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Counting partitions by genus.

A compendium of results

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Abstract

This article is devoted to the enumeration of set partitions, according to their length, number of parts, cyclic type, and genus. Genus dependent Bell, Stirling numbers, and Faà di Bruno coefficients are introduced. Besides attempting to summarize what is already known on the subject, we obtain new generic results (in particular for partitions into two parts, for arbitrary genus), and present computer generated new data extending the number of terms known for sequences or families of such coefficients; this also allows us to propose new conjectures.

Keywords: set-partitions.

Mathematics Subject Classification: 05A18, 60Cxx, 05A15.

Introduction

This is the second paper in a series devoted to the combinatorics of set partitions and their enumeration according to their genus. In a previous paper [20], functional equations were written between generating functions (G.F.) of partitions, enabling one to count partitions in genus 0, 1 and 2. In the present paper, which is completely independent, our goal is different. We wish to collect as much data as possible on that combinatorics. Accordingly, our paper gathers classical and known results as well as new data, obtained by computer “brute force” calculations and a few exact new results. In many cases, these data suggest conjectures and extrapolations, that we mark with the sign $\stackrel{?}{=}$.

This endeavor has benefited in a tremendous way from the existence of the On Line Encyclopedia of Sequences [16]. Several unexpected connections and identifications have been made possible thanks to this irreplaceable and unique source ¹.

Our paper is organized as follows. In section 1, we recall some basic definitions: total numbers of partitions are given by Bell numbers, and by Stirling numbers when the number of parts is fixed. The key notion of genus is also recalled. In Sect. 2, the Bell numbers are refined by fixing the genus of partitions, and by including partitions with or without singletons. Explicit expressions for their G.F. are given in genus 0 to 3, and the general form conjectured in higher genus. The same steps are repeated in Sect. 4 for the Stirling numbers, with again exact or conjectured results for their counting and G.F.

The rest of the paper is devoted to the counting of partitions of given cyclic type. In Sect. 5, we review three families of partitions for which this counting is generically known: the famous non-crossing partitions (i.e., of genus 0); the partitions into pairs, i.e., of type $[2^k]$, and arbitrary genus; and the partitions into two parts, for which we obtain a result in arbitrary genus, which is new, to the best of our knowledge. Section 6 gathers data on various types of partitions for which we have only partial results and conjectures: types $[p^k]$ for varying p or k and three-part partitions. Finally Tables in Appendix contain the numbers of partitions of the set $\{1, \dots, n\}$ of arbitrary genus up to $n = 15$.

It is our hope that these results will inspire some readers, lead to new results ... and become soon obsolete!

1 Bell, Stirling, and Faà di Bruno numbers

1.1 Equivalence relations on a set with n elements

Any equivalence relation on a set is specified, and conversely, by a partition α of this set (or set-partition, for short).

The number of equivalence relations on a set with n elements is given by the Bell numbers, which obey the recurrence:

$$B_{n+1} = \sum_{p=0}^n \binom{n}{p} B_p, \text{ with } B_1 = 1, \quad (1)$$

hence

$$B_n = \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{\ell^n}{\ell!}, \quad \text{OEIS sequence A000110.} \quad (2)$$

¹Sometimes we shall refer to some OEIS sequence because it is known to count the partitions under study, but sometimes it only means that the sequence is recognized as one present in OEIS, but not necessarily recognized as relevant for the chosen context.

The exponential generating function of the Bell numbers is

$$\mathcal{B}(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}. \quad (3)$$

1.2 Equivalence relations on a set with n elements, with k equivalence classes

One may impose that the equivalence relations have a given number, say k , of equivalence classes, i.e., that the partition has k parts.

The number of such relations is given by the Stirling number of the second kind, $S_{n,k}$, which obeys the recurrence relation:

$$S_{n,k} = k S_{n-1,k} + S_{n-1,k-1}, \quad n > 1, \text{ with } S_{1,k} = 0, \quad k > 1, \text{ and } S_{1,1} = 1. \quad (4)$$

Explicit form

$$S_{n,k} = \frac{1}{k!} \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} s^n, \quad (5)$$

OEIS sequence A008277.

The exponential generating function of the Stirling numbers $S_{n,k}$ is

$$\mathcal{S}(x, y) = e^{y(e^x - 1)}. \quad (6)$$

Sum rule:

$$B_n = \sum_{k=0}^n S_{n,k}. \quad (7)$$

1.3 Set with n elements, with k equivalence classes of specified cardinalities

One may further impose that the chosen equivalence classes have specified cardinalities.

The set of cardinalities of the classes of the equivalence relation defined by the set-partition α determines a partition $[\alpha] \vdash n$ of the integer n , called the *type* of the partition. It is usual to denote this integer partition as follows: $[\alpha] = [1^{\alpha_1}, \dots, n^{\alpha_n}]$. It can be represented as a Ferrers diagram or as a Young diagram. The number of equivalence relations on a set with n elements having equivalent classes with cardinalities specified by $[\alpha]$ will be denoted by $C_{n,[\alpha]}$. These numbers are sometimes called the Faà di Bruno coefficients.

$$C_{n,[\alpha]} = \frac{n!}{\prod_{\ell=1}^n \alpha_{\ell}! (\ell!)^{\alpha_{\ell}}}. \quad (8)$$

Sum rule: Calling $|\alpha| = \sum_{\ell} \alpha_{\ell}$ the number of parts of the integer partition $[\alpha]$, we have obviously

$$\sum_{\substack{[\alpha] \\ |\alpha|=k}} C_{n,[\alpha]} = S_{n,k}. \quad (9)$$

1.4 Genus of partitions on a cyclically ordered set

If the underlying set of n elements is totally ordered (for definiteness one may take $\{1, 2, 3, \dots, n\}$), or if it is cyclically ordered, one may introduce a new structure, finer than the ones already considered, by determining the genus of set partitions (a non-negative integer). With α a partition of $\{1, 2, 3, \dots, n\}$, we associate a *permutation* τ of \mathcal{S}_n : its cycles are the parts of α , whose elements are in increasing order. We also consider the cyclic permutation $\sigma := (1, 2, \dots, n)$. Then following [?, ?, 12], the genus $g(\alpha)$ is defined by

$$n + 2 - 2g = \#\text{cy}(\tau) + \#\text{cy}(\sigma) + \#\text{cy}(\sigma \circ \tau^{-1}) \quad (10)$$

or in the present case,

$$-2g = |\alpha| - 1 - n + \#cy(\sigma \circ \tau^{-1}) \quad (11)$$

since here $\#cy(\sigma) = 1$ and $\#cy(\tau) = \sum \alpha_\ell = |\alpha|$. As is well known and recalled in [20], Eq. (10) is just Euler's formula for the genus of the associated map, with $f := \#cy(\sigma \circ \tau^{-1})$ the number of faces of that map. Since f is a positive integer, and denoting $k = |\alpha|$ the number of parts of the integer partition $[\alpha]$, we have

$$g = \frac{n - k + 1 - f}{2} \leq \frac{(n - k)}{2}. \quad (12)$$

Also note that one-part partitions ($k = 1$) have necessarily genus 0. Hence for $g > 0$, $k \geq 2$ and

$$n \geq 2g + k \geq 2g + 2. \quad (13)$$

Remark. Assuming the existence of an order on the underlying set is not really a restriction since one can always choose one. Each family of set-partitions (or of equivalence relations) previously considered will be itself decomposed according to the genus, and we shall introduce notations $B_n^{(g)}$, $S_{n,k}^{(g)}$ and $C_{n,[\alpha]}^{(g)}$, with $\sum_g B_n^{(g)} = B_n$, $\sum_g S_{n,k}^{(g)} = S_{n,k}$ and $\sum_g C_{n,[\alpha]}^{(g)} = C_{n,[\alpha]}$.

1.5 Partitions with no singletons

If a set partition has no singleton, its associated equivalence relation is such that no element is isolated. Equivalently, each part of the partition contains at least 2 elements. By adding the constraint that the families of partitions considered previously should have no singleton one can define “associated”² Bell numbers \hat{B}_n , and “associated” Stirling numbers (of the second kind) $\hat{S}_{n,k}$. Obviously,

$$\sum_{k=1}^n \hat{S}_{n,k} = \hat{B}_n. \quad (14)$$

Since the notation $C_{n,[\alpha]}$ already incorporates the partition type, there is no “hat” version of the Faà di Bruno coefficients: either $[\alpha]$ contains singletons, or it does not.

One obtains the following (almost) obvious identities :

Associated Bell numbers \hat{B}_n . See OEIS A000296.

$$B_n = \hat{B}_n + \hat{B}_{n+1} \quad (15)$$

$$\hat{B}_n = \sum_{j=0}^{n-2} (-1)^j B_{n-1-j}. \quad (16)$$

Their exponential generating function is

$$\hat{B}(x) = e^{e^x - x - 1}. \quad (17)$$

Associated Stirling numbers of the second kind $\hat{S}_{n,k}$. See OEIS A008299 (see also A134991 where they are called Ward numbers).

$$\hat{S}_{n,k} = \sum_{\ell=0}^k (-1)^\ell \binom{n}{\ell} S_{n-\ell, k-\ell} \quad (18)$$

²More generally one could introduce s -associated Bell or Stirling numbers by imposing that each part contains at least s elements but in the present paper we consider only the case $s = 2$. See [2]

Conversely,

$$S_{n,k} = \sum_{\ell=0}^{k-1} \binom{n}{\ell} \widehat{S}_{n-\ell, k-\ell} \quad (19)$$

Their exponential generating function is

$$\widehat{S}(x, y) = e^{y(e^x - x - 1)}. \quad (20)$$

They can be expressed in terms of the second-order Eulerian numbers $E^{(2)}$ by

$$\widehat{S}_{n,k} = \sum_{\ell=0}^{n-k} \binom{\ell}{n-2k} E_{n-k, n-k-\ell}^{(2)} \quad (21)$$

The $E^{(2)}$'s can themselves be expressed in terms of Stirling numbers of the second kind by

$$E_{n,k}^{(2)} = \sum_{j=0}^k (-1)^{k-j} \binom{2n+1}{k-j} S_{n+j, j} \quad (22)$$

From (21) and (22) one can recover (18).

Notice that

$$C_{n, [1^r, \alpha']} = \binom{n}{r} C_{n-r, [\alpha']}, \quad (23)$$

where α' has no singleton.

One can also impose a genus restriction on the partitions without singletons, and as singletons do not affect the genus, one is therefore led to consider numbers $\widehat{B}_n^{(g)}$ and $\widehat{S}_{n,k}^{(g)}$ with, of course $\sum_g \widehat{B}_n^{(g)} = \widehat{B}_n$ and $\sum_g \widehat{S}_{n,k}^{(g)} = \widehat{S}_{n,k}$. Moreover

$$C_{n, [1^r, \alpha']}^{(g)} = \binom{n}{r} C_{n-r, [\alpha']}^{(g)}. \quad (24)$$

We shall return to these sequences in the next section.

2 Genus dependent Bell numbers $B_n^{(g)}$

2.1 Unconstrained partitions: basic numbers $B_n^{(g)}$

Genus 0. Known as Catalan numbers. See OEIS A000108.

$$\begin{aligned} B_n^{(0)} = C_n &:= \frac{1}{(n+1)!} \frac{(2n)!}{n!} \\ &= \{1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \dots\} \end{aligned} \quad (25)$$

$$\text{Ordinary G.F. is } B^{(0)}(x) = \frac{1 - \sqrt{1 - 4x}}{2x}. \quad (26)$$

Genus 1. See OEIS A002802.

$$B_n^{(1)} = \frac{1}{2^4 3} \frac{1}{(2n-3)(2n-1)} \frac{1}{(n-4)!} \frac{(2n)!}{n!}, \quad (27)$$

$$\text{Ordinary G.F. is } B^{(1)}(x) = \frac{x^4}{(1-4x)^{5/2}}, \quad \text{see [6]}. \quad (28)$$

Genus 2. The first terms are listed in OEIS A297179 (the formula below seems to be new).

$$B_n^{(2)} = \frac{1}{2^9 3^2 5} \frac{(5n^3 - 39n^2 + 88n - 84)}{(2n-7)(2n-5)(2n-3)(2n-1)} \frac{1}{(n-6)!} \frac{(2n)!}{n!}, \quad (29)$$

$$\text{Ordinary G.F. is } B^{(2)}(x) = \frac{x^6 (1 + 6x - 19x^2 + 21x^3)}{(1-4x)^{11/2}}, \quad \text{see [6]}. \quad (30)$$

Genus 3. We conjecture that

$$B_n^{(3)} \stackrel{?}{=} \frac{1}{2^{13} 3^4 5^7} \frac{(35n^6 - 819n^5 + 7589n^4 - 36009n^3 + 93464n^2 - 129060n + 95040)}{(2n-11)(2n-9)(2n-7)(2n-5)(2n-3)(2n-1)} \frac{1}{(n-8)!} \frac{(2n)!}{n!}, \quad (31)$$

$$\text{Ordinary G.F. is } B^{(3)}(x) \stackrel{?}{=} \frac{x^8 (1 + 60x - 66x^2 - 130x^3 + 1065x^4 - 2262x^5 + 1738x^6)}{(1-4x)^{17/2}}. \quad (32)$$

This suggests for any $g > 0$ the following Ansatz for the ordinary G.F.

$$B^{(g)}(x) \stackrel{?}{=} \frac{x^{2g+2} P^{(g)}(x)}{(1-4x)^{(6g-1)/2}} \quad (33)$$

with an overall power of x dictated by (13) and a polynomial $P^{(g)}$ of degree $3(g-1)$.

We shall see in the sequel a repeated appearance of formulae of that type, in particular with the universal ‘‘critical exponent’’ $(6g-1)/2$ in the denominator.

For genus $g \geq 4$, we have incomplete results that corroborate this Ansatz:

Genus 4. The formula below is conjectured, and one should compute $B_n^{(4)}$ for $n = 16, 17, 18, 19$ to determine all the coefficients a_i

$$B_n^{(4)} = \{1, 352, 19261, 541541, 10571561, 162718556\} \text{ for } n = 10, \dots, 15. \quad (34)$$

Ordinary G.F. should be

$$B^{(4)}(x) \stackrel{?}{=} \frac{x^{10} (1 + 306x + 4035x^2 - 16669x^3 + 63735x^4 - 136164x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9)}{(1-4x)^{23/2}}$$

Similarly, we propose for $g = 5, 6$

$$\begin{aligned} B^{(5)}(x) &\stackrel{?}{=} \frac{x^{12}(1 + 1320x + 75068x^2 + 218300x^3 + \dots)}{(1-4x)^{29/2}} \\ B^{(6)}(x) &\stackrel{?}{=} \frac{x^{14}(1 + 5406x + \dots)}{(1-4x)^{35/2}}. \end{aligned} \quad (35)$$

The first non trivial coefficient in the numerator of $B^{(g)}(x)$, $g > 0$, appears to be always divisible by 6: $6\{1, 10, 51, 220, 901, \dots\}$, for $g = 2, 3, \dots$, and we conjecture that this sequence is given by

$$\frac{(d(g) + 8g + 2)(6g - 2)!}{C(g)(3g - 1)!} - 2(6g - 1)$$

in terms of $C(g)$: $C = 12, 30240, 518918400, 28158588057600, 3497296636753920000, \dots$,
and $d(g)$: $d = 0, 10, 68, 318, 1336, 5426, \dots$, given by

$$C(g) = 3 \times 2^{2g-1} \frac{(2g)! (6g-5)!!}{g! (2g-3)!!} = 12(2g-1) \frac{(6g-5)!}{(3g-3)!} \quad (36)$$

$$d(g) = \frac{1}{3}(4^{1+g} - 1 - 3(6g-1)). \quad (37)$$

Warning. The sequence $B_n^{(0)}$, shifted in such a way that it starts with 1 for $n = 1$, (resp. $B_n^{(1)}$, shifted in such a way that it starts with 1 for $n = 3$) gives also the number of rooted bicolored unicellular maps of genus 0 (resp. of genus 1) on n edges. However this coincidence fails at genus 2 and above. Rooted bicolored unicellular maps are studied by Goupil et al. [8].

2.2 Partitions with no singletons: associated numbers $\widehat{B}_n^{(g)}$

For all g one has the recurrence

$$\widehat{B}_n^{(g)} = B_n^{(g)} - \sum_{s=1}^n \binom{n}{s} \widehat{B}_{n-s}^{(g)} \text{ with } \widehat{B}_n^{(g)} = 0 \text{ for } n < 2g + 2, \text{ and } \widehat{B}_{2g+2}^{(g)} = 1. \quad (38)$$

Genus 0. See OEIA A005043 (Riordan numbers).

$$\widehat{B}_n^{(0)} = \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{j}{\lfloor j/2 \rfloor}. \quad \text{Ordinary G.F. is } \frac{2}{1+x+\sqrt{(1-3x)(1+x)}} = \frac{1-\sqrt{\frac{1-3x}{1+x}}}{2x}, \quad (39)$$

0, 1, 1, 3, 6, 15, 36, 91, 232, 603, 1585, 4213, 11298, 30537, 83097, 227475, ...

For genus $g > 0$, we have again a general Ansatz for the ordinary G.F. of $\widehat{B}_n^{(g)}$

$$\widehat{B}^{(g)}(x) \stackrel{?}{=} \frac{x^{2(g+1)}(1+x)^{g-1} \widehat{P}^{(g)}(x)}{\Delta(x)^{\frac{(6g-1)}{2}}} \quad (40)$$

where $\Delta(x) = (1-3x)(1+x)$ is the discriminant of the algebraic equation satisfied by $\widehat{B}^{(0)}(x)$, namely $\widehat{B}^{(0)}(x) = 1 + (x\widehat{B}^{(0)}(x))^2 / (1 - x\widehat{B}^{(0)}(x))$, see [20], and $\widehat{P}^{(g)}$ is a polynomial of degree $3(g-1)$, see below.

Genus 1. See OEIS A245551.

$$\widehat{B}_n^{(1)} = \sum_{\ell=0}^{n-4} \frac{(-1)^{n-\ell} 3^{\ell-2} (2\ell+3)!! (2n-2\ell-5)!!}{2^{n-4} \ell! (-\ell+n-4)!}, \quad \text{G.F. } \widehat{B}^{(1)}(x) \text{ with } \widehat{P}^{(1)}(x) = 1, \quad (41)$$

0, 0, 0, 1, 5, 25, 105, 420, 1596, 5880, 21120, 74415, 258115, 883883, 2994355, 10051860, ...

Genus 2.

$\widehat{B}_n^{(2)} = 0, 0, 0, 0, 0, 1, 21, 203, 1512, 9513, 53592, 278355, 1359072, 6318312, 28227199, 122005884, \dots$

$$\text{G.F. } \widehat{B}^{(2)}(x) \text{ with } \widehat{P}^{(2)}(x) = (1+9x-4x^2+9x^3), \quad \text{see [20]}. \quad (42)$$

Genus 3.

$\widehat{B}_n^{(3)} = 0, 0, 0, 0, 0, 0, 1, 85, 1725, 21615, 208230, 1685112, 12028588, 78029380, 469278810, \dots$

$$\text{G.F. } \widehat{B}^{(3)}(x) \text{ with } \widehat{P}^{(3)}(x) \stackrel{?}{=} (1+66x+249x^2+226x^3+894x^4-480x^5+406x^6). \quad (43)$$

Genus 4.

$\widehat{B}_n^{(4)} = 0, 0, 0, 0, 0, 0, 1, 341, 15103, 318318, 4615611, 52720668, \dots$

$$\text{G.F. } \widehat{B}^{(4)}(x) \text{ with } P^{(4)}(x) \stackrel{?}{=} 1+315x+6519x^2+20228x^3+65718x^4+95247x^5+\dots$$

with 4 terms missing.

3 Genus dependent Stirling numbers $S_{n,k}^{(g)}$

3.1 Partitions with k parts : the numbers $S_{n,k}^{(g)}$

Genus 0. Known as the Narayana numbers. See OEIS A001263.

$$S_{n,k}^{(0)} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1}. \quad (44)$$

Their two-variable G.F. reads

$$S^{(0)}(x, y) = \frac{1 + x - xy - \sqrt{(1 + x - xy)^2 - 4x}}{2x}. \quad (45)$$

Genus 1. Conjectured by Yip [19], proved by Cori and Heteyi [5].

$$S_{n,k}^{(1)} = \frac{1}{6} \binom{n}{2} \binom{n-2}{k-2} \binom{n-2}{k} = \frac{1}{6} \binom{k}{2} \binom{n}{k} \binom{n-2}{k}, \quad (46)$$

$$\text{G.F. } S^{(1)}(x, y) = \frac{x^4 y^2}{((1 + x - xy)^2 - 4x)^{5/2}}. \quad (47)$$

Genus 2. Obtained by Cori-Heteyi. See OEIS A297178.

$$\begin{aligned} S_{n,k}^{(2)} = & 8\gamma[n-10, k-6] - 4\gamma[n-10, k-5] - 15\gamma[n-10, k-4] + 10\gamma[n-10, k-3] + \gamma[n-10, k-2] \\ & - 4\gamma[n-9, k-5] + 39\gamma[n-9, k-4] - 10\gamma[n-9, k-3] - 4\gamma[n-9, k-2] - 15\gamma[n-8, k-4] \\ & - 10\gamma[n-8, k-3] + 6\gamma[n-8, k-2] - 4\gamma[n-7, k-2] + 10\gamma[n-7, k-3] + \gamma[n-6, k-2] \end{aligned} \quad (48)$$

where

$$\gamma(n, k) = \frac{\binom{n+10}{5} \binom{n+5}{k} \binom{n+5}{n-k}}{\binom{10}{5}}. \quad (49)$$

The result for $S_{n,k}^{(2)}$ may be simplified in the form

$$S_{n,k}^{(2)} = \frac{1}{30240} \chi^{(2)} * \left(\frac{\Gamma(n-t)\Gamma(n-t+5)}{\Gamma(k-s-1)\Gamma(k-s+4)\Gamma(n-k+s-t-3)\Gamma(n-k+s-t+2)} \right)_{\substack{0 \leq t \leq 4 \\ 0 \leq s \leq t}} \quad (50)$$

with a triangular array of constants $\chi^{(2)}$ given by

$$\chi^{(2)}(t, s) = \begin{array}{cccc} & & & & 1 \\ & & & & -4 & 10 \\ & & 6 & -10 & -15 & \\ & -4 & -10 & 39 & -4 & \\ 1 & 10 & -15 & -4 & 8 & \end{array} \quad (51)$$

and the $*$ product means a summation over $0 \leq s \leq t \leq 4$ of the product of the two factors.

Then the G.F. reads

$$S^{(2)}(x, y) = \frac{x^6 y^2 p^{(2)}(x, y)}{((1 + x - xy)^2 - 4x)^{11/2}} \quad (52)$$

$$\begin{aligned} p^{(2)}(x, y) &= \sum_{\substack{0 \leq t \leq 4 \\ 0 \leq s \leq t}} \chi^{(2)}(t, s) x^t y^s \\ &= 1 - x(4 - 10y) + x^2(6 - 10y - 15y^2) - x^3(4 + 10y - 39y^2 + 4y^3) \\ &\quad + x^4(1 + 10y - 15y^2 - 4y^3 + 8y^4) \end{aligned} \quad (53)$$

as first derived by Cori and Heteyi [6].

Observations and conjectures.

For $k = 2$, we have an expression that follows from an exact formula for the two-part partitions of arbitrary genus, see below Sect. 4.3 :

$$S_{n,2}^{(g)} = \binom{n}{2g+2} = \frac{1}{\binom{2g+2}{g+1}} \binom{n}{g+1} \binom{n-g-1}{g+1}, \quad (54)$$

All known data for $k = 3$ and $n \leq 15$, $g \leq 6$ are consistent with

$$S_{n,3}^{(g)} \stackrel{?}{=} \frac{4^{g+1} - 1}{3} \frac{(n-g-1)}{g+2} \binom{n}{2g+3} = \frac{4^{g+1} - 1}{3 \binom{2g+3}{g+1}} \binom{n}{g+2} \binom{n-g-1}{g+2}. \quad (55)$$

In particular

$$S_{n,3}^{(2)} \stackrel{?}{=} \frac{21}{4} \binom{n}{7} (n-3) = \frac{3}{5} \binom{n}{4} \binom{n-3}{4} \text{ and } S_{n,3}^{(3)} \stackrel{?}{=} 17(n-4) \binom{n}{9}. \quad (56)$$

Also at given g , for the lowest $n = 2g + 3$, $S_{2g+3,3}^{(g)} = \frac{4^{g+1}-1}{3}$, OEIS A002450.

Observation: The result obtained in (50) for $S_{n,k}^{(2)}$ can be generalized in terms of an expression that encodes all (presently) known results for $S_{n,k}^{(g)}$, $g \geq 1$.

This expression is as follows

$$S_{n,k}^{(g)} \stackrel{?}{=} \frac{1}{C(g)} \chi^{(g)} * \left(\frac{\Gamma(n-t+g-2)\Gamma(n-t+4g-3)}{\Gamma(k-s-1)\Gamma(k-s+3g-2)\Gamma(n-k+s-t-2g+1)\Gamma(n-k+s-t+g)} \right)_{\substack{0 \leq t \leq 4g-4 \\ 0 \leq s \leq t}} \quad (57)$$

$$= \frac{1}{C(g)} \chi^{(g)} * \left((3g-1)! \binom{n-k+s-t+g-1}{n-k+s-t-2g} \binom{n-t+g-3}{k-s-2} \binom{n-t+4g-4}{n-k+s-t+g-1} \right)_{\substack{0 \leq t \leq 4g-4 \\ 0 \leq s \leq t}} \quad (58)$$

with the integer constant $C(g)$ given in (36) and coefficients $\chi^{(g)}$ given for $g = 3, 4$ by

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where the entries of $\chi^{(3)}$ have been determined from a subset of the existing data, but some of those of $\chi^{(4)}$ are still undetermined at this stage.

Observations:

- Each column of the arrays $\chi^{(g)}$ is symmetric.
- The entries of the first column of $\chi^{(g)}$, which is of length $4g - 3$, are binomial coefficients with alternated signs $(-1)^t \binom{4(g-1)}{t}$.
- The second column is the product of the line $4(g - 1)$ of the triangular array OEIS A144431 (a “sub-Pascal array”) by the coefficient $d(g)$ given above in (37).
- This second column can also be obtained as $d(g)$ times an appropriate line of a matrix defined as the inverse of the matrix of partial sums of the signed Pascal triangle (see A059260).
- The last $(g - 1)$ lines of the array $\chi^{(g)}$ have a vanishing sum (a justification is given below).
- The last line of the array $\chi^{(g)}$ is conjectured to be given by (60) (details are given below).

At genus g , the first non-zero coefficients $S_{n,k}^{(g)}$ appear for $(n, k) = (2g + 2, 2)$, and their “experimental” values up to $n = 4(2g - 1)$, $k = 2(2g - 1)$ can be used to determine the constants $\chi^{(g)}$ but one can lower these two integers by making use of the previous observations.

As already mentioned, (58), evaluated at $g = 1, 2$ gives back $S_{n,k}^{(1)}$ and $S_{n,k}^{(2)}$; moreover its evaluation at $g \geq 3$ is compatible with all presently known “experimental” results, with the Ansatz (55), and the sum over k of $S_{n,k}^{(3)}$ is indeed equal to $B_n^{(3)}$. This justifies the following conjectures.

Genus $g = 3$ conjecture (weak form). The expression (58), with $g = 3$, gives $S_{n,k}^{(3)}$ for all n and k .
Genus g conjecture (strong form). The expression (58), together with an appropriate triangular array of constants $\chi^{(g)}$ gives $S_{n,k}^{(g)}$ for all $n, k, g > 0$.

The corresponding Ansatz on the G.F. reads

$$\text{G.F.} \quad S^{(g)}(x, y) \stackrel{?}{=} \frac{x^{2g+2}y^2 p^{(g)}(x, y)}{((1+x-xy)^2 - 4x)^{(6g-1)/2}} \quad (59)$$

with $p^{(g)}(x, y) = \sum_{\substack{0 \leq t \leq 4(g-1) \\ 0 \leq s \leq t}} \chi^{(g)}(t, s) x^t y^s$.

We have $S^{(g)}(x, 1) = B^{(g)}(x)$, $p^{(g)}(x, 1) = P^{(g)}(x)$. The latter polynomial being (conjectured) of degree $3(g - 1)$ in x , this tells us that the last $(g - 1)$ lines of the array $\chi^{(g)}$ have a vanishing sum.

A further conjecture, in accordance with the existing data, is that the terms of highest degree in x , viz $4(g - 1)$, of $p^{(g)}(x, y)$ are of the form

$$[p^{(g)}(x, y)]_{x^{4(g-1)}} = (1-y)^{2(g-1)} \left[(1-y)^{4g+1} y^{-2g-3} \sum_{j=0}^{2g-2} \frac{2 s_{2g+2+j, j+1}}{(2g+j+2)(2g+j+1)} y^{-j} \right]_+ \quad (60)$$

where $[\cdot]_+$ is the polynomial part in y of the expression and $s_{p,q}$ are the Stirling numbers of the first kind. See OEIS A185259 where these polynomials are tabulated. If true, this conjecture determines the last line of $\chi^{(4)}$ to be

$$\chi^{(4)}(13, s) = \{1, 318, 6831, 6072, -99693, 103950, 196581, -413820, 155628, 146168, -117876, 7776, 8064\}.$$

Particular cases : $S_{n,k}^{(3)}$ and $S_{n,k}^{(4)}$ for small k . The above general conjecture for $S_{n,k}^{(g)}$ leads, when $g = 3, 4$, and small values of $k = 2, 3, 4$, to simple enough formulas that are displayed below. For $k = 2$, they follow from (54). One can check that they are compatible with the known (experimental) values of $S_{n,k}^{(3)}$, up to $n = 15$, (see tables the appendix).

Genus 3.

$$S_{n,2}^{(3)} = \binom{n}{8} = (0, 0, 0, 0, 0, 0, 0, 1, 9, 45, 165, 495, 1287, 3003, 6435, \dots) \quad \text{see OEIS A000581}$$

$$S_{n,3}^{(3)} \stackrel{?}{=} 17(n-4) \binom{n}{9} = (0, 0, 0, 0, 0, 0, 0, 0, 85, 1020, 6545, 29920, 109395, 340340, 935935, \dots)$$

$$S_{n,4}^{(3)} \stackrel{?}{=} \frac{5}{3}(32n^2 - 288n + 613) \binom{n}{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1555, 24145, 194150, 1085370, 4759755, 17482465, \dots)$$

Genus 4.

$$S_{n,2}^{(4)} = \binom{n}{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 11, 66, 286, 1001, 3003, \dots) \quad \text{see OEIS A001287}$$

$$S_{n,3}^{(4)} \stackrel{?}{=} \frac{341}{6}(n-5) \binom{n}{11} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 341, 4774, 35464, 186186, 775775, \dots)$$

$$S_{n,4}^{(4)} \stackrel{?}{=} \frac{11}{2}(65n^2 - 715n + 1842) \binom{n}{12} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 14421, 252538, 2288286, 14369355, \dots)$$

3.2 Partitions with no singletons: associated numbers $\widehat{S}_{n,k}^{(g)}$

Genus 0. See OEIS A108263.

$$\widehat{S}_{n,k}^{(0)} = \frac{1}{(n-k+1)} \binom{n-k-1}{n-2k} \binom{n}{k}. \quad (61)$$

$$\text{Ordinary G.F. is } \frac{1+x-\sqrt{(1-x)^2-4x^2y}}{2x(xy+1)}. \quad (62)$$

Genus 1.

$$\widehat{S}_{n,k}^{(1)} = \frac{1}{6} \binom{k}{2} \binom{n}{k} \binom{n-k}{k}, \quad (63)$$

$$\text{Ordinary G.F. is } \frac{x^4y^2}{((1-x)^2-4x^2y)^{5/2}}. \quad (64)$$

Genus 2 and above. It is conjectured that for generic genus $g > 0$, the G.F. has the form

$$\widehat{S}^{(g)}(x, y) \stackrel{?}{=} \frac{x^{2g+2}y^2\widehat{p}^{(g)}(x, y)}{((1-x)^2-4x^2y)^{(6g-1)/2}} \quad (65)$$

with $\widehat{p}^{(g)}(x, y)$ a polynomial of degree $4(g-1)$ in x . For instance

$$\widehat{p}^{(2)}(x, y) = 1 + 2x(-2+7y) + x^2(6-22y+21y^2) + x^3(-4+2y+7y^2) + x^4(1+6y-19y^2+21y^3), \quad (66)$$

as derived in [20].

Remark. Note that $p^{(2)}(x, 0) = \widehat{p}^{(2)}(x, 0) = (1-x)^4$ so that the term of order y^2 in $S^{(2)}(x, y)$ or $\widehat{S}^{(2)}(x, y)$, i.e., the G.F. of genus 2 partitions into two parts with or without singleton is $x^6/(1-x)^7 = \frac{1}{2} \sum_{n=4}^{\infty} \frac{n}{3} x^n \sum_{p=1}^{n-1} \binom{p-1}{2} \binom{n-p-1}{2}$, in agreement with formula (83) below.

By the same token, we may assert that $p^{(g)}(x, 0) = \widehat{p}^{(g)}(x, 0) = (1-x)^{4(g-1)}$ so that the term of order y^2 in $S^{(g)}(x, y)$ or $\widehat{S}^{(g)}(x, y)$ is $x^{2g+2}/(1-x)^{2g+3} = \frac{1}{2} \sum_{n=2g}^{\infty} \frac{n}{g+1} x^n \sum_{p=1}^{n-1} \binom{p-1}{g} \binom{n-p-1}{g}$. This implies that

$$\widehat{S}_{n,2}^{(0)} = S_{n,2}^{(0)} - n = \frac{1}{2}n(n-3) \quad \text{and} \quad \widehat{S}_{n,2}^{(g)} = S_{n,2}^{(g)} = \binom{n}{2g+2} \quad \text{for } g > 0 \quad (67)$$

in agreement with the result (54). These numbers can be recognized as the elements of the array OEIS A275514.

Particular case $n = 2k$.

Since we assume in this section that there are no singletons and since k denotes the number of parts, the equality $n = 2k$ implies that each part is of length 2, so that the type is determined: $[\alpha] = [2^k]$, and $\widehat{S}_{2k,k} = C_{2k,[2^k]}$. Because of this coincidence, we postpone the study of this particular case to the next section, which is devoted to the study of coefficients $C_{n,[\alpha]}^{(g)}$.

4 Genus dependent Faà di Bruno coefficients $C_{n,[\alpha]}^{(g)}$.

Part I. Fully solved cases.

The genus dependent Faà di Bruno coefficients $C_{n,[\alpha]}^{(g)}$ are explicitly known in many specific cases, for particular types $[\alpha] = [\dots \ell^{\alpha_\ell} \dots]$ and/or for particular values of the genus g , most of them

discussed and summarized in section 5. However, to the best of our knowledge, they are generically known in only three families of cases, two of them are classic – the cases of genus 0, all types; and the partitions of type $[2^p]$, all g –, and the third one is new, the partitions into two parts, i.e., of type $[p, n - p]$; all g . We review these three cases in this section. In addition, the G.F. of all types of partitions are explicitly known for genus 1 and 2, see [20], although the extraction of explicit formulae for the Faà di Bruno coefficients is arduous.

4.1 The particular case $g = 0$

$C_{n,[\alpha]}^{(0)}$ is the number of *non crossing* partitions (also called planar partitions) of type $[\alpha]$.

$$C_{n,[\alpha]}^{(0)} = \frac{n!}{(n+1 - \sum \alpha_\ell)! \prod_\ell \alpha_\ell!} = \frac{1}{n+1} \binom{n+1}{\alpha_1, \dots, \alpha_n, n+1 - \sum \alpha_j} \quad (68)$$

where the symbol $\binom{\cdot}{\cdot}$ denotes a multinomial coefficient. It was first derived by Kreweras [13], and reappeared later in the context of large random matrices [1] and of free probabilities and their free (or non crossing) cumulants [15]. One may also collect these expressions into a G.F.

$$Z^{(0)}(x) = 1 + \sum_{n \geq 1} x^n \sum_{[\alpha] \vdash n} C_{n,[\alpha]}^{(0)} \prod_{\ell=1}^n \kappa_\ell^{\alpha_\ell} \quad (69)$$

where the κ_ℓ are new indeterminates, from which we may also construct the function $W(x) = \sum_{\ell \geq 1} \kappa_\ell x^\ell$. Then, it was shown in [1] that (68) is equivalent to the following functional relation between $Z^{(0)}$ and W

$$Z^{(0)}(x) = 1 + W(x Z^{(0)}(x)), \quad (70)$$

see also [7] for a nice graphical interpretation of that identity.

As recalled above, the genus 1 and 2 G.F. have been constructed in [20]. We don't repeat these formulae here but we shall use them in the following to substantiate some remarks and conjectures.

4.2 Type $[\alpha] = [2^k]$. So $n = 2k$ (k parts of length 2) and $g \leq \frac{k}{2}$.

If we focus on the terms with $[\alpha] = [2^k]$, it suffices to specialize the indeterminates κ to $\kappa_\ell = \kappa_2 \delta_{\ell,2}$. By a small abuse of notation, we still use $Z^{(g)}(x)$ and $W(x)$ for these specialized G.F.

As already mentioned, $C_{2k,[2^k]}^{(g)}$ is known for all g and coincides with $\widehat{S}_{2k,k}^{(g)}$. This famous case was first solved by Walsh and Lehman [17, 18] by combinatorial methods; then by Harer and Zagier [9], in the context of the virtual Euler characteristics of the moduli space of curves, by means of matrix integrals; and by Jackson by a character theoretic approach [11]. It has been the object of an abundant literature since then, a good review of which is given in [14].

$$\widehat{S}_{2k,k}^{(g)} = C_{2k,[2^k]}^{(g)} = \frac{(2k)!}{(k+1)!(k-2g)!} \left[\left(\frac{u/2}{\tanh u/2} \right)^{k+1} \right]_{u^{2g}} \quad (71)$$

where the notation $[Y]_{u^k}$ means the coefficient of u^k in expression Y . The first terms are given by the following table:

g	0	1	2	3	4
$k = 1$	1				
$k = 2$	2	1			
$k = 3$	5	10			
$k = 4$	14	70	21		
$k = 5$	42	420	483		
$k = 6$	132	2310	6468	1485	
$k = 7$	429	12012	66066	56628	
$k = 8$	1430	60060	570570	1169740	225225

The $g = 0$ column is, by (68): $C_{2k,[2^k]}^{(0)} = \frac{1}{k+1} \binom{2k}{k} = C_k$ (Catalan numbers)³, whose G.F. is

$$Z^{(0)}(u) = 1 + \sum_{k=1}^{\infty} C_{2k,[2^k]}^{(0)} u^k = \frac{1 - \sqrt{1 - 4u}}{2u} \quad (72)$$

which satisfies

$$Z^{(0)}(u) = 1 + u(Z^{(0)}(u))^2. \quad (73)$$

(This is the equation (70) expressed here for $W(x) = \kappa_2 x^2$ in the variable $u = \kappa_2 x^2$.)

The $g = 1$ column is $\frac{(2k-1)!}{6(k-2)!(k-1)!} = \binom{2k-1}{3} C_{k-2} = \frac{(k+1)k(k-1)}{12} C_k$, see OEIS A002802.

The k -th row's sum is, by (8), given by $(2k-1)!!$, viz $\{1, 1, 3, 15, 105, 945, \dots\}$.

More generally,

$$C_{2k,[2^k]}^{(g)} = \frac{1}{2^g} C_k R_g(k) \quad (74)$$

with $R_g(k)$ a polynomial of degree $3g$ in k [9,17] which (for $g > 0$) vanishes for $k = -1, 0, \dots, 2g-1$, and whose form can be made explicit [3, 8]. There are several expressions for the values that it takes when its argument k is an arbitrary non-negative integer. One of them, in terms of *unsigned* Stirling numbers of the *first* kind $c_{p,q}$ ⁴, reads as follows [4]:

$$R_g(k) = \sum_{s=0}^k \binom{k}{s} \sum_{j=0}^{k+2-2g} (-1)^{s+1-j} c_{k-s+1, k+2-2g-j} c_{s+1, j}. \quad (75)$$

As noticed in [9], the numbers $C_{2k,[2^k]}^{(g)}$ (called $\epsilon_g(k)$ there) satisfy a recurrence formula

$$(k+1)C_{2k,[2^k]}^{(g)} = 2(2k-1)C_{2(k-1),[2^{k-1}]}^{(g)} + \frac{1}{2}(2k-1)(2k-2)(2k-3)C_{2(k-2),[2^{k-2}]}^{(g-1)} \quad (76)$$

or in terms of the polynomials R introduced in Eq. (74),

$$R_g(k) = R_g(k-1) + \binom{k}{2} R_{g-1}(k-2). \quad (77)$$

See also [3] for other recurrences...

Proposition 1. *The generating function of the $C_{2k,[2^k]}^{(g)}$ for $g > 0$ is of the form*

$$Z^{(g)}(u) := \sum_k C_{2k,[2^k]}^{(g)} u^k = \frac{u^{2g} Q^{(g)}(u)}{(1-4u)^{(6g-1)/2}}, \quad (78)$$

where $Q^{(g)}(u)$ is a polynomial of degree $g-1$ in u satisfying $Q^{(g)}(0) = \frac{(4g)!}{2^{2g}(2g+1)!}$.

Proof. One finds by explicit calculation that $Q^{(1)}(u) = 1$ and then proceeds by induction. Equation (76) implies that $Q^{(g)}(u)$ satisfies the following recurrence formula

$$\begin{aligned} & (1-4u)u \frac{d}{du} Q^{(g)}(u) + (2g+1+4u(g-1))Q^{(g)}(u) = \\ & \left(3 \binom{4g-1}{3} + (192g^3 - 384g^2 + 108g - 6)u + (384g^3 - 1536g^2 + 1440g + 192)u^2 + (256g^3 - 1920g^2 + 4736g - 3840)u^3 \right) Q^{(g-1)}(u) \\ & + \left(48g^2 - 24g + 3 + (192g^2 - 480g - 24)u + (192g^2 - 1152g + 1728)u^2 \right) u(1-4u) \frac{d}{du} Q^{(g-1)}(u) \\ & + 24(g+u(2g-7))u^2(1-4u)^2 \frac{d^2}{du^2} Q^{(g-1)}(u) + 4u^3(1-4u)^3 \frac{d^3}{du^3} Q^{(g-1)}(u). \end{aligned} \quad (79)$$

³It therefore coincides with $B_k^{(0)}$.

⁴ $c_{p,q}$ is the number of permutations of p elements that have q distinct cycles. They are positive integers such that $s_{p,q} = (-1)^{p-q} c_{p,q}$ where the $s_{p,q}$, the Stirling numbers of the first kind, obey $\sum_{p \geq 1} S_{n,p} s_{p,q} = \delta_{n,q}$.

Assume that $Q^{(g-1)}(u)$ is a polynomial of degree $(g-2)$ which satisfies $Q^{(g-1)}(0) = \frac{(4(g-1))!}{2^{2(g-1)}(2g-1)!}$. The r.h.s. of (79) is a priori of degree $g+1$ but one checks that it is in fact of degree g , making it possible to find a polynomial solution of degree $(g-1)$ in u for $Q^{(g)}$. Moreover the equation (79) for $u=0$ fixes the ratio $Q^{(g)}(0)/Q^{(g-1)}(0)$ in agreement with the values above. \square

Note that the expression $(1-4u)$ in the denominator of (78) is –once again– nothing else than the discriminant of the equation (73) satisfied by $Z^{(0)}$.

The first $Q^{(g)}$ read

$$\begin{aligned} Q^{(1)}(u) &= 1 ; & Q^{(2)}(u) &= 21(1+u) ; & Q^{(3)}(u) &= 11(135+558u+158u^2) ; \\ Q^{(4)}(u) &= 11 \times 13(1575+13689u+18378u^2+2339u^3) ; & & & & (80) \\ Q^{(5)}(u) &= 3 \times 13 \times 17 \times 19(4725+67620u+201348u^2+132356u^3+9478u^4) ; \dots \end{aligned}$$

with $Q^{(g)}(0) = \{1, 21, 1485, 225225, \dots\} = C_{4g, [2^{2g}]}^{(g)} = \frac{1}{2^g} C_{2g} R_g(2g) = \frac{(4g)!}{2^{2g}(2g+1)!}$, see A035319.

4.3 Partitions into two parts: type $[\alpha] = [p, n-p]$.

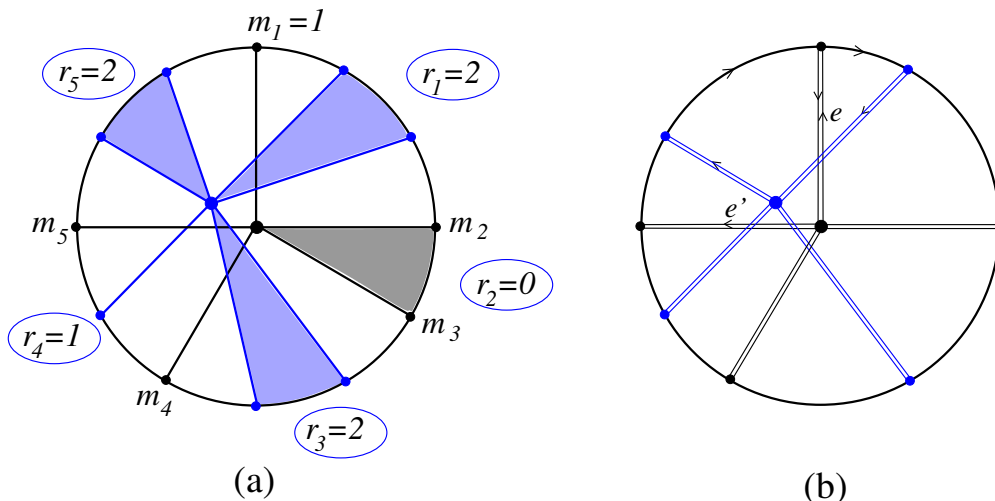


Figure 1: (a) Partition with $n = 12, p = 5, f = 5, s = 1, g = 3$;

(b) removing of “trivial faces” and doubling the edges to make the faces more visible, $f' = 1$

We first recall the main lines of the diagrammatic representation of a partition, see for example [20] for more details. Consider some partition α of the set $\{1, \dots, n\}$. We first draw n points on a circle numbered from 1 to n and distributed clockwise. With each part of length ℓ of α , is associated a ℓ -valent vertex, in short a ℓ -vertex, whose edges are numbered clockwise by the elements of the part. The vertices are arranged inside the disc. Edges are then drawn by matching the indices. If the n points are thought of as forming a single additional n -vertex, this defines a map, which has $|\alpha| + 1$ vertices, n edges and a number f of faces. The Euler characteristics of that map and its genus match the formulae given in Sect. 1.4 in terms of permutations σ and τ .

For example, Fig 1(a) shows a diagrammatic representation of a partition of $\{1, \dots, 12\}$ into two parts $\alpha = (\{1, 4, 5, 8, 10\}, \{2, 3, 6, 7, 9, 11, 12\})$. The map has 5 faces and its genus is thus 3. This can be seen either by applying Eq. (11) with $\sigma = (1, 2, \dots, 12)$, $\tau = (1, 4, 5, 8, 10), (2, 3, 6, 7, 9, 11, 12)$, and by calculating the product $\sigma \circ \tau^{-1} = (1, 11, 10, 9, 8, 6, 4, 2)(3)(5)(7)(12)$, whose cyclic parts label the faces of the associated map, or diagrammatically, as it will be done below.

Lemma. (i) For the partition of type $[\alpha] = [p, n - p]$, $g \leq \min(p - 1, n - p - 1)$.

(ii) Let s be the number of pairs of edges joining the p -vertex to consecutive points on the circle, then $g = p - s - 1$.

Proof. Let us make use of the representation exemplified in Fig. 1. There $n = 12$ and $p = 5$. In general, the diagram is completely determined by choosing p points m_1, m_2, \dots, m_p out of the n points on the circle, and attaching to them the edges coming from the p -vertex (in black), while respecting their cyclic order. These p points on the circle will be called “black points”. A $(n - p)$ -vertex (in blue) is then connected to the $n - p$ other (blue) points. Let r_i the number of points between m_i and m_{i+1} , $i = 1, 2, \dots, p$, (with the convention $m_{p+1} \equiv m_1$). Thus $\sum_i r_i = n - p$. Let us compute a lower bound on the number of faces of the resulting diagram. For each of the s indices i for which $r_i = 0$, (i.e., $m_{i+1} = m_i + 1$) there is one face attached to the p -vertex (in our example there is only one such index and the associated face on the figure is grey shaded). For each i for which $r_i \geq 1$, there are $r_i - 1$ such “trivial” faces attached to the second vertex (in our example there are three such, that are the blue shaded ones in Fig. 1). In addition to these $s + \sum_i (r_i - 1)$ trivial faces, there are $f' \geq 1$ other faces (unshaded in Fig. 1), and thus the total number of faces is

$$f = \sum_{r_i=0} 1 + \sum_{r_i \geq 1} (r_i - 1) + f' = s + ((n - p) - (p - s)) + f' = n - 2p + 2s + f' \quad (81)$$

hence

$$g = \frac{n - 1 - f}{2} = p - s - \frac{f' + 1}{2} \leq p - 1. \quad (82)$$

Because of the symmetry between the two vertices, one has as well $g \leq n - p - 1$, whence the inequality of the first part of the lemma. Note that this bound on g is in general stronger than the one $\lfloor \frac{n-2}{2} \rfloor$ written in (12).

Clearly the minimum value $g = 0$ is reached for $f' = 1$, $s = p - 1$, i.e., with the p -vertex being attached to p consecutive points on the circle.

In fact the previous inequality $f' \geq 1$ is an equality. In other words, there is always only one more face in the argument above and $g = p - s - 1$, which is the second part of the lemma. This is readily seen once the trivial faces attached to either the p - or the $(n - p)$ -vertex are erased, together with the appropriate edges and points on the circle. After that removal of all trivial faces, the blue and black points alternate on the circle. Then we claim that the remaining edges make a circuit that visits alternately the two vertices and delineates a single face. To see this, pick some edge e starting from, say, the black vertex, and follow a circuit, keeping the face to your right. The next vertex to be encountered is the blue one, since otherwise, we would have still a trivial face attached to the black vertex. Keep going: from the blue vertex we go to a blue point on the circle, then to its clockwise neighbour which is black, then we return to the original black vertex along the original edge e (traveled in the opposite way), and leave that vertex along the edge e' , which is the (anti-clockwise) neighbour of e on the vertex. Iterating these steps, we encounter alternately the two vertices along successive pairs of neighbouring edges, and when the circuit closes, all edges have been traveled and only one face has been delineated. See Fig. 1(b) for an illustration of this general result in a particular case. \square

Some readers may prefer to see the equality $f' = 1$ expressed in group theoretical terms. It reads as follows: Let τ be a permutation of $\{1, 2, \dots, n\}$ defined as the product of two disjoint increasing cycles, and σ be the circular permutation $(1, 2, \dots, n)$, then the cyclic decomposition of the product $\sigma \circ \tau^{-1}$ contains only one non-trivial cycle.

Proposition 2. For general n, g and $p \geq 2$, (and $n \neq 2p$, otherwise, if $n = 2p$, multiply by $\frac{1}{2}$)

$$C_{n, [p, n-p]}^{(g)} = \frac{n}{g+1} \binom{p-1}{g} \binom{n-p-1}{g}. \quad (83)$$

Note that this result is symmetric under $p \leftrightarrow n - p$ as it should, and that the bound $g \leq \min(p - 1, n - p - 1)$ is manifest. An alternative expression is

$$C_{n,[p,n-p]}^{(g)} = \frac{n}{p} \binom{p}{p-1-g} \binom{n-p-1}{g}. \quad (84)$$

Proof. Following the lines of Lemma 1, a configuration is completely determined by the choice of the p points m_i on the circle, subject to the condition that there are s pairs of adjacent points. Thus

$$C_{n,[p,n-p]}^{(g)} = \#\{m_1, \dots, m_p \in \{1, n\} \mid \#\{i \mid m_{i+1} = m_i + 1\} = s\}, \quad (85)$$

where $g = p - s - 1$. This number may be easily computed by a transfer matrix technique⁵. Suppose the black and blue points of the circle are representing two states of a periodic system on a circle, and assign a weight $1, t, t^2u$ to a transition (i.e., an arc on the circle) between respectively black-black, black-blue and blue-blue points. The matrix

$$M = \begin{pmatrix} 1 & t \\ t & t^2u \end{pmatrix}$$

describes the possible transitions between these states. The number $C_{n,[p,n-p]}^{(g)}$ is then the coefficient of $t^{2p}u^{p-1-g}$ in

$$t_n = \text{tr } M^n. \quad (86)$$

Let $z := t^2$. By virtue of the characteristic equation of M , the numbers t_n satisfy the recurrence relations

$$t_n = t_1 t_{n-1} + z(1 - u)t_{n-2} \quad (87)$$

with $t_0 = 2$ and $t_1 = 1 + zu$, whence $t_2 = 1 + 2z + z^2u^2$, $t_3 = 1 + 3z + 3z^2u + z^3u^3$. It follows from (87) that

$$[t_n]_{z^p u^{p-1-g}} = [t_{n-1}]_{z^p u^{p-1-g}} + [t_{n-1}]_{z^{p-1} u^{p-2-g}} + [t_{n-2}]_{z^{p-1} u^{p-1-g}} - [t_{n-2}]_{z^{p-1} u^{p-2-g}} \quad (88)$$

from which all $[t_n]_{z^p u^{p-1-g}}$ may be reconstructed for $p \leq n - 1, 0 \leq g \leq p - 1$.

Let $D_{n,p,g} := \frac{n}{g+1} \binom{p-1}{g} \binom{n-p-1}{g}$, one may check that the D 's satisfy the same relation, namely

$$D_{n,p,g} = D_{n-1,p,g} + D_{n-1,p-1,g} + D_{n-2,p-1,g-1} - D_{n-2,p-1,g}, \quad (89)$$

as well as

$$D_{2,1,0} = [t_2]_{zu^0} = 2, \quad D_{2,1,1} = [t_2]_{zu^{-1}} = 0, \quad D_{3,2,0} = [t_3]_{z^2u} = 3, \quad D_{3,2,1} = [t_3]_{z^2u^0} = 0, \quad D_{2,1,0} = [t_2]_{zu^0} = 0.$$

Thus $C_{n,[p,n-p]}^{(g)}$ is given by that expression $D_{n,p,g}$, which completes the proof of Proposition 1. \square

Remarks

1. The formula (83), originally proposed for $p \geq 2$, extends trivially to all p : from (24), we learn that $C_{n,[1,n-1]}^{(g)} = n C_{n-1,[n-1]}^{(g)} = n \delta_{g,0}$, in accordance with the rhs of (83) evaluated at $g = 0$.

2. One may check that the expression of $C_{n,[p,n-p]}^{(g)}$ is consistent with the Faà di Bruno coefficient (8): $\sum_{g=0}^{p-1} C_{n,[p,n-p]}^{(g)} = \binom{n}{p}$. Once again, this is trivially true for $p = 1$ and, for $2 \leq p < n - p$, this is an easy consequence of the celebrated Vandermonde identity, namely:

$$\sum_{k=0}^m \binom{n}{k} \binom{\ell}{m-k} = \binom{\ell+n}{m}. \quad (90)$$

⁵We are quite grateful to Philippe Di Francesco for this suggestion

Here, using (84) together with (90), one finds: $\frac{n}{p} \sum_{g=0}^{p-1} \binom{p}{p-1-g} \binom{n-p-1}{g} = \frac{n}{p} \binom{n-1}{p-1} = \binom{n}{p}$. \square

3. Generating function of the $C_{[p,n-p]}^{(g)}$. One may build a G.F. for the $C_{[p,n-p]}^{(g)}$ adapted to their symmetry under $p \leftrightarrow n - p$:

$$Z^{(g)}(x, v) := \frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^{n-1} C_{[p,n-p]}^{(g)} x^n v^{2p-n} \quad (91)$$

One finds

$$Z^{(g)}(x, v) = \mathcal{Z}^{(g)}(x, 1/v) = \frac{(2 - x(v + 1/v))x^{2g+2}}{2(1 - xv)^{g+2}(1 - x/v)^{g+2}}. \quad (92)$$

In particular, for $v = 1$, one recovers the G.F. of the $S_{n,2}^{(g)}$ already encountered in the Remark at the end of Sect. 3.2, $\sum_n S_{n,2}^{(g)} x^n = \frac{x^{2g+2}}{(1-x)^{2g+3}}$.

The previous Proposition is illustrated here by the explicit computation of $C_{n,[n-5,5]}^{(g)}$ for low n

g	0	1	2	3	4
$n = 6$	6				
$n = 7$	7	14			
$n = 8$	8	32	16		
$n = 9$	9	54	54	9	
$n = 10$	5	40	60	20	1
$n = 11$	11	110	220	110	11
$n = 12$	12	144	360	240	36
$n = 13$	13	182	546	455	91
$n = 14$	14	224	784	784	196
$n = 15$	15	270	1080	1260	378

Case $[\alpha] = [p^2]$. So $n = 2p$ (2 parts of length p), and $g = \frac{2p-1-f}{2} \leq p - 1$.

This is of course a particular case of the type $[\alpha] = [p, n - p]$ considered in this section. Here, the Faà di Bruno coefficients are the numbers “of ways to put p identical objects into $g + 1$ of altogether p distinguishable boxes”, see OEIS A103371. Notice that the last writing below is indeed consistent with (83). One has

$$C_{2p,[p^2]}^{(g)} = \binom{p-1}{p-g-1} \binom{p}{p-g-1} = \binom{p-1}{g} \binom{p}{g+1} = \frac{p}{g+1} \binom{p-1}{g} \quad (93)$$

For small values of p and g these coefficients are gathered with those of the cases $[\alpha] = [p^k]$ studied in section 5.1.

The general formula being given above we only notice that the $g = 0$ sequence is just p and that the $g = 1$ sequence defines the pentagonal pyramidal numbers that we shall meet again in section 5.1 —see our comments there. Notice also that the sum over g is $\binom{2p-1}{p} = \{1, 3, 10, 35, 126, 462 \dots\}$, see OEIS A001700, and that the penultimate term in each row, $\{2, 6, 12, 20, 30, \dots\}$, is equal to $p(p - 1)$, indeed, for $g = p - 2$, $C_{2p,[p^2]}^{(p-2)} = \binom{p-1}{p-2} \binom{p}{p-1} = p(p - 1)$.

5 Genus dependent Faà di Bruno coefficients $C_{n,[\alpha]}^{(g)}$. Part II. A compilation of partial results.

5.1 About types $[\alpha] = [p^k]$ for given $p = 2, 3, 4, \dots$ as a function of k

The results for $C_{n,[\alpha]}^{(g)}$ when $[\alpha] = [p^k]$ are gathered in Table 1. For given p and k the values are listed vertically (downward) according to the genus g , for $g = 0, 1, 2, \dots$. Here $n = kp$, (i.e., k parts of length p); we have $g = (k(p-1) + 1 - f)/2$, therefore $g \leq k(p-1)/2$. These values have been obtained by an explicit determination of the genus for computer generated set partitions, or obtained from general theorems.

Table 1. Table of coefficients $C_{n,[\alpha]}^{(g)}$ for $n = kp$, $[\alpha] = [p^k]$.

	$k=1$	$k=2$	$k=3$	$k=4$
$p=2$	1	2, 1	5, 10	14, 70, 21
$p=3$	1	3, 6, 1	12, 102, 144, 22	55, 1212, 6046, 7163, 924
$p=4$	1	4, 18, 12, 1	22, 432, 2007, 2604, 710	140, 7236, 108090, 592824, 1180364, 688270, 50701
$p=5$	1	5, 40, 60, 20, 1	35, 1240, 12060, 41820, 51565, 18540, 866	285, 26800, 809960, ...
$p=6$	1	6, 75, 200, 150, 30, 1	51, 2850, 47475, 316700, 905415, 1076238, 462375, 47752	506, 75450, 3837575, ...
$p=7$	1	7, 126, 525, 700, 315, 42, 1	70, 5670, 144270, ...	819, 177660, 13656006, ...

	$k=5$	$k=6$	$k=7$
$p=2$	42, 420, 483	132, 2310, 6468, 1485	429, 12012, 66066, 56628
$p=3$	273, 12330, 149674, 576660, 593303, 69160	1428, 114888, 2771028, ...	7752, 1011486, 42679084, ...
$p=4$	969, 103680, 3588318, ...	7084, 1359882, 90800208, ...	53820, 16846704, 1929948363, ...
$p=5$	2530, 495200, 34034480, ...	23751, 8373000, 1097464620, ...	231880, 133685440, 29830376800, ...
$p=6$	5481, 1707000, 195525750, ...	62832, 35331000, 7670848500, ...	749398, 690413850, 254134018600, ...
$p=7$	10472, 4755870, 818352528, ...	141778, 116450460, 37838531178, ...	1997688, 2691733464, 1479039054696, ...

In this section we describe some generic features of the sequences that are obtained for increasing values of p , for various choices of $[\alpha] = [p^k]$. The situation where $[\alpha] = [p^2]$, which is a particular case of the type $[\alpha] = [p, n-p]$ considered in section 4.3 – a “solved case” – (see eqs. (83) or (84)), was already discussed at the end of 4.3.

We recall from (8) that:

- the k -th row’s sum (over g) in the Table of $[p^k]$ is $(pk)!/(k!(p!)^k)$, see OEIS A025035, A025036, A025037, A025038, A025039, for $p = 3, \dots, 7$.
- and from (68) that $C_{p.k,[p^k]}^{(0)} = \frac{1}{pk+1} \binom{pk+1}{k} = \frac{1}{(p-1)k+1} \binom{pk}{k}$.

Genus $g = 1$: Observations and conjectures.

For $k = 2$, it follows from Prop. 1 below that

$$C_{2p,[p^2]}^{(1)} = \frac{p(p-1)^2}{2} \tag{94}$$

which are the “pyramidal pentagonal numbers”, $\{0, 1, 6, 18, 40, 75, 126, 196, 288, 405, \dots\}$, see OEIS A002411. Furthermore, we observe that these pyramidal pentagonal numbers factorise the coefficients $C_{pk,[p^k]}^{(1)}$

$$C_{pk,[p^k]}^{(1)} \stackrel{?}{=} C_{2p,[p^2]}^{(1)} \phi(p, k) \tag{95}$$

$$\begin{array}{cccccc}
& 0 & 1 & 10 & 70 & 420 & \dots \\
& 0 & 1 & 17 & 202 & 2055 & \dots \\
& 0 & 1 & 24 & 402 & 5760 & \dots \\
\phi(p, k) = & 0 & 1 & 31 & 670 & 12380 & \dots \\
& 0 & 1 & 38 & 1006 & 22760 & \dots \\
& 0 & 1 & 45 & 1410 & 37745 & \dots \\
& \vdots & & & & &
\end{array} \quad 2 \leq p \leq 7 \quad (96)$$

in which the third column is an arithmetic series $7p - 4$, the fourth $34p^2 - 38p + 10$, etc. In other words, the above tables are compatible with the following expressions⁶:

$$\begin{aligned}
C_{3p,[p^3]}^{(1)} &= \frac{p(p-1)^2}{2}(7p-4), \\
C_{4p,[p^4]}^{(1)} &= \frac{p(p-1)^2}{2}(34p^2 - 38p + 10), \\
C_{5p,[p^5]}^{(1)} &= \frac{p(p-1)^2}{2} \frac{5}{6}(169p^3 - 279p^2 + 146p - 24), \\
C_{6p,[p^6]}^{(1)} &= \frac{p(p-1)^2}{2} \left(533p^4 - 1160p^3 + \frac{1813}{2}p^2 - \frac{599}{2}p + 35 \right), \\
C_{7p,[p^7]}^{(1)} &= \frac{p(p-1)^2}{2} \frac{7}{120} (32621p^5 - 87970p^4 + 91335p^3 - 45410p^2 + 10744p - 960).
\end{aligned} \quad (97)$$

The constant terms in the polynomial ϕ appear to be (up to a sign $(-1)^k$) the ‘‘tetrahedral (or triangular pyramidal) numbers’’: $k(k^2 - 1)/6$, A000292.

Genus $g = 2$: Observations and conjectures.

$$\begin{aligned}
C_{3p,[p^3]}^{(2)} &\stackrel{?}{=} \frac{1}{8}p(p-1)^2(p-2)(27 - 55p + 26p^2), \\
C_{4p,[p^4]}^{(2)} &\stackrel{?}{=} \frac{1}{6}p(p-1)^2p - 1(287 - 1248p + 1908p^2 - 1218p^3 + 274p^4), \\
C_{5p,[p^5]}^{(2)} &\stackrel{?}{=} \frac{1}{144}p(p-1)^2(-30576 + 194318p - 467213p^2 + 532986p^3 - 288895p^4 + 59500p^5).
\end{aligned} \quad (98)$$

In each case, the conjecture has been tested on at least two more values than those used in the extrapolation.

5.2 Cases $[\alpha] = [n - p - q, p, q]$. *Partition of n into $k = 3$ parts.*

All the data that have been collected in that case, when the genus is 0 or 1, see the Tables in the Appendix, are consistent with the following Ansätze

$$\begin{aligned}
C_{n,[n-p-q,p,q]}^{(0)} &= n(n-1) \\
C_{n,[n-p-q,p,q]}^{(1)} &\stackrel{?}{=} (-5(n-1)^2 + 3(p^2 + q^2 + r^2 - 1) + 6pqr + (r+p)(r+q)(p+q)) \frac{n}{2}
\end{aligned} \quad (99)$$

$$\quad (100)$$

where in the last expression, the symmetry in the exchange of p, q and $r := n - p - q$ is manifest; these expressions have to be multiplied by $\frac{1}{2}$ if two of the three integers $p, q, n - p - q$ are equal, and by $\frac{1}{6}$ if $p = q = n - p - q$.

⁶These expressions (eqs 97) were presented as conjectures when the first version of our paper was posted as a preprint to arXiv but they have been subsequently proved by A. Hock: see our comments in the acknowledgments section.

6 Tables

Coefficients $C_{n, [\alpha]}^{(g)}$ for $2 \leq n \leq 15$ and $\alpha_1 = 0$

Partitions without singletons, ordered by increasing number of parts $|\alpha|$ and then by lexicographic order on $[\alpha]$

	g	0	1	2	3	4	5
n = 2	$[2]$	1	0	0	0	0	0
n = 3	$[3]$	1	0	0	0	0	0
n = 4	$[4]$	1	0	0	0	0	0
	$[2^2]$	2	1	0	0	0	0
n = 5	$[5]$	1	0	0	0	0	0
	$[2, 3]$	5	5	0	0	0	0
n = 6	$[6]$	1	0	0	0	0	0
	$[2, 4]$	6	9	0	0	0	0
	$[3^2]$	3	6	1	0	0	0
	$[2^3]$	5	10	0	0	0	0
n = 7	$[7]$	1	0	0	0	0	0
	$[2, 5]$	7	14	0	0	0	0
	$[3, 4]$	7	21	7	0	0	0
	$[2^2, 3]$	21	70	14	0	0	0
n = 8	$[8]$	1	0	0	0	0	0
	$[2, 6]$	8	20	0	0	0	0
	$[3, 5]$	8	32	16	0	0	0
	$[4^2]$	4	18	12	1	0	0
	$[2^2, 4]$	28	128	54	0	0	0
	$[2, 3^2]$	28	152	100	0	0	0
	$[2^4]$	14	70	21	0	0	0
n = 9	$[9]$	1	0	0	0	0	0
	$[2, 7]$	9	27	0	0	0	0
	$[3, 6]$	9	45	30	0	0	0
	$[4, 5]$	9	54	54	9	0	0
	$[2^2, 5]$	36	207	135	0	0	0
	$[2, 3, 4]$	72	531	603	54	0	0
	$[3^3]$	12	102	144	22	0	0
	$[3, 2^3]$	84	630	546	0	0	0
n = 10	$[10]$	1	0	0	0	0	0
	$[2, 8]$	10	35	0	0	0	0
	$[3, 7]$	10	60	50	0	0	0
	$[4, 6]$	10	75	100	25	0	0
	$[5^2]$	5	40	60	20	1	0
	$[2^2, 6]$	45	310	275	0	0	0
	$[2, 3, 5]$	90	830	1340	260	0	0
	$[2, 4^2]$	45	450	840	240	0	0
	$[3^2, 4]$	45	510	1115	430	0	0
	$[2^3, 4]$	120	1165	1685	180	0	0
	$[2^2, 3^2]$	180	1985	3565	570	0	0
	$[2^5]$	42	420	483	0	0	0

n = 11

	g	0	1	2	3	4	5
	$[11]$	1	0	0	0	0	0
	$[2, 9]$	11	44	0	0	0	0
	$[3, 8]$	11	77	77	0	0	0
	$[4, 7]$	11	99	165	55	0	0
	$[5, 6]$	11	110	220	110	11	0
	$[2^2, 7]$	55	440	495	0	0	0
	$[2, 3, 6]$	110	1210	2530	770	0	0
	$[2, 4, 5]$	110	1375	3564	1793	88	0
	$[3^2, 5]$	55	770	2277	1452	66	0
	$[3, 4^2]$	55	825	2684	2035	176	0
	$[2^3, 5]$	165	1936	3905	924	0	0
	$[2^2, 3, 4]$	495	7007	19085	8063	0	0
	$[2, 3^3]$	165	2607	8195	4433	0	0
	$[3, 2^4]$	330	4620	10395	1980	0	0

n = 12

	g	0	1	2	3	4	5
	$[12]$	1	0	0	0	0	0
	$[2, 10]$	12	54	0	0	0	0
	$[3, 9]$	12	96	112	0	0	0
	$[4, 8]$	12	126	252	105	0	0
	$[5, 7]$	12	144	360	240	36	0
	$[6^2]$	6	75	200	150	30	1
	$[2^2, 8]$	66	600	819	0	0	0
	$[2, 3, 7]$	132	1680	4308	1800	0	0
	$[2, 4, 6]$	132	1968	6510	4740	510	0
	$[2, 5^2]$	66	1032	3672	3072	474	0
	$[3^2, 6]$	66	1092	4062	3640	380	0
	$[3, 4, 5]$	132	2436	10500	12084	2568	0
	$[4^3]$	22	432	2007	2604	710	0
	$[2^3, 6]$	220	3000	7730	2910	0	0
	$[2^2, 3, 5]$	660	11232	40716	28968	1584	0
	$[2^2, 4^2]$	330	5988	23877	20097	1683	0
	$[2, 3^2, 4]$	660	13218	59076	59442	6204	0
	$[3^4]$	55	1212	6046	7163	924	0
	$[2^4, 4]$	495	8616	28590	14274	0	0
	$[2^3, 3^2]$	990	19104	73050	45456	0	0
	$[2^6]$	132	2310	6468	1485	0	0

$n = 13$		g						$n = 14$							
								g	0	1	2	3	4	5	6
$[\alpha]$		0	1	2	3	4	5	$[\alpha]$	0	1	2	3	4	5	6
[13]		1	0	0	0	0	0	[14]	1	0	0	0	0	0	0
[2, 11]		13	65	0	0	0	0	[2, 12]	14	77	0	0	0	0	0
[3, 10]		13	117	156	0	0	0	[3, 11]	14	140	210	0	0	0	0
[4, 9]		13	156	364	182	0	0	[4, 10]	14	189	504	294	0	0	0
[5, 8]		13	182	546	455	91	0	[5, 9]	14	224	784	784	196	0	0
[6, 7]		13	195	650	650	195	13	[6, 8]	14	245	980	1225	490	49	0
[2 ² , 9]		78	793	1274	0	0	0	[7 ²]	7	126	525	700	315	42	1
[2, 3, 8]		156	2249	6825	3640	0	0	[2 ² , 10]	91	1022	1890	0	0	0	0
[2, 4, 7]		156	2691	10803	10335	1755	0	[2, 3, 9]	182	2926	10248	6664	0	0	0
[2, 5, 6]		156	2912	13130	15470	4238	130	[2, 4, 8]	182	3556	16758	19894	4655	0	0
[3 ² , 7]		78	1482	6630	7670	1300	0	[2, 5, 7]	182	3934	21420	32620	13034	882	0
[3, 4, 6]		156	3393	18161	28145	9945	260	[2, 6 ²]	91	2030	11550	18900	8645	826	0
[3, 5 ²]		78	1768	9984	16796	7046	364	[3 ² , 8]	91	1946	10157	14406	3430	0	0
[4 ² , 5]		78	1872	11271	20904	10322	598	[3, 4, 7]	182	4536	28938	56434	28280	1750	0
[2 ³ , 7]		286	4420	13819	7215	0	0	[3, 5, 6]	182	4858	33824	75040	48174	6090	0
[2 ² , 3, 6]		858	16900	76037	76765	9620	0	[4 ² , 6]	91	2562	18893	45654	33285	4620	0
[2 ² , 4, 5]		858	18720	97539	125606	27547	0	[4, 5 ²]	91	2660	20496	52710	42679	7490	0
[2, 3 ² , 5]		858	20488	117806	175916	45292	0	[2 ³ , 8]	364	6265	22981	15435	0	0	0
[2, 3, 4 ²]		858	21684	133887	222781	71240	0	[2 ² , 3, 7]	1092	24290	129948	170380	34650	0	0
[3 ³ , 4]		286	7878	53404	100997	37635	0	[2 ² , 4, 6]	1092	27587	176351	307020	114940	3640	0
[2 ⁴ , 5]		715	14612	63323	53053	3432	0	[2 ² , 5 ²]	546	14343	96726	182756	80094	3913	0
[2 ³ , 3, 4]		2860	68172	363610	419068	47190	0	[2, 3 ² , 6]	1092	29988	209510	415870	178920	5460	0
[2 ² , 3 ³]		1430	37336	221715	298649	41470	0	[2, 3, 4, 5]	2184	65674	513450	1200738	696794	43680	0
[3, 2 ⁵]		1287	30030	138138	100815	0	0	[2, 4 ³]	364	11529	95613	243180	162099	12740	0
								[3 ³ , 5]	364	11844	100660	262696	173348	11648	0
								[3 ² , 4 ²]	546	18690	168273	475195	356951	31395	0
								[2 ⁴ , 6]	1001	23240	123214	146020	21840	0	0
								[2 ³ , 3, 5]	4004	111608	754614	1286642	365652	0	0
								[2 ³ , 4 ²]	2002	58912	427777	809823	278061	0	0
								[2 ² , 3 ² , 4]	6006	191968	1525454	3268552	1314320	0	0
								[2, 3 ⁴]	1001	34692	301070	725193	339444	0	0
								[2 ⁵ , 4]	2002	56378	354263	473242	60060	0	0
								[2 ⁴ , 3 ²]	5005	153832	1075165	1663893	255255	0	0
								[2 ⁷]	429	12012	66066	56628	0	0	0

$n = 15$

g	0	1	2	3	4	5	6
$[\alpha]$							
[15]	1	0	0	0	0	0	0
[2, 13]	15	90	0	0	0	0	0
[3, 12]	15	165	275	0	0	0	0
[4, 11]	15	225	675	450	0	0	0
[5, 10]	15	270	1080	1260	378	0	0
[6, 9]	15	300	1400	2100	1050	140	0
[7, 8]	15	315	1575	2625	1575	315	15
$[2^2, 11]$	105	1290	2700	0	0	0	0
[2, 3, 10]	210	3720	14760	11340	0	0	0
[2, 4, 9]	210	4575	24720	35070	10500	0	0
[2, 5, 8]	210	5145	32760	61215	32340	3465	0
[2, 6, 7]	210	5430	37200	78000	50550	8610	180
$[3^2, 9]$	105	2490	14835	24920	7700	0	0
[3, 4, 8]	210	5880	43470	102165	66675	6825	0
[3, 5, 7]	210	6420	53010	146040	127290	26940	450
$[3, 6^2]$	105	3300	28200	81500	77075	19380	650
$[4^2, 7]$	105	3375	29385	87510	84975	19575	300
[4, 5, 6]	210	7185	67260	221310	252090	79575	3000
$[5^3]$	35	1240	12060	41820	51565	18540	866
$[2^3, 9]$	455	8610	36190	29820	0	0	0
$[2^2, 3, 8]$	1365	33705	208320	336210	96075	0	0
$[2^2, 4, 7]$	1365	38955	293670	644535	347175	25650	0
$[2^2, 5, 6]$	1365	41580	342000	853500	579405	74040	0
$[2, 3^2, 7]$	1365	42105	344595	853035	522450	38250	0
[2, 3, 4, 6]	2730	94290	886620	2671710	2296950	354000	0
$[2, 3, 5^2]$	1365	48825	479040	1530360	1449285	274905	0
$[2, 4^2, 5]$	1365	51240	529425	1816410	1913445	417840	0
$[3^3, 6]$	455	16905	171635	570805	550950	90650	0
$[3^2, 4, 5]$	1365	55020	612705	2303715	2692020	641475	0
$[3, 4^3]$	455	19215	224925	902400	1158840	321790	0
$[2^4, 7]$	1365	35250	219660	337050	82350	0	0
$[2^3, 3, 6]$	5460	172350	1398740	3159450	1515700	54600	0
$[2^3, 4, 5]$	5460	188025	1708035	4524870	2847420	185640	0
$[2^2, 3^2, 5]$	8190	304335	2994180	8823360	6327465	461370	0
$[2^2, 3, 4^2]$	8190	319605	3329865	10604790	8605395	780780	0
$[2, 3^3, 4]$	5460	229365	2581215	9060870	8289600	854490	0
$[3^5]$	273	12330	149674	576660	593303	69160	0
$[2^5, 5]$	3003	97140	761880	1493520	482292	0	0
$[2^4, 3, 4]$	15015	554250	5104260	12450900	5524200	0	0
$[2^3, 3^3]$	10010	399660	4013730	10965465	5632135	0	0
$[2^6, 3]$	5005	180180	1471470	2622620	450450	0	0

Table of $S_{n,k}^{(g)}$, from $n = 1$ to $n = 15$

This table can be obtained either directly, by generating for each n all partitions with a fixed number of parts, then calculating their genus, or, indirectly, by summing appropriate rows of the table of the genus dependent Faà di Bruno coefficients, while taking into account singletons (since the following is a table for $S_{n,k}^{(g)}$, not for $\widehat{S}_{n,k}^{(g)}$). For each value of n the number of parts, k appears vertically, and the genus $g = 0, 1, \dots$ increases in each row.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	2	1	2	3	1	2	6	1	2	15, 15, 1	2	21, 35, 7	2	28, 70, 28, 1	2	36, 126, 84, 9	3	336, 1470, 1134, 85	4	1176, 4410, 2184	2
3	3	1	2	3	3	6	6	1	3	50, 40	3	105, 175, 21	3	196, 560, 210	3	490, 1050, 161	4	1764, 4410, 777	5	1176, 1470	3
4	4	1	2	3	4	1	4	10	4	50, 15	4	175, 175	4	490, 560	4	196, 70	5	336, 126	6	36	4
5	5	1	2	3	4	5	1	5	15	5	105, 35	5	21	5	28	6	1	8	1	5	5
6	6	1	2	3	4	5	6	1	6	1	7	1	8	1	9	1	6	6	6	6	6
7	7	1	2	3	4	5	6	7	1	8	1	9	1	10	1	11	1	7	7	7	7
8	8	1	2	3	4	5	6	7	8	1	9	1	10	1	11	1	8	8	8	8	8
9	9	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	9	9	9	9
10	10	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	10	10	10	10
11	11	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	11	11	11	11
12	12	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	12	12	12	12
13	13	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	13	13	13	13
14	14	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	14	14	14	14
15	15	1	2	3	4	5	6	7	8	9	1	10	1	11	1	12	1	15	15	15	15

Table of $\widehat{S}_{n,k}^{(g)}$, from $n = 2$ to $n = 15$

This table can be obtained either directly, by generating, for each n , all partitions without singletons, with a fixed number of parts, then calculating their genus, or, indirectly, by summing appropriate rows of the table of the genus dependent Faà di Bruno coefficients (without singletons). For each value of n the number of parts, k appears vertically, and the genus $g = 0, 1, \dots$ increases in each row. Warning: the table starts at $n = 2$.

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
1 1	1 1	1 1 2 2, 1	1 1 2 5, 5	1 1 2 9, 15, 1 3 5, 10	1 1 2 14, 35, 7 3 21, 70, 14	1 1 2 20, 70, 28, 1 3 56, 280, 154 4 14, 70, 21	1 1 2 27, 126, 84, 9 3 120, 840, 882, 76 4 84, 630, 546
$n = 10$	$n = 11$	$n = 12$	$n = 13$				
1 1 2 35, 210, 210, 45, 1 3 225, 2100, 3570, 930 4 300, 3150, 5250, 750 5 42, 420, 483	1 1 2 44, 330, 462, 165, 11 3 385, 4620, 11550, 6050, 330 4 825, 11550, 31185, 13420 5 330, 4620, 10395, 1980	1 1 2 54, 495, 924, 495, 66, 1 3 616, 9240, 31878, 27940, 4642 4 1925, 34650, 137445, 118580, 10395 5 1485, 27720, 101640, 59730 6 132, 2310, 6468, 1485	1 1 2 65, 715, 1716, 1287, 286, 13 3 936, 17160, 78078, 102960, 34606, 1352 4 4004, 90090, 492492, 709280, 191334 5 5005, 120120, 648648, 770770, 92092 6 1287, 30030, 138138, 100815				
$n = 14$	$n = 15$						
1 1 2 77, 1001, 3003, 3003, 1001, 91, 1 3 1365, 30030, 174174, 322322, 182182, 21658 4 7644, 210210, 1513512, 3273270, 1797796, 112476 5 14014, 420420, 3132129, 6236230, 2319317 6 7007, 210210, 1429428, 2137135, 315315 7 429, 12012, 66066, 56628	1 1 2 90, 1365, 5005, 6435, 3003, 455, 15 3 1925, 50050, 360360, 890890, 760760, 182910, 5446 4 13650, 450450, 4129125, 12512500, 11606595, 2238600 5 34398, 1261260, 12381369, 37087050, 28261233, 2406040 6 28028, 1051050, 9879870, 24909885, 11638627 7 5005, 180180, 1471470, 2622620, 450450						

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After the present article was made available as a preprint (its version 1 on arXiv) we received several comments from A. Hock suggesting that several partitions functions obtained in the present paper could be re-written by using changes of functions suggested by the formalism of topological recursion. This is in particular so for $g = 1$ in the case of partitions into three equal parts where expressions (97) presented as conjectures in the first version of our paper could be proved by A. Hock by using this method. We understand that this discussion will be the subject of a forthcoming publication [10].

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