxwell theorem is an elegant characterization of the  
 367 Gaussian distribution originally due to the astronomer J.F.W. Herschel, who was studying  
 368 two-dimensional errors in astronomical observations. Ten years later, James Clerk Maxwell  
 369 presented a three-dimensional version of the same argument to show that the stationary prob-  
 370 ability distribution of the velocities of molecules in a gas follows a Gaussian distribution under  
 371 the assumptions that (P1) the components of the random vector in an orthogonal coordinate  
 372 system are independent, and (P2) the distribution only depends on the magnitude of the  
 373 vector [29, 35, 39]. A modern statement of the theorem is that if the distribution of a random  
 374 vector with independent components is invariant under rotations, then the components must  
 375 be identically distributed as a Gaussian distribution.

376 Here, we will follow the Herschel-Maxwell derivation as it appears in Jaynes [35], which  
 377 relies on the assumption of the existence of densities of the components of the random variables  
 378 and solving a functional equation. This derivation closely follows Maxwell's own argument.  
 379 Let  $(u, v)$  be a Cartesian coordinate system in the Euclidean plane. We seek to determine

380 a continuous joint probability distribution  $p(u, v)dudv$  that satisfies assumptions P1 and P2.  
 381 By assumption P1, we have

$$382 \quad (7.1) \quad p(u, v)dudv = h(u)du h(v)dv$$

383 for some function  $h$ . By assumption P2, we have

$$384 \quad (7.2) \quad p(u, v)dudv = g(r)rdrd\theta$$

385 as the density cannot depend on the angle  $\theta$  when expressed in polar coordinates  $(r, \theta)$ . (7.1)  
 386 and (7.2) combine to give the functional equation

$$387 \quad (7.3) \quad h(u)h(v) = g\left(\sqrt{u^2 + v^2}\right),$$

388 which admits the general solution

$$389 \quad (7.4) \quad h(u) = \sqrt{\frac{\lambda}{\pi}}e^{-\lambda u^2}, \quad p(u, v) = \frac{\lambda}{\pi}e^{-\lambda(u^2+v^2)},$$

390 for any  $\lambda > 0$  upon insisting that the distribution be normalized.

391 To replicate this derivation in the hyperbolic plane, we must first specify what we mean by  
 392 an ‘orthogonal coordinate system’ in the hyperbolic plane. The most natural choice for such a  
 393 coordinate system seems to be offered by Lobachevsky coordinates, determined by specifying  
 394 a directed geodesic ( $u$ -axis) through the origin. The Lobachevsky coordinates of a point  $x$   
 395 then found by dropping a perpendicular to the  $u$ -axis. The signed distance from the foot of the  
 396 perpendicular to the origin is the  $u$ -coordinate of the point, and the  $v$ -coordinate is the signed  
 397 distance along the perpendicular to the  $u$ -axis (with the distance taken to be positive on one  
 398 side of the  $u$ -axis and negative on the other). With respect to such a coordinate system, we  
 399 seek a joint probability distribution  $p(x)\text{vol}(dx) = p(u, v)\text{vol}(dx) = h(u)h(v)\text{vol}(dx)$ , where  
 400  $\text{vol}(dx)$  denotes the Riemannian volume form in the hyperbolic plane (P1). Moreover, (P2)  
 401 generalizes to the assumption that  $p(x)$  can only depend on the Riemannian distance to the  
 402 origin  $r = d(o, x)$ , which once again yields the functional equation  $h(u)h(v) = g(r)$ . As this  
 403 equation holds for all  $(u, v)$ , we can take any point on the  $u$ -axis to obtain  $g(u) = h(u)h(0)$ .  
 404 That is, we have the functional equation

$$405 \quad (7.5) \quad h(u)h(v) = h(0)h(r)$$

406 where  $(u, v)$  and  $r$  are related by

$$407 \quad (7.6) \quad \cosh(r) = \cosh(u) \cosh(v)$$

408 according to the hyperbolic Pythagorean theorem. Setting  $\tilde{h} = (h \circ \text{arcosh})/h(0)$ , (7.5) be-  
 409 comes  $\tilde{h}(\cosh u)\tilde{h}(\cosh v) = \tilde{h}(\cosh u \cosh v)$ , which admits the general solution  $\tilde{h}(\cosh u) =$   
 410  $(\cosh u)^{-a}$ , for  $a \in \mathbb{R}$  (i.e.,  $\tilde{h}$  is a power function). Thus,

$$411 \quad h(r) = h(0)(\cosh(r))^{-a},$$

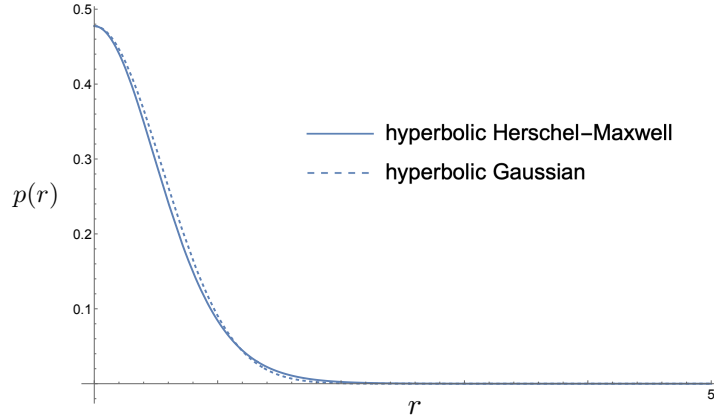


Figure 2: Plots of the normalized hyperbolic Herschel-Maxwell (solid curve, (7.7)) and Gaussian (dashed curve, (7.8)) densities as functions of  $r = d(o, x)$  for parameters  $a = 4$  and  $\lambda = 1.66$ .

412 which yields the probability density

$$413 \quad p(x) \text{vol}(dx) = h(0)^2 (\cosh(r))^{-a} \text{vol}(dx) = h(0)^2 (\cosh(r))^{-a} \sinh(r) dr d\theta,$$

414 in geodesic polar coordinates. Note that this density is normalizable only if  $a > 1$ . The  
415 constant  $h(0)$  can be determined through normalizing the distribution, which yields

$$416 \quad (7.7) \quad p(x) \text{vol}(dx) = \frac{1}{\pi} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a-1}{2}\right)} (\cosh(d(o, x)))^{-a} \text{vol}(dx).$$

417 We call this distribution the Herschel-Maxwell distribution in the hyperbolic plane. It can  
418 be viewed as one of several possible generalizations of the Gaussian distribution centered at  
419 the point  $o$  to the hyperbolic plane alongside the distinct distributions derived from the heat  
420 kernel interpretation of the Gaussian [28, 40] and

$$421 \quad (7.8) \quad p(x) \text{vol}(dx) = \frac{1}{\pi} \sqrt{\frac{\lambda}{\pi}} \frac{e^{-\frac{1}{4\lambda}}}{\text{erf}\left(\frac{1}{2\sqrt{\lambda}}\right)} e^{-\lambda d(o, x)^2} \text{vol}(dx),$$

422 which inherits an important characteristic property of Euclidean Gaussian distributions from  
423 statistical inference: the maximum likelihood estimate of the mean reduces to a (Riemannian)  
424 center of mass computation [44, 45, 47, 46, 15]. Figure 2 compares plots of the density functions  
425  $p(r) \text{vol}(dx)$  in (7.7) and (7.8) as functions of  $r = d(o, x)$  for parameters  $a = 4$  and  $\lambda = 1.66$ .

426 Finally, we note that the hyperbolic Herschel-Maxwell distribution can be generalized  
427 to  $n$ -dimensional hyperbolic space if we insist that the components of the point  $x$  in any  
428 Lobachevsky coordinate system in *any* hyperbolic plane containing  $o$  and  $x$  are independent  
429 and the distribution only depends on the geodesic distance  $r = d(o, x)$ . Reasoning as before,  
430 we obtain

$$431 \quad (7.9) \quad p(x) \text{vol}(dx) = \frac{1}{\omega_{n-1}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+1-n}{2}\right) \Gamma\left(\frac{n}{2}\right)} (\cosh(d(o, x)))^{-a} \text{vol}(dx),$$

432 where  $\omega_{n-1}$  denotes the area of the unit sphere in  $\mathbb{R}^n$  and the normalization factor in (7.9) is  
 433 determined by integration with respect to the volume form  $\text{vol}(dx)$  in  $n$ -dimensional hyperbolic  
 434 space. The distribution is well-defined for  $a > n - 1$ .

435 *Remark 7.1.* We can repeat the derivation of the hyperbolic Herschel-Maxwell distribu-  
 436 tion presented here in the case of spherical geometry, with the main difference being the  
 437 replacement of the hyperbolic Pythagorean theorem with the spherical Pythagorean theorem  
 438

$$439 \quad (7.10) \quad \cos(\theta) = \cos(u) \cos(v)$$

440 where  $o, x$  denote points on the 2-sphere  $S^2$ ,  $\theta = d(o, x)$ , and  $(u, v)$  is an orthogonal coordinate  
 441 system centered on  $o$ , whereby the coordinates of a point  $x$  are obtained by dropping a  
 442 perpendicular to a specified geodesic through  $o$  and measuring the signed lengths  $u$  along the  
 443 geodesic from  $o$  to the foot of the perpendicular and  $v$  along the perpendicular to the geodesic  
 444 from  $x$ . This yields the spherical Herschel-Maxwell kernel

$$445 \quad (7.11) \quad k(x, y) = |\cos(d(x, y))|^a,$$

446 where  $a \geq 0$  to ensure the corresponding distribution is finite. Note that by representing  
 447  $x, y \in S^2$  by unit vectors  $w_x, w_y \in \mathbb{R}^3$ , we have  $\cos(d(x, y)) = \langle w_x, w_y \rangle$ , and so (7.11) takes  
 448 the form  $k(x, y) = |\langle w_x, w_y \rangle|^a$ . Thus, for non-negative even integer  $a$ , the spherical Herschel-  
 449 Maxwell distribution can be identified with a homogeneous polynomial kernel  $\langle x, y \rangle^n$ , which  
 450 is known to be positive-definite if and only if  $n$  is a non-negative integer [49].

## 451 **8. Proofs of the main results.**

### 452 **8.1. Proof of Theorem 3.1.**

453 **8.1.1. Notation.** The following notation will be used in the proof.

- 454 • Let  $W$  denote the Weyl group, which arises from the adjoint action of  $H$  on  $\mathfrak{a}$  [32].  
 455 Also, let  $\alpha$  denote the measure on  $\mathfrak{a}^*$ , which is given by  $\alpha(dt) = c_M |c(t)|^{-2} dt$ . Then,  
 456  $L_W^2(\mathfrak{a}^*)$  stands for the space of  $W$ -invariant square-integrable functions (with respect  
 457 to the measure  $\alpha$ ).
- 458 • Denote  $\Phi_*$  the set of functions  $\varphi_t$  in (3.1) where  $t \in \mathfrak{a}^*$ . This is a proper subset of the  
 459 set  $\Phi$  of all positive-definite spherical functions, which appears in Godement's theorem  
 460 (Section 2, in particular (2.4)).
- 461 • Functions or measures on  $\mathfrak{a}^*$  can be pulled back to  $\Phi_*$ , assuming they are  $W$ -invariant.  
 462 This is because  $\varphi_{t'} = \varphi_t$  if and only if  $t' = w \cdot t$  for some  $w \in W$ . Accordingly, the  
 463 measure  $\alpha$  on  $\mathfrak{a}^*$  can be identified with a measure on  $\Phi_*$ , also denoted  $\alpha$ . Similarly,  
 464 if  $\hat{f}(t)$  is a spherical transform, as in (3.2),  $\hat{f}(\varphi)$  is just  $\hat{f}(t)$  when  $\varphi \in \Phi_*$  and  $\varphi = \varphi_t$   
 465 for some  $t \in \mathfrak{a}^*$ .
- 466 • Finally,  $C_H^c(M)$  is the space of  $H$ -invariant continuous, compactly supported functions,  
 467 with complex values, defined on  $M$ .

468 **8.1.2. The if part.** For square-integrable functions, the spherical transform is only defined  
 469 in an abstract sense. It is a linear isometry between the Hilbert spaces  $L_H^2(M)$  and  $L_W^2(\mathfrak{a}^*)$ ,

470  $f \mapsto \hat{f}$  [30][52]. However, if  $\hat{f}$  is integrable, the inversion formula (3.3) holds

$$471 \quad (8.1) \quad f(x) = \int_{\mathfrak{a}^*} \hat{f}(t) \varphi_t(x) \alpha(dt) \quad \text{for all } x \in M$$

472 and this implies that  $f$  is continuous, since each  $\varphi_t$  is continuous and verifies  $|\varphi_t(x)| \leq 1$ ,  
473 being a positive-definite function with  $\varphi_t(o) = 1$ . The fact that  $f$  is positive-definite follows  
474 from Godement's theorem. Indeed, (8.1) can be written

$$475 \quad (8.2) \quad f(x) = \int_{\Phi_*} \varphi(x) (\hat{f}\alpha)(d\varphi)$$

476 where  $(\hat{f}\alpha)(d\varphi) = \hat{f}(\varphi) \alpha(d\varphi)$ . Then, since  $\hat{f}(\varphi) \geq 0$  for almost all  $\varphi \in \Phi_*$  and  $\hat{f}$  is integrable,  
477 with respect to  $\alpha$ ,  $\hat{f}\alpha$  is a finite positive measure on  $\Phi_*$ . Moreover, since  $\Phi_*$  is a (measurable)  
478 subset of  $\Phi$ , (8.2) is exactly of the form (2.4), but with the measure  $\mu_f$  supported on  $\Phi_*$ ,  
479 where it is equal to  $\hat{f}\alpha$ . This shows that  $f$  is positive-definite, and even determines the finite  
480 positive measure  $\mu_f$  (recall that  $\mu_f$  is unique).

481 **8.1.3. The only-if part.** If  $f$  is  $H$ -invariant, continuous and positive-definite, then one  
482 has

$$483 \quad (8.3) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) \geq 0$$

484 for any  $\psi \in C_H^c(M)$ , where  $*$  denotes the convolution and  $\psi^\dagger(g \cdot o) = \bar{\psi}(g^{-1} \cdot o)$  [22] (Page  
485 286). This yields a scalar product on  $C_H^c(M)$ ,

$$486 \quad (8.4) \quad \langle \psi_1, \psi_2 \rangle_f = \int_M (\psi_1 * \psi_2^\dagger)(x) f(x) \text{vol}(dx).$$

487 It will shortly be proved that this scalar product admits a spectral representation,

$$488 \quad (8.5) \quad \langle \psi_1, \psi_2 \rangle_f = \langle \hat{\psi}_1, \hat{\psi}_2 \rangle_{\hat{f}}$$

489 where the scalar product on the right-hand side is the usual “ $L^2$ ” scalar product, with respect  
490 to the measure  $\hat{f}\alpha$  on  $\Phi_*$ . By [27] (Plancherel theorem, Page 101), this implies the measure  
491  $\mu_f$  (in (2.4)) is supported on  $\Phi_*$ , where it is equal to  $\hat{f}\alpha$ . Since  $\mu_f$  is a finite positive measure,  
492  $\hat{f}(t) \geq 0$  for almost all  $t$  and  $\hat{f}$  is integrable.

493 To prove (8.5), consider the left-hand side of (8.3). Both  $\psi$  and  $\psi^\dagger$  belong to  $C_H^c(M)$ .  
494 The same is therefore true of their convolution. Since the spherical transform of  $\psi^\dagger$  is the  
495 complex conjugate of the spherical transform of  $\psi$ , the convolution property of spherical  
496 transforms (see [52] (Page 88)) implies that

$$497 \quad (8.6) \quad \widehat{\psi * \psi^\dagger}(t) = |\hat{\psi}(t)|^2.$$

498 Since both  $f$  and  $\psi * \psi^\dagger$  are in  $L_H^2(M)$ , the Plancherel formula (3.4) yields

$$499 \quad (8.7) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 \hat{f}(t) \alpha(dt) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 (\hat{f}\alpha)(dt).$$



500 By (8.4), and by pulling back the integral of the right-hand side to  $\Phi_*$ , this is

$$501 \quad (8.8) \quad \langle \psi, \psi \rangle_f = \int_{\Phi_*} |\hat{\psi}(\varphi)|^2 (\hat{f}\alpha)(d\varphi) = \langle \hat{\psi}, \hat{\psi} \rangle_{\hat{f}}$$

502 so that (8.5) follows immediately, using the polarization identity for scalar products. ■

503 **8.1.4. Alternative proof of Theorem 3.1.** The only-if part of the proposition admits at  
 504 least one more proof. Indeed [22] (Page 302), for any continuous positive-definite  $f \in L_H^2(M)$ ,  
 505 it is possible to construct  $\phi \in L_H^2(M)$  such that  $f = \phi * \phi^\dagger$ . If one can apply the convolution  
 506 property as in (8.6), it would follow that  $\hat{f}(t) = |\hat{\phi}(t)|^2$ , so  $\hat{f}(t)$  is positive for all  $t$  (not just  
 507 for almost all  $t$ ), and  $\hat{f}$  is integrable.

508 The convolution property does not apply directly because  $\phi$  is not integrable. Letting  
 509  $(\phi_n; n = 1, 2, \dots)$  belong to  $C_H^c(M)$  and converge to  $\phi$  in  $L_H^2(M)$ , the functions  $f_n = \phi_n * \phi_n$   
 510 belong to  $C_H^c(M)$  and converge uniformly to  $f$ , as can be shown by using Young's convolution  
 511 inequality [33] (Page 296).

512 Now, as in (8.6), one has  $\hat{f}_n(t) = |\hat{\phi}_n(t)|^2$  for each  $n$ . However, since  $\hat{\phi}_n$  converges to  $\hat{\phi}$   
 513 in  $L_W^2(\mathfrak{a}^*)$ , it follows that  $|\hat{\phi}_n|^2$  converges to  $|\hat{\phi}|^2$  in  $L_W^1(\mathfrak{a}^*)$  (this is the space of  $W$ -invariant  
 514 integrable functions) [9] (Page 298).

Therefore, a representation of  $f$  under the form (3.3) can be obtained by noting, for  $x \in M$ ,

$$\lim f_n(x) = \lim \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}_n(t)|^2 \alpha(dt) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}(t)|^2 \alpha(dt)$$

515 where the first equality follows from (3.3) and  $\hat{f}_n(t) = |\hat{\phi}_n(t)|^2$ , and the second equality from  
 516  $|\hat{\phi}_n|^2 \rightarrow |\hat{\phi}|^2$  in  $L_W^1(\mathfrak{a}^*)$  and  $|\varphi_t(x)| \leq 1$  (this is a consequence of  $\varphi_t$  being positive-definite).

However, the limit of  $f_n(x)$  is just  $f(x)$ , since the  $f_n$  converge uniformly to  $f$ . Thus,

$$f(x) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}(t)|^2 \alpha(dt).$$

517 Comparing to (3.3), it follows that  $\hat{f}(t) = |\hat{\phi}(t)|^2$ , as required. ■

518 **8.2. Proof of Proposition 6.1.** Let  $R$  be such that  $r \leq R$  implies  $|\cosh(r) - 1| < 1$ .  
 519 Then [51] (Page 142)

$$520 \quad (8.9) \quad P_{-\frac{1}{2}+it}(\cosh(r)) = 1 - (1/4 + t^2)(r^2/4) + \varepsilon(t, r) \quad \text{for } r \leq R$$

521 with remainder  $|\varepsilon(t, r)| \leq C(T, R)r^4$  for all  $t \leq T$  and  $r \leq R$ , where  $C(T, R) > 0$  is a constant.  
 522 Replacing this into (4.2), the following decomposition is obtained,

$$523 \quad (8.10) \quad \hat{f}(t) = \hat{f}_a(t) + \hat{f}_b(t) + \hat{f}_c(t)$$

524 in terms of the following integrals

$$525 \quad (8.11) \quad \hat{f}_a(t) = \int_0^R e^{-\lambda r^2} [1 - (1/4 + t^2)(r^2/4)] \sinh(r) dr,$$

$$526 \quad (8.12) \quad \hat{f}_b(t) = \int_0^R e^{-\lambda r^2} \varepsilon(t, r) \sinh(r) dr,$$

$$527 \quad (8.13) \quad \hat{f}_c(t) = \int_R^\infty e^{-\lambda r^2} P_{-\frac{1}{2}+it}(\cosh(r)) \sinh(r) dr.$$

528 Noting that  $|P_{-\frac{1}{2}+it}(\cosh(r))| \leq 1$  [51] (Page 161), a simple calculation yields

$$529 \quad (8.14) \quad |\hat{f}_c(t)| \leq \int_R^\infty e^{-\lambda r^2} \sinh(r) dr = O(e^{-\lambda R^2}).$$

On the other hand,

$$(C(T, R))^{-1} |\hat{f}_b(t)| \leq \int_0^R e^{-\lambda r^2} r^4 \sinh(r) dr \leq \int_0^\infty e^{-\lambda r^2} r^4 \sinh(r) dr.$$

However, by performing an elementary change of variables and then assuming that  $\lambda \geq 1$ , this implies the following bound,

$$(C(T, R))^{-1} |\hat{f}_b(t)| \leq \lambda^{-\frac{5}{2}} \int_0^\infty e^{-u^2} u^4 \sinh(u) du$$

530 which means that

$$531 \quad (8.15) \quad |\hat{f}_b(t)| = O(\lambda^{-\frac{5}{2}}).$$

532 Finally,  $\hat{f}_a(t)$  can be extended to an integral from 0 to  $\infty$ , with an exponentially small error,

$$533 \quad (8.16) \quad \hat{f}_a(t) = Z(\lambda) - (1/4 + t^2) V(\lambda) + O(e^{-\lambda R^2})$$

where  $Z(\lambda)$  and  $V(\lambda)$  are the following integrals

$$Z(\lambda) = \int_0^\infty e^{-\lambda r^2} \sinh(r) dr, \quad V(\lambda) = \frac{1}{4} \int_0^\infty e^{-\lambda r^2} r^2 \sinh(r) dr.$$

534 By a direct evaluation

$$535 \quad (8.17) \quad Z(\lambda) = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \exp\left(\frac{1}{4\lambda}\right) \operatorname{erf}\left(\frac{1}{2\sqrt{\lambda}}\right) = \frac{\lambda^{-1}}{2} + \frac{\lambda^{-2}}{6} + O(\lambda^{-3})$$

536 where  $\operatorname{erf}$  denotes the error function [5] (Page 35), and then by noting that  $4V(\lambda) = -Z'(\lambda)$   
537 (the prime denotes derivation),

$$538 \quad (8.18) \quad V(\lambda) = \frac{\lambda^{-2}}{8} + O(\lambda^{-3}).$$

539 Therefore, replacing (8.17) and (8.18) into (8.16),

$$540 \quad (8.19) \quad \hat{f}_a(t) = \frac{\lambda^{-1}}{2} - \left(t^2 - \frac{13}{12}\right) \frac{\lambda^{-2}}{8} + O(\lambda^{-3}).$$

541 To conclude the proof, replace (8.14), (8.15) and (8.19) into (8.10). This yields

$$542 \quad (8.20) \quad \hat{f}(t) \sim \frac{\lambda^{-1}}{2} - \left(t^2 - \frac{13}{12}\right) \frac{\lambda^{-2}}{8} \quad \text{as } \lambda \rightarrow \infty$$

543 ( $a \sim b$  means  $a/b$  converges to 1), where the convergence is uniform in  $t \leq T$ . It is now enough  
544 to note that the right-hand side of (8.20) is positive for all  $t \leq T$  when  $\lambda \geq T^2/4$ . ■

545 **8.3. Proof of Lemma 6.4.** Note that  $\sinh(r) \sin(tr)$  is an even function of  $r$ . Moreover,  
546 since  $g(r)$  is also even, (4.5) can be written

$$547 \quad \hat{f}(t) = \frac{4\pi}{t} \int_0^\infty g(r) \sinh(r) \sin(tr) dr$$

$$548 \quad = \frac{2\pi}{t} \int_{-\infty}^{+\infty} g(r) \sinh(r) \sin(tr) dr$$

In other words,

$$\hat{f}(t) = \frac{2\pi}{t} \operatorname{Im} \left\{ \int_{-\infty}^{+\infty} g(r) \sinh(r) e^{itr} dr \right\}.$$

Using the definition of the hyperbolic sine, this becomes

$$\hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \left\{ \int_{-\infty}^{+\infty} g(r) e^{(it+1)r} dr - \int_{-\infty}^{+\infty} g(r) e^{(it-1)r} dr \right\}$$

and then, by definition of the moment-generating function  $G$ ,

$$\hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{G(it+1) - G(it-1)\}$$

549 which is the same as (6.2). ■

550 **8.4. Proof of Lemma 6.5.** It will be enough to prove that there exists a totally geodesic  
551 submanifold  $M_o$  of  $M$ , such that  $M_o$  is a hyperbolic plane. Indeed,  $M$  is a Hadamard manifold  
552 (a simply connected complete Riemannian manifold with negative sectional curvatures [32]  
553 (Chapter VI)). Therefore, if  $M_o$  is a totally geodesic submanifold, it is also an isometrically  
554 embedded metric space. The proof will rely very heavily on the algebraic description of  
555 symmetric spaces, detailed in [32] (in particular Chapters V and VI).

556 Recall that  $M$  is determined by a symmetric pair  $(G, H)$ . The Lie algebra  $\mathfrak{g}$  of  $G$  is a  
557 real semisimple Lie algebra, and admits a Cartan involution  $\theta$  with corresponding Cartan  
558 decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$  ( $\mathfrak{h}$  is the Lie algebra of  $H$ ). If  $B$  is the Killing form of  $\mathfrak{g}$ , then

$$559 \quad (8.21) \quad \langle \xi, \eta \rangle = -B(\xi, \theta(\eta)) \quad \xi, \eta \in \mathfrak{g}$$

560 defines a scalar product on  $\mathfrak{g}$ . Let  $\mathfrak{a}$  be a maximal Abelian subspace of  $\mathfrak{p}$ . The linear maps  
 561  $\text{ad}_a : \mathfrak{g} \rightarrow \mathfrak{g}$  where  $\text{ad}_a(\xi) = [a, \xi]$  for  $a \in \mathfrak{a}$  are jointly diagonalisable, with real eigenvalues.  
 562 These eigenvalues are linear forms  $\lambda : \mathfrak{a} \rightarrow \mathbb{R}$ , called the roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ .

563 Fix a non-zero root  $\lambda$ , and  $a_\lambda \in \mathfrak{a}$  such that  $\lambda(a) = B(a_\lambda, a)$  for  $a \in \mathfrak{a}$ . Then, let  $\xi_\lambda$  be a  
 564 joint eigenvector,  $\text{ad}_a(\xi_\lambda) = \lambda(a)\xi_\lambda$  for  $a \in \mathfrak{a}$ . Note that

$$565 \quad (8.22) \quad [a, \theta(\xi_\lambda)] = \theta([\theta(a), \xi_\lambda]) = -\theta([a, \xi_\lambda])$$

566 for any  $a \in \mathfrak{a}$ , where the first equality follows because  $\theta$  is an involution, and the second  
 567 because  $a \in \mathfrak{a}$ , a subspace of  $\mathfrak{p}$ , and  $\mathfrak{p}$  is by definition the  $(-1)$ -eigenspace of  $\theta$ . However,

$$568 \quad (8.23) \quad \theta([a, \xi_\lambda]) = \lambda(a)\theta(\xi_\lambda)$$

569 by the definition of  $\xi_\lambda$  and linearity of  $\theta$ . Thus, (8.22) and (8.23) imply that  $\text{ad}_a(\theta(\xi_\lambda)) =$   
 570  $-\lambda(a)\theta(\xi_\lambda)$  for any  $a \in \mathfrak{a}$ . Denoting  $\theta(\xi_\lambda)$  by  $\xi_{-\lambda}$  and recalling the definition of  $a_\lambda \in \mathfrak{a}$ , it is  
 571 now seen that the following commutation relations hold

$$572 \quad (8.24) \quad [a_\lambda, \xi_\lambda] = \|a_\lambda\|^2 \xi_\lambda \quad [a_\lambda, \xi_{-\lambda}] = -\|a_\lambda\|^2 \xi_{-\lambda}$$

573 where  $\|\cdot\|$  is the norm with respect to the scalar product (8.21). It will now be proved that

$$574 \quad (8.25) \quad [\xi_\lambda, \xi_{-\lambda}] = -\|\xi_\lambda\|^2 a_\lambda.$$

575 First, any  $a \in \mathfrak{a}$  commutes with  $[\xi_\lambda, \xi_{-\lambda}]$ . Indeed,

$$576 \quad [a, [\xi_\lambda, \xi_{-\lambda}]] = [[a, \xi_\lambda], \xi_{-\lambda}] + [\xi_\lambda, [a, \xi_{-\lambda}]]$$

$$577 \quad = \lambda(a)[\xi_\lambda, \xi_{-\lambda}] - \lambda(a)[\xi_\lambda, \xi_{-\lambda}]$$

578 where the first equality follows from Jacobi's identity for Lie brackets, and the second because  
 579  $\xi_\lambda$  and  $\xi_{-\lambda}$  are joint eigenvectors with eigenvalues  $\lambda(a)$  and  $-\lambda(a)$ . Second,  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{p}$ .  
 580 Indeed,  $\theta([\xi_\lambda, \xi_{-\lambda}]) = [\theta(\xi_\lambda), \theta(\xi_{-\lambda})]$ , but  $\theta(\xi_\lambda) = \xi_{-\lambda}$  by definition, and  $\theta(\xi_{-\lambda}) = \xi_\lambda$  since  $\theta$   
 581 is an involution. Therefore,  $\theta([\xi_\lambda, \xi_{-\lambda}]) = -[\xi_\lambda, \xi_{-\lambda}]$  which means  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{p}$ . However,  $\mathfrak{a}$   
 582 is maximal Abelian in  $\mathfrak{p}$ , and this yields  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{a}$ .

Finally, to obtain (8.25), note that

$$B([\xi_\lambda, \xi_{-\lambda}], a) = B(\xi_{-\lambda}, [a, \xi_\lambda]) = B(\xi_{-\lambda}, \xi_\lambda)\lambda(a)$$

where the first equality holds since  $B$  is ad-invariant, and the second since  $\xi_\lambda$  is a joint  
 eigenvector with eigenvalue  $\lambda(a)$ . Now, the fact that  $B$  is symmetric and the definition of  $a_\lambda$   
 imply  $B([\xi_\lambda, \xi_{-\lambda}], a) = B(\xi_\lambda, \xi_{-\lambda})B(a_\lambda, a)$ , so that

$$[\xi_\lambda, \xi_{-\lambda}] = B(\xi_\lambda, \xi_{-\lambda}) a_\lambda$$

583 and (8.25) follows after recalling (8.21) and  $\xi_{-\lambda} = \theta(\xi_\lambda)$ .

584 Moving on, let  $\mathfrak{g}_\circ$  denote the Lie subalgebra of  $\mathfrak{g}$ , generated by  $\{\xi_{-\lambda}, a_\lambda, \xi_\lambda\}$ . The Lie  
 585 bracket of  $\mathfrak{g}_\circ$  is completely determined by (8.24) and (8.25). It is straightforward to see that  
 586  $\mathfrak{g}_\circ \simeq \mathfrak{sl}(2, \mathbb{R})$  (the Lie algebra of  $2 \times 2$  zero-trace real matrices). The isomorphism is given by

$$587 \quad (8.26) \quad \xi_{-\lambda} \longleftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} ; \quad a_\lambda \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \xi_\lambda \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

588 after proper rescaling of  $\{\xi_{-\lambda}, a_\lambda, x_{-\lambda}\}$  (this needs to ensure that  $\|a_\lambda\|^2 = 2$  and  $\|\xi_\lambda\|^2 = 1$ ).  
 589 Let  $G_o$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{g}_o$  and let  $M_o$  be the orbit of  $o$   
 590 under  $G_o$  (recall that  $o \in M$  has stabiliser  $H$  in  $G$  and can be thought of as the “point of  
 591 origin”). Then,  $M_o$  is a totally geodesic submanifold of  $M$ , because  $\mathfrak{g}_o$  is a Lie subalgebra of  
 592  $\mathfrak{g}$  [32] (Page 189). Moreover,  $M_o$  is a Riemannian homogeneous space under the action of  $G_o$ ,  
 593  $M_o \simeq G_o/H_o$  where  $H_o = G_o \cap H$  is the stabiliser of  $o$  in  $G_o$ . To see that  $M_o$  is a hyperbolic  
 594 plane (up to an irrelevant scaling factor), note first that  $G_o \simeq SL(2, \mathbb{R})$ . It then remains to  
 595 check that  $H_o \simeq SO(2)$ , since the Riemannian homogeneous space  $SL(2, \mathbb{R})/SO(2)$  is just a  
 596 hyperbolic plane [51]. To do so, let  $\mathfrak{h}_o$  be the Lie algebra of  $H_o$ . Then,  $\mathfrak{h}_o = \mathfrak{g}_o \cap \mathfrak{h}$ . However,  
 597 since  $\mathfrak{h}$  is the (+1)-eigenspace of  $\theta$ , it is easy to see that  $\mathfrak{h}_o$  is generated by  $\tau = \xi_\lambda + \xi_{-\lambda}$ .  
 598 Since  $\tau$  corresponds to the generator of  $SO(2) \subset SL(2, \mathbb{R})$ , under the isomorphism (8.26),  
 599  $H_o \simeq SO(2)$ , as required. ■

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725 **Appendix A. L<sup>1</sup>-Godement theorem.** The L<sup>2</sup>-Godement theorem, stated back in Sec-  
 726 tion 3, has the following variant, which will be called the L<sup>1</sup>-Godement theorem. It seems  
 727 useful to have this at hand, to deal with kernels  $k$  such that the function  $f(x) = k(o, x)$   
 728 is integrable but not square-integrable.

729 **Theorem A.1 (L<sup>1</sup>-Godement Theorem).** *A continuous function  $f \in L^1_H(M)$  is positive-*  
 730 *definite if and only if  $\hat{f}(t) \geq 0$  for all  $t \in \mathfrak{a}^*$  and  $\hat{f}$  is integrable with respect to the measure*  
 731  *$|c(t)|^{-2}dt$ .*

732 Here,  $L^1_H(M)$  denotes the space of  $H$ -invariant integrable functions (with respect to  $\text{vol}$ ). The  
 733 following proof is modeled on the one in Section 8.1. In particular, the notation is the same  
 734 as in Paragraph 8.1.1.

735 **A.1. The if part.** It will be enough to prove that the inversion formula (3.3) holds

$$736 \quad (\text{A.1}) \quad f(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \alpha(dt) \quad \text{for all } x \in M.$$

737 The fact that  $f$  is continuous and positive-definite then follows exactly as in Paragraph 8.1.2.  
 738 Denote by  $G_s(x) = G_s(o, x)$  the heat kernel of  $M$ . For  $s > 0$ ,  $G_s$  has spherical transform [26]

$$739 \quad (\text{A.2}) \quad \hat{G}_s(t) = (Z(s))^{-1} \exp[-s(\|t\|^2 + \|\rho\|^2)]$$

where  $Z(s)$  is a normalizing constant, which guarantees that  $G_s$  integrates to 1 over  $M$ , and  
 where  $\rho$  is half the sum of positive roots (as in Formula (3.1)). Recall that the heat semigroup  
 is strongly continuous on  $L^1_H(M)$  [21] (Page 148). This means the convolutions  $f_s = G_s * f$   
 converge to  $f$  in  $L^1_H(M)$  as  $s \rightarrow 0$ . Of course,  $G_s$  satisfies the inversion formula (3.3),

$$G_s(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{G}_s(t) \alpha(dt).$$

740 Therefore, by taking the convolution under the integral

$$741 \quad (\text{A.3}) \quad f_s(x) = \int_{\mathfrak{a}^*} (f * \varphi_t)(x) \hat{G}_s(t) \alpha(dt) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \hat{G}_s(t) \alpha(dt)$$

where the second equality follows because  $\varphi_t$  is an eigenfunction of convolution operators [52]  
 (Page 67). Now, from the assumption that  $\hat{f}$  is integrable (*i.e.*  $\hat{f} \in L^1_W(\mathfrak{a}^*)$ ), and from (A.2),  
 the dominated convergence theorem can be applied in (A.3), in order to obtain

$$\lim_{s \rightarrow 0} f_s(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \alpha(dt)$$

742 as  $s \rightarrow 0$ . But this limit is identical to  $f(x)$ , so (A.1) is proved.

743 **A.2. The only-if part.** As in Paragraph 8.1.3, the aim is to prove (8.7),

$$744 \quad (\text{A.4}) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 \hat{f}(t) \alpha(dt)$$



which then implies the spectral representation (8.5). Since  $\psi * \psi^\dagger \in C_H^c(M)$ , it follows from (3.3) and (8.6) that

$$(\psi * \psi^\dagger)(x) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\psi}(t)|^2 \alpha(dt).$$

745 Replacing this under the integral on the left-hand side of (A.4), the right-hand side is obtained  
 746 by using the definition (3.2) of  $\hat{f}$ . Now, everything follows from (8.5) as in Paragraph 8.1.3.  
 747 ■

748 **Appendix B. The hyperbolic secant in three dimensions.** Recall the hyperbolic secant  
 749 kernel from 4.3.2. This is here considered in the case where  $M$  is a hyperbolic space. The aim  
 750 is to apply the  $L^2$ -Godement theorem, with the help of the spherical transform pair (4.5).

Setting  $g(r) = (\cosh(r))^{-a}$  and  $f(x) = g(d(o, x))$ , it follows that  $f$  is continuous and in  $L_H^2(M)$  when  $a > 1$ . This is because,

$$\int_M |f(x)|^2 = 4\pi \int_0^\infty (\cosh(r))^{-2a} \sinh^2(r) dr = \int_1^\infty u^{-2a} (u^2 - 1)^{\frac{1}{2}} du < \infty$$

as follows by integrating in geodesic spherical coordinates. Thus, the  $L^2$ -Godement theorem can be applied to  $f$ . The spherical transform  $\hat{f}$  can be found from Lemma 6.4, at least for integer values of  $a$ . Indeed, the moment-generating function  $G$ , required in the lemma, reads

$$G(s) = \int_{-\infty}^{+\infty} (\cosh(r))^{-a} e^{sr} dr.$$

751 When  $a$  is an (of course, positive) integer, this can be obtained using the residue theorem [4].  
 752 Precisely, if  $a$  is odd,  $a = 2n + 1$ ,

$$753 \quad (\text{B.1}) \quad G(i\sigma) = \frac{\pi \operatorname{sech}(\pi\sigma/2)}{(2n)!} \prod_{m=0}^{n-1} [\sigma^2 + (2m+1)^2]$$

754 and if  $a$  is even,  $a = 2n$ ,

$$755 \quad (\text{B.2}) \quad G(i\sigma) = \frac{\pi\sigma \operatorname{csch}(\pi\sigma/2)}{(2n-1)!} \prod_{m=1}^{n-1} [\sigma^2 + (2m)^2]$$

756 where  $\operatorname{sech} = 1/\cosh$  and  $\operatorname{csch} = 1/\sinh$  are the hyperbolic secant and cosecant, and where  
 757 the imaginary part of  $\sigma$  is  $\geq -1$ . With  $G$  given by (B.1) and (B.2), the spherical transform  $\hat{f}$   
 758 can be computed using (6.2). Letting  $H(\sigma) = G(i\sigma)$ , this yields

$$759 \quad (\text{B.3}) \quad \hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{H(t-i) - H(t+i)\}.$$

760 **Lemma B.1.** *If  $z \in \mathbb{C} \setminus \mathbb{Z}$  and  $n$  is a positive integer, then*

$$761 \quad \Gamma(n+z)\Gamma(n-z) = -\frac{\pi}{z \sin(\pi z)} \prod_{m=0}^{n-1} (m^2 - z^2),$$

$$762 \quad \Gamma\left(n + \frac{1}{2} + z\right) \Gamma\left(n + \frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)} \prod_{m=0}^{n-1} \left[ \left(m + \frac{1}{2}\right)^2 - z^2 \right].$$

763 *Proof.* The proof of the first equality follows by induction using Euler's reflection formula  
 764  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$  for  $z \notin \mathbb{Z}$  and the identity  $\Gamma(z+1) = z\Gamma(z)$ . The second equality  
 765 follows in the same manner using the formula  $\Gamma(1/2+z)\Gamma(1/2-z) = \pi/\cos(\pi z)$ , which itself  
 766 follows from replacing  $z$  with  $z+1/2$  in Euler's reflection formula. ■

767 **Proposition B.2.** *The hyperbolic secant kernel is positive-definite in three-dimensional hy-*  
 768 *perbolic space whenever  $a$  is a positive integer.*

769 *Proof.* If  $a$  is odd with  $a = 2n + 1$ , then

$$770 \quad H(\sigma) = G(i\sigma) = \frac{4^n \pi \operatorname{sech}(\pi\sigma/2)}{(2n)!} \prod_{m=0}^{n-1} \left[ \left(m + \frac{1}{2}\right)^2 - \left(\frac{i\sigma}{2}\right)^2 \right]$$

$$771 \quad = \frac{4^n}{(2n)!} \Gamma\left(n + \frac{1}{2} + i\frac{\sigma}{2}\right) \Gamma\left(n + \frac{1}{2} - i\frac{\sigma}{2}\right)$$

772 using Lemma B.1 with  $z = i\sigma/2$ . Thus,

$$773 \quad H(t-i) - H(t+i) = \frac{4^n}{(2n)!} \left[ \Gamma\left(n+1 + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) - \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n+1 - \frac{it}{2}\right) \right]$$

$$774 \quad = \frac{4^n}{(2n)!} \left[ \left(n + \frac{it}{2}\right) \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) - \left(n - \frac{it}{2}\right) \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) \right]$$

$$775 \quad = i \frac{4^n t}{(2n)!} \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) = i \frac{4^n t}{(2n)!} \left| \Gamma\left(n + \frac{it}{2}\right) \right|^2$$

776 where we have used the fact that  $t \in \mathbb{R}$  and  $\overline{\Gamma(z)} = \Gamma(\bar{z})$  for  $z \in \mathbb{C}$ . Thus, (B.3) reduces to

$$777 \quad \hat{f}(t) = \frac{4^n \pi}{(2n)!} \left| \Gamma\left(n + \frac{it}{2}\right) \right|^2 \geq 0$$

778 for any  $t \in \mathbb{R}$ . If  $a$  is even with  $a = 2n$ , a similar calculation using Lemma B.1 yields

$$779 \quad \hat{f}(t) = \frac{4^n n \pi}{(2n)!} \left| \Gamma\left(n - \frac{1}{2} + \frac{it}{2}\right) \right|^2 \geq 0$$

780 for any  $t \in \mathbb{R}$ , which completes the proof. ■