



Liquid Sloshing Damping in an Elastic Container

Thomas Miras, Jean-Sébastien Schotté, Roger Ohayon

► To cite this version:

Thomas Miras, Jean-Sébastien Schotté, Roger Ohayon. Liquid Sloshing Damping in an Elastic Container. Journal of Applied Mechanics, 2012, 79 (1), <10.1115/1.4005189>. <hal-04269031>

HAL Id: hal-04269031

<https://hal.science/hal-04269031v1>

Submitted on 13 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Liquid Sloshing Damping in an Elastic Container

Thomas Miras, Jean-Sébastien Schotté

ONERA, The French Aerospace Lab, BP 72, 92322 Chatillon Cedex, France

Roger Ohayon

CNAM, Structural Mechanics and Coupled System Laboratory, 292 rue Saint Martin, Paris, 75141 France

It is proposed to investigate in this paper the damped vibrations of an incompressible liquid contained in a deformable tank. A linearized formulation describing the small movements of the system is presented. At first, a diagonal damping is introduced in the reduced equations of the hydroelastic sloshing problem. We obtain a non-classically damped coupled system with a damping matrix that is not symmetric. Then, by projecting the system onto its complex modes, the frequency and time responses for different type of loads are built. A numerical application is illustrated on a test case.

Keywords: hydroelasticity, sloshing, damping, complex modes, modal synthesis, time response, frequency response.

1 Introduction

Space structures, such as satellites, probes, or space stations generally contain large amounts of liquids, which can be propellants, cooling liquids, etc. The motion of these liquids can influence the vibrational behavior of the main structure and can potentially disturb the trajectory controller or the stabilization procedures. For instance, a flutter analysis with sloshing effects is studied in Ref. [1].

Although the liquid sloshing phenomenon has been studied for many years [2–4], it is still a vibrant topic [5,6]. Among the numerous possible approaches of this problem, the one we propose here is adapted to study fluid-structure coupled vibrations of small amplitudes and particularly the modal analysis of deformable structures containing liquids. There exists, indeed, a need for some industrial sectors such as aerospace, building, transport, etc. (where consequences of the liquid movements can be dramatic) to estimate the evolution of the inner liquid eigenmodes and eigenfrequencies of such systems, right from the design phase.

This study is in line with previous works that have been done in this field by the authors [7–9].

The accurate modeling of all the dissipative sources in the liquid (viscosity, capillarity, wetting, contamination, etc.) would require solving the Navier-Stokes equation in time domain. However, this approach is not adapted when only the vibrational behavior of the system is required. For this reason, we would like to extend to damped systems the fluid potential approach, which is generally used to treat linearized and undamped fluid-structure coupled problems.

In Sec. 2, we first present a linearized formulation of the hydroelastic problem with sloshing for an incompressible and inviscid fluid under gravity contained in an elastic tank. Then, we introduce a Caughey damping in the conservative matrix equation of the fluid. We obtain a dissipative coupled problem with a non-classical damping matrix. We link the modal damping coefficients of the fluid to their physical meaning in Sec. 3.

In Sec. 4 we use the complex eigenmodes properties to reduce the associated state-space problem. We give in Sec. 5 an expression of the time response and frequency response of the coupled system using a modal synthesis method for different types of loads. Finally, we apply this formulation to a test case in Sec. 6.

2 Coupled System Equations of Motion

2.1 Nondissipative Fluid. Let us consider a homogeneous, inviscid, incompressible liquid (and thus irrotational) with small amplitude vibrations around the equilibrium position. The system is supposed to be at a constant temperature and in a uniform gravity field. We are reminded of the local dynamic equations and initial conditions (see Eqs. (1a)–(1f) and Fig. 1) [10,11]. To discretize the problem, we write the weak formulation multiplying Eq. (1a) by a test function $\delta\varphi$ and integrating on Ω_F . $\mathbf{C}_\varphi = \{\varphi \in H^1(\Omega_F)\}$ is the admissible space of regular functions

$$\Delta\varphi(M, t) = 0 \text{ in } \Omega_F \quad (1a)$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = \mathbf{u} \cdot \mathbf{n} \text{ in } \Sigma \quad (1b)$$

$$\frac{\partial\varphi}{\partial z} = -\frac{\ddot{\varphi}}{g} - \frac{1}{A(\Gamma)} \int_{\Sigma} \mathbf{u} \cdot \mathbf{n} \, d\Sigma \text{ in } \Gamma \quad (1c)$$

$$\int_{\Gamma} \varphi \, d\Gamma = 0 \quad (1d)$$

$$\varphi(M, t = 0) = \varphi_{ini} \quad (1e)$$

$$\dot{\varphi}(M, t = 0) = \dot{\varphi}'_{ini} \quad (1f)$$

- \mathbf{u}_F is the fluid displacement.
- φ : displacement potential of the fluid with $\mathbf{u}_F = \nabla\varphi$.
- $\ddot{\varphi}$: second derivative of φ with respect to time.
- \mathbf{u} : displacement vector of the structure.
- \mathbf{n} : outwards unit normal to the interface Σ .
- z : vertical coordinate.
- (M, t) : space and time variables.
- Σ : fluide-structure interface.
- Σ_u : fixed interface.
- Ω_F : fluid volume.
- Γ : free surface.
- $A(\Gamma)$: free surface area.

$$\varphi \text{ and } \mathbf{C}_\varphi^* = \{\varphi \in \mathbf{C}_\varphi / \int_{\Gamma} \varphi \, d\Gamma = 0\}$$

$$\forall t \in \mathbb{R}^+, \exists \varphi \in \mathbf{C}_\varphi^*, \forall \delta\varphi \in \mathbf{C}_\varphi^* : \int_{\Omega_F} \Delta\varphi \delta\varphi \, d\Omega_F = 0 \quad (2)$$

The time t is considered as a parameter and the test function $\delta\varphi$ does not depend on this parameter. Using Green formula and Eqs. (1b)–(1d) we obtain

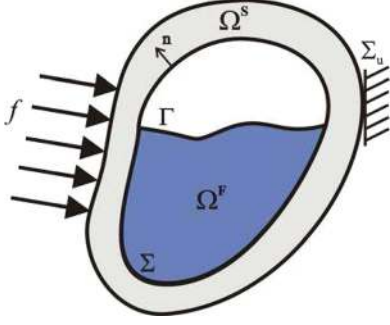


Fig. 1 Description of the system and notations

$\forall t \in \mathbb{R}^+, \exists \varphi \in \mathbf{C}_\varphi^*, \forall \delta\varphi \in \mathbf{C}_\varphi^* :$

$$\int_{\Omega_F} \nabla \varphi \cdot \nabla \delta\varphi d\Omega_F + \frac{1}{g} \int_{\Gamma} \ddot{\varphi} \delta\varphi d\Gamma - \int_{\Sigma} (\mathbf{u} \cdot \mathbf{n}) \delta\varphi d\Gamma = 0$$

with: $\varphi(M, 0) = \varphi_{\text{ini}}, \dot{\varphi}(M, 0) = \dot{\varphi}'_{\text{ini}}$ (3)

We introduce the operators \mathcal{F} , \mathcal{S} , and \mathcal{C} defined as follows (with ρ_F the mass density of the fluid):

$$\mathcal{F}(\varphi, \delta\varphi) = \rho_F \int_{\Omega_F} \nabla \varphi \cdot \nabla \delta\varphi d\Omega_F \quad (4)$$

$$\mathcal{S}(\ddot{\varphi}, \delta\varphi) = \frac{\rho_F}{g} \int_{\Gamma} \ddot{\varphi} \delta\varphi d\Gamma \quad (5)$$

$$\mathcal{C}(\mathbf{u}, \delta\varphi) = \rho_F \int_{\Sigma} (\mathbf{u} \cdot \mathbf{n}) \delta\varphi d\Gamma \quad (6)$$

which are, respectively, associated to the liquid kinetic energy, the sloshing potential energy of the liquid, and the fluid-structure coupling operator. Then, Eq. (3) can be rewritten as

$$\mathcal{F}(\varphi, \delta\varphi) + \mathcal{S}(\ddot{\varphi}, \delta\varphi) = \mathcal{C}(\mathbf{u}, \delta\varphi) \quad (7)$$

We can show that \mathcal{F} and \mathcal{S} are symmetric and positive bilinear forms. The coupling operator also occurs in the weak formulation of the structure supposed slightly damped

$\forall t \in \mathbb{R}^+, \exists \mathbf{u} \in \mathbf{C}_u^*, \forall \delta\mathbf{u} \in \mathbf{C}_u^* :$

$$\mathcal{K}(\mathbf{u}, \delta\mathbf{u}) + \mathcal{D}_S(\dot{\mathbf{u}}, \delta\mathbf{u}) + \mathcal{M}(\ddot{\mathbf{u}}, \delta\mathbf{u}) + \mathcal{C}(\delta\mathbf{u}, \varphi) = f(\delta\mathbf{u})$$

with: $\mathbf{u}(M, 0) = \mathbf{u}_{\text{ini}}, \dot{\mathbf{u}}(M, 0) = \dot{\mathbf{u}}'_{\text{ini}}$ (8)

$\mathbf{C}_u = \{\mathbf{u} \in H^1(\Omega_S)^3\}$ is the admissible space of regular functions \mathbf{u} and $\mathbf{C}_u^* = \{\mathbf{u} \in \mathbf{C}_u / \mathbf{u} = 0 \text{ on } \Sigma_u\}$. \mathcal{D}_S is the damping operator of the structure (with constant coefficients), \mathcal{M} is the mass operator, and f represents the linear operator of external forces applied to the structure. \mathcal{K} is the elastogravity operator that contains the stiffness operator and prestresses operators associated with gravity [11,12].

After discretization of the weak formulations (3) and (8) by the finite element method, we write ϕ and U the nodal unknowns of φ and \mathbf{u}

$$\mathbf{K}U + \mathbf{D}_S \dot{U} + \mathbf{M} \ddot{U} + \mathbf{C} \ddot{\phi} = \mathbf{f} \quad (9)$$

$$\mathbf{F} \phi + \mathbf{S} \ddot{\phi} - \mathbf{C}^T U = 0 \quad (10)$$

$$\phi(M, 0) = \phi_{\text{ini}}, \dot{\phi}(M, 0) = \dot{\phi}'_{\text{ini}} \quad (11)$$

$$U(M, 0) = U_{\text{ini}}, \dot{U}(M, 0) = \dot{U}'_{\text{ini}} \quad (12)$$

with $U = 0$ on Σ_u and $b^T \Phi = 0$, which is the discretized form of the constraint $\int_{\Gamma} \varphi d\Gamma = 0$. \mathbf{K} , \mathbf{M} , \mathbf{D}_S , \mathbf{F} , \mathbf{S} , and \mathbf{C} are the matrices associated with the operators of the same name.

2.2 Dissipative Fluid. We propose here to introduce the fluid damping directly in the matrix equation instead of in the local Eqs. (1). Hence, in Eq. (10), we introduce a viscous damping matrix \mathbf{D}_F . We suppose here that dissipative phenomena do not introduce new coupling effects between the liquid and structure deformation.

$$\mathbf{K}U + \mathbf{D}_S \dot{U} + \mathbf{M} \ddot{U} + \mathbf{C} \ddot{\phi} = \mathbf{f} \quad (13)$$

$$\mathbf{F} \phi + \mathbf{D}_F \dot{\phi} + \mathbf{S} \ddot{\phi} - \mathbf{C}^T U = 0 \quad (14)$$

$$\phi(M, 0) = \phi_{\text{ini}}, \dot{\phi}(M, 0) = \dot{\phi}'_{\text{ini}} \quad (15)$$

$$U(M, 0) = U_{\text{ini}}, \dot{U}(M, 0) = \dot{U}'_{\text{ini}} \quad (16)$$

We will not build the matrix \mathbf{D}_F introducing a physical modeling of damping but we suppose that a diagonal damping model is suitable to model this damping operator proposed by Caughey [13]. \mathbf{F} and \mathbf{S} play the role of the stiffness operator and mass operator of the fluid, respectively; by analogy with Caughey, the matrix \mathbf{D}_F takes the form

$$\mathbf{D}_F = \sum_{j=0}^{m-1} c_j (\mathbf{F} \mathbf{S}^{-1})^j \mathbf{S} \quad (17)$$

where m can be chosen arbitrarily. The matrix \mathbf{D}_F written this way is symmetric and is diagonalizable on the basis of the real sloshing eigenmodes of the fluid.

2.3 Projection on the Sloshing Modes

2.3.1 Decomposition of the Fluid Potential. It is possible to approximate the damped response of the liquid φ with good accuracy using a linear combination of the sloshing modes φ_i completed by a quasi-static response of the fluid φ^s to the structure deformation [10,11].

$$\varphi(M, t) = \varphi^s(\mathbf{u}(M, t)) + \sum_{i=1}^n \kappa_i(t) \varphi_i(M) \quad (18)$$

κ_i is the general coordinate associated to φ_i . Using ϕ^s and ϕ_i the nodal unknowns of φ^s and φ_i , Eq. (18) can be written as

$$\phi = \phi^s + \Phi \kappa \quad (19)$$

where Φ is the modal matrix containing the n selected modes $\{\varphi_i\}_{i \in [1, n]}$. κ is the vector containing the general coordinates κ_i associated to these modes.

2.3.2 Sloshing Modes. The harmonic movements of an incompressible heavy fluid in a rigid motionless cavity are called *sloshing modes*. They are the solutions $\varphi_i(M, t) = \varphi_i(M) \cos(\omega_i t)$ of Eq. (1) when $\mathbf{u} = 0$ (fixed wall condition)

$$\Delta \varphi_i = 0 \quad \text{sur } \Omega_F \quad (20a)$$

$$\frac{\partial \varphi_i}{\partial \mathbf{n}} = 0 \quad \text{in } \Sigma \quad (20b)$$

$$\frac{\partial \varphi_i}{\partial z} = \frac{\omega_i^2}{g} \varphi_i \quad \text{in } \Gamma \quad (20c)$$

$$\varphi_i \in \mathbf{C}_\varphi^* \quad (20d)$$

where ω_i is the real eigenpulsation associated with φ_i .

Then, using Eq. (7), the sloshing modes are obtained from the first-order eigenvalue problem associated with the following variational formulation:

$$\exists (\varphi_i, \omega_i) \in \mathbf{C}_\varphi^* \times \mathbb{R} / \forall \delta\varphi \in \mathbf{C}_\varphi^*, \mathcal{F}(\varphi_i, \delta\varphi) - \omega_i^2 \mathcal{S}(\varphi_i, \delta\varphi) = 0 \quad (21)$$

As introduced in Sec. 2.1, the bilinear forms \mathcal{F} and \mathcal{S} being symmetric and positive, there exists an infinite set of solutions $(\varphi_i, \omega_i \geq 0)$ that form a basis for the space \mathbf{C}_φ^* . Once normalized, those solutions verify the orthogonality conditions:

$$\mathcal{F}(\varphi_i, \varphi_j) = \rho_F \int_{\Omega_F} \nabla \varphi_i \cdot \nabla \varphi_j d\Omega_F = \mu_i \delta_{i,j} \quad (22)$$

$$\mathcal{S}(\varphi_i, \varphi_j) = \frac{\rho_F}{g} \int_{\Gamma} \varphi_i \varphi_j d\Gamma = \frac{\mu_i}{\omega_i^2} \delta_{i,j} \quad (23)$$

which can be written in the discretized form

$$\Phi^T \mathbf{F} \Phi = \Upsilon = \begin{pmatrix} \ddots & & 0 \\ & \mu_i & \\ 0 & & \ddots \end{pmatrix} \quad (24a)$$

$$\Phi^T \mathbf{S} \Phi = \Upsilon \Omega^{-2} = \begin{pmatrix} \ddots & & 0 \\ & \frac{\mu_i}{\omega_i^2} & \\ 0 & & \ddots \end{pmatrix} \quad (24b)$$

Ω and Υ are diagonal matrices that contain, respectively, the pulsations ω_i and the modal masses μ_i . Φ is the modal matrix introduced in Eq. (19).

In order to express the damping term $\Phi^T \mathbf{D}_F \Phi$ we can first demonstrate, reasoning by recurrence, the following relation:

$$\Phi^T (\mathbf{F} \mathbf{S}^{-1})^i = \Omega^{2i} \Phi^T \quad (25)$$

Then, using the expression of the damping matrix \mathbf{D}_F in Eq. (17) we have

$$\begin{aligned} \Phi^T \mathbf{D}_F \Phi &= \sum_{i=0}^{m-1} a_i \Phi^T (\mathbf{F} \mathbf{S}^{-1})^i \mathbf{S} \Phi \\ &= \Upsilon \sum_{i=0}^{m-1} a_i \Omega^{2(i-1)} \end{aligned} \quad (26)$$

To simplify relation (26), we will consider the following:

$$\forall i \in [1..m], \quad 2\zeta_i = \frac{1}{\omega_i} \sum_{j=0}^{m-1} a_j \omega_i^{2j} \quad (27)$$

where ζ_i is the modal damping coefficient associated with φ_i . Written in a matrix form, expression (27) gives

$$2Z = \Omega^{-1} \sum_{i=0}^{m-1} a_i \Omega^{2i} \quad (28)$$

where Z is a diagonal matrix containing the coefficients ζ_i . We deduce from (26) and (28) an expression of $\Phi^T \mathbf{D}_F \Phi$

$$\Phi^T \mathbf{D}_F \Phi = 2\Upsilon \Omega^{-1} Z \quad (29)$$

2.3.3 Quasi-Static Solution. We choose a quasi-static solution $\varphi^s = \varphi_0$, which is the hydrostatic response of the fluid.¹ From Eq. (1) we get the local equations verified by φ_0

$$\Delta \varphi_0 = 0 \quad \text{sur } \Omega_F \quad (30a)$$

$$\frac{\partial \varphi_0}{\partial \mathbf{n}} = \mathbf{u} \cdot \mathbf{n} \quad \text{in } \Sigma \quad (30b)$$

$$\frac{\partial \varphi_0}{\partial z} = -\frac{1}{\mathcal{A}(\Gamma)} \int_{\Sigma} \mathbf{u} \cdot \mathbf{n} d\Sigma - \varphi'_{\text{ini}} \quad \text{in } \Gamma \quad (30c)$$

$$\varphi_0 \in \mathbf{C}_\varphi^* \quad (30d)$$

with the same approach from Eq. (7), we write the weak formulation associated with these local equations:

$$\exists \varphi_0 \in \mathbf{C}_\varphi^* / \forall \delta \varphi \in \mathbf{C}_\varphi^*, \mathcal{F}(\varphi_0, \delta \varphi) = \mathcal{C}(\mathbf{u}, \delta \varphi) \quad (31)$$

2.3.4 Conjuguate Relations Between φ_0 and φ_i . Applying the Green formula on φ_0 and φ_i we obtain the following relation:

$$\int_{\Omega_F} (\varphi_0 \Delta \varphi_i - \varphi_i \Delta \varphi_0) d\Omega = \int_{\partial \Omega_F} (\varphi_0 (\nabla \varphi_i \cdot \mathbf{n}) - \varphi_i (\nabla \varphi_0 \cdot \mathbf{n})) d\sigma \quad (32)$$

Then, using (30) and (20) we have

$$\begin{aligned} 0 &= \int_{\Gamma} \left(\varphi_0 \frac{\partial \varphi_i}{\partial z} - \varphi_i \frac{\partial \varphi_0}{\partial z} \right) d\Gamma + \int_{\Sigma} \left(\varphi_0 \frac{\partial \varphi_i}{\partial \mathbf{n}} - \varphi_i \frac{\partial \varphi_0}{\partial \mathbf{n}} \right) d\Sigma \\ 0 &= \int_{\Gamma} \varphi_0 \frac{\omega_i^2}{g} \varphi_i d\Gamma - \int_{\Sigma} \varphi_i \mathbf{u} \cdot \mathbf{n} d\Sigma \end{aligned} \quad (33)$$

We deduce from expression (33) the first conjuguate relation

$$\mathcal{S}(\varphi_0, \varphi_i) = \frac{1}{\omega_i^2} \mathcal{C}(\mathbf{u}, \varphi_i) \quad (34)$$

Moreover, if we consider in Eq. (31) $\delta \varphi = \varphi_i \in \mathbf{C}_\varphi^*$, we obtain the second conjuguate relation

$$\mathcal{F}(\varphi_0, \varphi_i) = \mathcal{C}(\mathbf{u}, \varphi_i) \quad (35)$$

Finally, both expressions can be written in the discretized form

$$\Phi^T \mathbf{S} \phi_0 = \Upsilon \Omega^{-2} \Phi^T \mathbf{C} \mathbf{U} \quad (36a)$$

$$\Phi^T \mathbf{F} \phi_0 = \Phi^T \mathbf{C}^T \mathbf{U} \quad (36b)$$

We deduce from (17), (25), and (36), an expression of the damping term $\Phi^T \mathbf{D}_F \phi_0$

$$\begin{aligned} \Phi^T \mathbf{D}_F \phi_0 &= \sum_{i=0}^{m-1} a_i \Phi^T (\mathbf{F} \mathbf{S}^{-1})^i \mathbf{S} \phi_0 \\ &= \sum_{i=0}^{m-1} a_i \Omega^{2i} \Phi^T \mathbf{S} \phi_0 \\ &= \sum_{i=0}^{m-1} a_i \Omega^{2(i-1)} \Phi^T \mathbf{C} \mathbf{U} \\ &= 2\Omega^{-1} Z \Phi^T \mathbf{C} \mathbf{U} \end{aligned} \quad (37)$$

2.3.5 Reduced Equations. We now introduce Eq. (19) in Eqs. (13) and (14). With Eqs. (24), (29), (37) and the conjugate relations (36), we obtain the reduced equations for the fluid and the structure. The hydrostatic response φ_0 being a linear function of the structure displacement \mathbf{u} , we define the operator \mathcal{M}_0^F

$$\mathcal{M}_0^F(\mathbf{u}, \delta \mathbf{u}) = \mathcal{C}(\varphi_0(\mathbf{u}), \delta \mathbf{u}) \quad (38)$$

We call \mathcal{M}_0^F the added mass operator as it has all the properties of a mass operator. The discretized form of Eq. (38) is given in

¹In Ref. 10, Morand and Ohayon also propose to use φ^∞ , the solution of the hydroelastic problem without gravity.

Ref. 10. Introducing the unknown (U, κ) of the coupled problem projected on the sloshing modes we have

$$\underbrace{\begin{bmatrix} K & 0 \\ 0 & \gamma\Omega^2 \end{bmatrix}}_{\hat{K}} \underbrace{\begin{pmatrix} U \\ \kappa \end{pmatrix}}_{\psi} + \underbrace{\begin{bmatrix} D_s & 0 \\ -2\Omega Z' \Phi' C & 2\gamma\Omega Z \end{bmatrix}}_{\hat{D}} \underbrace{\begin{pmatrix} \dot{U} \\ \dot{\kappa} \end{pmatrix}}_{\dot{\psi}} + \underbrace{\begin{bmatrix} M + M_0^F & -C\Phi \\ -\Phi' C & \gamma \end{bmatrix}}_{\hat{M}} \underbrace{\begin{pmatrix} \ddot{U} \\ \ddot{\kappa} \end{pmatrix}}_{\ddot{\psi}} = \underbrace{\begin{bmatrix} f \\ 0 \end{bmatrix}}_{\hat{F}} \quad (39)$$

where M_0^F is the matrix associated with the operator \mathcal{M}_0^F .

We will write in the rest of this article the stiffness, the damping, and the mass of the coupled system with the symbol $\hat{\cdot}$. Then, Eq. (39) can be written as

$$\begin{aligned} \hat{K}\psi(t) + \hat{D}\dot{\psi}(t) + \hat{M}\ddot{\psi}(t) &= \hat{F}(t) \\ \psi(M, 0) &= \psi_{\text{ini}} = \begin{pmatrix} U_{\text{ini}} \\ \kappa_{\text{ini}} \end{pmatrix} \\ \dot{\psi}(M, 0) &= \dot{\psi}'_{\text{ini}} = \begin{pmatrix} U'_{\text{ini}} \\ \kappa'_{\text{ini}} \end{pmatrix} \end{aligned} \quad (40)$$

ψ being the unknown of the coupled system and \hat{F} the associated force. We deduce κ_{ini} and κ'_{ini} from the relation (19).

3 Damping Coefficient Estimation

In this section, we build the matrix Z , linking the modal damping coefficients ζ_i to their physical meaning and the different dissipative phenomenon in the fluid.

The coefficients ζ_i can be expressed as a function of the energy dissipation in each mode i . Assuming that the damping is small ($\zeta_i \ll 1$), by analogy with a one-degree-of-freedom system, we have

$$\zeta_i \simeq \frac{-\Delta E_i}{4\pi E_i} \quad (41)$$

with ΔE_i the loss of energy during a period by the mode i and E_i the total energy of the mode. This total energy can be calculated as the maximum of the kinetic energy of the fluid (or the maximum of the potential energy). It is then possible to demonstrate that [10]

$$E_i = \frac{1}{2} \rho_F \omega_i^2 \int_{\Omega_F} (\nabla \varphi_i)^2 d\Omega = \frac{1}{2} \frac{\rho_F \omega_i^4}{g} \int_{\Gamma} \varphi_i^2 d\Gamma = \frac{1}{2} \mu_i \omega_i^2 \quad (42)$$

In Ref. [14], Henderson and Miles enumerate the different sources of liquid surface-wave damping

- viscous dissipation at the fluid-structure interface,
- viscous dissipation in the interior fluid,
- viscous dissipation at the free surface, and
- capillary hysteresis at the contact line (triple line between gas, liquid, and solid)

Among these sources, we will consider, first, the viscous dissipation in the interior fluid ΔE_i^Ω and at the fluid-structure interface ΔE_i^Σ . Thus, we will have in Eq. (41)

$$\Delta E_i = \Delta E_i^\Sigma + \Delta E_i^\Omega \quad (43)$$

To calculate the modal damping coefficients, we use a modal strain energy approach (MSE). This method consists in assuming that the damped and undamped sloshing modes are sufficiently

similar in order to use the eigenmodes and eigenpulsations (ϕ_i, ω_i) of the conservative problem to estimate the energy loss associated to the damped modes [15,16].

To justify this hypothesis, we follow the reasoning in Ref. [17] which presents a study of the velocity field in a weakly viscous and incompressible fluid in a deformable container. We can show that the fluid velocity \mathbf{v} can be written as

$$\mathbf{v} = \nabla \chi + \mathbf{rot} \mathbf{A} \quad (44)$$

where χ and \mathbf{A} are, respectively, scalar and vector potentials. This decomposition is also called the Helmholtz decomposition. The existence and unicity of such potentials are discussed in Ref. 18. Then, we use the perturbation theory to approximate the full solution \mathbf{v} and develop the weakly damped potentials as a series in the small parameter ϵ with

$$\epsilon = \frac{l_\nu}{L} \quad (45)$$

where L is the characteristic length of the problem and l_ν represents the characteristic length of viscous phenomenon

$$l_\nu(\omega) = \sqrt{\frac{2\nu}{\omega}} \quad (46)$$

ν being the kinetic viscosity coefficient of the fluid. The first-order development is obtained by truncating the series:

$$\chi = \chi_0 + \epsilon \chi_1 + o(\epsilon) \text{ et } \mathbf{A} = \mathbf{A}_0 + \epsilon \mathbf{A}_1 + o(\epsilon) \quad (47)$$

Then, inserting Eq. (44) with the development Eq. (47) in the linearized Navier Stokes equation [17] we can show that $\mathbf{rot} \mathbf{A}_0$ decays exponentially when the study point moves away from the container's wall. When the viscosity ν tends to 0 (ϵ tends to 0), the velocity field of the viscous liquid tends to the velocity field of an inviscid liquid everywhere except in the boundary layer at the fluid-structure interface (whose thickness also tends to 0). Hence, for small displacements, the weakly damped fluid can be considered as quasi-irrotational except in the boundary layer. However, this region is sufficiently thin to consider that the volume of the irrotational fluid and the total fluid volume can be confused [19,20].

3.1 Viscous Dissipation at the Fluid-Structure Interface. The viscous dissipation of the liquid in contact with the solid walls is generated in the boundary layer. The boundary-layer thickness is sufficiently thin and we can calculate the properties of the flow with a good accuracy considering that the walls are locally planar and the flow is assumed to be parallel to the walls. It is then possible to estimate an average of the energy lost during a period of oscillation by viscous effect in the boundary layer [14]:

$$\Delta E_i^\Sigma = -\frac{\rho\pi}{2} l_\nu(\omega_i) \omega_i^2 \int_{\Sigma} (\nabla \varphi_i)^2 dS \quad (48)$$

3.2 Viscous Dissipation in the Interior Fluid. Although the viscous dissipation at the fluid-structure interface and the viscous dissipation in the interior fluid are respectively proportional to ϵ and ϵ^2 , their contributions are sometimes comparable [21,22]. Adding this dissipative term can improve the comparison of the damping coefficients calculated with the corresponding analytical result (see Sec. 3.3). It is possible to estimate an average of the energy lost during a period of oscillation by viscous effect in the interior fluid (which is considered as incompressible and irrotational) by Refs. [22] and [23]

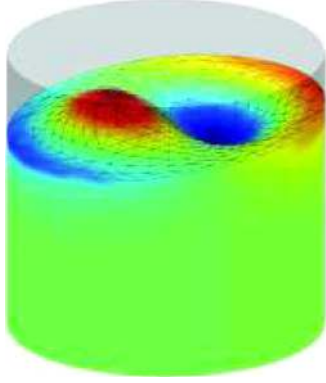


Fig. 2 Example of a sloshing mode

Table 1 Comparison between damping coefficients obtained analytically and numerically

Mode number	Finite element results (10^{-3})	Analytical results (10^{-3})
1-2	5.0	4.9
3-4	6.2	6.1
5	2.9	2.9
6-7	7.1	7.1
8-9	7.8	7.8

$$\Delta E_i^\Omega = -\frac{\rho\pi}{2} l_\nu^2(\omega_i)\omega_i^2 \int_{\partial\Omega_f} \nabla((\nabla\phi_i)^2) \cdot \mathbf{n} dS \quad (49)$$

3.3 Validation. We consider here the test case proposed by Henderson and Miles [14]: a small cylindrical container (diameter: 5.532 cm, height: 3.80 cm), which is also studied in other articles [19,21]. An example of a sloshing mode is presented Fig. 2 and a comparison between the analytical results from [19] and the numerical results obtained from a finite element calculation of the integrals in Eqs. (48) and (49) is given by Table 1. We notice a good approximation between the two approaches.

4 Nonsymmetric Quadratic Eigenvalue Problem

4.1 Laplace Transform. The matrices \hat{M} , \hat{D} , and \hat{K} of Eq. (31) are now entirely built and we now want to calculate ψ . Using the Laplace transform we have

$$\mathcal{L}[\psi(t)] = \psi(s) \quad (50)$$

$$\mathcal{L}[\dot{\psi}(t)] = s\psi(s) - \psi_{\text{ini}} \quad (51)$$

$$\mathcal{L}[\ddot{\psi}(t)] = s^2\psi(s) - s\psi_{\text{ini}} - \psi'_{\text{ini}} \quad (52)$$

$$\mathcal{L}[\hat{F}(t)] = \hat{F}(s) \quad (53)$$

Then, we obtain from (40)

$$(s^2\hat{M} + s\hat{D} + \hat{K})\psi(s) = \hat{F}(s) + \underbrace{(\hat{D} + s\hat{M})\psi_{\text{ini}} + \hat{M}\psi'_{\text{ini}}}_{\text{Initial Conditions}} \quad (54)$$

A costly way to obtain $\psi(s)$ consists in solving this linear matrix system for each desired s . We will prefer here a faster way to calculate $\psi(s)$ and $\psi(t)$ using a modal synthesis method (Secs. 4 and 5)

by computing first the eigenmodes of the eigenvalue problem associated with Eq. (54). We will finally compare both approaches.

4.2 Eigenvalue Problem and Properties. We consider now the second-order eigenvalue problem associated with Eq. (54); the left and right eigenvalue problems can be written as

$$\text{Find } (x_i, p_i) / (\hat{M}p_i^2 + \hat{D}p_i + \hat{K})x_i = 0 \quad x_i \neq 0 \quad (55a)$$

$$\text{Find } (y_i, p_i) / y_i^*(\hat{M}p_i^2 + \hat{D}p_i + \hat{K}) = 0 \quad y_i \neq 0 \quad (55b)$$

x_i , y_i , and p_i are, respectively, the right and left eigenvectors, and the associated eigenvalues. y_i^* is the conjugate transpose of y_i ; p_i , x_i , and y_i are generally complex.

First, we will present some noteworthy properties of the second-order eigenvalue problem (also called quadratic eigenvalue problem) [24,25]. Then we will consider the first-order state-space representation of the system. The results are used in Sec. 5 to calculate the time response and frequency response.

\hat{K} and \hat{M} are real and symmetric matrices of size $N \times N$. The properties of the eigenvectors and eigenvalues will depend more particularly upon the property of the damping matrix \hat{D} . In our case, \hat{D} is a real and nonsymmetric matrix. Thus, the eigenvectors x_i and y_i , and solutions of the quadratic eigenvalue problems will be different and do not necessary diagonalize simultaneously the three matrices \hat{K} , \hat{M} , and \hat{D} .

Let us introduce the polynomial matrix of degree two $P(p)$ (p is a complex variable)

$$P(p) = p^2M + pD + K \quad (56)$$

We denote by $S(P)$ the spectrum of $P(p)$ with

$$S(P) = \{p \in \mathbb{C}, \det(P(p)) = 0\} \quad (57)$$

$S(P)$ is the set of eigenvalues of $P(p)$ and $\det(P(p))$ is the associated characteristic polynomial which is defined by [24]

$$\det(P(p)) = \det(M)p^{2N} + \text{lower order terms} \quad (58)$$

Consequently, when M is nonsingular, $P(p)$ has $2N$ finite eigenvalues in \mathbb{C} . When M is singular, $P(p)$ has r eigenvalues in \mathbb{C} with $r < 2N$ and $2N - r$ infinite eigenvalues. Moreover, $P(p)$ may have two distinct eigenvalues having the same eigenvector, in that case, the number of independent eigenvectors will be inferior to $2N$ [24,26].

Because \hat{K} , \hat{D} , and \hat{M} are real, the spectrum of $P(p)$ is symmetric about the real axis of the complex plane: The eigenvalues are either real or occur in complex conjugate pairs. Complex eigenvalues are associated with vibrational modes and real ones can be related to supercritical damping (damping ratio above 1) or a particular constitutive law of the material. Because we consider here only weakly damped cases, we will only consider complex conjugate solutions.

A common technique consists in employing the first-order state-space representation of the system. From a quadratic eigenvalue problem we obtain a generalized eigenvalue problem which

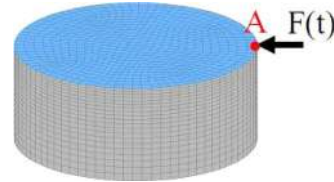


Fig. 3 Test tank. Height: 2 m, Radius: 2 m.

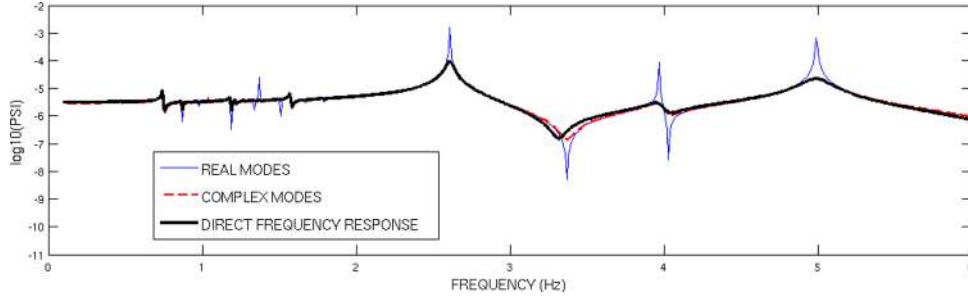


Fig. 4 Comparison between the frequency responses in displacement of the point A from 0 to 6 Hz in the radial direction for an excitation at the same point and in the same direction

is of first order but has twice the size of the initial problem. We consider the variable Θ , given by

$$\Theta(M, t) = \begin{pmatrix} \psi(M, t) \\ \dot{\psi}(M, t) \end{pmatrix} \quad (59)$$

Thus, the problem in Eq. (40) can be written as

$$\begin{aligned} \mathbf{B}\dot{\Theta}(M, t) - \mathbf{A}\Theta(M, t) &= \mathbf{C} \\ \Theta(M, 0) &= \Theta_{\text{ini}} = \begin{pmatrix} \psi_{\text{ini}} \\ \dot{\psi}_{\text{ini}} \end{pmatrix} \end{aligned} \quad (60)$$

with

$$\mathbf{B} = \begin{bmatrix} -\hat{K} & 0 \\ 0 & \hat{M} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & -\hat{K} \\ -\hat{K} & -\hat{D} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 \\ \hat{F} \end{bmatrix}$$

As \hat{D} is not symmetric, \mathbf{A} is not symmetric either. Left and right eigenvalue problems associated with Eq. (60) can be written as

$$\text{Find } (X_i, p_i) / (p_i \mathbf{B} - \mathbf{A})X_i = 0 \quad X_i \neq 0 \quad (61)$$

$$\text{Find } (Y_i, p_i) / Y_i^*(p_i \mathbf{B} - \mathbf{A}) = 0 \quad Y_i \neq 0 \quad (62)$$

By analogy with Eq. (56) we define the polynomial matrix Q

$$Q(p) = p\mathbf{B} - \mathbf{A} \quad (63)$$

There are many ways to build the matrices \mathbf{A} and \mathbf{B} , indeed, the one we chose makes Q an equivalent of P ; we have

$$\begin{bmatrix} P(p) & 0 \\ 0 & I_N \end{bmatrix} = E(p)Q(p)F(p) \quad (64)$$

where I_N is the identity of size N . $E(p)$ and $F(p)$ are $2N \times 2N$ matrices with constant nonzero determinants [24]. Then, regarding Eq. (64), we notice that the spectrum of Q and P are the same. Moreover, with \mathbf{A} and \mathbf{B} written with this form, eigenvectors (Y_i, X_i) can be expressed as functions of (x_i, y_i, p_i)

$$X_i = \begin{pmatrix} x_i \\ p_i x_i \end{pmatrix} \quad Y_i = \begin{pmatrix} y_i \\ \bar{p}_i y_i \end{pmatrix} \quad (65)$$

Indeed, developing Eqs. (61) and (62) with the relations (65), we obtain on the top block the problems in Eqs. (55a) and (55b).

In the rest, we define the modal matrix \mathbf{x} of size $N \times m$ (m being the number of modes calculated and $m < 2N$) containing the eigenvectors x_i , solutions of Eq. (55a) with positive imaginary part. Similarly, \mathbf{y} contains the left eigenvectors y_i with positive imaginary part and λ is the diagonal matrix that contains the m corresponding eigenvalues. Let \mathbf{X} and \mathbf{Y} be the modal matrices

containing the selected eigenvectors X_i and Y_i ; we can then write using Eq. (65)

$$\mathbf{X} = \begin{bmatrix} \mathbf{x} & \bar{\mathbf{x}} \\ \mathbf{x}\lambda & \bar{\mathbf{x}}\bar{\lambda} \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} \mathbf{y} & \bar{\mathbf{y}} \\ \mathbf{y}\lambda & \bar{\mathbf{y}}\bar{\lambda} \end{bmatrix} \quad (66)$$

We consider here the case where M is a singular matrix and $\text{rank}(P(p)) = N$ (P is said to be regular). Moreover, all the eigenvalues of P are considered as semisimple,² then, the eigenvalues of Q are semisimple too and the matrices \mathbf{A} and \mathbf{B} are diagonalizable (see lemma 5 in Ref. 2). The eigenvectors X_i and Y_i are said to be bi-orthogonal in the sense that

$$\mathbf{Y}^* \mathbf{B} \mathbf{X} = \mathbf{R} = \begin{bmatrix} \mathbf{r} & 0 \\ 0 & \bar{\mathbf{r}} \end{bmatrix} \quad \text{and} \quad \mathbf{Y}^* \mathbf{A} \mathbf{X} = \mathbf{R} \mathbf{L} = \begin{bmatrix} \mathbf{r}\lambda & 0 \\ 0 & \bar{\mathbf{r}}\bar{\lambda} \end{bmatrix} \quad (67)$$

5 Time Response and Frequency Response

We now solve Eq. (60) by using the Laplace transform with

$$\mathcal{L}[\Theta(t)] = \Theta(s) \quad (68)$$

We write a solution of the state equation Θ as a linear combination of the right eigenvectors X_i multiplied by modal coordinates Q_i as follows:

$$\Theta(s) = \mathbf{X}Q(s) = \sum_{i=1}^{2m} X_i Q_i(s) \quad (69)$$

Then we have

$$\mathcal{L}[\dot{\Theta}(t)] = s\Theta(s) - \Theta_{\text{ini}} = s\mathbf{X}Q(s) - \mathbf{X}Q_{\text{ini}} \quad (70)$$

$$\mathcal{L}[F(t)] = F(s) \quad (71)$$

Pre-multiplying Eq. (60) by \mathbf{Y}^* and using the relation (69) between Θ and \mathbf{X} yields

$$\mathbf{Y}^*(s\mathbf{B} - \mathbf{A})\mathbf{X}Q(s) = \mathbf{Y}^*\mathbf{B}\Theta_{\text{ini}} + \mathbf{Y}^*\mathbf{C}F(s) \quad (72)$$

The relations previously described in Eq. (67) simplify Eq. (72) and we have

$$Q(s) = (sI_{2N} - \mathbf{L})^{-1} \mathbf{R}^{-1} (\mathbf{Y}^*\mathbf{B}\Theta_{\text{ini}} + \mathbf{Y}^*\mathbf{C}F(s)) \quad (73)$$

We then deduce $\Theta(s)$ from Eq. (69)

$$\Theta(s) = \mathbf{X}(sI_{2N} - \mathbf{L})^{-1} \mathbf{R}^{-1} (\mathbf{Y}^*\mathbf{B}\Theta_{\text{ini}} + \mathbf{Y}^*\mathbf{C}F(s)) \quad (74)$$

²Let α be the algebraic multiplicity of the eigenvalue p_i , α is the order of the corresponding zero in $\det(P(p))$. The geometric multiplicity β of p_i is the dimension of the associated characteristic subspace. For a semisimple eigenvalue we have: $\alpha = \beta$.

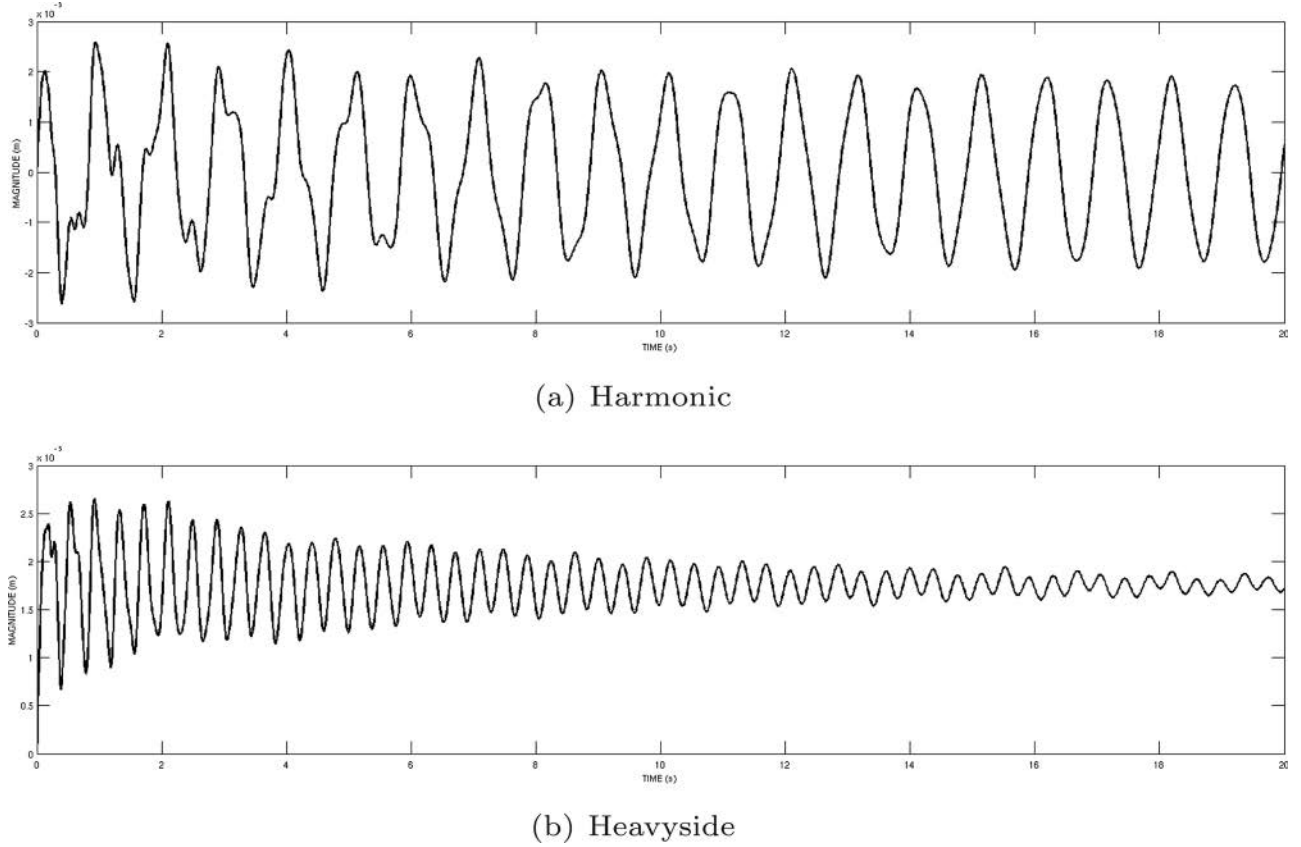


Fig. 5 Time response of point A in the radial direction for a harmonic excitation at the frequency $f_1 = 0.63$ Hz and a Heavyside; $F_0 = 600$ N

The relations between Θ and ψ are given by Eq. (60). Then, by developing Eq. (74), it is possible to give an expression of $\psi(s)$ using the modal matrices \mathbf{x} and \mathbf{y}

$$\psi(s) = -\hat{K}T_1\psi_{\text{ini}} + \hat{M}T_2\psi'_{\text{ini}} + T_2F(s) \quad (75)$$

with

$$T_1 = \mathbf{x}(sI - \boldsymbol{\lambda})^{-1}\mathbf{r}^{-1}\mathbf{y}^* + \bar{\mathbf{x}}(sI - \bar{\boldsymbol{\lambda}})^{-1}\bar{\mathbf{r}}^{-1}\bar{\mathbf{y}}^* \quad (76)$$

$$T_2 = \mathbf{x}(sI - \boldsymbol{\lambda})^{-1}\mathbf{r}^{-1}\boldsymbol{\lambda}\mathbf{y}^* + \bar{\mathbf{x}}(sI - \bar{\boldsymbol{\lambda}})^{-1}\bar{\mathbf{r}}^{-1}\bar{\boldsymbol{\lambda}}\bar{\mathbf{y}}^* \quad (77)$$

The first two terms of Eq. (75) will give (in the time domain) the free response of the system to an initial perturbation (in displacement and velocity). The third term depends on the load type and will give (in the time domain) the transitory and forced response of the system.

The frequency response between the entry node k and the exit node l is given by Eq. (78) considering zero initial condition and $s = i\omega$, where ω is the pulsation of excitation

$$\psi_{kl}(\omega) = E_k^T (\mathbf{x}(i\omega I - \boldsymbol{\lambda})^{-1}\mathbf{r}^{-1}\boldsymbol{\lambda}\mathbf{y}^* + \bar{\mathbf{x}}(i\omega I - \bar{\boldsymbol{\lambda}})^{-1}\bar{\mathbf{r}}^{-1}\bar{\boldsymbol{\lambda}}\bar{\mathbf{y}}^*) E_l \quad (78)$$

where E_i is a column vector composed of zeros and a single coefficient equals to 1 on the line number i (each node and direction of the system can be associated with a unique number).

It is also possible to obtain from Eq. (75) the time response using the inverse Laplace transform. This response can be written (using zero initial conditions) as

$$\psi(t) = \mathcal{L}^{-1}[\psi(s)] = 2\text{Re} \left(\int_0^t \mathbf{x}e^{\boldsymbol{\lambda}(t-\tau)} \mathbf{r}^{-1}\boldsymbol{\lambda}\mathbf{y}F(\tau) d\tau \right) \quad (79)$$

This expression can be calculated analytically for a simple excitation type (harmonic or Heavyside for instance); using the classical form of a sum we have

$$\text{Harmonic excitation: } F(t) = F_0 \cos(\omega t)$$

$$\psi(t) = \text{Re} \sum_{i=1}^{2m} \frac{p_i x_i y_i^*}{(i\omega - p_i) r_i} F_0 (e^{i\omega t} - e^{p_i t}) \quad (80)$$

$$\text{Heavyside excitation: } F(t) = F_0 H(t)$$

$$\psi(t) = \sum_{i=1}^{2m} \frac{x_i y_i^*}{r_i} F_0 (e^{p_i t} - 1) \quad (81)$$

6 Test Case

In order to apply this formulation to a test case, we consider a cylindrical container made of steel full of water and elastic. The tank bottom is fixed so that the rigid body movements are removed. The excitation is made at the point A on the top edge of the tank and in the radial direction (see Fig. 3). We study here the displacement of point A in the radial direction.

6.1 Frequency Response. In Sec. 5, we give the frequency response expressed with the complex modes of the coupled system x_i and y_i . These modes are computed in NASTRAN with the complex Lanczos algorithm. We compare this modal method with the direct frequency response, which serves as reference solution, given by

$$\psi_{kl}(\omega) = E_k^T \left(-\omega^2 \hat{M} + i\omega \hat{D} + \hat{K} \right)^{-1} E_l \quad (82)$$

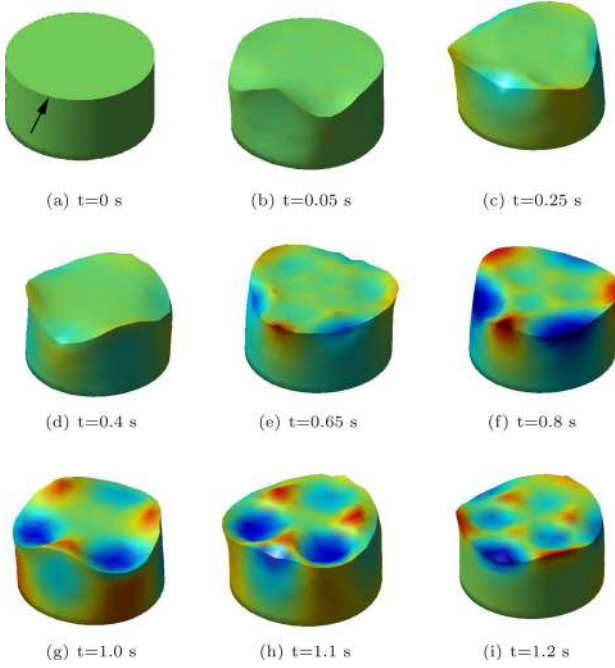


Fig. 6 Snapshots of the liquid deformation at different instants for a harmonic excitation at the frequency 0.99 Hz (black arrow in Fig. 6(a)). The colors represent the pressure fluctuations in the fluid.

Figure 4 presents a comparison between the direct frequency response and the responses obtained by modal synthesis with the complex modes and the real modes (the real modes are obtained considering $\hat{D} = 0$ in Eq. (55)). We notice that the modal synthesis performed using complex modes gives a good approximation of the direct frequency response for low frequency modes (i.e., sloshing modes) and hydro-elastic modes (from 2 Hz).

6.2 Time Response. From expressions (80) and (81), we represent the displacement of the point A as a function of time for different type of excitation: harmonic or Heavyside (see Fig. 5). The harmonic excitation is at frequency $f_1 = 0.99$ Hz, which is close to a sloshing frequency of the liquid.

We also represent, in Fig. 6, the transitory response of the fluid for the same harmonic load (sketched by the black arrow in Figs. 3 or 6(a)) with the same amplitude and at frequency f_1 (in those pictures the structure is hidden) and the colors represent the fluid pressure variations. In the first snapshot, the fluid is at rest. From Fig. 6(b), we can see the fluid starting to be excited by the force applied on the top edge of the cylinder.

7 Conclusion

The linearized formulation we present here is adapted for small excitation amplitudes and for the use of a modal approach (it is not directly applicable to another sloshing problem, for instance the nonlinear problem of slosh-induced impact pressure in a ship cargo tank [6]).

We first build the coupled fluid-structure system, introducing a diagonal damping model for the incompressible fluid. The associated eigenvalue problem is quadratic with a nonsymmetric damping operator. The properties of its eigenmodes are presented. Then, we project the reduced coupled system on its complex modes. The computed eigenmodes are not orthogonal and do not diagonalize the matrices \hat{M} , \hat{K} , and \hat{D} . However, using a particular

state-space representation, it is possible to find properties of bi-orthogonality of the left and right eigenmodes of the system, and solve the problem with a modal synthesis approach.

Thus, a method to compute the linearized time response or the frequency response of a dissipative and nonsymmetric coupled system is proposed. This formulation allow us to compute with good accuracy those responses for any point of the structure or the fluid, for different excitation types, in a faster way than a matrix inversion for all frequency steps or using an iterative time integration scheme.

References

- [1] Chiu, E., Schotté, J.-S., Farhat, C., and Ohayon, R., 2011, "Flutter Analysis of a Wing-Store Configuration With Slosh Effects," *Proceedings of the 30th International Forum of Aeroelasticity and Structural Dynamics*, Paris, France.
- [2] Abramson, H. N., 1966, "The Dynamic Behaviour of Liquids in Moving Containers," NASA Report No. SP-106.
- [3] Moiseyev, N. N., and Romyantsev, V. V., 1968, *Dynamic Stability of Bodies Containing Fluid*, (Springer-Verlag Applied Physics and Engineering, Vol. 6), Springer-Verlag, Berlin.
- [4] Smith, C. C., 1948, "The Effect of Fuel Sloshing on the Lateral Stability of a Free-Flying Airplane Model," Langley Memorial Aeronautical Laboratory, Technical Report No. NACA RL L8C16.
- [5] Ibrahim, R. A., 2005, *Liquid Sloshing Dynamics: Theory and Applications*, Cambridge University Press, Cambridge, England.
- [6] Faltinsen, O. M., and Timokha, A. N., 2009, *Sloshing*, Cambridge University Press, Cambridge, England.
- [7] El-Kamali, M., 2010, "Ballotement des Liquides Avec Tension Superficielle: Etudes Dynamique et Statique," Ph.D. thesis, Conservatoire National des Arts et Métiers, Paris, France.
- [8] El-Kamali, M., Schotté, J.-S., and Ohayon, R., 2010, "Three Dimensional Modal Analysis of Sloshing Under Surface Tension," *Int. J. Numer. Methods Fluids*, **65**, pp. 87–105.
- [9] Schotté, J.-S., 2001, "Influence de la Gravité sur les Vibrations Linéaires d'une Structure Élastique Contenant un Liquide Incompressible," Ph.D. thesis, Conservatoire National des Arts et Métiers, Paris, France.
- [10] Morand, H. J.-P., and Ohayon, R., 1995, *Fluid-Structure Interaction: Applied Numerical Methods*, Wiley, New York.
- [11] Schotté, J.-S., and Ohayon, R., 2009, "Various Modelling Levels to Represent Internal Liquids in the Vibratory Analysis of Complex Structures," *Comp. Methods Appl. Mech. Eng.*, **198**, pp. 1913–1925.
- [12] Schotté, J.-S., and Ohayon, R., 2005, "Incompressible Hydroelastic Vibrations: Finite Element Modelling of the Elastogravity Operator," *Comput. Struct.*, **83**, pp. 209–219.
- [13] Caughey, T., 1960, "Classical Normal Modes in Damped Linear Systems," *J. Appl. Mech.*, **27**, pp. 269–271.
- [14] Henderson, D. M., and Miles, J. W., 1994, "Surface-Wave Damping in a Circular Cylinder With a Fixed Contact Line," *J. Fluid Mech.*, **275**, pp. 285–299.
- [15] Johnson, C. D., Kienholz, D. A., and Rogers, L. C., 2003, "Finite Element Prediction of Damping in Beams With Constrained Viscoelastic Layers," *Shock Vib. Bull.*, **51**, pp. 71–82.
- [16] Tsai, M. H., and Chang, K. C., 2000, "A Study on Modal Strain Energy Method for Viscoelastically Damped Structures," *Proceedings of the Twelfth World Conference on Earthquake Engineering*, Auckland, New Zealand.
- [17] Anquez, L., 1974, "Etude de l'Amortissement des Vibrations Dans un Liquide Peu Visqueux," *Acta Astronaut.*, **1**, pp. 172–184.
- [18] Girault, V., and Raviart, P.-A., 1981, *Finite Element Approximation of the Navier-Stokes Equations*, Springer-Verlag, Berlin.
- [19] Wang, W., Li, J., and Wang, T., 2008, "Modal Analysis of Liquid Sloshing With Different Contact Line Boundary Conditions Using FEM," *J. Sound Vib.*, **317**, pp. 739–759.
- [20] Kidambi, R., 2009, "Capillary Damping of Inviscid Surface Waves in a Circular Cylinder," *J. Fluid Mech.*, **627**, pp. 323–340.
- [21] Martel, C., Nicolás, J. A., and Vega, J. M., 1998, "Surface-Wave Damping in a Brimful Circular Cylinder," *J. Fluid Mech.*, **360**, pp. 213–228.
- [22] Miles, J. W., and Henderson, D. M., 1998, "A Note on Interior vs. Boundary-Layer Damping of Surface Waves in a Circular Cylinder," *J. Fluid Mech.*, **364**, pp. 319–323.
- [23] Case, K. M., and Parkinson, W. C., 1957, "Damping of Surface Waves in an Incompressible Liquid," *J. Fluid Mech.*, **2**, pp. 172–184.
- [24] Tisseur, F., and Meerbergen, K., 2001, "The Quadratic Eigenvalue Problem," *SIAM Rev.*, **43**, pp. 235–286.
- [25] Saad, Y., 1992, *Numerical Methods for Large Eigenvalue Problems*, Halsted, New York.
- [26] Gohberg, I., Lancaster, P., and Rodman, L., 1982, *Matrix Polynomials*, Academic, New York.