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# Invariant kernels on Riemannian symmetric spaces : a harmonic-analytic approach

Nathael Da Costa\*, Cyrus Mostajeran\*, Juan-Pablo Ortega\*, and Salem Said†

**Abstract.** This work aims to prove that the classical Gaussian kernel, when defined on a non-Euclidean symmetric space, is never positive-definite for any choice of parameter. To achieve this goal, the paper develops new geometric and analytical arguments. These provide a rigorous characterization of the positive-definiteness of the Gaussian kernel, which is complete but for a limited number of scenarios in low dimensions that are treated by numerical computations. Chief among these results are the  $L^p$ -Godement theorems (where  $p = 1, 2$ ), which provide verifiable necessary and sufficient conditions for a kernel defined on a symmetric space of non-compact type to be positive-definite. A celebrated theorem, sometimes called the Bochner-Godement theorem, already gives such conditions and is far more general in its scope, but is especially hard to apply. Beyond the connection with the Gaussian kernel, the new results in this work lay out a blueprint for the study of invariant kernels on symmetric spaces, bringing forth specific harmonic analysis tools that suggest many future applications.

**Key words.** Riemannian manifold, symmetric space, harmonic analysis, Bochner's theorem, Godement theorem, positive-definite kernel

**MSC codes.** 43A35, 43A85, 43A90, 46E22, 53C35, 53Z50

**1. Introduction.** Positive-definite functions play a fundamental role in harmonic analysis and the theory of group representations, as well as in probability and statistical inference [22, 28, 43, 44]. In machine learning, these functions are of paramount importance in the context of kernel-based methods. Indeed, kernels are most often required to be positive-definite for Reproducing Kernel Hilbert Space (RKHS) methods and associated linear geometry and classification algorithms to apply [6, 38]. While many positive-definite kernels are known in Euclidean spaces, this is not yet true for non-Euclidean metric spaces, such as non-trivial Riemannian manifolds. Even when such kernels are known, they are far from easy to work with (in the first place, to evaluate). For example, just one evaluation of the heat kernel on a Riemannian symmetric space may require an elaborate Monte Carlo scheme [1, 2].

This being the case, one hopes to develop a tractable means of constructing closed-form positive-definite kernels on Riemannian manifolds or at least to develop an explicit criterion to verify whether a given closed-form expression yields a positive-definite kernel or not. This should apply, in particular, to the so-called distance kernels, which are of the form  $k(x, y) = g(d(x, y))$ , with  $g$  a suitable function and  $d(x, y)$  the Riemannian distance.

The results developed in this work serve exactly this purpose. They are motivated by the study of the Gaussian kernel

$$(1.1) \quad k(x, y) = \exp(-\lambda d^2(x, y)) \quad \text{where } \lambda > 0.$$

In particular, they aim to show that this kernel is never positive-definite (meaning it is not

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38 positive-definite for any value of  $\lambda$ ) when defined on a non-Euclidean Riemannian symmetric  
 39 space  $M$ . In general, when  $M$  is a non-Euclidean Riemannian manifold, it is known that  
 40 there always exists some  $\lambda$  such that the Gaussian kernel is not positive-definite [26, 16, 15].  
 41 However, whether there exists any subset of values of the parameter  $\lambda$  for which the Gaussian  
 42 kernel is positive-definite is an open problem of theoretical and practical significance as noted  
 43 in the literature [33]. For instance, [40] shows that the Gaussian kernel may be positive-  
 44 definite for some non-trivial collection of parameters in certain non-Euclidean geometries.  
 45 Recently, it has been shown that if  $M$  is compact and not simply connected (*e.g.* a torus or  
 46 a real projective space), then the Gaussian kernel is never positive-definite [29]. While these  
 47 general results rely on somewhat complicated geometric arguments, they will be considerably  
 48 extended here, in the case of symmetric spaces, thanks to the introduction of new analytical  
 49 tools.

50 A typical result of choice in this circle of ideas is Bochner’s theorem, one of the most  
 51 famous theorems in harmonic analysis [17]. It states that a function defined on a locally  
 52 compact Abelian group is positive-definite if and only if it is the inverse Fourier transform  
 53 of some positive measure. For example, this theorem guarantees the existence of a spectral  
 54 power measure for any wide-sense stationary signal (Wiener-Khinchin theorem).

55 The generalization of Bochner’s theorem to symmetric spaces is due to Godement [19]. It  
 56 will be referred to as Godement’s theorem in this work, although it is sometimes called the  
 57 Bochner-Godement theorem elsewhere. This theorem is recalled in Section 2, where we explain  
 58 why it can be challenging to apply in practice. Focusing on non-compact symmetric spaces,  
 59 the  $L^2$ -Godement theorem (Section 3) and the  $L^1$ -Godement theorem (Appendix A), both  
 60 state that in the presence of additional integrability assumptions, the difficulties inherent in  
 61 its application are greatly alleviated: it turns out that a function is positive-definite if and only  
 62 if its spherical transform is positive and integrable (the definition of the spherical transform  
 63 is recalled in Section 3).

64 Basic examples of applications of the  $L^2$ -Godement theorem are provided in Section 4.  
 65 Sections 5 and 6 then return to the Gaussian kernel (1.1). Section 5 proves that the Gaussian  
 66 kernel is never positive-definite on a compact symmetric space, expanding on [12] and [29],  
 67 which prove the corresponding result on the circle and nonsimply connected closed Riemannian  
 68 manifolds, respectively. Section 6 takes up the case of non-compact symmetric spaces (under  
 69 the general assumptions of Sections 2 and 3, compact or non-compact have the same meaning  
 70 as “of compact or non-compact type” — no Euclidean spaces are allowed).

71 Section 6 provides a rigorous proof that deals with the case of non-compact symmetric  
 72 spaces in all but a few limited scenarios in low dimensions, which are investigated compu-  
 73 tationally instead. First, in the case of the hyperbolic plane, a numerical evaluation shows  
 74 that the positivity condition in the  $L^2$ -Godement theorem does not hold, for any  $\lambda$  within  
 75 a certain continuous range, starting near  $\lambda = 0$ . However (Lemma 6.5), any non-compact  
 76 symmetric space contains an isometrically embedded hyperbolic plane, and this implies the  
 77 Gaussian kernel fails to be positive-definite on any non-compact symmetric space, for that  
 78 same range of values of  $\lambda$ . Section 6 also proves (with no recourse to numerical work) that  
 79 the Gaussian kernel is never positive-definite when defined on a three-dimensional hyperbolic  
 80 space. Again, this extends to any symmetric space that contains an isometrically embed-  
 81 ded hyperbolic space (*e.g.* the spaces of  $n \times n$  real, complex, or quaternion, positive-definite

82 matrices, with the requirement that  $n \geq 4$  for the real case).

83 Section 7 introduces an alternative to the Gaussian kernel (1.1) for hyperbolic spaces,  
84 giving it the name of hyperbolic secant or Herschel-Maxwell kernel. Once more, this is a  
85 distance kernel

$$86 \quad (1.2) \quad k(x, y) = (\cosh(d(x, y)))^{-a} \quad \text{where } a > 0.$$

87 This kernel is derived from first principles, similar to the famous Herschel-Maxwell derivation  
88 [27]. In contrast with the Gaussian kernel, it is shown to always be positive-definite on the  
89 hyperbolic plane and positive-definite on the hyperbolic space for integer values of  $a$ .

90 The  $L^p$ -Godement theorems (where  $p = 1, 2$ ) are most easily applied to distance kernels  
91 (for example, to the Gaussian or Herschel-Maxwell kernels). A detailed examination of their  
92 application to more general invariant kernels, which are not merely functions of distance, is  
93 one of the main tasks to be carried out in future work.

94 **2. Godement's theorem.** Let  $M$  be a set and let  $k : M \times M \rightarrow \mathbb{C}$ . Assume that  $k$  satisfies  
95

$$96 \quad (2.1) \quad \sum_{i,j=1}^N k(x_i, x_j) c_i \bar{c}_j \geq 0$$

97 for all positive integer  $N$ , where  $x_i \in M$ ,  $c_i \in \mathbb{C}$ , and the bar denotes complex conjugation.  
98 Then,  $k$  is said to be a positive-definite kernel on  $M$ . Without loss of generality, it is enough  
99 to consider Hermitian kernels, that is,  $k(x, y)$  is the conjugate of  $k(y, x)$ . In addition, assume  
100 that a group  $G$  acts transitively on  $M$ ; this action is denoted by  $g \cdot x$ , for  $g \in G$  and  $x \in M$ .  
101 If  $k$  satisfies that

$$102 \quad (2.2) \quad k(g \cdot x, g \cdot y) = k(x, y),$$

103 for all  $g \in G$  and  $x, y \in M$ , then  $k$  is said to be a  $G$ -invariant (or just an invariant) positive-  
104 definite kernel. Let  $o \in M$  and denote by  $H$  the stabiliser of  $o$  in  $G$ . A function  $f : M \rightarrow \mathbb{C}$  is  
105 called  $H$ -invariant if  $f(h \cdot x) = f(x)$  for all  $h \in H$  and  $x \in M$ . Using such a function, we can  
106 define  $k_f : M \times M \rightarrow \mathbb{C}$ ,

$$107 \quad (2.3) \quad k_f(x, y) = f(g_1^{-1} g_2 \cdot o) \text{ for } x = g_1 \cdot o \text{ and } y = g_2 \cdot o.$$

108 Indeed,  $H$ -invariance of  $f$  implies the right-hand side of (2.3) depends only on  $x$  and  $y$  (and not  
109 on the choice of  $g_1$  and  $g_2$ ). Now,  $k_f$  is  $G$ -invariant (it satisfies (2.2)). If  $k_f$  happens to be  
110 positive-definite (to satisfy (2.1)), then  $f$  is said to be a positive-definite function.

111 All invariant positive-definite kernels can be represented in this way, since (2.1) and (2.2)  
112 imply that  $k = k_f$ , where  $f(x) = k(o, x)$  and  $f$  is  $H$ -invariant and positive-definite. In  
113 particular, if  $k = k_f$ , then  $k$  is positive-definite if and only if  $f$  is positive-definite.

114 Note that the setup that we just presented implies that the set  $M$  is necessarily isomorphic  
115 to the homogeneous manifold  $G/H$ , where  $G$  is the group that acts transitively on  $M$  and  $H$   
116 the isotropy group of some element  $o \in M$ . Conversely, given any group  $G$  and an arbitrary  
117 subgroup  $H$ , we can cast the homogeneous space  $G/H$  in this framework by setting  $M = G/H$ ,

118  $o = eH$ , and considering the transitive left action of  $G$  on  $G/H$ . Given these observations,  
 119 in the sequel,  $M$  will be a Riemannian symmetric space and  $(G, H)$  a symmetric pair.  $G$  is  
 120 assumed to be semisimple, of either compact or non-compact type, and with finite center [24].  
 121 Godement's theorem generalizes Bochner's classical theorem to this setting [19]. Roughly  
 122 speaking, it states that an  $H$ -invariant continuous function  $f$  is positive-definite if and only if  
 123 it is a positive combination of positive-definite spherical functions. In [7], Godement's theorem  
 124 is called the Bochner-Godement theorem.

125 Recall that a spherical function  $\varphi : M \rightarrow \mathbb{C}$  is an  $H$ -invariant function, such that  $\varphi(o) = 1$   
 126 and  $\varphi$  is moreover a joint eigenfunction of all the  $G$ -invariant differential operators on  $M$  [22].  
 127 The set  $\Phi$  of positive-definite spherical functions on  $M$  is locally compact for the topology  
 128 of uniform convergence on compact subsets of  $M$  [19]. Godement's theorem states that an  
 129  $H$ -invariant continuous  $f$  is positive-definite if and only if

$$130 \quad (2.4) \quad f(x) = \int_{\Phi} \varphi(x) \mu_f(d\varphi)$$

131 where  $\mu_f$  is a unique finite positive measure on  $\Phi$ .

132 It is complicated to use Godement's theorem to verify whether a given  $f$  is positive-  
 133 definite. When  $M$  is non-compact, the set  $\Phi$  may be challenging to determine [22] (Page 484),  
 134 and there is no straightforward way of determining  $\mu_f$  from  $f$ . The following section identifies  
 135 a special case of Godement's theorem where these issues do not arise.

136 **3.  $L^2$ -Godement theorem.** The idea is to combine Godement's theorem with the so-called  
 137 spherical transform. Roughly speaking, the spherical transform can be seen as a radial Fourier  
 138 transform for  $H$ -invariant functions [41, 42]. When  $M = \mathbb{R}^2$ , it boils down to a zero-order  
 139 Hankel transform.

140 Assume that  $M$  is a non-compact symmetric space. Pick an Iwasawa decomposition of  
 141 the Lie algebra  $\mathfrak{g}$  of  $G$ , say  $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{h}$  (where  $\mathfrak{n}$  is nilpotent,  $\mathfrak{a}$  is Abelian, and  $\mathfrak{h}$  is the Lie  
 142 algebra of  $H$  [24]). For each  $g \in G$ ,  $g = n(g) \exp(a(g)) h(g)$ , where  $n(g) \in N$ ,  $a(g) \in \mathfrak{a}$  and  
 143  $h(g) \in H$  are unique ( $N$  is the Lie subgroup of  $G$  with Lie algebra  $\mathfrak{n}$ ).

144 For each  $t \in \mathfrak{a}^*$  (the dual of  $\mathfrak{a}$  as a real vector space), there is a spherical function  
 145  $\varphi_t : M \rightarrow \mathbb{C}$ , given by the following integral formula [22],

$$146 \quad (3.1) \quad \varphi_t(g \cdot o) = \int_H \exp[(it + \rho)(a(hg))] dh \quad \text{for } g \in G.$$

147 Here,  $\rho$  is half the sum of the positive roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ , and  $dh$  denotes the  
 148 normalized Haar measure on  $H$  (a gentle introduction to these concepts is provided in [41, 42]).  
 149 Define the spherical transform of an  $H$ -invariant  $f : M \rightarrow \mathbb{C}$  as  $\hat{f} : \mathfrak{a}^* \rightarrow \mathbb{C}$ ,

$$150 \quad (3.2) \quad \hat{f}(t) = \int_M f(x) \varphi_{-t}(x) \text{vol}(dx)$$

151 where  $\text{vol}$  denotes the volume measure of  $M$ . This is well-defined when  $f$  is continuous and  
 152 compactly supported, in which case it admits an inversion formula,

$$153 \quad (3.3) \quad f(x) = c_M \int_{\mathfrak{a}^*} \hat{f}(t) \varphi_t(x) |c(t)|^{-2} dt$$

154 where  $c(t)$  is a certain function, called the Harish-Chandra function (for an overview, see [23]),  
 155  $dt$  denotes the Lebesgue measure, and  $c_M$  is a constant that is independent of the function  $f$ .

156 Defined as above, the spherical transform extends to the space  $L^2_H(M)$  of  $H$ -invariant  
 157 square-integrable functions (with respect to  $\text{vol}$ ) and satisfies the Plancherel formula

$$158 \quad (3.4) \quad \int_M |f(x)|^2 \text{vol}(dx) = c_M \int_{\mathfrak{a}^*} |\hat{f}(t)|^2 |c(t)|^{-2} dt$$

159 for any  $f \in L^2_H(M)$ . Here, the following proposition will be called the  $L^2$ -Godement theorem.  
 160 It is a special case of Godement's theorem, which holds for square-integrable functions.

161 **Theorem 3.1 ( $L^2$ -Godement Theorem).** *A continuous function  $f \in L^2_H(M)$  is positive-*  
 162 *definite if and only if  $\hat{f}(t) \geq 0$  for almost all  $t \in \mathfrak{a}^*$  and  $\hat{f}$  is integrable with respect to the*  
 163 *measure  $|c(t)|^{-2} dt$ .*

164 The connection with Godement's theorem, from the previous Section 2, comes from the fact  
 165 that all of the functions  $\varphi_t$  in (3.1) are positive-definite [22] (Page 484). The set of  $\varphi_t$  where  
 166  $t \in \mathfrak{a}^*$  is therefore a subset of  $\Phi$  (in fact, it is a proper subset). The proof of Theorem 3.1 is  
 167 given in Section 8.1, where a second alternative proof is also discussed.

168 The following section spells out the spherical transform pair (3.2)–(3.3) in two basic cases,  
 169 where  $M$  is a hyperbolic plane or a hyperbolic space of dimension equal to three, and provides  
 170 two examples of applications of Theorem 3.1.

## 171 4. Basic examples.

172 **4.1. Hyperbolic plane.**  $M$  is a two-dimensional simply connected space with constant  
 173 negative curvature equal to  $-1$ . One may take  $G = SL(2, \mathbb{R})$  and  $H = SO(2)$  (in the usual  
 174 notation,  $SL(2, \mathbb{R}) = 2 \times 2$  real unit-determinant matrices,  $SO(2) = 2 \times 2$  rotation matrices).  
 175 A detailed exposition may be found in [41]. For any  $H$ -invariant function  $f : M \rightarrow \mathbb{C}$  there  
 176 exists  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ , with  $f(x) = g(r)$  where  $r = d(o, x)$ , the distance in  $M$  between  $x$  and  
 177 the “point of origin”  $o$  (the choice of this point is arbitrary). In particular, the spherical  
 178 functions (3.1) are given in terms of Legendre functions [22, 41],

$$179 \quad (4.1) \quad \varphi_t(x) = P_{-\frac{1}{2}+it}(\cosh(r))$$

180 for  $t \in \mathbb{R}$  (with regard to (3.1),  $\mathfrak{a}$  is one-dimensional, and therefore  $\mathfrak{a} \simeq \mathfrak{a}^* \simeq \mathbb{R}$ . Accordingly,  
 181  $t \in \mathfrak{a}^*$  becomes  $t \in \mathbb{R}$ ), and it should be noted that  $\varphi_t = \varphi_{-t}$  (see [41] (Page 141)).

182 The spherical transform pair (3.2)–(3.3) can be expressed as follows (see [41] (Page 149))

$$183 \quad (4.2) \quad \hat{f}(t) = 2\pi \int_0^\infty g(r) P_{-\frac{1}{2}+it}(\cosh(r)) \sinh(r) dr,$$

$$184 \quad (4.3) \quad g(r) = \frac{1}{2\pi} \int_0^\infty \hat{f}(t) P_{-\frac{1}{2}+it}(\cosh(r)) \alpha(t) dt,$$

185 where  $f(x) = g(r)$  as above, and  $\alpha(t) = t \tanh(\pi t)$ . As pointed out in [41], (4.2)–(4.3) is  
 186 known as a Mehler-Fock transform pair in the mathematical physics literature.

187 **4.2. Hyperbolic space.**  $M$  is now of dimension 3 instead of 2. One may take  $G = SL(2, \mathbb{C})$   
 188 and  $H = SU(2)$  (in other words,  $SL(2, \mathbb{C}) = 2 \times 2$  complex unit determinant matrices, and  
 189  $SU(2) =$  unitary matrices in  $SL(2, \mathbb{C})$ ). As before, any  $H$ -invariant function  $f$  is of the form  
 190  $f(x) = g(r)$ . The spherical functions (3.1) are given by [22] (Page 432)

$$191 \quad (4.4) \quad \varphi_t(x) = \frac{\sin(tr)}{t \sinh(r)}$$

192 Again,  $\varphi_t = \varphi_{-t}$ . The spherical transform pair (3.2)–(3.3) is essentially a sine transform pair,

$$193 \quad (4.5) \quad \hat{f}(t) = \frac{4\pi}{t} \int_0^\infty g(r) \sinh(r) \sin(tr) dr$$

$$194 \quad (4.6) \quad g(r) \sinh(r) = \frac{1}{2\pi^2} \int_0^\infty t \hat{f}(t) \sin(tr) dt$$

195 as can be seen using (4.4) and computing the integral (3.2) in geodesic spherical coordinates.

196 **4.3. Distance kernels.** For the hyperbolic plane and space that we studied in the previous  
 197 examples, any  $H$ -invariant function  $f$  is of the form  $f(x) = g(d(o, x))$ . It follows from (2.3)  
 198 that  $k_f(x, y) = g(d(x, y))$ . But, as stated in Section 2, any invariant kernel  $k$  is of the form  
 199  $k = k_f$  and it is hence a distance kernel

$$200 \quad (4.7) \quad k(x, y) = g(d(x, y)),$$

201 for some  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ . The positive-definiteness of  $k$  implies that  $g$  is necessarily real-valued.  
 202 In other words, any positive-definite kernel is a real-valued distance kernel. This statement  
 203 holds whenever  $M$  is a rank-one symmetric space. This class of symmetric spaces includes  
 204 spheres, hyperbolic spaces, projective and hyperbolic projective spaces of any dimension [24].

**4.3.1. Wishart kernel.** Let  $M$  be the hyperbolic plane, and let  $g(r) = \exp(-2a \cosh(r))$ ,  
 where  $a > 0$ , in (4.7). It is well known that  $M$  can be identified with the space of  $2 \times 2$   
 unit-determinant positive-definite matrices [41]. Thinking of  $x \in M$  as a matrix of this kind,  
 and choosing  $o$  the identity matrix,  $d(o, x) = r$  implies the trace of  $x$  is  $\text{tr}(x) = 2 \cosh(r)$  [41]  
 (Page 149, Exercice 20). Thus, by (4.7)

$$k(o, x) = \exp(-a \text{tr}(x))$$

Recalling the action  $g \cdot x = gxg^\dagger$  (a matrix product, with  $\dagger$  the transpose [41]) and  $o =$  identity,  
 and setting  $x = g_1 \cdot o$  and  $y = g_2 \cdot o$ , it follows from (2.2) that

$$k(x, y) = k(o, g_1^{-1} g_2 \cdot o) = \exp(-a \text{tr}(x^{-1}y))$$

205 This is the unnormalized density of a Wishart distribution and  $k$  is the Wishart kernel. Now,  
 206 the  $L^2$ -Godement theorem can be used to show that this kernel is not positive-definite, for any  
 207 value of  $a$ . The spherical transform  $\hat{f}(t)$  is given in [41] (Page 151),

$$208 \quad (4.8) \quad g(r) = \exp(-2a \cosh(r)) \text{ in (4.2)} \implies \hat{f}(t) = \left(\frac{4\pi}{a}\right)^{\frac{1}{2}} K_{\text{it}}(2a)$$



209 where  $K_{it}$  is a modified Bessel function of the second kind with imaginary order. For fixed  
 210  $a$ ,  $K_{it}(2a)$  has infinitely many simple zeros in  $t$  [3]. In particular,  $\hat{f}(t)$  does not satisfy the  
 211 positivity condition in the  $L^2$ -Godement theorem (Theorem 3.1), so  $k$  is not positive-definite  
 212 (the theorem can be applied because  $f(x) = k(o, x)$  is continuous and in  $L^2_H(M)$  for any  $a$ ).

213 Furthermore, since the Wishart kernel is never positive-definite on the hyperbolic plane,  
 214 and since a hyperbolic space contains infinitely many isometric copies of the hyperbolic plane,  
 215 the Wishart kernel is never positive-definite on the hyperbolic space. This statement can be  
 216 made rigorous using the embedding lemma below (Lemma 5.2).

**4.3.2. Hyperbolic secant kernel.** Let  $M$  be the hyperbolic plane and  $g(r) = (\cosh(r))^{-a}$   
 where  $a > 1/2$ . Then,  $f(x) = g(d(o, x))$  is clearly continuous, and  $f \in L^2_H(M)$ , because

$$\int_M |f(x)|^2 \text{vol}(dx) = 2\pi \int_0^\infty (\cosh(r))^{-2a} \sinh(r) dr = 2\pi \int_1^\infty u^{-2a} du < \infty$$

217 as follows by integrating in geodesic spherical coordinates. From [41] (Page 151)

$$218 \quad (4.9) \quad g(r) = (\cosh(r))^{-a} \text{ in (4.2)} \implies \hat{f}(t) = 2^{a-1} \frac{\sqrt{\pi}}{\Gamma(a)} |\Gamma((a - 1/2 + it)/2)|^2$$

which is positive for all  $t$  ( $\Gamma$  is the Gamma function). Now, by the  $L^2$ -Godement theorem, the  
 hyperbolic secant kernel

$$k(x, y) = (\cosh(d(x, y)))^{-a}$$

219 is positive-definite on the hyperbolic plane for any  $a > 1/2$ . This example of a positive-definite  
 220 kernel on the hyperbolic plane does not seem to be known in the literature. We believe it is  
 221 reasonable to name it the hyperbolic secant kernel.

222 The case where  $M$  is the hyperbolic space, rather than the hyperbolic plane, is investigated  
 223 in Appendix B. Positive-definiteness seems to be, once more, the general rule.

224 **5. Compact symmetric spaces.** This section considers the case in which  $M$  is a compact  
 225 symmetric space (for example, a sphere, a projective space, or a Grassmannian). The following  
 226 proposition will now be proved.

227 **Proposition 5.1.** *Let  $M$  be a compact symmetric space. The Gaussian kernel (1.1) is not*  
 228 *positive-definite on  $M$ , for any value of  $\lambda$ .*

229 In other words, the Gaussian kernel is never positive-definite on a compact symmetric space.  
 230 The case where  $M$  is not simply connected (real projective space or real Grassmannian, etc)  
 231 is already taken care of in [29]. The case where  $M$  is simply connected will now be proved.  
 232 First, we prove the following result, referred to as the Embedding Lemma.

233 **Lemma 5.2 (Embedding Lemma).** *Assume  $M_o$  and  $M$  are metric spaces, with  $M_o$  isomet-*  
 234 *rically embedded in  $M$ . Let  $k$  be a distance kernel (i.e. a kernel of the form (4.7)). If  $k$  is*  
 235 *positive-definite on  $M$ , then it is positive-definite on  $M_o$ .*

236 Here, isometric embedding means that the distance function in  $M_o$  is the restriction of the  
 237 distance function in  $M$  (this is the isometric embedding of metric spaces, a stronger property  
 238 than the isometric embedding of Riemannian manifolds). Due to (4.7), this implies the kernel



239  $k$  on  $M_o$  is just the restriction of the same kernel  $k$  from  $M$ . The fact that positive-definiteness  
 240 is hereditary from  $M$  to  $M_o$  is immediate from (2.1).

241 A corollary of the embedding lemma is that the Gaussian kernel (1.1) cannot be positive-  
 242 definite on  $M$  if it is not positive-definite on  $M_o$ .

243 **Lemma 5.3.** *Any compact simply connected symmetric space  $M$  contains an isometrically*  
 244 *embedded circle  $\sigma$  (the embedding being an isometric embedding of metric spaces).*

245 *Proof.* Assume that  $M$  is an irreducible symmetric space [24, Chapter VIII]. Denote  $\kappa^2$   
 246 the maximum of the sectional curvatures of  $M$ . Then,  $M$  admits a simple periodic geodesic  
 247  $\sigma$  of length  $2\pi/\kappa$  [24] (Page 334). Let  $x$  be a point on  $\sigma$  and choose a parameterisation  
 248  $\sigma : [0, 2\pi] \rightarrow M$  with  $\sigma(0) = x$ . Then, let  $\sigma(\pi) = x'$ .

249 It can be show that  $\sigma$  is length-minimising from  $x$  to  $x'$  (when restricted to  $[0, \pi]$ ) and  
 250 also from  $x'$  to  $x$  (when restricted to  $[\pi, 2\pi]$ ). Recall that the cut-locus of  $x$  in  $M$  is identical  
 251 to its first conjugate locus [11]. By definition of  $\kappa$ ,  $x$  has no conjugate points along  $\sigma$  before  
 252  $\sigma(\pi) = x'$  [9, Morse-Schönberg Theorem, Page 86]. Thus  $\sigma$  is length-minimising from  $x$  to  $x'$ .  
 253 An identical argument shows  $\sigma$  is also length-minimising from  $x'$  to  $x$ .

254 Denote  $\sigma = \sigma([0, 2\pi])$  (the image of the geodesic curve  $\sigma$  in  $M$ ). This is the isometrically  
 255 embedded circle announced in Lemma 5.3. To see this, choose any  $y \in \sigma$ ,  $y = \sigma(t)$ . If  
 256  $t \in [0, \pi]$  the distance in  $M$  between  $x$  and  $y$  is  $d(x, y) = t$ . On the other hand, if  $t \in [\pi, 2\pi]$ ,  
 257  $d(x, y) = 2\pi - t$  (note both of these claims result from the length-minimising property of  $\sigma$ ,  
 258 just obtained). In other words, the restriction of the distance function from  $M$  to  $\sigma$  is the same  
 259 as the distance function of a circle (identified with the interval  $[0, 2\pi]$  with glued endpoints).

260 The assumption that  $M$  is irreducible can now be removed since any simply connected  
 261 symmetric space is a direct product of irreducible symmetric spaces [24, Page 381]. ■

262 We recall now (see [12]) that the Gaussian kernel is never positive-definite on the circle.  
 263 With this in mind, Lemmas 5.2 and 5.3 yield Proposition 5.1.

264 Even though it should be possible to prove Proposition 5.1 using an analytic approach  
 265 via Godement's theorem, the geometric approach used above is more elementary and easier  
 266 to visualise.

267 **6. Non-compact symmetric spaces.** In this section,  $M$  will be a non-compact symmetric  
 268 space, as in Section 3 (for example, a hyperbolic space, a space of positive-definite matrices,  
 269 *etc*). Using a mixture of numerical and analytical results, we show that the Gaussian kernel  
 270 systematically fails to be positive-definite on  $M$ . First, the  $L^2$ -Godement theorem is applied  
 271 to the two basic cases where  $M$  is a hyperbolic plane or a three-dimensional hyperbolic space.  
 272 Specifically, from (1.1), the Gaussian kernel corresponds to  $g(r) = e^{-\lambda r^2}$ , and upon replacing  
 273 this into (4.2) and (4.5), the following observations are made:

- 274 • In both cases, the spherical transform  $\hat{f}(t)$  is positive when  $t$  is small in comparison  
 275 to  $\lambda$  (Proposition 6.1, and the remark after Formula (6.1)).
- 276 • In (4.2), a numerical evaluation shows that  $\hat{f}(t)$  oscillates between positive and negative  
 277 values, for any  $\lambda$  within a certain continuous range, starting near  $\lambda = 0$  (Figure 1).
- 278 • In (4.5),  $\hat{f}(t)$  has the same sign as  $\sin(t/2\lambda)$ . The Gaussian kernel is, therefore, never  
 279 positive-definite on the hyperbolic space (Proposition 6.3).

280 Second, the general case uses the Embedding Lemma 5.2.

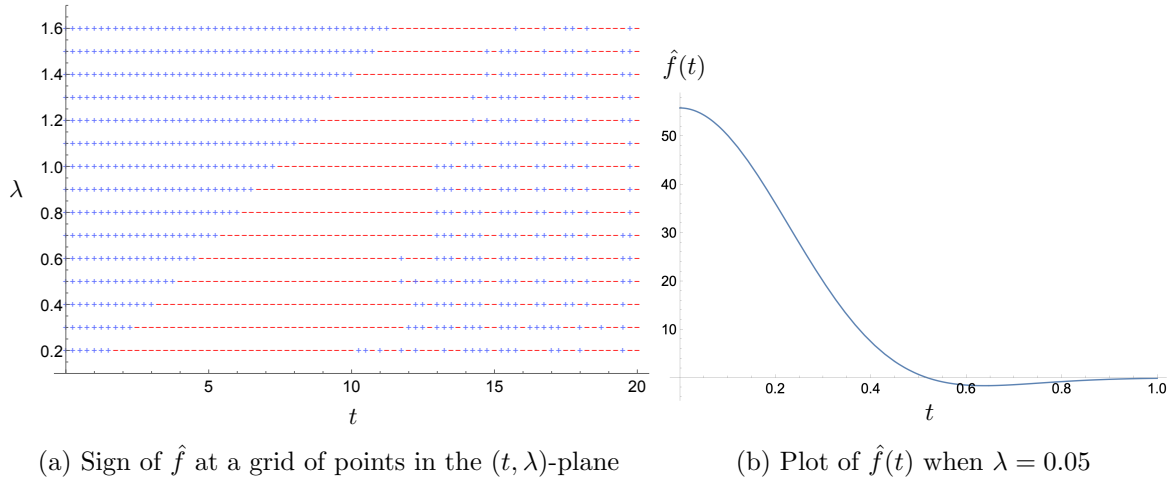


Figure 1: Spherical transform for the Gaussian kernel on the hyperbolic plane

281 **6.1. Hyperbolic plane.** For the Gaussian kernel on the hyperbolic plane, the behavior of  
 282 the spherical transform  $\hat{f}(t)$ , when  $t$  is small in comparison to  $\lambda$ , is the following.

283 **Proposition 6.1.** *Let  $g(r) = e^{-\lambda r^2}$  in (4.2). For any  $T > 0$ , there exists  $\lambda_T > 0$  such that*  
 284  *$\lambda > \lambda_T$  implies  $\hat{f}(t) \geq 0$  for all  $t \leq T$ .*

285 The proof of Proposition 6.1 is given in Section 8.2. The behavior of  $\hat{f}(t)$  for larger values of  
 286  $t$  was investigated numerically, with the results reported in Figure 1.

287 Figure 1a shows the sign of  $\hat{f}$  in the  $(t, \lambda)$ -plane. The blue (positive) area on the left  
 288 confirms Proposition 6.1. It is followed to the right by an alternation of red and blue areas,  
 289 which shows that  $\hat{f}(t)$  oscillates between positive and negative values (note that the  $\lambda$  axis  
 290 begins at  $\lambda = 0.2$ , since the numerical evaluation of the integral (4.2) becomes unstable for  
 291 much smaller values of  $\lambda$ ). Figure 1b shows an individual plot of  $\hat{f}(t)$ , obtained for a fixed  
 292  $\lambda = 0.05$ .

293 It is clear from Figure 1a that the  $L^2$ -Godement theorem implies the Gaussian kernel is  
 294 not positive-definite on the hyperbolic plane, for any  $\lambda$  in the range of the  $\lambda$ -axis of this figure.

295 **Remark 6.2.** Proposition 6.1 continues to hold for a general non-compact  $M$ , as above.  
 296 The proof of this general statement is not provided in order to maintain a reasonable length.

297 **6.2. Hyperbolic space.** In this case,  $\hat{f}(t)$  can be found in closed form.

$$298 \quad (6.1) \quad g(r) = e^{-\lambda r^2} \text{ in (4.5)} \implies \hat{f}(t) = \left(\frac{2\pi}{t}\right) \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}} e^{(1-t^2)/4\lambda} \sin\left(\frac{t}{2\lambda}\right)$$

299 This shows that the sign of  $\hat{f}(t)$  is determined by  $\sin(t/2\lambda)$ . This remains positive for  $t \leq 2\pi\lambda$ ,  
 300 and then oscillates indefinitely. From the  $L^2$ -Godement theorem, the following is immediate.

301 **Proposition 6.3.** *The Gaussian kernel (1.1) is not positive-definite on the hyperbolic space*  
 302 *for any value of  $\lambda$ .*

303 The expression (6.1) can be obtained from the following general lemma, proved in Section 8.3.

304 **Lemma 6.4.** *In (4.5), assume that  $g(r)$  is an even function. Then,*

305 (6.2) 
$$\hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{G(it + 1) - G(it - 1)\}$$

where  $\operatorname{Im}$  denotes the imaginary part and  $G$  is the moment-generating function

$$G(s) = \int_{-\infty}^{+\infty} g(r) e^{sr} dr$$

Indeed, (6.1) follows from (6.2) using the expression of the moment-generating function of a univariate normal distribution,

$$G(s) = \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}} e^{s^2/4\lambda}$$

306 **6.3. The general case.** Let  $M$  be any non-compact symmetric space, as in Section 3  
 307 (so the isometry group  $G$  of  $M$  is semisimple, of non-compact type, and has finite center [24]).  
 308 The following lemma will be proved in Section 8.4.

309 **Lemma 6.5.** *Under the assumptions just stated,  $M$  contains an isometrically embedded*  
 310 *hyperbolic plane.*

311 The numerical results of Paragraph 6.1 show that the Gaussian kernel is not positive-definite  
 312 on the hyperbolic plane, at least for any  $\lambda$  in the range appearing in Figure 1a. The Embedding  
 313 Lemma 5.2 and Lemma 6.5 therefore imply that the Gaussian kernel is not positive-definite  
 314 on any non-compact Riemannian symmetric space, for  $\lambda$  in this same range (the additional  
 315 assumptions of Lemma 6.5 merely serve to prevent  $M$  from being Euclidean).

316 On the other hand, Proposition 6.3 shows the Gaussian kernel is never positive-definite  
 317 on the hyperbolic space. By the Embedding Lemma, it is therefore never positive-definite on  
 318 symmetric spaces that contain an isometrically embedded hyperbolic space. Examples of such  
 319 symmetric spaces include all spaces of  $n \times n$  real, complex, or quaternion, positive-definite  
 320 matrices (with the requirement that  $n \geq 4$  for the real case) [30].

321 Hopefully, the numerical treatment of the hyperbolic plane case in Paragraph 6.1 will soon  
 322 be replaced with a complete analytical treatment, similar to the one that we conducted for  
 323 the hyperbolic space case in Paragraph 6.2. This is the only step still missing to obtain a  
 324 fully analytical proof of the statement that the Gaussian kernel is never positive-definite on  
 325 any non-Euclidean symmetric space.

326 **7. The hyperbolic secant revisited: non-Euclidean Herschel-Maxwell distributions.** Re-  
 327 call the hyperbolic secant kernel on the hyperbolic plane from Section 4.3.2, which was shown  
 328 to be positive-definite in contrast to the Gaussian kernel. Here, we will present a new inter-  
 329 pretation of the hyperbolic secant kernel as the Herschel-Maxwell kernel on the hyperbolic  
 330 plane. In Euclidean space, the Herschel-Maxwell theorem is an elegant characterization of  
 331 the Gaussian distribution originally due the astronomer J.F.W. Herschel who was studying  
 332 two-dimensional errors in astronomical observations. Ten years later, James Clerk Maxwell  
 333 presented a three-dimensional version of the same argument to show that the stationary prob-  
 334 ability distribution of the velocities of molecules in a gas follows a Gaussian distribution under

335 the assumptions that (P1) the components of the random vector in an orthogonal coordinate  
 336 system are independent, and (P2) the distribution only depends on the magnitude of the  
 337 vector [21, 27, 31]. A modern statement of the theorem is that if the distribution of a random  
 338 vector with independent components is invariant under rotations, then the components must  
 339 be identically distributed as a Gaussian distribution.

340 Here, we will follow the Herschel-Maxwell derivation as it appears in Jaynes [27], which  
 341 relies on the assumption of the existence of densities of the components of the random variables  
 342 and solving a functional equation. This derivation follows closely Maxwell's own argument.  
 343 Let  $(u, v)$  be a Cartesian coordinate system in the Euclidean plane. We seek to determine a  
 344 joint probability distribution  $p(u, v)dudv$  that satisfies assumptions P1 and P2. By assumption  
 345 P1, we have

$$346 \quad (7.1) \quad p(u, v)dudv = h(u)du h(v)dv$$

347 for some function  $h$ . By assumption P2, we have

$$348 \quad (7.2) \quad p(u, v)dudv = g(r)rdrd\theta$$

349 as the density cannot depend on the angle  $\theta$  when expressed in polar coordinates  $(r, \theta)$ . (7.1)  
 350 and (7.2) combine to give the functional equation

$$351 \quad (7.3) \quad h(u)h(v) = g\left(\sqrt{u^2 + v^2}\right),$$

352 which admits the general solution

$$353 \quad (7.4) \quad h(u) = \sqrt{\frac{\lambda}{\pi}}e^{-\lambda u^2}, \quad p(u, v) = \frac{\lambda}{\pi}e^{-\lambda(u^2+v^2)},$$

354 for any  $\lambda > 0$  upon insisting that the distribution be normalized.

355 To replicate this derivation in the hyperbolic plane, we must first specify what we mean  
 356 by an 'orthogonal coordinate system' in the hyperbolic plane. The most natural choice for  
 357 such a coordinate system seems to be Lobachevsky coordinates, determined by specifying a  
 358 directed geodesic ( $u$ -axis) through the origin. The Lobachevsky coordinates of a point  $x$  are  
 359 then found by dropping a perpendicular to the  $u$ -axis. The signed distance from the foot of the  
 360 perpendicular to the origin is the  $u$ -coordinate of the point and the  $v$ -coordinate is the signed  
 361 distance along the perpendicular to the  $u$ -axis (with the distance taken to be positive on one  
 362 side of the  $u$ -axis and negative on the other). With respect to such a coordinate system, we  
 363 seek a joint probability distribution  $p(x)\text{vol}(dx) = p(u, v)\text{vol}(dx) = h(u)h(v)\text{vol}(dx)$ , where  
 364  $\text{vol}(dx)$  denotes the Riemannian volume form in the hyperbolic plane (P1). Moreover, (P2)  
 365 generalizes to the assumption that  $p(x)$  can only depend on the Riemannian distance to the  
 366 origin  $r = d(o, x)$ , which once again yields the functional equation  $h(u)h(v) = g(r)$ . As this  
 367 equation holds for all  $(u, v)$ , we can take any point on the  $u$ -axis to obtain  $g(u) = h(u)h(0)$ .  
 368 That is, we have the functional equation

$$369 \quad (7.5) \quad h(u)h(v) = h(0)h(r)$$

370 where  $(u, v)$  and  $r$  are related by

$$371 \quad (7.6) \quad \cosh(r) = \cosh(u) \cosh(v)$$

372 according to the hyperbolic Pythagorean theorem. Setting  $\tilde{h} = (h \circ \operatorname{arcosh})/h(0)$ , (7.5) be-  
 373 comes  $\tilde{h}(\cosh u)\tilde{h}(\cosh v) = \tilde{h}(\cosh u \cosh v)$ , which admits the general solution  $\tilde{h}(\cosh u) =$   
 374  $(\cosh u)^{-a}$ , for  $a \in \mathbb{R}$  (i.e.,  $\tilde{h}$  is a power function). Thus,

$$375 \quad h(r) = h(0)(\cosh(r))^{-a},$$

376 which yields the probability density

$$377 \quad p(x) \operatorname{vol}(dx) = h(0)^2 (\cosh(r))^{-a} \operatorname{vol}(dx) = h(0)^2 (\cosh(r))^{-a} \sinh(r) dr d\theta,$$

378 in geodesic polar coordinates. Note that this density is normalizable only if  $a > 1$ . The  
 379 constant  $h(0)$  can be determined through normalizing the distribution, which yields

$$380 \quad (7.7) \quad p(x) \operatorname{vol}(dx) = \frac{1}{\pi} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a-1}{2}\right)} (\cosh(d(o, x)))^{-a} \operatorname{vol}(dx).$$

381 We call this distribution the Herschel-Maxwell distribution in the hyperbolic plane. It can  
 382 be viewed as one of several possible generalizations of the Gaussian distribution centered at  
 383 the point  $o$  to the hyperbolic plane alongside the distinct distributions derived from the heat  
 384 kernel interpretation of the Gaussian [20, 32] and

$$385 \quad (7.8) \quad p(x) \operatorname{vol}(dx) = \frac{1}{\pi} \sqrt{\frac{\lambda}{\pi}} \frac{e^{-\frac{1}{4\lambda}}}{\operatorname{erf}\left(\frac{1}{2\sqrt{\lambda}}\right)} e^{-\lambda d(o, x)^2} \operatorname{vol}(dx),$$

386 which inherits an important characteristic property of Euclidean Gaussian distributions from  
 387 statistical inference: the maximum likelihood estimate of the mean reduces to a (Riemannian)  
 388 center of mass computation [34, 35, 37, 36, 10]. Figure 2 compares plots of the density functions  
 389  $p(r) \operatorname{vol}(dx)$  in (7.7) and (7.8) as functions of  $r = d(o, x)$  for parameters  $a = 4$  and  $\lambda = 1.66$ .

390 Finally, we note that the hyperbolic Herschel-Maxwell distribution can be generalized  
 391 to  $n$ -dimensional hyperbolic space if we insist that the components of the point  $x$  in any  
 392 Lobachevsky coordinate system in *any* hyperbolic plane containing  $o$  and  $x$  are independent  
 393 and the distribution only depends on the geodesic distance  $r = d(o, x)$ . Reasoning as before,  
 394 we obtain

$$395 \quad (7.9) \quad p(x) \operatorname{vol}(dx) = \frac{1}{\omega_{n-1}} \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+1-n}{2}\right) \Gamma\left(\frac{n}{2}\right)} (\cosh(d(o, x)))^{-a} \operatorname{vol}(dx),$$

396 where  $\omega_{n-1}$  denotes the area of the unit sphere in  $\mathbb{R}^n$  and the normalization factor in (7.9) is  
 397 determined by integration with respect to the volume form  $\operatorname{vol}(dx)$  in  $n$ -dimensional hyperbolic  
 398 space. The distribution is well-defined for  $a > n - 1$ .

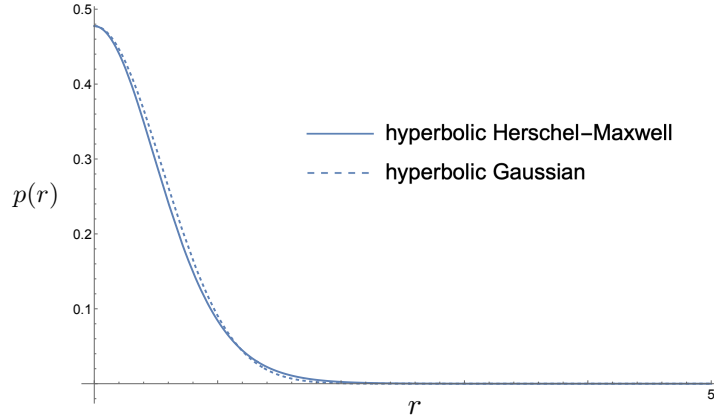


Figure 2: Plots of the normalized hyperbolic Herschel-Maxwell (solid curve, 7.7) and Gaussian (dashed curve, 7.8) densities as functions of  $r = d(o, x)$  for parameters  $a = 4$  and  $\lambda = 1.66$ .

399 *Remark 7.1.* We can repeat the derivation of the hyperbolic Herschel-Maxwell distribution  
 400 presented here in the case of spherical geometry with the main difference being the replacement  
 401 of the hyperbolic Pythagorean theorem with the spherical Pythagorean theorem

$$402 \quad (7.10) \quad \cos(\theta) = \cos(u) \cos(v)$$

403 where  $o, x$  denote points on the 2-sphere  $S^2$ ,  $\theta = d(o, x)$ , and  $(u, v)$  is an orthogonal coordinate  
 404 system centered on  $o$ , whereby the coordinates of a point  $x$  are obtained by dropping a  
 405 perpendicular to a specified geodesic through  $o$  and measuring the signed lengths  $u$  along the  
 406 geodesic from  $o$  to the foot of the perpendicular and  $v$  along the perpendicular to the geodesic  
 407 from  $x$ . This yields the spherical Herschel-Maxwell kernel

$$408 \quad (7.11) \quad k(x, y) = |\cos(d(x, y))|^a,$$

409 where  $a > 0$  to ensure the corresponding distribution is normalizable. Note that by represent-  
 410 ing  $x, y \in S^2$  by unit vectors  $w_x, w_y \in \mathbb{R}^3$ , we have  $\cos(d(x, y)) = \langle w_x, w_y \rangle$ , and thus (7.11)  
 411 takes the form  $k(x, y) = |\langle w_x, w_y \rangle|^a$ . That is, the spherical Herschel-Maxwell distribution can  
 412 be identified with a homogeneous polynomial kernel, which is known to be positive-definite if  
 413 and only if the power  $a$  is a positive integer [39].

## 414 8. Proofs of the main results.

### 415 8.1. Proof of Theorem 3.1.

416 **8.1.1. Notation.** The following notation will be used in the proof.

- 417 • Let  $W$  denote the Weyl group, which arises from the adjoint action of  $H$  on  $\mathfrak{a}$  [24].  
 418 Also, let  $\alpha$  denote the measure on  $\mathfrak{a}^*$ , which is given by  $\alpha(dt) = c_M |c(t)|^{-2} dt$ . Then,  
 419  $L_W^2(\mathfrak{a}^*)$  stands for the space of  $W$ -invariant square-integrable functions (with respect  
 420 to the measure  $\alpha$ ).
- 421 • Denote  $\Phi_*$  the set of functions  $\varphi_t$  in (3.1) where  $t \in \mathfrak{a}^*$ . This is a proper subset of the  
 422 set  $\Phi$  of all positive-definite spherical functions, which appears in Godement's theorem  
 423 (Section 2, in particular (2.4)).

- 424 • Functions or measures on  $\mathfrak{a}^*$  can be pulled back to  $\Phi_*$ , assuming they are  $W$ -invariant.  
 425 This is because  $\varphi_{t'} = \varphi_t$  if and only if  $t' = w \cdot t$  for some  $w \in W$ . Accordingly, the  
 426 measure  $\alpha$  on  $\mathfrak{a}^*$  can be identified with a measure on  $\Phi_*$ , also denoted  $\alpha$ . Similarly,  
 427 if  $\hat{f}(t)$  is a spherical transform, as in (3.2),  $\hat{f}(\varphi)$  is just  $\hat{f}(t)$  when  $\varphi \in \Phi_*$  and  $\varphi = \varphi_t$   
 428 for some  $t \in \mathfrak{a}^*$ .
- 429 • Finally,  $C_H^c(M)$  is the space of  $H$ -invariant continuous, compactly supported functions,  
 430 with complex values, defined on  $M$ .

431 **8.1.2. The if part.** For square-integrable functions, the spherical transform is only defined  
 432 in an abstract sense. It is a linear isometry between the Hilbert spaces  $L_H^2(M)$  and  $L_W^2(\mathfrak{a}^*)$ ,  
 433  $f \mapsto \hat{f}$  [22][42]. However, if  $\hat{f}$  is integrable, the inversion formula (3.3) holds

$$434 \quad (8.1) \quad f(x) = \int_{\mathfrak{a}^*} \hat{f}(t) \varphi_t(x) \alpha(dt) \quad \text{for all } x \in M$$

435 and this implies that  $f$  is continuous, since each  $\varphi_t$  is continuous and verifies  $|\varphi_t(x)| \leq 1$ ,  
 436 being a positive-definite function with  $\varphi_t(o) = 1$ . The fact that  $f$  is positive-definite follows  
 437 from Godement's theorem. Indeed, (8.1) can be written

$$438 \quad (8.2) \quad f(x) = \int_{\Phi_*} \varphi(x) (\hat{f}\alpha)(d\varphi)$$

439 where  $(\hat{f}\alpha)(d\varphi) = \hat{f}(\varphi) \alpha(d\varphi)$ . Then, since  $\hat{f}(\varphi) \geq 0$  for almost all  $\varphi \in \Phi_*$  and  $\hat{f}$  is integrable,  
 440 with respect to  $\alpha$ ,  $\hat{f}\alpha$  is a finite positive measure on  $\Phi_*$ . Moreover, since  $\Phi_*$  is a (measurable)  
 441 subset of  $\Phi$ , (8.2) is exactly of the form (2.4), but with the measure  $\mu_f$  supported on  $\Phi_*$ ,  
 442 where it is equal to  $\hat{f}\alpha$ . This shows that  $f$  is positive-definite, and even determines the finite  
 443 positive measure  $\mu_f$  (recall that  $\mu_f$  is unique).

444 **8.1.3. The only-if part.** If  $f$  is  $H$ -invariant, continuous and positive-definite, then one  
 445 has

$$446 \quad (8.3) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) \geq 0$$

447 for any  $\psi \in C_H^c(M)$ , where  $*$  denotes the convolution and  $\psi^\dagger(g \cdot o) = \bar{\psi}(g^{-1} \cdot o)$  [14] (Page  
 448 286). This yields a scalar product on  $C_H^c(M)$ ,

$$449 \quad (8.4) \quad \langle \psi_1, \psi_2 \rangle_f = \int_M (\psi_1 * \psi_2^\dagger)(x) f(x) \text{vol}(dx)$$

450 It will shortly be proved that this scalar product admits a spectral representation,

$$451 \quad (8.5) \quad \langle \psi_1, \psi_2 \rangle_f = \langle \hat{\psi}_1, \hat{\psi}_2 \rangle_{\hat{f}}$$

452 where the scalar product on the right-hand side is the usual “ $L^2$ ” scalar product, with respect  
 453 to the measure  $\hat{f}\alpha$  on  $\Phi_*$ . By [19] (Plancherel theorem, Page 101), this implies the measure  
 454  $\mu_f$  (in (2.4)) is supported on  $\Phi_*$ , where it is equal to  $\hat{f}\alpha$ . Since  $\mu_f$  is a finite positive measure,  
 455  $\hat{f}(t) \geq 0$  for almost all  $t$  and  $\hat{f}$  is integrable.



456 To prove (8.5), consider the left-hand side of (8.3). Both  $\psi$  and  $\psi^\dagger$  belong to  $C_H^c(M)$ .  
 457 The same is therefore true of their convolution. The spherical transform of  $\psi^\dagger$  being the  
 458 complex conjugate of the spherical transform of  $\psi$ , it follows by the convolution property of  
 459 spherical transforms [42] (Page 88)

$$460 \quad (8.6) \quad \widehat{\psi * \psi^\dagger}(t) = |\hat{\psi}(t)|^2$$

461 Since both  $f$  and  $\psi * \psi^\dagger$  are in  $L_H^2(M)$ , the Plancherel formula (3.4) yields

$$462 \quad (8.7) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 \hat{f}(t) \alpha(dt) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 (\hat{f}\alpha)(dt)$$

463 By (8.4), and by pulling back the integral of the right-hand side to  $\Phi_*$ , this is

$$464 \quad (8.8) \quad \langle \psi, \psi \rangle_f = \int_{\Phi_*} |\hat{\psi}(\varphi)|^2 (\hat{f}\alpha)(d\varphi) = \langle \hat{\psi}, \hat{\psi} \rangle_{\hat{f}}$$

465 so that (8.5) follows immediately, using the polarization identity for scalar products. ■

466 **8.1.4. Alternative proof of Theorem 3.1.** The only-if part of the proposition admits at  
 467 least one more proof. Indeed [14] (Page 302), for any continuous positive-definite  $f \in L_H^2(M)$ ,  
 468 it is possible to construct  $\phi \in L_H^2(M)$  such that  $f = \phi * \phi^\dagger$ . If one can apply the convolution  
 469 property as in (8.6), it would follow that  $\hat{f}(t) = |\hat{\phi}(t)|^2$ , so  $\hat{f}(t)$  is positive for all  $t$  (not just  
 470 for almost all  $t$ ), and  $\hat{f}$  is integrable.

471 The convolution property does not apply directly, because  $\phi$  is not integrable. Letting  
 472  $(\phi_n; n = 1, 2, \dots)$  belong to  $C_H^c(M)$  and converge to  $\phi$  in  $L_H^2(M)$ , the functions  $f_n = \phi_n * \tilde{\phi}_n$   
 473 belong to  $C_H^c(M)$  and converge uniformly to  $f$ , as can be shown by using Young's convolution  
 474 inequality [25] (Page 296).

475 Now, as in (8.6), one has  $\hat{f}_n(t) = |\hat{\phi}_n(t)|^2$  for each  $n$ . However, since  $\hat{\phi}_n$  converge to  $\hat{\phi}$   
 476 in  $L_W^2(\mathfrak{a}^*)$ , it follows that  $|\hat{\phi}_n|^2$  converge to  $|\hat{\phi}|^2$  in  $L_W^1(\mathfrak{a}^*)$  (this is the space of  $W$ -invariant  
 477 integrable functions) [8] (Page 289).

Therefore, a representation of  $f$  under the form (3.3) can be obtained by noting, for  $x \in M$ ,

$$\lim f_n(x) = \lim \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}_n(t)|^2 \alpha(dt) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}(t)|^2 \alpha(dt)$$

478 where the first equality follows from (3.3) and  $\hat{f}_n(t) = |\hat{\phi}_n(t)|^2$ , and the second equality from  
 479  $|\hat{\phi}_n|^2 \rightarrow |\hat{\phi}|^2$  in  $L_W^1(\mathfrak{a}^*)$  and  $|\varphi_t(x)| \leq 1$  (this is a consequence of  $\varphi_t$  being positive-definite).

However, the limit of  $f_n(x)$  is just  $f(x)$ , since the  $f_n$  converge uniformly to  $f$ . Thus,

$$f(x) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\phi}(t)|^2 \alpha(dt)$$

480 Comparing to (3.3), it follows that  $\hat{f}(t) = |\hat{\phi}(t)|^2$ , as required. ■

481 **8.2. Proof of Proposition 6.1.** Let  $R$  be such that  $r \leq R$  implies  $|\cosh(r) - 1| < 1$ .  
 482 Then [41] (Page 142)

$$483 \quad (8.9) \quad P_{-\frac{1}{2}+it}(\cosh(r)) = 1 - (1/4 + t^2)(r^2/4) + \varepsilon(t, r) \quad \text{for } r \leq R$$

484 with remainder  $|\varepsilon(t, r)| \leq C(T, R)r^4$  for all  $t \leq T$  and  $r \leq R$ , where  $C(T, R) > 0$  is a constant.  
 485 Replacing this into (4.2), the following decomposition is obtained,

$$486 \quad (8.10) \quad \hat{f}(t) = \hat{f}_a(t) + \hat{f}_b(t) + \hat{f}_c(t)$$

487 in terms of the following integrals

$$488 \quad (8.11) \quad \hat{f}_a(t) = \int_0^R e^{-\lambda r^2} [1 - (1/4 + t^2)(r^2/4)] \sinh(r) dr$$

$$489 \quad (8.12) \quad \hat{f}_b(t) = \int_0^R e^{-\lambda r^2} \varepsilon(t, r) \sinh(r) dr$$

$$490 \quad (8.13) \quad \hat{f}_c(t) = \int_R^\infty e^{-\lambda r^2} P_{-\frac{1}{2}+it}(\cosh(r)) \sinh(r) dr$$

491 Noting that  $|P_{-\frac{1}{2}+it}(\cosh(r))| \leq 1$  [41] (Page 161), a simple calculation yields

$$492 \quad (8.14) \quad |\hat{f}_c(t)| \leq \int_R^\infty e^{-\lambda r^2} \sinh(r) dr = O\left(e^{-\lambda R^2}\right)$$

On the other hand,

$$(C(T, R))^{-1} |\hat{f}_b(t)| \leq \int_0^R e^{-\lambda r^2} r^4 \sinh(r) dr \leq \int_0^\infty e^{-\lambda r^2} r^4 \sinh(r) dr$$

However, by performing an elementary change of variables, and then assuming that  $\lambda \geq 1$ , this implies the following bound,

$$(C(T, R))^{-1} |\hat{f}_b(t)| \leq \lambda^{-\frac{5}{2}} \int_0^\infty e^{-u^2} u^4 \sinh(u) du$$

493 which means that

$$494 \quad (8.15) \quad |\hat{f}_b(t)| = O\left(\lambda^{-\frac{5}{2}}\right)$$

495 Finally,  $\hat{f}_a(t)$  can be extended to an integral from 0 to  $\infty$ , with an exponentially small error,

$$496 \quad (8.16) \quad \hat{f}_a(t) = Z(\lambda) - (1/4 + t^2) V(\lambda) + O\left(e^{-\lambda R^2}\right)$$

where  $Z(\lambda)$  and  $V(\lambda)$  are the following integrals

$$Z(\lambda) = \int_0^\infty e^{-\lambda r^2} \sinh(r) dr \quad V(\lambda) = \frac{1}{4} \int_0^\infty e^{-\lambda r^2} r^2 \sinh(r) dr$$

497 By a direct evaluation

$$498 \quad (8.17) \quad Z(\lambda) = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \exp\left(\frac{1}{4\lambda}\right) \operatorname{erf}\left(\frac{1}{2\sqrt{\lambda}}\right) = \frac{\lambda^{-1}}{2} + \frac{\lambda^{-2}}{6} + O(\lambda^{-3})$$

499 where  $\operatorname{erf}$  denotes the error function [5] (Page 35), and then by noting that  $4V(\lambda) = -Z'(\lambda)$   
500 (the prime denotes derivation),

$$501 \quad (8.18) \quad V(\lambda) = \frac{\lambda^{-2}}{8} + O(\lambda^{-3})$$

502 Therefore, replacing (8.17) and (8.18) into (8.16),

$$503 \quad (8.19) \quad \hat{f}_a(t) = \frac{\lambda^{-1}}{2} - \left(t^2 - \frac{13}{12}\right) \frac{\lambda^{-2}}{8} + O(\lambda^{-3})$$

504 To conclude the proof, replace (8.14), (8.15) and (8.19) into (8.10). This yields

$$505 \quad (8.20) \quad \hat{f}(t) \sim \frac{\lambda^{-1}}{2} - \left(t^2 - \frac{13}{12}\right) \frac{\lambda^{-2}}{8} \quad \text{as } \lambda \rightarrow \infty$$

506 ( $a \sim b$  means  $a/b$  converges to 1), where the convergence is uniform in  $t \leq T$ . It is now enough  
507 to note that the right-hand side of (8.20) is positive for all  $t \leq T$  when  $\lambda \geq T^2/4$ . ■

508 **8.3. Proof of Lemma 6.4.** Note that  $\sinh(r) \sin(tr)$  is an even function of  $r$ . Moreover,  
509 since  $g(r)$  is also even, (4.5) can be written

$$510 \quad \hat{f}(t) = \frac{4\pi}{t} \int_0^{\infty} g(r) \sinh(r) \sin(tr) dr$$

$$511 \quad = \frac{2\pi}{t} \int_{-\infty}^{+\infty} g(r) \sinh(r) \sin(tr) dr$$

In other words,

$$\hat{f}(t) = \frac{2\pi}{t} \operatorname{Im} \left\{ \int_{-\infty}^{+\infty} g(r) \sinh(r) e^{itr} dr \right\}$$

Using the definition of the hyperbolic sine, this becomes

$$\hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \left\{ \int_{-\infty}^{+\infty} g(r) e^{(it+1)r} dr - \int_{-\infty}^{+\infty} g(r) e^{(it-1)r} dr \right\}$$

and then, by definition of the moment-generating function  $G$ ,

$$\hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{G(it+1) - G(it-1)\}$$

512 which is the same as (6.2). ■

513 **8.4. Proof of Lemma 6.5.** It will be enough to prove that there exists a totally-geodesic  
 514 submanifold  $M_o$  of  $M$ , such that  $M_o$  is a hyperbolic plane. Indeed,  $M$  is a Hadamard manifold  
 515 (a simply connected complete Riemannian manifold with negative sectional curvatures [24]  
 516 (Chapter VI)). Therefore, if  $M_o$  is a totally geodesic submanifold, it is also an isometrically  
 517 embedded metric space. The proof will rely very heavily on the algebraic description of  
 518 symmetric spaces, detailed in [24] (in particular Chapters V and VI).

519 Recall that  $M$  is determined by a symmetric pair  $(G, H)$ . The Lie algebra  $\mathfrak{g}$  of  $G$  is a  
 520 real semisimple Lie algebra, and admits a Cartan involution  $\theta$  with corresponding Cartan  
 521 decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$  ( $\mathfrak{h}$  the Lie algebra of  $H$ ). If  $B$  is the Killing form of  $\mathfrak{g}$ , then

$$522 \quad (8.21) \quad \langle \xi, \eta \rangle = -B(\xi, \theta(\eta)) \quad \xi, \eta \in \mathfrak{g}$$

523 defines a scalar product on  $\mathfrak{g}$ . Let  $\mathfrak{a}$  be a maximal Abelian subspace of  $\mathfrak{p}$ . The linear maps  
 524  $\text{ad}_a : \mathfrak{g} \rightarrow \mathfrak{g}$  where  $\text{ad}_a(\xi) = [a, \xi]$  for  $a \in \mathfrak{a}$  are jointly diagonalisable, with real eigenvalues.  
 525 These eigenvalues are linear forms  $\lambda : \mathfrak{a} \rightarrow \mathbb{R}$ , called the roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ .

526 Fix a non-zero root  $\lambda$ , and  $a_\lambda \in \mathfrak{a}$  such that  $\lambda(a) = B(a_\lambda, a)$  for  $a \in \mathfrak{a}$ . Then, let  $\xi_\lambda$  be a  
 527 joint eigenvector,  $\text{ad}_a(\xi_\lambda) = \lambda(a)\xi_\lambda$  for  $a \in \mathfrak{a}$ . Note that

$$528 \quad (8.22) \quad [a, \theta(\xi_\lambda)] = \theta([\theta(a), \xi_\lambda]) = -\theta([a, \xi_\lambda])$$

529 for any  $a \in \mathfrak{a}$ , where the first equality follows because  $\theta$  is an involution, and the second  
 530 because  $a \in \mathfrak{a}$ , a subspace of  $\mathfrak{p}$ , and  $\mathfrak{p}$  is by definition the  $(-1)$ -eigenspace of  $\theta$ . However,

$$531 \quad (8.23) \quad \theta([a, \xi_\lambda]) = \lambda(a)\theta(\xi_\lambda)$$

532 by the definition of  $\xi_\lambda$  and linearity of  $\theta$ . Thus, (8.22) and (8.23) imply that  $\text{ad}_a(\theta(\xi_\lambda)) =$   
 533  $-\lambda(a)\theta(\xi_\lambda)$  for any  $a \in \mathfrak{a}$ . Denoting  $\theta(\xi_\lambda)$  by  $\xi_{-\lambda}$  and recalling the definition of  $a_\lambda \in \mathfrak{a}$ , it is  
 534 now seen that the following commutation relations hold

$$535 \quad (8.24) \quad [a_\lambda, \xi_\lambda] = \|a_\lambda\|^2 \xi_\lambda \quad [a_\lambda, \xi_{-\lambda}] = -\|a_\lambda\|^2 \xi_{-\lambda}$$

536 where  $\|\cdot\|$  is the norm with respect to the scalar product (8.21). It will now be proved that

$$537 \quad (8.25) \quad [\xi_\lambda, \xi_{-\lambda}] = -\|\xi_\lambda\|^2 a_\lambda$$

538 First, any  $a \in \mathfrak{a}$  commutes with  $[\xi_\lambda, \xi_{-\lambda}]$ . Indeed,

$$539 \quad [a, [\xi_\lambda, \xi_{-\lambda}]] = [[a, \xi_\lambda], \xi_{-\lambda}] + [\xi_\lambda, [a, \xi_{-\lambda}]] \\ 540 \quad = \lambda(a)[\xi_\lambda, \xi_{-\lambda}] - \lambda(a)[\xi_\lambda, \xi_{-\lambda}]$$

541 where the first equality follows from Jacobi's identity for Lie brackets, and the second because  
 542  $\xi_\lambda$  and  $\xi_{-\lambda}$  are joint eigenvectors with eigenvalues  $\lambda(a)$  and  $-\lambda(a)$ . Second,  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{p}$ .  
 543 Indeed,  $\theta([\xi_\lambda, \xi_{-\lambda}]) = [\theta(\xi_\lambda), \theta(\xi_{-\lambda})]$ , but  $\theta(\xi_\lambda) = \xi_{-\lambda}$  by definition, and  $\theta(\xi_{-\lambda}) = \xi_\lambda$  since  $\theta$   
 544 is an involution. Therefore,  $\theta([\xi_\lambda, \xi_{-\lambda}]) = -[\xi_\lambda, \xi_{-\lambda}]$  which means  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{p}$ . However,  $\mathfrak{a}$   
 545 is maximal Abelian in  $\mathfrak{p}$ , and this yields  $[\xi_\lambda, \xi_{-\lambda}] \in \mathfrak{a}$ .

Finally, to obtain (8.25), note that

$$B([\xi_\lambda, \xi_{-\lambda}], a) = B(\xi_{-\lambda}, [a, \xi_\lambda]) = B(\xi_{-\lambda}, \xi_\lambda) \lambda(a)$$

where the first equality holds since  $B$  is ad-invariant, and the second since  $\xi_\lambda$  is a joint eigenvector with eigenvalue  $\lambda(a)$ . Now, the fact that  $B$  is symmetric and the definition of  $a_\lambda$  imply  $B([\xi_\lambda, \xi_{-\lambda}], a) = B(\xi_\lambda, \xi_{-\lambda}) B(a_\lambda, a)$ , so that

$$[\xi_\lambda, \xi_{-\lambda}] = B(\xi_\lambda, \xi_{-\lambda}) a_\lambda$$

546 and (8.25) follows after recalling (8.21) and  $\xi_{-\lambda} = \theta(\xi_\lambda)$ .

547 Moving on, let  $\mathfrak{g}_o$  denote the Lie subalgebra of  $\mathfrak{g}$ , generated by  $\{\xi_{-\lambda}, a_\lambda, \xi_\lambda\}$ . The Lie  
548 bracket of  $\mathfrak{g}_o$  is completely determined by (8.24) and (8.25). It is straightforward to see that  
549  $\mathfrak{g}_o \simeq \mathfrak{sl}(2, \mathbb{R})$  (the Lie algebra of  $2 \times 2$  zero-trace real matrices). The isomorphism is given by

$$550 \quad (8.26) \quad \xi_{-\lambda} \longleftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} ; \quad a_\lambda \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \xi_\lambda \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

551 after proper rescaling of  $\{\xi_{-\lambda}, a_\lambda, \xi_\lambda\}$  (this needs to ensure that  $\|a_\lambda\|^2 = 2$  and  $\|\xi_\lambda\|^2 = 1$ ).  
552 Let  $G_o$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{g}_o$  and let  $M_o$  be the orbit of  $o$   
553 under  $G_o$  (recall that  $o \in M$  has stabiliser  $H$  in  $G$  and can be thought of as the “point of  
554 origin”). Then,  $M_o$  is a totally geodesic submanifold of  $M$ , because  $\mathfrak{g}_o$  is a Lie subalgebra of  
555  $\mathfrak{g}$  [24] (Page 189). Moreover,  $M_o$  is a Riemannian homogeneous space under the action of  $G_o$ ,  
556  $M_o \simeq G_o/H_o$  where  $H_o = G_o \cap H$  is the stabiliser of  $o$  in  $G_o$ . To see that  $M_o$  is a hyperbolic  
557 plane (up to an irrelevant scaling factor), note first that  $G_o \simeq SL(2, \mathbb{R})$ . It then remains to  
558 check that  $H_o \simeq SO(2)$ , since the Riemannian homogeneous space  $SL(2, \mathbb{R})/SO(2)$  is just a  
559 hyperbolic plane [41]. To do so, let  $\mathfrak{h}_o$  be the Lie algebra of  $H_o$ . Then,  $\mathfrak{h}_o = \mathfrak{g}_o \cap \mathfrak{h}$ . However,  
560 since  $\mathfrak{h}$  is the (+1)-eigenspace of  $\theta$ , it is easy to see that  $\mathfrak{h}_o$  is generated by  $\tau = \xi_\lambda + \xi_{-\lambda}$ .  
561 Since  $\tau$  corresponds to the generator of  $SO(2) \subset SL(2, \mathbb{R})$ , under the isomorphism (8.26),  
562  $H_o \simeq SO(2)$ , as required. ■

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664 **Appendix A. L<sup>1</sup>-Godement theorem.** The L<sup>2</sup>-Godement theorem, stated back in Sec-  
 665 tion 3, has the following variant, which will be called the L<sup>1</sup>-Godement theorem. It seems  
 666 useful to have this at hand, to deal with kernels  $k$  such that the function  $f(x) = k(o, x)$   
 667 is integrable but not square-integrable.

668 **Theorem A.1 (L<sup>1</sup>-Godement theorem).** *A continuous function  $f \in L^1_H(M)$  is positive def-*  
 669 *inite if and only if  $\hat{f}(t) \geq 0$  for all  $t \in \mathfrak{a}^*$  and  $\hat{f}$  is integrable with respect to the measure*  
 670  *$|c(t)|^{-2}dt$ .*

671 Here,  $L^1_H(M)$  denotes the space of  $H$ -invariant integrable functions (with respect to  $\text{vol}$ ). The  
 672 following proof is modelled on the one in Section 8.1. In particular, the notation is the same  
 673 as in Paragraph 8.1.1.

674 **A.1. The if part.** It will be enough to prove that the inversion formula (3.3) holds

$$675 \quad (\text{A.1}) \quad f(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \alpha(dt) \quad \text{for all } x \in M$$

676 The fact that  $f$  is continuous and positive-definite then follows exactly as in Paragraph 8.1.2.  
 677 Denote  $G_s(x) = G_s(o, x)$  the heat kernel of  $M$ . For  $s > 0$ ,  $G_s$  has spherical transform [18]

$$678 \quad (\text{A.2}) \quad \hat{G}_s(t) = (Z(s))^{-1} \exp[-s(\|t\|^2 + \|\rho\|^2)]$$

where  $Z(s)$  is a normalizing constant, which guarantees  $G_s$  integrates to 1 over  $M$ , and where  
 $\rho$  is half the sum of positive roots (as in Formula (3.1)). Recall that the heat semigroup is  
 strongly continuous on  $L^1_H(M)$  [13] (Page 148). This means the convolutions  $f_s = G_s * f$   
 converge to  $f$  in  $L^1_H(M)$  as  $s \rightarrow 0$ . Of course,  $G_s$  satisfies the inversion formula (3.3),

$$G_s(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{G}_s(t) \alpha(dt)$$

679 Therefore, by taking the convolution under the integral

$$680 \quad (\text{A.3}) \quad f_s(x) = \int_{\mathfrak{a}^*} (f * \varphi_t)(x) \hat{G}_s(t) \alpha(dt) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \hat{G}_s(t) \alpha(dt)$$

where the second equality follows because  $\varphi_t$  is an eigenfunction of convolution operators [42]  
 (Page 67). Now, from the assumption that  $\hat{f}$  is integrable (*i.e.*  $\hat{f} \in L^1_W(\mathfrak{a}^*)$ ), and from (A.2),  
 the dominated convergence theorem can be applied in (A.3), in order to obtain

$$\lim_{s \rightarrow 0} f_s(x) = \int_{\mathfrak{a}^*} \varphi_t(x) \hat{f}(t) \alpha(dt)$$

681 as  $s \rightarrow 0$ . But this limit is identical to  $f(x)$ , so (A.1) is proved.

682 **A.2. The only-if part.** As in Paragraph 8.1.3, the aim is to prove (8.7),

$$683 \quad (\text{A.4}) \quad \int_M (\psi * \psi^\dagger)(x) f(x) \text{vol}(dx) = \int_{\mathfrak{a}^*} |\hat{\psi}(t)|^2 \hat{f}(t) \alpha(dt)$$

which then implies the spectral representation (8.5). Since  $\psi * \psi^\dagger \in C_H^c(M)$ , it follows from (3.3) and (8.6) that

$$(\psi * \psi^\dagger)(x) = \int_{\mathfrak{a}^*} \varphi_t(x) |\hat{\psi}(t)|^2 \alpha(dt)$$

684 Replacing this under the integral on the left-hand side of (A.4), the right-hand side is obtained  
 685 by using the definition (3.2) of  $\hat{f}$ . Now, everything follows from (8.5) as in Paragraph 8.1.3.  
 686 ■

687 **Appendix B. The hyperbolic secant in three dimensions.** Recall the hyperbolic secant  
 688 kernel from 4.3.2. This is here considered in the case where  $M$  is a hyperbolic space. The aim  
 689 is to apply the  $L^2$ -Godement theorem, with the help of the spherical transform pair (4.5).

Setting  $g(r) = (\cosh(r))^{-a}$  and  $f(x) = g(d(o, x))$ , it follows that  $f$  is continuous and in  $L_H^2(M)$  when  $a > 1$ . This is because,

$$\int_M |f(x)|^2 = 4\pi \int_0^\infty (\cosh(r))^{-2a} \sinh^2(r) dr = \int_1^\infty u^{-2a} (u^2 - 1)^{\frac{1}{2}} du < \infty$$

as follows by integrating in geodesic spherical coordinates. Thus, the  $L^2$ -Godement theorem can be applied to  $f$ . The spherical transform  $\hat{f}$  can be found from Lemma 6.4, at least for integer values of  $a$ . Indeed, the moment-generating function  $G$ , required in the lemma, reads

$$G(s) = \int_{-\infty}^{+\infty} (\cosh(r))^{-a} e^{sr} dr$$

690 When  $a$  is (of course, positive) integer, this can be obtained using the residue theorem [4].  
 691 Precisely, if  $a$  is odd,  $a = 2n + 1$ ,

$$692 \quad (\text{B.1}) \quad G(i\sigma) = \frac{\pi \operatorname{sech}(\pi\sigma/2)}{(2n)!} \prod_{m=0}^{n-1} [\sigma^2 + (2m+1)^2]$$

693 and if  $a$  is even,  $a = 2n$ ,

$$694 \quad (\text{B.2}) \quad G(i\sigma) = \frac{\pi\sigma \operatorname{csch}(\pi\sigma/2)}{(2n-1)!} \prod_{m=1}^{n-1} [\sigma^2 + (2m)^2]$$

695 where  $\operatorname{sech} = 1/\cosh$  and  $\operatorname{csch} = 1/\sinh$  are the hyperbolic secant and cosecant, and where  
 696 the imaginary part of  $\sigma$  is  $\geq -1$ . With  $G$  given by (B.1) and (B.2), the spherical transform  $\hat{f}$   
 697 can be computed using (6.2). Letting  $H(\sigma) = G(i\sigma)$ , this yields

$$698 \quad (\text{B.3}) \quad \hat{f}(t) = \frac{\pi}{t} \operatorname{Im} \{H(t-i) - H(t+i)\}$$

699 **Lemma B.1.** *If  $z \in \mathbb{C} \setminus \mathbb{Z}$  and  $n$  is a positive integer, then*

$$700 \quad \Gamma(n+z)\Gamma(n-z) = -\frac{\pi}{z \sin(\pi z)} \prod_{m=0}^{n-1} (m^2 - z^2)$$

$$701 \quad \Gamma\left(n + \frac{1}{2} + z\right) \Gamma\left(n + \frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)} \prod_{m=0}^{n-1} \left[\left(m + \frac{1}{2}\right)^2 - z^2\right]$$

702 *Proof.* The proof of the first equality follows by induction using Euler's reflection formula  
 703  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$  for  $z \notin \mathbb{Z}$  and the identity  $\Gamma(z+1) = z\Gamma(z)$ . The second equality  
 704 follows in the same manner using the formula  $\Gamma(1/2+z)\Gamma(1/2-z) = \pi/\cos(\pi z)$ , which itself  
 705 follows from replacing  $z$  with  $z+1/2$  in Euler's reflection formula. ■

706 **Proposition B.2.** *The hyperbolic secant kernel is positive definite in three-dimensional hy-*  
 707 *perbolic space whenever  $a$  is a positive integer.*

708 *Proof.* If  $a$  is odd with  $a = 2n + 1$ , then

$$709 \quad H(\sigma) = G(i\sigma) = \frac{4^n \pi \operatorname{sech}(\pi\sigma/2)}{(2n)!} \prod_{m=0}^{n-1} \left[ \left(m + \frac{1}{2}\right)^2 - \left(\frac{i\sigma}{2}\right)^2 \right]$$

$$710 \quad = \frac{4^n}{(2n)!} \Gamma\left(n + \frac{1}{2} + i\frac{\sigma}{2}\right) \Gamma\left(n + \frac{1}{2} - i\frac{\sigma}{2}\right)$$

711 using Lemma B.1 with  $z = i\sigma/2$ . Thus,

$$712 \quad H(t-i) - H(t+i) = \frac{4^n}{(2n)!} \left[ \Gamma\left(n+1 + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) - \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n+1 - \frac{it}{2}\right) \right]$$

$$713 \quad = \frac{4^n}{(2n)!} \left[ \left(n + \frac{it}{2}\right) \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) - \left(n - \frac{it}{2}\right) \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) \right]$$

$$714 \quad = i \frac{4^n t}{(2n)!} \Gamma\left(n + \frac{it}{2}\right) \Gamma\left(n - \frac{it}{2}\right) = i \frac{4^n t}{(2n)!} \left| \Gamma\left(n + \frac{it}{2}\right) \right|^2$$

715 where we have used the fact that  $t \in \mathbb{R}$  and  $\overline{\Gamma(z)} = \Gamma(\bar{z})$  for  $z \in \mathbb{C}$ . Thus, (B.3) reduces to

$$716 \quad \hat{f}(t) = \frac{4^n \pi}{(2n)!} \left| \Gamma\left(n + \frac{it}{2}\right) \right|^2 \geq 0$$

717 for any  $t \in \mathbb{R}$ . If  $a$  is even with  $a = 2n$ , a similar calculation using Lemma B.1 yields

$$718 \quad \hat{f}(t) = \frac{4^n n \pi}{(2n)!} \left| \Gamma\left(n - \frac{1}{2} + \frac{it}{2}\right) \right|^2 \geq 0,$$

719 for any  $t \in \mathbb{R}$ , which completes the proof. ■