Tailored vertex ordering for faster triangle listing in large graphs

Fabrice Lécuyer, Louis Jachiet, Clémence Magnien, Lionel Tabourier

Tailored vertex ordering for faster triangle listing in large graphs

Fabrice Lécuyer* Louis Jachiet† Clémence Magnien* Lionel Tabourier*

Abstract

Listing triangles is a fundamental graph problem with many applications, and large graphs require fast algorithms. Vertex ordering allows the orientation of edges from lower to higher vertex indices, and state-of-the-art triangle listing algorithms use this to accelerate their execution and to bound their time complexity. Yet, only basic orderings have been tested. In this paper, we show that studying the precise cost of algorithms instead of their bounded complexity leads to faster solutions. We introduce cost functions that link ordering properties with the running time of a given algorithm. We prove that their minimization is NP-hard and propose heuristics to obtain new orderings with different trade-offs between cost reduction and ordering time. Using datasets with up to two billion edges, we show that our heuristics accelerate the listing of triangles by an average of 38% when the ordering is already given as an input, and 16% when the ordering time is included. (arxiv version: doi.org/10.48550/arXiv.2203.04774)

1 Introduction

1.1 Context and problem.

Small connected subgraphs are key to identifying families of real-world networks [22] and are used for descriptive or predictive purposes in various fields such as biology [31, 24], linguistics [4] or engineering [33]. In sociology in particular, characterizing networks with specific structural patterns has been a focus of interest for a long time, as it is even present in the works of early 20^{th} century sociologists such as Simmel [29]. Consequently, it is a common practice in social network analysis to describe interactions between individuals using local patterns [13, 35]. Recently, the ability to count and list small size patterns efficiently allowed the characterization of various types of social networks on a large scale [8, 6]. In particular, listing elementary motifs such as triangles and 3-motifs is a stepping stone in the analysis of the structure of networks and their dynamics [12]. For instance, the closure of a triplet of nodes to form a triangle is supposed to be a driving force of social networks evolution [18, 30].

The task of listing triangles may seem simple, but web crawlers and social platforms generate graphs that are so large that scalability becomes a challenge. Thus, a lot of effort has been dedicated to efficient in-memory triangle listing. Note that methods exist for graphs that do not fit in main memory: some use I/O-efficient accesses to the disk [9], while others partition the graph and process each part separately [2]. However, such approaches induce a costly counterpart that makes them much less efficient than in-memory listing methods. It is also worth noticing that exact or approximate methods designed for triangle *counting* [1, 34, 14] can generally not be adapted to triangle *listing*.

An efficient algorithm for triangle listing has been proposed early on in [7]. Based on the observation that real-world graphs generally have a heterogeneous degree distribution, later contributions [28, 16] showed how ordering vertices by degree or core value accelerates the listing. Such orderings create an orientation of edges so that nodes that are costly to process are not processed many times. A unifying description of this method has been proposed in [23] and it has been successfully extended to larger cliques [10, 20, 32]. However, only degree and core orderings have been exploited, but their properties are not specifically tailored for the triangle listing problem. Other types of orderings benefited other problems such as graph compression [5, 11] or cache optimization [36, 17]. The main purpose of this work is thus to find a general method to design efficient vertex orderings for triangle listing.

1.2 Contributions.

In this work, we show how vertex ordering directly impacts the running time of the two fastest existing triangle listing algorithms. First, we introduce cost functions that relate the vertex ordering and the running time of each algorithm. We prove that finding an optimal ordering that minimizes either of these costs is NP-hard. Then, we expose a gap in the combinations of algorithm and ordering considered in the literature, and we bridge it with three heuristics producing orderings with low corresponding costs. Our heuristics reach a compromise between their running time and the quality of the ordering obtained, in order to address two distinct tasks: listing triangles with or without taking into account the ordering time. Finally, we show that our resulting combinations of algorithm and ordering outperform state-of-the-art running times for either task. We release an efficient open-source implementation[1] of all considered methods.

Section 2 presents state-of-the-art methods to list triangles. In Section 3, we analyze the cost induced by a given ordering on these algorithms and propose several heuristics to reduce it; the proofs of NP-hardness are in

*Sorbonne Université, CNRS, LIP6, F-75005 Paris, France
†LTCI, Télécom Paris, Institut Polytechnique de Paris

[1]Open-source `c++` implementation available at: `https://github.com/lecfab/volt`

Appendix A and B. The experiments of Section 4 show that our methods are efficient in practice and improve the state of the art.

1.3 Notations.

We consider an unweighted undirected simple graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges. The set of neighbors of a vertex u is denoted $N_u = \{v, \{u, v\} \in E\}$, and its degree is $d_u = |N_u|$. An ordering π is a permutation over the vertices that gives a distinct index $\pi_u \in [\![1, n]\!]$ to each vertex u. In the directed acyclic graph (DAG) $G_\pi = (V, E_\pi)$, for $\{u, v\} \in E$, E_π contains (u, v) if $\pi_u < \pi_v$, and (v, u) otherwise. In such a directed graph, the set N_u of neighbors of u is partitioned into its predecessors N_u^- and successors N_u^+. We define the indegree $d_u^- = |N_u^-|$ and the outdegree $d_u^+ = |N_u^+|$; their sum is $d_u^- + d_u^+ = d_u$. A triangle of G is a set of vertices $\{u, v, w\}$ such that $\{u, v\}, \{v, w\}, \{u, w\} \in E$. A k-clique is a set of k fully-connected vertices. The coreness c_u of vertex u is the highest value k such that u belongs to a subgraph of G where all vertices have degree at least k; the core value or degeneracy $c(G)$ of G is the maximal c_u for $u \in V$. A core ordering π verifies $\pi_u \leq \pi_v \Leftrightarrow c_u \leq c_v$. Core value and core ordering can be computed in linear time [3].

2 State of the art

2.1 Triangle listing algorithms.

Ortmann and Brandes [23] have identified two families of triangle listing algorithms: *adjacency testing*, and *neighborhood intersection*. The former sequentially considers each vertex u as a seed, and processes all pairs $\{v, w\}$ of its neighbors; if they are themselves adjacent, $\{u, v, w\}$ is a triangle. Algorithms `tree-lister` [15], `node-iterator` [28] and `forward` [28] belong to this category. In contrast, the neighborhood intersection family methods sequentially considers each edge (u, v) as a seed; each common neighbor w of u and v forms a triangle $\{u, v, w\}$. Algorithms `edge-iterator` [28], `compact-forward` [16] and `K3` [7] belong to this category, as well as some algorithms that list larger cliques [21, 10, 20].

In naive versions of both adjacency testing and neighborhood intersection, finding a triangle (u, v, w) does not prevent from finding triangle (v, w, u) at a later step. The above papers avoid this unwanted redundancy by using an ordering, explicitly or not. We use the framework developed in [23]: a total ordering π is defined over the vertices, and the triple (u, v, w) is only considered a valid triangle if $\pi_u < \pi_v < \pi_w$. This guarantees that each triangle is listed only once: as illustrated in Figure 1, vertices in any triangle of the DAG G_π appear in one and only one of 3 positions: u is first, v is second, w is third; the same holds for edges: L is the long edge, and S_1 and S_2 are the first and second short edges. It leads to 3 variants of adjacency testing (seed vertex v or w instead of u) and of neighborhood intersection (seed edge L or S_2 instead of S_1).

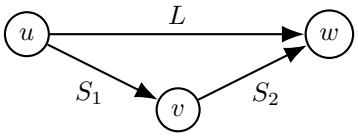


Figure 1: **Directed triangle** with the unified notations proposed in [23]. The edges are directed according to an ordering π such that $\pi_u < \pi_v < \pi_w$.

Choosing the right data-structure is key to the performance of algorithms. All triangle listing algorithms have to visit the neighborhoods of vertices. Using hash table or binary tree to store them is very effective: they respectively allow for constant and logarithmic search on average. However, because of high constants, they are reportedly slow in terms of actual running time [28]. A faster structure is the boolean array used in `K3` for neighborhood intersection. It registers the elements of N_u^+ in a boolean table B so that, for each neighbor v of u, it is possible to check in constant time if a neighbor w of v is also a neighbor of u. This is the structure used by the fastest methods [23, 10].

In the rest of this paper, we therefore only consider triangle listing algorithms that use neighborhood intersection and a boolean array. We present the two that we will study in Algorithms 1 and 2 with the notations of Figure 1 for the vertices. They initialize the boolean array B to false (line 1), consider a first vertex (line 2) and store its neighbors in B (line 3); then, for each of its neighbors (line 4), they check if their neighbors (line 5) are in B (line 6), in which case the three vertices form a triangle (line 7). B is reset (line 8) before continuing with the next vertex. The Algorithm 1 corresponds to `L+n` in [23]; we call it `A++` because of the two "+" (referring to out-degrees) involved in its complexity. The Algorithm 2 corresponds to `S`$_1$`+n` in [23]; we call it `A+-` [2]. Their complexities are given in Property 1. Since they depend on the indegree and outdegree of vertices, the choice of ordering will impact the running time of the algorithms.

Property 1 (Complexity of `A++` and `A+-`) *The time complexity of* `A++` *is* $\Theta(\sum_{u \in V} {d_u^+}^2)$. *The time complexity of* `A+-` *is* $\Theta\left(m + \sum_{v \in V} d_v^+ d_v^-\right)$.

Proof: In both algorithms, the boolean table B requires n initial values, m set and m reset operations, which is $\Theta(m)$ assuming that $n \in \mathcal{O}(m)$. In `A++`, a given vertex u appears in the loop of line 4 as many times as it has a successor w; every time, a loop over each of its successors v is performed. In total, u is involved in $\Theta({d_u^+}^2)$ operations. Similarly, in `A+-`, a given vertex v appears in the loop of line 4 as many times as it has a predecessor u; every time, a loop over each of its successors w is performed. In total, v is involved

[2] A third natural variant exists: `A--` or `S`$_2$`+n`. We ignore it here since its complexity is equivalent to the one of `A++`.

Algorithm 1 – `A++` (or `L+n`)

```
1: for each vertex v do B[v] ← False
2: for each vertex w do
3:     for v ∈ N_w^- do B[v] ← True
4:     for u ∈ N_w^- do
5:         for v ∈ N_u^+ do
6:             if B[v] then
7:                 output triangle{u, v, w}
8:     for v ∈ N_w^- do B[v] ← False
```

Complexity:

$$\Theta\Big(m + \sum_{(u,w)\in E_\pi} d_u^+\Big) = \Theta\Big(\sum_{u\in V} {d_u^+}^2\Big)$$

Algorithm 2 – `A+-` (or $\mathtt{S}_1$`+n`)

```
1: for each vertex w do B[w] ← False
2: for each vertex u do
3:     for w ∈ N_u^+ do B[w] ← True
4:     for v ∈ N_u^+ do
5:         for w ∈ N_v^+ do
6:             if B[w] then
7:                 output triangle{u, v, w}
8:     for w ∈ N_u^+ do B[w] ← False
```

Complexity:

$$\Theta\Big(m + \sum_{(u,v)\in E_\pi} d_v^+\Big) = \Theta\Big(m + \sum_{v\in V} d_v^+ d_v^-\Big)$$

in $\Theta(d_v^+ d_v^-)$ operations. The term m is omitted in the complexity of `A++` as $\sum_{u\in V} {d_u^+}^2 \geq \sum_{u\in V} d_u^+ = m$, but not in `A+-` as $\sum_{v\in V} d_v^+ d_v^-$ can be lower than m. □

2.2 Orderings and complexity bounds.

Ortmann and Brandes [23] order the vertices by non-decreasing degree or core value. In their experimental comparison, they test several algorithms as well as `A++` and `A+-`, each with degree ordering, core ordering, and with the original ordering of the dataset. They conclude that the fastest method is `A++` with core or degree ordering: core is faster to list triangles when the ordering is given as an input, and degree is faster when the time to compute the ordering is also included.

Danisch *et al.* [10] also use core ordering in the more general problem of listing k-cliques. For triangles ($k = 3$), their algorithm is equivalent to `A+-`, and they show that using core ordering outperforms the methods of [7, 16, 21].

With these two orderings, it is possible to obtain upper-bounds for the time complexity in terms of graph properties. Chiba and Nishizeki [7] show that `K3` with degree ordering has a complexity in $\mathcal{O}(m \cdot \alpha(G))$, where $\alpha(G)$ is the arboricity of graph G. With core ordering, `node-iterator-core` [28] and `kClist` [10] have complexity $\mathcal{O}(m \cdot c(G))$, where $c(G)$ is the core value of graph G. These bounds are considered equal in [23], following the proof in [37] that $\alpha(G) \leq c(G) \leq 2\alpha(G) - 1$. However, we focus in this work on the complexities expressed in Algorithms `A++` and `A+-` as we will see that they describe the running time more accurately.

3 New orderings to reduce the cost of triangle listing

3.1 Formalizing the cost of triangle listing algorithms.

In this section, we discuss how to design vertex orderings to reduce the cost of triangle listing algorithms. For this purpose, we introduce the following costs that appear in the complexity formulas of Algorithms 1 and 2. Recall that the initial graph is undirected and that the orientation of the edges is given by the ordering π, which partitions neighbors into successors and predecessors.

Definition 1 (Cost induced by an ordering) *Given an undirected graph G, the costs C^{++} and C^{+-} induced by a vertex ordering π are defined by:*

$$C^{++}(\pi) = \sum_{u\in V} d_u^+ d_u^+ \qquad C^{+-}(\pi) = \sum_{u\in V} d_u^+ d_u^-$$

The fastest methods in the state of the art are `A++` with core or degree ordering [23], and `A+-` with core ordering [10]. The intuition of both orderings is that high degree vertices are ranked after most of their neighbors in π so that their outdegree in G_π is lower. This reduces the cost C^{++}, which in turn reduces the number of operations required to list all the triangles as well as the actual running time of `A++`. In [23], it is mentioned that core ordering performs well with `A+-` as a side effect.

To our knowledge, no previous work has designed orderings with a low C^{+-} cost and used them with `A+-`. We will show that such orderings can lower the computational cost further. Yet, optimizing C^{+-} or C^{++} is computationally hard because of Theorem 1:

Theorem 1 (NP-hardness) *Given a graph $G = (V, E)$, it is NP-hard to find an ordering π on V that minimizes $C^{+-}(\pi)$ or that minimizes $C^{++}(\pi)$.*

Proof: For the hardness of C^{+-}, a proof was already known from [26] but never published as far as we know; we give a new and simpler proof in Appendix A. We prove the result for C^{++} in Appendix B. □

3.2 Distinguishing two tasks for triangle listing.

Triangle listing typically consists of the following steps: loading a graph, computing a vertex ordering, and listing the triangles. Time measurements in [16, 10, 20] only take the last step into account, while [28, 23] also

include the other steps. We therefore address two distinct tasks in our study: we call **mere-listing** the task of listing the triangles of an already loaded graph with a given vertex ordering; we call **full-listing** the task of loading a graph, computing a vertex ordering, and listing its triangles.

In the rest of the paper, we use the notation *task-order-algorithm*: for instance, mere-core-`A+-` refers to the mere-listing task with core ordering and algorithm `A+-`. Using this notation, the fastest methods identified in the literature are mere-core-`A+-` in [10], mere-core-`A++` and full-degree-`A++` in [23]. We use all three methods as benchmarks in our experiments of Section 4.

With mere-listing, the ordering time is not taken into account, which allows to spend a long time to find an ordering with low cost. On the other hand, full-listing favors quickly obtained orderings even if their induced cost is not the lowest. For this reason, there is a time-quality trade-off for cost-reducing heuristics.

3.3 Reducing C^{+-} along a time-quality trade-off.

We remind that two efficient algorithms are identified in the literature for triangle listing (see Algorithms 1 and 2). Their number of operations are respectively C^{++} and C^{+-}. However, the orderings that have been considered (degree and core) induce a low C^{++} cost, but not necessarily a low C^{+-} cost.

Our goal here is therefore to design a procedure that takes a graph as input and produces an ordering π with a low induced cost $C^{+-}(\pi)$. Because of Theorem 1, finding an optimal solution is not realistic for graphs with millions of edges. We therefore present three heuristics aiming at reducing the C^{+-} value, exploring the trade-off between quality in terms of C^{+-} and ordering time.

3.3.1 *Neigh* heuristic.

We define the *neighborhood optimization* method, a greedy reordering where each vertex is placed at the optimal index with respect to its neighbors, as illustrated in Figure 2. First, notice that changing an index π_u only affects $C^{+-}(\pi)$ if the position of u with respect to at least one of its neighbors changes; otherwise the in- and outdegrees of all vertices remain unchanged. Starting from any ordering π, the algorithm described in Algorithm 3 considers each vertex u one by one (line 3) and, for each $p \in [\![1, d_u]\!]$, it computes $C^{+-}(p)$, the value of C^{+-} when u is just after its p-th neighbor in π, as well as $C^{+-}(0)$ when u is before all its neighbors. The position p_* that induces the lowest value of C^{+-} is selected (line 5) and the ordering is updated (line 6). The process is repeated until C^{+-} reaches a local minimum, or until the relative improvement is under a threshold ϵ (last line). The resulting π induces a low C^{+-} cost.

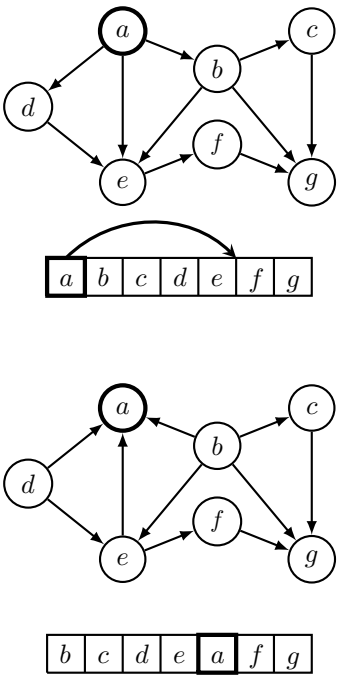


Figure 2: **Example of update in the *Neigh* heuristic**: vertex a is moved to a position among its neighbors that induces the lowest cost. The tables indicates how the ordering is updated. The edge in the DAG are reoriented accordingly. Here, the ordering at the top has $C^{+-} = 9$ while the ordering at the bottom has $C^{+-} = 6$. For this graph, the optimal C^{+-} cost is 3 (with ordering e, g, f, a, c, d, b).

Algorithm 3 Neighborhood optimization (*Neigh* heuristic)

Input: graph G, initial ordering π, threshold $\epsilon \geq 0$

1: **repeat**
2: $\quad C_0 = C^{+-}(\pi)$
3: $\quad$ **for** each vertex u of G **do**
4: $\quad\quad$ sort N_u according to π
5: $\quad\quad$ $p_* = \text{argmin}_{p \in [\![0, d_u]\!]} \{C^{+-}(p)\}$
6: $\quad\quad$ update ordering π to put u in position p_*
7: **while** $C^{+-}(\pi) < (1 - \epsilon) \cdot C_0$

For a vertex u, sorting the neighborhood according to π takes $\mathcal{O}(d_u \log d_u)$ operations; finding the best position takes $\Theta(d_u)$ because it only depends on the values d_v^+ and d_v^- of each neighbor v of u. With a linked list, π is updated in constant time. If Δ is the highest degree in the graph, one iteration over all the vertices thus takes $\mathcal{O}(m \log \Delta)$, which leads to a total complexity $\mathcal{O}(Im \log \Delta)$ if the improvement threshold ϵ is reached after I iterations. Notice that on all the tested datasets the process reaches $\epsilon = 10^{-2}$ after less than ten iterations.

This heuristic has several strong points: it can be used for other objective functions, for instance C^{++}; it is greedy, so the cost keeps improving until the process stops; if the initial ordering already induces a low C^{+-} cost, the heuristic can only improve it; it is stable in practice, which means that starting from several random orderings give similar final costs; and we show in

Section 4 that it allows for the fastest mere-listing.

In spite of its log-linear complexity, this heuristic can take longer than the actual task of listing triangles in practice, which is an issue for the full-listing task. We therefore propose the following faster heuristics in the case of the full-listing task.

3.3.2 *Check* heuristic.

This heuristic is inspired by core ordering, where vertices are repeatedly selected according to their current degree [3]. It considers all vertices by decreasing degree and *checks* whether it is better to put a vertex at the beginning or at the end of the ordering. More specifically, π is obtained as follows: before placing vertex u, let V_b (resp. V_e) be the vertices that have been placed at the beginning (resp. at the end) of the ordering, and $V_?$ those that are yet to place. The neighbors of u are partitioned in $N_b = N_u \cap V_b$, $N_e = N_u \cap V_e$ and $N_? = N_u \cap V_?$. We consider two options to place u: either just after the vertices in V_b ($\pi_u = |V_b| + 1$), or just before the vertices in V_e ($\pi_u = n - |V_e|$). In either case, u has all vertices of N_b as predecessors, and all vertices of N_e as successors. In the first case, vertices in $N_?$ become successors, which induces a C^{+-} cost $C_b = |N_b| \cdot (|N_e| + |N_?|)$. In the second, the cost is $C_e = (|N_b| + |N_?|) \cdot |N_e|$. The option with the smaller cost is selected. Sorting the vertices by degree requires $\mathcal{O}(n)$ steps with bucket sort. Maintaining the sizes of N_b, N_e, $N_?$ for each vertex requires one update for each edge. Therefore, the complexity is $\mathcal{O}(m+n)$, or $\mathcal{O}(m)$ assuming that $n \in \mathcal{O}(m)$.

3.3.3 *Split* heuristic.

Finally, we propose a heuristic that is faster to achieve but compromises on the quality of the resulting ordering. Degree ordering has been identified as the best solution for mere-listing with algorithm-`A++` [23]. We adapt it for C^{+-} by *splitting* vertices alternatively at the beginning and at the end of the ordering π. More precisely, a non-increasing degree ordering δ is computed, then the vertices are split according to their parity: if u has index $\delta_u = 2i + 1$ then $\pi_u = i + 1$; if $\delta_u = 2i$, then $\pi_u = n + 1 - i$. Thus, high degree vertices will have either few predecessors or few successors, which ensures a low C^{+-} cost. With the graph of Figure 2, supposing that we start from the non-decreasing degree ordering (e, b, g, a, f, d, c), which has $C^{+-} = 7$, the *Split* method leads to (e, g, f, c, d, a, b), which has $C^{+-} = 4$. The complexity of this method is in $\mathcal{O}(n)$ like the degree ordering.

4 Experiments

4.1 Experimental setup.

4.1.1 Datasets.

We use the 12 real-world graphs described in Table 1. Loops have been removed and the directed graphs have been transformed into undirected graphs by keeping one edge when one existed in either or both directions.

Table 1: **Datasets used for the experiments**, ranked by number of edges. They represent either web networks ★, social networks ▲ or citation networks ■.

dataset [source]	vertices	edges	triangles
skitter ★[19]	1,696,415	11,095,298	28,769,868
patents ■[19]	3,774,768	16,518,947	7,515,023
baidu ★[25]	2,141,301	17,014,946	25,207,196
pokec ▲[19]	1,632,804	22,301,964	32,557,458
socfba ▲[25]	3,097,166	23,667,394	55,606,428
LJ ▲[19]	4,036,538	34,681,189	177,820,130
wiki ★[19]	2,070,486	42,336,692	145,707,846
orkut ▲[19]	3,072,627	117,185,083	627,584,181
it ★[5]	41,291,318	1,027,474,947	48,374,551,054
twitter ▲[5]	41,652,230	1,202,513,046	34,824,916,864
friendster ▲[19]	124,836,180	1,806,067,135	4,173,724,142
sk ★[5]	50,636,151	1,810,063,330	84,907,041,475

4.1.2 Software and hardware.

We release a uniform open-source implementation [3] of `A++` and `A+-` algorithms, as well as the different ordering strategies that we discussed in Section 3. Our implementation allows to run either algorithm in parallel, which is possible because each iteration of the main loop is independent from the others. Among orderings however, only degree and *Split* are easily parallelizable; to be consistent, we use a single thread to compare the different methods. The code is in `c++` and uses `gnu make 4` and the compiler `g++ 8.2` with optimization flag `Ofast` and `openmp` for parallelisation. We run all the programs on a `sgi ub2000 intel xeon e5-4650L @2.6 GHz, 128Gb ram` running `linux suse 12.3`.

Regarding the state of the art, the most competitive implementation available for triangle listing is `kClist` in `c` [10], which has already been shown to outperform previous programs [21, 16]. It lists k-cliques using a core ordering and a recursive algorithm that is equivalent to `A+-` for $k = 3$. We compared our implementation to `kClist` in various settings and found that ours is 14% faster on average, presumably because it does not use recursion. Moreover, the paper that identified core-`A++` and degree-`A++` as the fastest methods [23] does not provide the corresponding code. Therefore, we only use our own implementation of `A+-` and `A++` in the rest of this paper: we exclusively focus on the speedup caused by the vertex ordering, separating it from the speedup originating from the implementation.

4.2 Cost and running time are linearly correlated.

In order to show that the cost functions C^{++} and C^{+-} are good estimates of the running time, we measure the correlation between the running time of mere-listing and the corresponding cost induced by various orderings (core, degree, our heuristics, but also breadth- and depth-first search, random ordering, etc). In Figure 3, we see that the running time for a given dataset correlates almost linearly to the corresponding cost: the

[3] `https://github.com/lecfab/volt`

lines represent linear regressions. It only presents some of the datasets for readability, but the correlation is above 0.82 on all of them. In other words, the execution time of a listing algorithm is almost a linear function of the cost induced by the ordering, which is why reducing this cost actually improves the running time, as we will see.

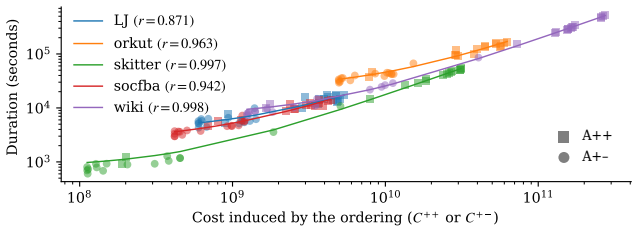


Figure 3: **Algorithm running time vs the cost induced by the ordering.** Each mark represents an ordering: circles are for cost C^{+-} and algorithm `A+-`, squares are for cost C^{++} and algorithm `A++`. Each color represents a dataset: the line of linear regressions and associated correlation coefficients r show the proportionality between cost and time.

4.3 *Neigh* outperforms previous mere-listing methods.

We compare our methods to the state of the art for mere-listing (core-`A+-` in [10] and core-`A++` in [23]) and for full-listing (degree-`A++` in [23]) in Figure 4. The top charts present the running time of the three state-of-the-art methods for all datasets, for the mere-listing task (left) and the full-listing task (right). We can see that there is no clear winner for mere-listing: both `A++` methods have a very similar duration, but core-`A+-` can be between 1.4 times faster and 2.4 times slower depending on the dataset. This explains why [23] and [10] did not agree on the fastest method.

On the other hand, our heuristics *Neigh*, *Check* and *Split* manage to produce orderings significantly lower C^{+-} costs. This translates directly into short running times for mere-listing with `A+-`. To compare our contributions with the state of the art, we take for each dataset the fastest of the three existing methods. The bottom left chart of Figure 4 shows the speedup of our methods compared to the fastest existing one. Exact runtimes of the best existing and of the methods proposed in this work are reported in Table 2.

The main result is that *Neigh*-`A+-` is always faster than the best previous method. The speedup is 1.38 on average and ranges from only 1.02 on *twitter* to 1.71 on the *it* dataset. *Check*-`A+-` is almost as good, with a 1.32 average speedup ranging from 1.10 to 1.60; it is even faster than *Neigh*-`A+-` on two of the datasets. *Split*-`A+-` is a little slower, which is expected because this ordering is designed to be obtained quickly and does not reduce C^{+-} as efficiently as our other heuristics. However it still consistently outperforms all the previous methods, with a 1.20 average speedup.

	mere-listing		**full-listing**	
dataset	existing	this paper	existing	this paper
skitter	1.00s	0.71s	1.91s	1.75s
patents	2.40s	1.67s	5.71s	5.15s
baidu	3.68s	2.87s	6.38s	5.77s
pokec	4.87s	3.44s	7.91s	7.21s
socfba	5.52s	3.98s	8.92s	7.79s
LJ	6.23s	4.79s	10.91s	9.88s
wiki	10.82s	8.22s	16.23s	15.65s
orkut	42.11s	33.09s	57.47s	51.60s
it	3m13	1m53	4m09	2m45
twitter	12m31	11m20	15m21	14m08
friendster	42m36	30m31	55m47	48m13
sk	5m10	3m06	6m47	4m31

Table 2: **Duration of triangle listing of existing methods against methods of this paper.** For each dataset, we compare the fastest state-of-the-art method against the fastest of our methods. Recall that mere-listing only takes into account the runtime of the listing algorithm (`A++` or `A+-`) while full-listing also counts the graph loading time and the ordering time.

4.4 *Split* outperforms previous full-listing methods.

For full-listing, the top right chart of Figure 4 compares the three state-of-the-art methods and shows that degree-`A++` is the fastest for almost all datasets. This result is consistent with the result reported in [23], that specifically addresses full-listing. The bottom right chart shows the speedup of our three methods compared to the fastest state-of-the-art method. Note that the *Neigh* heuristic is not competitive here (speedup under one) since its ordering time is long compared to other methods.

The main result is that *Split*-`A+-` is always faster than previous methods. The speedup compared to existing methods is 1.16 on average, and it ranges from 1.04 on *wiki* to 1.50 on *it* dataset. *Check* also gives very good results: on medium datasets, it is a bit slower than degree-`A++`, but it outperforms all state-of-the-art methods on large datasets (*it, twitter, friendster, sk*), and it even beats *Split* on three of them. This hints at a transition effect: the *Check* ordering has a lower C^{+-} value but it takes $\mathcal{O}(m)$ steps to compute, while *Split* only needs $\mathcal{O}(n)$; for larger datasets, the listing step prevails, so the extra time spent to compute *Check* becomes profitable.

Conclusion

In this work, we address the issue of in-memory triangle listing in large graphs. We formulate explicitly the computational costs of the most efficient existing algorithms, and investigate how to order vertices to minimize these costs. After proving that the optimization problems are NP-hard, we propose scalable heuristics that are specifically tailored to reduce the costs induced by the orderings. We show experimentally that these methods outperform the current state of the art for both the mere-listing and the full-listing tasks.

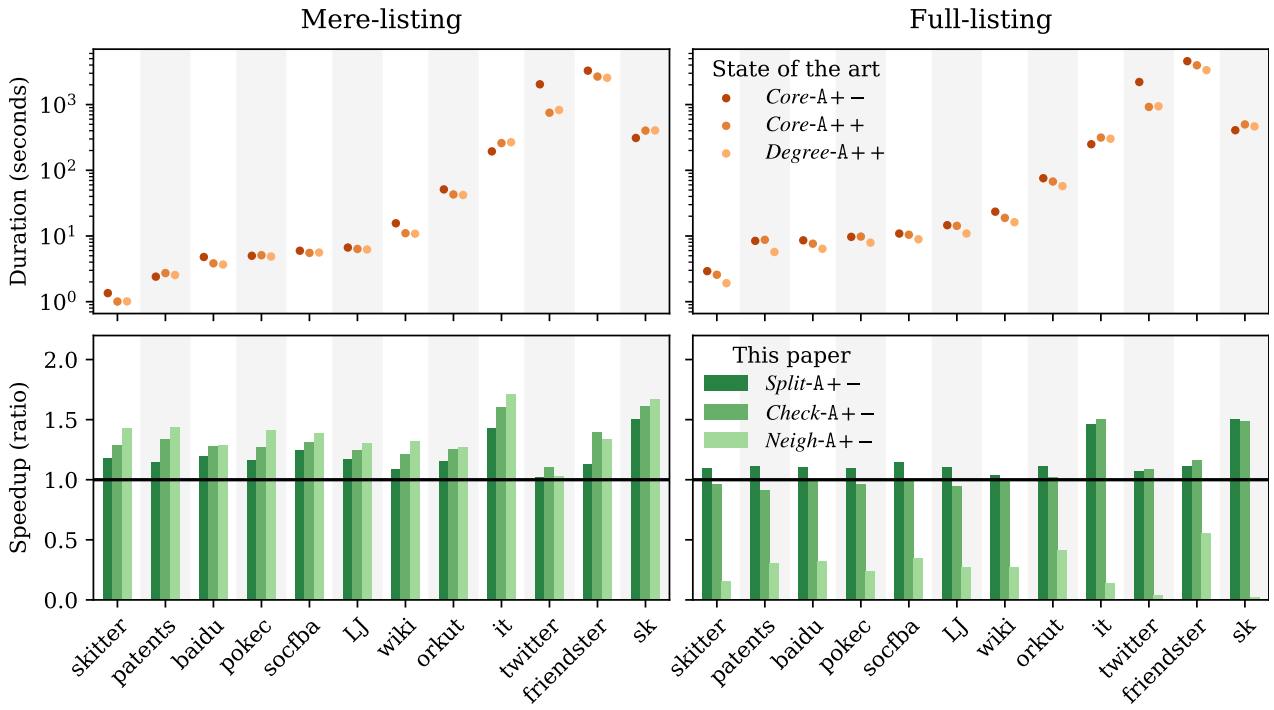


Figure 4: **Comparison of state-of-the-art methods and speedup of our methods.** The top charts show the runtime of the three state-of-the-art methods; depending on the dataset, the fastest method is not always the same. The bottom charts show the speedup of our three methods against the fastest existing method of each dataset. On the left, for mere-listing, we see that our three heuristics consistently outperform the three state-of-the-art methods, and that *Neigh* or *Check* are the fastest. On the right, for full-listing, *Neigh* is not efficient but *Split* is always faster than existing methods and *Check* is faster on bigger datasets.

Our results also emphasize a limitation in the possible acceleration: while it is certainly possible to keep improving the mere-listing step, a significant part of full-listing is spent on other steps: computing the ordering, but also loading the graph or writing the output. It seems, however, that the mere-listing step takes more importance as graphs grow larger, which makes our listing methods all the more relevant for future, larger datasets. A natural extension of this work is to use similar vertex ordering heuristics in the more general case of clique listing. Formulating appropriate cost functions for clique listing algorithms is not straightforward and requires studying precisely the different possibilities to detect all the vertices of a clique.

Acknowledgements

We express our heartfelt thanks to Maximilien Danisch who initiated this project. We also thank Alexis Baudin, Esteban Bautista, Katherine Byrne and Matthieu Latapy for their valuable comments. This work is funded by the ANR (French National Agency of Research) partly by Limass project (under grant ANR-19-CE23-0010) and partly by ANR FiT LabCom.

References

[1] M. Al Hasan and V. S. Dave. Triangle counting in large networks: a review. *WIREs Data Mining and Knowledge Discovery*, 2018.

[2] S. Arifuzzaman, M. Khan, and M. Marathe. Fast parallel algorithms for counting and listing triangles in big graphs. *TKDD*, 2019.

[3] V. Batagelj and M. Zaversnik. An O(m) algorithm for cores decomposition of networks. *arXiv*, 2003.

[4] C. Biemann, L. Krumov, S. Roos, and K. Weihe. Network motifs are a powerful tool for semantic distinction. In *Towards a Theoretical Framework for Analyzing Complex Linguistic Networks*. 2016.

[5] P. Boldi and S. Vigna. The webgraph framework I: compression techniques. In *WWW*, 2004.

[6] R. Charbey and C. Prieur. Stars, holes, or paths across your facebook friends: A graphlet-based characterization of many networks. *Network Science*, 2019.

[7] N. Chiba and T. Nishizeki. Arboricity and subgraph listing algorithms. *SIAM Journal on Computing*, 1985.

[8] S. Choobdar, P. Ribeiro, S. Bugla, and F. Silva. Comparison of co-authorship networks across scientific fields using motifs. In *ASONAM*, 2012.

[9] S. Chu and J. Cheng. Triangle listing in massive networks and its applications. In *SIGKDD*, 2011.

[10] M. Danisch, O. D. Balalau, and M. Sozio. Listing k-cliques in sparse real-world graphs. In *WWW*, 2018.

[11] L. Dhulipala, I. Kabiljo, B. Karrer, G. Ottaviano, S. Pupyrev, and A. Shalita. Compressing graphs and indexes with recursive graph bisection. In *KDD*, 2016.

[12] K. Faust. A puzzle concerning triads in social networks: Graph constraints and the triad census. *Social Networks*, 32(3):221–233, 2010.

[13] P. W. Holland and S. Leinhardt. Local structure in social networks. *Sociological methodology*, 1976.

[14] L. Hu, L. Zou, and Y. Liu. Accelerating triangle counting on gpu. SIGMOD/PODS, 2021.

[15] A. Itai and M. Rodeh. Finding a minimum circuit in a graph. *SIAM Journal on Computing*, 1978.

[16] M. Latapy. Main-memory triangle computations for very large (sparse (power-law)) graphs. *Theoretical Computer Science*, 2008.

[17] E. Lee, J. Kim, K. Lim, S. Noh, and J. Seo. Preselect static caching and neighborhood ordering for bfs-like algorithms on disk-based graph engines. 2019.

[18] J. Leskovec, L. Backstrom, R. Kumar, and A. Tomkins. Microscopic evolution of social networks. In *SIGKDD*, 2008.

[19] J. Leskovec and A. Krevl. SNAP Datasets: Stanford large network dataset collection, 2014.

[20] R.-H. Li, S. Gao, L. Qin, G. Wang, W. Yang, and J. X. Yu. Ordering heuristics for k-clique listing. *Proc. VLDB Endow.*, 2020.

[21] K. Makino and T. Uno. New algorithms for enumerating all maximal cliques. In *SWAT*. 2004.

[22] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon. Network motifs: simple building blocks of complex networks. *Science*, 2002.

[23] M. Ortmann and U. Brandes. Triangle listing algorithms: Back from the diversion. In *Proc. ALENEX*. SIAM, 2014.

[24] N. Pržulj. Biological network comparison using graphlet degree distribution. *Bioinformatics*, 2007.

[25] R. A. Rossi and N. K. Ahmed. The network data repository with interactive graph analytics and visualization. In *AAAI*, 2015.

[26] M. Rudoy. `https://cstheory.stackexchange.com/q/38274`, 2017.

[27] T. J. Schaefer. The complexity of satisfiability problems. STOC. ACM, 1978.

[28] T. Schank and D. Wagner. Finding, counting and listing all triangles in large graphs, an experimental study. In *WEA*. Springer, 2005.

[29] G. Simmel. *Soziologie*. Duncker & Humblot Leipzig, 1908.

[30] S. Sintos and P. Tsaparas. Using strong triadic closure to characterize ties in social networks. In *SIGKDD*, 2014.

[31] O. Sporns, R. Kötter, and K. J. Friston. Motifs in brain networks. *PLoS biology*, 2004.

[32] T. Uno. Implementation issues of clique enumeration algorithm. *Progress in Informatics*, 2012.

[33] S. Valverde and R. V. Solé. Network motifs in computational graphs: A case study in software architecture. *Physical Review E*, 2005.

[34] L. Wang, Y. Wang, C. Yang, and J. D. Owens. A comparative study on exact triangle counting algorithms on the gpu. HPGP '16, 2016.

[35] S. Wasserman, K. Faust, et al. Social network analysis: Methods and applications. 1994.

[36] H. Wei, J. X. Yu, C. Lu, and X. Lin. Speedup graph processing by graph ordering. SIGMOD, 2016.

[37] X. Zhou and T. Nishizeki. Edge-coloring and f-coloring for various classes of graphs. *J. Graph Algorithms Appl.*, 1999.

A NP-hardness of the C^{+-} problem

Given a graph G and an order $\prec$ on the vertices of G, we define $succ_{\prec}(u)$ (respectively $pred_{\prec}(u)$) as the set of neighbors v of u such that $u \prec v$ (resp. $v \prec u$). For any subset of vertices W, we note $C^{+-}_{\prec}(W) = \sum_{u \in W} |succ_{\prec}(u)| \cdot |pred_{\prec}(u)|$. Using this definition we formalize the following problem:

Problem 1 (C^{+-}) *Given an undirected graph $G = (V, E)$ and an integer K, is there an order $\prec$ on the vertices such that $C^{+-}_{\prec}(V) \leq K$?*

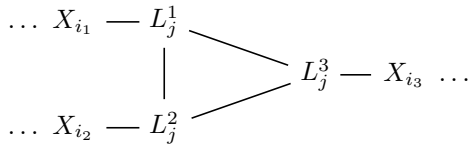
Problem 2 (NAE3SAT+) *Not-All-Equal Positive Three-Satisfiability. Given a formula $\phi = c_1 \wedge \cdots \wedge c_m$ in conjunctive normal form where each clause consists in three positive literals, is there an assignment to the variables satisfying ϕ such that in no clause all three literals have the same truth value?*

The NAE3SAT+ problem is known to be NP-complete by Schaefer's dichotomy theorem [27]. We will show that this problem can be reduced to the C^{+-} problem, thus proving that C^{+-} is NP-hard. Note that a proof was given in-hard [26] but, as far as we know, it has never been published. We give a new simpler proof of the following theorem:

Theorem 2 *C^{+-} is NP-hard.*

Definition 2 *Let ϕ be an instance of NAE3SAT+ with variables $x_1, \dots, x_n$ and clauses $c_1, \dots, c_m$, where clause c_j is of the form $l_j^1 \vee l_j^2 \vee l_j^3$. We define a graph G_ϕ by creating three connected vertices L_j^1, L_j^2, L_j^3 representing the literals of each clause c_j; additionally, a vertex X_i is created for each variable x_i and connected to all the L_j^a such that $l_j^a = x_i$. More formally, $G_\phi = (V_\phi, E_\phi)$ with:*

- $V_\phi = \{X_i \mid i \in [\![1, n]\!]\} \cup \{L_j^1, L_j^2, L_j^3 \mid j \in [\![1, m]\!]\}$
- $E_\phi = \big\{\{L_j^1, L_j^2\}, \{L_j^1, L_j^3\}, \{L_j^2, L_j^3\} \mid j \in [\![1, m]\!]\big\} \cup \big\{\{X_i, L_j^a\} \mid x_i = l_j^a\big\}$



Proposition 1 ($\Longrightarrow$) *Given an instance ϕ of NAE3SAT+ with m clauses and the associated graph G_ϕ, if ϕ is satisfiable then there exists an order $\prec$ on V_ϕ such that $C_\prec^{+-}(V_\phi) \leq 2m$.*

Proof: Let ϕ be a satisfiable instance of NAE3SAT+ with the above notations. Take a valid assignment and let us note k the number of variables set to true. There exist indices $i_1, \dots, i_n$ such that $x_{i_1}, \dots, x_{i_k} = true$ and $x_{i_{k+1}}, \dots, x_{i_n} = false$, and for each clause c_j, there are indices $t_j, a_j, f_j \in \{1, 2, 3\}$ such that $l_j^{t_j} = true$, $l_j^{f_j} = false$ and $l_j^{a_j}$ has any value. Now construct the following order on V_ϕ, so that true variables come first, then in each clause the false literal comes before the true one, and the false variables are at the end:

$X_1 \prec \cdots \prec X_k$	True variables
$\prec L_1^{f_1} \prec \cdots \prec L_m^{f_m}$	False literals
$\prec L_1^{a_1} \prec \cdots \prec L_m^{a_m}$	Other literals
$\prec L_1^{t_1} \prec \cdots \prec L_m^{t_m}$	True literals
$\prec X_{k+1} \prec \cdots \prec X_n$	False variables

If a given variable x_i is true, the associated vertex X_i has only successors, if it is false it has only predecessors, so in both cases $C_\prec^{+-}(\{X_i\}) = 0$. For a given clause c_j, the variable $l_j^{f_j}$ is false so the corresponding X_i is a successor of $L_j^{f_j}$, which also has successors $L_j^{a_j}$ and $L_j^{t_j}$, but no predecessor. Similarly, $L_j^{t_j}$ has no successor; thus $C_\prec^{+-}(\{L_j^{f_j}, L_j^{t_j}\}) = 0$. Now $L_j^{a_j}$ has one predecessor $L_j^{f_j}$, one successor $L_j^{t_j}$, and one neighbor X_i that is a predecessor if x_i is true, otherwise a successor; in both cases, $C_\prec^{+-}(\{L_j^{a_j}\}) = 2$. The only vertices with a non-negative cost are the $L_j^{a_j}$, so the sum over all m clauses gives $C_\prec^{+-}(V_\phi) = 2m$. □

Proposition 2 ($\Longleftarrow$) *Given an instance ϕ of NAE3SAT+ with m clauses and the associated graph G_ϕ, if there exists an order $\prec$ on V_ϕ such that $C_\prec^{+-}(V_\phi) \leq 2m$ then ϕ is satisfiable.*

Proof: Conversely, consider an order $\prec$ on V_ϕ such that $C_\prec^{+-}(V_\phi) \leq 2m$. For all j, define $f_j, a_j, t_j \in \{1, 2, 3\}$ such that $L_j^{f_j} \prec L_j^{a_j} \prec L_j^{t_j}$; then $L_j^{a_j}$ has one successor, one predecessor, and one other neighbor X_i, so its cost is 2. As G_ϕ contains m such independent triangles, $C_\prec^{+-}(\{L_1^{a_1}, \dots, L_m^{a_m}\}) = 2m$. To ensure $C_\prec^{+-}(V_\phi) \leq 2m$, all the other vertices must have either only predecessors or only successors. If vertex X_i has successors only, assign x_i to true; if X_i has predecessors only, assign x_i to false. For all j, $L_j^{f_j}$ has at least 2 successors ($L_j^{a_j}$ and $L_j^{t_j}$) so its corresponding X_i has to be a successor, which means $x_i = l_j^{f_j}$ is false; similarly, $l_j^{t_j}$ is true. Each clause thus has one true and one false literal, so ϕ is satisfied. □

B NP-hardness of the C^{++} problem

Order of elimination. In the main text of the paper, we search for a permutation π but the only important aspect of the permutation is that it defines an order on the vertices. For this NP-hardness proof, it will help think of the following equivalent but more "intuitive" formulation of the problem: we are looking for an order $\prec$ minimizing the cost function of interest. We can think of the order as an order in which we eliminate vertices and each time we eliminate a vertex with an outdegree d we pay a cost of d^2 and the cost of an order is the cost of eliminating all vertices.

For the formulation using orders it will help to look at the set of neighbors of u appearing after u in the order $\prec$, which we denote $succ_\prec(u)$ for an order $\prec$. Therefore $|succ_\prec(u)|^2$ is the cost that we pay when we remove u, which allows us to reformulate the C^{++} problem as:

Problem 3 (C^{++}) *For a given undirected graph $G = (V, E)$ and an integer K, does there exist an order $\prec$ of the vertices such that $\sum_{u \in V} |succ_\prec(u)|^2 \leq K$?*

The weighted-C^{++} problem. For the sake of simplicity our proof of completeness will rely on a second novel problem, the weighted-C^{++} problem, and we will show that C^{++} is NP-complete by exhibiting first a reduction between C^{++} and weighted-C^{++} and then a second reduction between weighted-C^{++} and the Set Cover problem (a well-known NP-complete problem). We now present the weighted-C^{++} problem:

Problem 4 (weighted-C^{++}) *Given an undirected graph $G = (V, E)$, a vertex-weighting function $w : V \to \mathbb{N}$ and an integer K, does there exist an order $\prec$ of the vertices such that $\sum_{u \in V} (|succ_\prec(u)| + w(u))^2 \leq K$?*

Terminology. Given a graph G with the vertex weighting function w and an order $\prec$, the *cost* is the function $\sum_{u\in V}(|succ_\prec(u)|+w(u))^2$ applied to the graph with that order. The *optimal cost* of a graph is the minimal cost achievable by any order. Notice that an instance of the weightless problem can be viewed as an instance of the weighted problem where all weights are 0.

B.1 Optimality criteria for orders.

One difficulty of the reduction proofs is to show that an order necessarily behaves in a controlled way. We see in this section several criteria that ensures that some order has an optimal cost.

We define the notion of multiset of costs that will help expressing optimality criteria for orders. Given a graph G and an order $\prec$, the multiset of costs $MC(G,\prec)$ is the multiset composed of the $(|succ_\prec(u)|+w(u))$ for each vertex u in G. The *linear cost* of a multiset M is the sum of elements in the multiset, i.e., $\sum_{c\in M} c$. The *cost* (or *squared cost*) of a multiset M is $\sum_{c\in M} c^2$.

Property 2 *For a graph G (weighted or not) the size and the linear sum of the multiset $MC(G,\prec)$ does not depend on the order $\prec$.*

Proof: By definition, the size of $MC(G,\prec)$ is $|V|$, the number of vertices in G, and its linear cost is $\sum_{c\in M} c = \sum_{u\in V}|succ_\prec(u)|+w(u) = |E|+\sum_{u\in V} w(u)$. □

Note that this allows us to talk about the *linear cost* of a graph G as the linear cost of any multiset of costs, corresponding to any of its order.

Property 3 *When there exists some $d\in\mathbb{N}$ such that $MC(G,\prec)$ contains only the values d and $d+1$ then the order $\prec$ is optimal.*

Proof: Let us consider an order $\prec$ as described above: it only contains a times d and b times $d+1$ for some d, and let us consider any optimal order $\prec'$ of G. Suppose that the multiset $MC(G,\prec')$ contains e and f such that $e < f-1$. Then replacing them by $e+1$ and $f-1$ reduces the cost because $(e^2+f^2)-\big((e+1)^2+(f-1)^2\big) = 2(f-1-e) > 0$. By iteratively applying this operation we end up with a multiset M that has the same size and the same linear cost but a lower cost than $MC(G,\prec')$ and contains a' times d' and b' times $d'+1$ for some d'.

Without loss of generality we can suppose that there is at least one d in $MC(G,\prec)$ which means that d is the quotient of the Euclidean division of the linear cost by $|V|$. For the same reason, d' is the quotient of the Euclidean division of the linear cost of $MC(G,\prec')$ by $|V|$. Because the linear cost of $MC(G,\prec)$ does not depend on $\prec$ this proves that $d'=d$ as well as $a=a'$ and $b=b'$, which in turn implies that the costs of $MC(G,\prec)$ and $MC(G,\prec')$ are similar, and thus $\prec$ is optimal. □

While the property above is true for any graph (weighted or not) it is not really useful for weightless graphs because, in a weightless graph, the last vertex u that we eliminate in the order $\prec$ has $|succ_\prec(u)|=0$, and more generally the vertex u_i which is ordered in the i-th position from the end, has $|succ_\prec(u_i)| < i$. The following property handles this case:

Property 4 *For a weightless graph, when there exists $d\in\mathbb{N}$ such that $MC(G,\prec)$ contains all integers from 0 to $d+1$ and at most once the integers 0 to $d-1$, then $\prec$ is an optimal order.*

Proof: The proof of optimality is similar to the proof of property 3: when the property does not hold, we can find two elements v_i and v_j with $v_i+2\le v_j$ and we can diminish the cost by setting $v_i = v_i+1$ and $v_j = v_j-1$. □

Finally, let us introduce the notion of *marginal cost* for a multiset.

Definition 3 (Marginal cost) *We introduce the* marginal cost *to measure how much a multiset deviates from the optimal repartition (as given in property 3). Formally, given a multiset M of size n, we can compute d such that the linear cost of M is $n\times d+v$ where $1\le v\le n$. The* marginal cost c_m *of M is then:*

$$c_m = \sum_{u\in M}\max\Big(0, u-(d+1)\Big)$$

Note that we can equivalently define the marginal cost of M as $\sum_{u\in M} u - \sum_{u\in M} min(d+1,u)$. We know from property 3 that the multiset M' that minimizes the cost with the same linear cost and the same size only contains d and $d+1$ (with at least one $d+1$ since $v>0$). In other words the marginal cost counts the number of elements larger than needed and how much they go over the average cost: if we have a $d+2$ it counts for 1, if we have a $d+5$ it counts for 4, etc.

Note that the marginal cost cannot be used directly to decide if an order is optimal. Indeed, consider the two following multisets: M composed of nine times the value 10 and one time the value 11 and M' composed of nine times the values 11 and one time the value 2. They have the same size, the same linear cost and the same marginal cost (which is 0) but M has a lower squared cost than M'.

The following property describes the minimal cost among all the multisets with the same size, the same linear cost and the same marginal cost:

Property 5 *Among all the multisets that have a size n, a linear cost of $d\times n+v$ with $1\le v\le n$ and a marginal cost of at least k (with $2k<v$), then the ones achieving the minimal cost are composed of k times the value $d+2$, $v-2k$ times the value $d+1$ and $(n-v+k)$ times the value d.*

Proof: Take such a multiset M with size n, linear cost of $d\times n+v$ and marginal cost of at least k, satisfying $2k<v$. Suppose that M contains a value $d-i$ with $i>0$. Because the linear cost of M is strictly larger than $d\times n$ we can find at least one value $d+j$

with $j > 0$ such that diminishing $d + j$ to $d + j - 1$ keeps the marginal cost above k.

Indeed, in a first case, there is at least one value $d + 1$ in M, and diminishing this value to d does not affect the marginal cost c_m. In the other case, let us consider the sum $\sum_{u \in M} min(d + 1, u)$, by definition of the marginal cost, there are no more than c_m elements in M larger than $d + 1$. All other elements are at most d with at least one at $d - i < d$. So, we have $\sum_{u \in M} min(d + 1, u) < nd + c_m$ and using $c_m = \sum_{u \in M} u - \sum_{u \in M} min(d + 1, u)$, we find that

$$c_m > nd + v - (nd + c_m) \Rightarrow 2c_m > v$$

and as we have made the assumption that $v > 2k$, we have $c_m > k$. Consequently, we can also find in this case a value $d + j$ with $j > 0$ such that diminishing $d + j$ to $d + j - 1$ keeps the marginal cost above k.

We can deduce from this observation that the multiset of size n, with a linear cost $d \times n + v$ with $1 \leq v \leq n$ and a marginal cost of at least k (with $2k < v$) that achieves the minimal cost has k times the value $d + 2$, $v - 2k$ times the value $d + 1$ and $(n - v + k)$ times the value d. □

This property can then be used to compare multisets and is summarized by the following property:

Property 6 *When M has a marginal cost of k then the cost of M is at least $2k$ larger than the balanced distribution (as given by property 3). This $2k$ bound is reached for the optimality criterion described in property 5.*

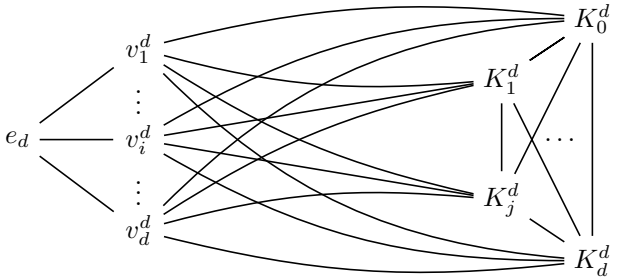
Proof: As seen before the optimal can be reached by taking two values $i, j \in M$ with $i + 2 \leq j$ and changing them to $i + 1$ and $j - 1$. This balancing operation can reduce by at most 1 the marginal cost but reduces the cost by $i^2 + j^2 - (i + 1)^2 - (j - 1)^2 = 2(j - i) - 2$ and since $j - i \geq 2$ this means the reduction is at least 2 and exactly 2 when $i + 2 = j$. Since we need at least k balancing operations to reach the optimal, this gives us at least a $2k$ reduction of the cost to reach the optimal. Notice that when dealing with an optimal multiset in the sense of property 5 we only combine a d with a $d + 2$ which gives us the exact bound. Conversely if we are not in the case of 5 we will have to combine something below (or equal to) d with something larger than $d + 3$ or do a combination that does not diminish the marginal cost (such as combining $d + 1$ and $d + 3$). □

B.2 Reduction between weighted-C^{++} and C^{++}.

Any instance of C^{++} can be seen as an instance of weighted-C^{++} where the weights are set to 0. For that purpose, the idea is to take a vertex u with some non null weight $w(u)$, link u to $w(u)$ vertices $v_1, \ldots, v_{w(u)}$ and make sure that we can guarantee that u appears before all the $v_1, \ldots, v_{w(u)}$ in any optimal order. We will thus exhibit a family of graphs to create such v_i vertices before showing that these v_i vertices can always appear after u in the order. Finally we will prove the full reduction.

B.2.1 The L_d family of graphs.

Let us consider the graph L_d parameterized by $d \in \mathbb{N}$ that contains a $(d + 1)$-clique K^d composed of the vertices $K_0^d, \ldots, K_d^d$, one vertex e_d that has d neighbors $v_1^d \ldots v_d^d$ and such that there is an edge between each v_i^d and each vertex of K^d. Consequently, there are three types of vertices in L_d: the vertex e_d, the vertices of type V (the $(v_i^d)_i$) and the vertices of type K (the $(K_i^d)_i$). Here is a depiction of L_d:



Best cost C_d for L_d. In the weightless case, the best cost C_d for L_d is induced by the order that starts with e_d followed by the v_i^d nodes and finally by the K_i^d nodes. Indeed, in that case the cost is d^2 for e_d, $(d + 1)^2$ for each v_i^d and i^2 for K_i^d (supposing we start with K_d^d and end with K_0^d). This is optimal by virtue of property 4.

Best cost for L_d with a weight 1 on e_d. If we add a weight 1 on e_d then the best cost can be achieved with the same order but this time the cost of e_d is increased from d^2 to $(d + 1)^2$ which means an increase of $2d + 1$. In other words, in that case, the best cost is $C_d + 2d + 1$. Note that here, the optimality cannot be deduced directly from property 4 as the property only applies to weightless graphs. However we prove that there is an optimal order starting with e_d.

For that, consider any order $\prec$ and let us show that $\prec$ can always can be improved to an order that places the vertex e_d in first position.

The order $\prec$ ranks three types of vertices: e_d, V nodes and K nodes, according to the description above. Let us first suppose that there is a vertex of type K before a vertex of type V before the vertex e_d. In that case the first i vertices are of type V (we can have $i = 0$), then we have $j + 1$ vertices of type K and then one vertex of type V. Let us consider how the cost changes by exchanging this last K with this last V, i.e., to change from $V^i K^j K V$ to $V^i K^j V K$. It is clear that the cost changes only for the exchanged V and K. Before the exchange the cost of V was $(d - j)^2$ and after it is $(d - j + 1)^2$ whereas for K it was $(2d - i - j)^2$

and after it is $(2d-i-j-1)^2$. Overall, if ΔC is the difference between the cost before and the cost after the exchange, we have:

$$\begin{aligned}\Delta C &= (2d-i-j)^2 - (2d-i-j-1)^2 \\ &\quad + (d-j)^2 - (d-j+1)^2 \\ &= 2d-2i-4 \\ &= 2(d-i-2)\end{aligned}$$

Therefore, unless $i+1=d$, the cost decreases which means that we can always move the V vertices at the beginning except for maybe one to improve the cost of the order. In the end we have that the beginning of an optimal sequence can be restricted to the form $V^iK^le_d$ or $V^{d-1}K^lVe_d$. In the first case, transforming $V^iK^le_d$ into $e_dV^iK^l$ decreases the score by (i^2+i). In the second case, transforming $V^{d-1}K^lVe_d$ into $e_dV^{d-1}K^lV$ decreases the score by d^2+d-2l (which is ≥ 0 because $l \leq d$). Thus, we can move in all cases e_d at the beginning of the order to improve the related cost.

We have proved that the best cost can be achieved by placing e_d at the beginning of the order. As the best cost C_d for the rest of the order is unaffected by the cost of the elimination of e_d first, we have that the best cost is C_d+2d+1.

B.2.2 Partitioned graphs.

Let us consider a graph G composed of two subgraphs G_1 and G_2 plus exactly one edge $\{e_1,e_2\}$ with $e_1 \in V_1$ and $e_2 \in V_2$. Any order $\prec$ on V induces an order on V_1, an order on V_2 and an order between e_1 and e_2. If another order $\prec'$ induces the same order on V_1, the same order on V_2 and the same order between e_1 and e_2, it has the same cost as $\prec$. Therefore, an optimal order for G can be seen as either an optimal for G_1 and an optimal order for G_2 where we add a weight of 1 on e_2 (if e_2 precedes e_1), or an optimal order for G_2 and an optimal order for G_1 where we add a weight of 1 on e_1 (if e_1 precedes e_2).

As a result, we obtain the following property:

Property 7 *If adding a weight 1 on e_1 in G_1 increases the best cost of G_1 by x and if adding a weight 1 on e_2 increases the best cost of G_2 by at most x, then the best cost of G is equal to the best cost of G_1 plus the best of G_2 where we add a weight of 1 on e_2.*

B.2.3 Finishing the reduction.

Property 8 *Let (G,K) be an instance of the weighted problem, we can compute an equivalent instance of the weightless problem in a time polynomial in the number of edges and vertices in G plus the sum of weights in G.*

Proof: If all the weights in G are zeros, the result is immediate. Let us suppose that there is a vertex u with a weight $w(u)>0$ and a degree $d-w(u)$. Let us consider the graph G' composed of G but where the weight of u is reduced by 1 plus a fresh copy of L_d and an edge between u and e_d. We claim that the best cost of G' is lower than $K+C_d$ if and only if the best cost of G is lower than K.

Indeed, we have shown that the graph L_d is such that adding a weight 1 on e_d increases the best cost from C_d to C_d+2d+1. We also know that the sum of the degree of u plus its cost is d therefore for any order $\prec$ adding a weight 1 on u increases the cost of $\prec$ by at most $2d+1$. By applying property 7 where G has the role of G_2 (u is e_2) and L_d of G_1 (e_d is e_1) and $x=2d+1$, we obtain that the best cost of G' is the best cost of G where node u has a weight increased by 1. In other words, it is equivalent in terms of best cost to handle the graph G or to handle the graph G' where the weight of node u has been decreased by 1 unit.

By applying $\sum_u w(u)$ times this property we obtain an instance (G',K') of the weightless problem which is equivalent to the instance of the weighted problem (G,K). This new instance has $\sum_u w(u) \times |L_{deg(u)+w(u)}|$ more vertices than the original one, but this is still polynomial in the size of G plus the sum of weights and the resulting instance can be computed in polynomial time. □

This proves that if the weighted-C^{++} problem is strongly NP-hard, the C^{++} problem is also NP-hard.

B.3 Reduction between the weighted-C^{++} and Set Cover.

Our reduction for the weighted case will be a *strong reduction*, meaning the version of the problem where the weights are polynomial in the size of the graph is still NP-hard. It will be based on the *Set cover problem*. We recall here the definition of this problem and invite the reader to check the literature for a proof of its NP-completeness:

Problem 5 (Set cover) *Given two integers n,k, we denote U the set of elements $\{1,\ldots,n\}$. Let P be a set of sets of elements of U, does there exist a subset $P' \subset P$ of size k such that $\cup_{S\in P'} S = U$?*

Let us fix an instance (P,n,k) of the Set Cover problem asking whether we can find k sets $S_1,\ldots,S_k$ in P such that $S_1 \cup \cdots \cup S_k = U$. We suppose, without loss of generality, that the instance is not trivial in the sense that $|P| \geq k$ (there are at least k sets in P), $\cup_{S\in P} S = U$ (each integer in U is contained in at least one $S \in P$) and that all sets $S \in P$ are such that $S \subseteq U$.

Let us exhibit a weighted graph G and a value V such that best cost for G is less than V if and only if $\{1,\ldots,n\}$ can be covered with k sets from P.

B.3.1 Construction of a weighted-C^{++} instance from a Set Cover instance.

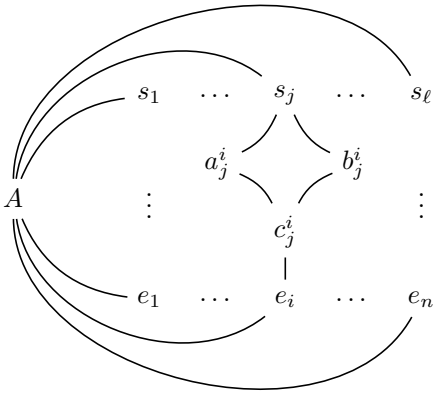
Our reduction will provide a graph G with a weight function w depending on a parameter d such that the Set Cover instance has a solution if and only if the best order has a multiset of costs containing at most k

values $d+2$ and all other values are either d or $d+1$ (we will explicit later the values of V and d).

Vertices of G. In G the vertices are: a special vertex A, n vertices $e_1, \ldots, e_n$ one for each $i \in \{1, \ldots, n\}$, ℓ vertices, $s_1, \ldots, s_\ell$ with one vertex s_j for each set $S_j \in P$ and finally three vertices a_j^i, b_j^i, c_j^i for each $i \in S_j$.

Edges of G. The vertex A has an edge with all vertices of the form s_j or e_i. For a pair (i,j) with $i \in S_j$, both a_j^i and b_j^i have an edge with s_j and c_j^i; in turn c_j^i has an edge with e_i.

Overall the graph G looks like this:



Weights of vertices in G. Recall that the cost of a vertex is the sum of its degree and its weight. In G, we set the weights so that each vertex has a cost of $d+2$, except for the c_j^i which have a cost of $d+3$ and the vertex A which has a cost of $d+1+n+k$. Parameter d needs to be large enough so that all weights are positive This is not constraining for vertices a_j^i, b_j^i and c_j^i. Vertices s_j have degree 1 plus twice the number of i appearing in set S_j, i.e., $1 + 2 \times |S_j|$, so it suffices that $d > 2 \times |S_j|$ for all $S_j \in P$. Vertices e_i have degree 1 plus the number of S_j sets where i can be found, which is at most $1+\ell$. Vertex A has degree $\ell + n$. Having the additional condition $d > \ell$ is sufficient to guarantee the constraint on A and e_i vertices.

Value of V. As we will show, when there is a Set Cover with k sets then we have an order $\prec$ for G such that $MC(G, \prec)$ contains k times the value $d+2$ (corresponding to the k selected sets), $\sum_{S \in P} |S| - n$ times the value d and all the other values are $d+1$. It implies that the cost $V = k(d+2)^2 + (\sum_{S \in P} |S| - n)d^2 + r(d+1)^2$, where r is the number of vertices in G minus k and minus $(\sum_{S \in P} |S| - n)$.

Note that, per property 5, this value V corresponds to the minimal cost for an order that has a marginal cost of k. Conversely, we will show that if there is a solution with a marginal cost of k or less then there is a Set Cover with k sets, proving that it is a reduction. Note that this converse direction is stronger than what is needed as there exists multisets with a marginal cost of k that do not match the minimal cost.

The general intuition underlying the equivalence between a solution (if any) of the Set Cover problem and a solution of the corresponding weighted-C^{++} problem is the following. The first k vertices s_j selected in the elimination order correspond to the S_j sets that cover U. Indeed, each of these vertices generate exactly a marginal cost of 1 and all other nodes according to the elimination order will not generate any marginal cost if we can eliminate all e_i nodes without adding any marginal cost. This condition is met if deleting the k first s_j nodes allows to decrease the cost of all e_i nodes by (at least) 1 unit, which means that we have deleted at least one triplet a_j^i, b_j^i, c_j^i related to node e_i. If so, we have found an elimination order with cost V as well as k sets $S_1 \ldots S_k \in P$ which cover U.

B.3.2 Proof that a solution to Set Cover implies a solution to C^{++}.

Suppose that we have a solution to Set Cover with the sets $S_{j_1}, \ldots, S_{j_k}$. Let us prove that our graph G has an elimination order where the cost of each vertex is d or $d+1$ or $d+2$ but with only k vertices with cost $d+2$.

The elimination order can be built by having j going through $j_1, \ldots, j_k$. For each j value, we eliminate first s_j for a cost of $d+2$, then we go through $i \in S_j$ and eliminate the corresponding a_j^i and b_j^i vertices (both at cost $d+1$ once s_j has been removed). Then we eliminate c_j^i (for a cost of d if e_i is already eliminated and $d+1$ otherwise). Finally, if e_i has not yet been eliminated by a previous j value, we eliminate it for a cost of $d+1$.

Once we have done this, the vertex A has lost $k+n$ neighbors: all the e_i and the k vertices s_j that we have selected. Its remaining cost is $d+1$ so we eliminate it, which in turn means that all the remaining s_j have a cost of $d+1$ and we can eliminate them all (with their a_j^i, b_j^i and c_j^i attached).

Overall the cost of this elimination order is exactly V.

B.3.3 Proof that a solution to weighted-C^{++} implies a solution to Set Cover.

Suppose that we have an order $\prec$ such that the total cost is below V. Since V is the optimal cost for a marginal cost of k, the order $\prec$ cannot have a marginal cost higher than k otherwise its cost would be higher than V (see property 6). Knowing that $\prec$ has a marginal cost of at most k, we will extract a solution to the corresponding Set Cover instance.

First we notice that when A is eliminated, its cost is $d+1+k-E_s+E_n+R_n$ where E_s the number of s_i eliminated, and R_n is the number of e_i remaining. However, as long as A is not eliminated, E_s is less than (or equal to) the marginal cost of all the vertices eliminated before A. Indeed, if s_j is eliminated while A is still present it is because we have paid a marginal

cost at least 1 to eliminate directly one of s_j or a_j^i or b_j^i or c_j^i for $i \in S_j$. That is true because if A is present, then all those vertices have a cost of $d+2$ except c_j^i that has a cost $d+3$ or $d+2$ depending on whether e_i is eliminated or not yet.

Overall when we eliminate A, we pay a marginal cost of $(k - E_s) + R_n$ where E_s is the number of sets eliminated and R_n is the number of integers not yet eliminated. The marginal cost of the order $\prec$ is at least the marginal cost of all vertices eliminated before A plus the marginal cost for A. Because the marginal cost of vertices removed before A is at least E_s, by adding the marginal cost of A we get a marginal cost larger than $E_s + (k - E_s) + R_n = k + R_n$ which can be equal to k only if $R_n = 0$ which means that all vertices e_i corresponding to integers $\{1 \dots n\}$ have been eliminated. Note that if a vertex e_i is directly eliminated without eliminating first a vertex s_j and a triplet a_j^i, b_j^i, c_j^i then we have to add a marginal cost of 1 specifically for this vertex e_i. But in that case, it means that the marginal cost of all vertices before A includes the cost of removing this e_i which means that we cannot have an overall marginal cost of k. Combining everything we get that if we have an order that has a marginal cost of k and thus a cost of at most V, then we have k sets $S_{i_1}, \dots, S_{i_k}$ covering all integers in U.