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Kaïs Ammari, Fathi Hassine, Luc Robbiano. Carleman Estimate and Application to the Stabilization of a Dissipative Hyperbolic System. Control and Inverse Problems The 2022 Spring Workshop in Monastir, Tunisia, Part F1467, 2023, Trends in Mathematics, 10.1007/978-3-031-35675-9_6 . hal-04268887

HAL Id: hal-04268887

<https://hal.science/hal-04268887>

Submitted on 3 Nov 2023

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Carleman Estimate and Application to the Stabilization of a Dissipative Hyperbolic System

KAÏS AMMARI, FATHI HASSINE, AND LUC ROBBIANO

Abstract. We study the problem of stabilization for the acoustic system with a spatially distributed damping. Without imposing any hypotheses on the structural properties of the damping term, we identify logarithmic decay of solutions with growing time. Logarithmic decay rate is shown by using a frequency domain method and combines a contradiction argument with the multiplier technique and a new Carleman estimate to carry out a special analysis for the resolvent.

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1. Introduction

We consider the following system of equations:

$$(1.1) \quad \begin{cases} \nabla \cdot u + r + b u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ r_t + \operatorname{div} u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u \cdot n = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(0, x) = u^0(x), \quad r(0, x) = r^0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^d , $d \geq 2$, with a smooth boundary Γ , $\operatorname{div} = \nabla \cdot$ is the divergence operator and $b \in L^\infty(\Omega)$, with $b \geq 0$ on Ω and such that

$$(1.2) \quad \exists b_- > 0 \text{ such that } b \geq b_- \text{ on } \omega.$$

Here $\omega \neq \emptyset$ stands for the open subset of Ω on which the feedback is active. As usual n denotes the unit outward normal vector along Γ .

The system of equations (1.1) is a linearization of the *acoustic equation* governing the propagation of acoustic waves in a compressible medium, see Lighthill [21, 22, 23], where $b u$ represents

2010 *Mathematics Subject Classification.* 35L04, 93B07, 93B52, 74H55.

Key words and phrases. logarithmic stability, Carleman estimate, resolvent estimate, dissipative hyperbolic system, acoustic equation.

a damping term of Brinkman type. This kind of damping arises also in the process of homogenization (see Allaire [1]), and is frequently used as a suitable *penalization* in fluid mechanics models, see Angot, Bruneau, and Fabrie [4]. Our main goal is to prove the logarithmic decay of solutions of (1.1) with growing time.

Let $L^2(\Omega)$ denote the standard Hilbert space of square integrable functions in Ω and its closed subspace $L_m^2(\Omega) = \{f \in L^2(\Omega) : \int_{\Omega} f(x) dx = 0\}$. To avoid abuse of notation, we shall write $\|\cdot\|$ for the $L^2(\Omega)$ -norm or the $L^2(\Omega)^d$ -norm.

Denoting $H = (L^2(\Omega))^d \times L_m^2(\Omega)$, we introduce the operator

$$A = \begin{pmatrix} 0 & \nabla \\ \operatorname{div} & 0 \end{pmatrix} : D(A) = \left\{ (u, r) \in H, (\nabla r, \operatorname{div} u) \in H, u \cdot n|_{\Gamma} = 0 \right\} \subset H \rightarrow H,$$

and

$$B = \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix} \in L((L^2(\Omega))^d, H), B^* = \begin{pmatrix} \sqrt{b^-} & 0 \end{pmatrix} \in L(H, (L^2(\Omega))^d).$$

We recall that for $u \in (L^2(\Omega))^d$ with $\operatorname{div} u \in L^2(\Omega)$, $u \cdot n|_{\Gamma}$ make sens in $H^{-1/2}(\Gamma)$ (see Girault-Raviart [13, Chp 1, Theorem 2.5]).

Accordingly, the problem (1.1) can be recasted in an abstract form:

$$(1.3) \quad \begin{aligned} Z_t(t) + AZ(t) + BB^*Z(t) &= 0, t > 0, \\ Z(0) &= Z^0, \end{aligned}$$

where $Z = (u, r)$, or, equivalently,

$$(1.4) \quad \begin{aligned} Z_t(t) &= A_d Z(t), t > 0, \\ Z(0) &= Z^0, \end{aligned}$$

with $A_d = -A - BB^*$ with $D(A_d) = D(A)$.

It can be shown (see [2]) that for any initial data $(u^0, r^0) \in D(A)$ the problem (1.1) admits a unique solution

$$(u, r) \in C([0, \infty); D(A)) \cap C^1([0, \infty); H).$$

Moreover, the solution (u, r) satisfies the energy identity

$$(1.5) \quad E(0) - E(t) = \int_0^t \int_{(L^2(\Omega))^d} \sqrt{b} |u(s)|^2 ds, \text{ for all } t \geq 0$$

with

$$(1.6) \quad E(t) = \frac{1}{2} \|(u(t), r(t))\|_H^2, \forall t \geq 0,$$

where we have denoted

$$\langle (u, r), (v, p) \rangle_H = \int_{\Omega} (u(x) \cdot v(x) + r(x)p(x)) dx, \|(u, r)\|_H = \sqrt{\int_{\Omega} |u(x)|^2 + r^2(x) dx}.$$

Using (1.5) and a standard density argument, we can extend the solution operator for data $(u^0, r^0) \in H$. Consequently, we associate with the problem (1.1) (or to the abstract Cauchy problems (1.3) or (1.4)) a semi-group that is globally bounded in H .

As the energy E is nonincreasing along trajectories, we want to determine the set of initial data (u^0, r^0) for which

$$(1.7) \quad E(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Such a question is of course intimately related to the structural properties of the function b , notably to the geometry of the set ω on which the damping is effective. In fact, when the damping term is globally distributed Ammari, Feireisl and Nicaise [2] showed an exponential decay rate of

the energy by the means of an observability inequality associated with the conservative problem of (1.1). Besides, it is also shown that if the damping coefficient is not uniformly positive definite (i.e. $\inf_{x \in \Omega} b(x) = 0$) then the system (1.1) is not exponentially stable. In this paper we consider a damping which is locally distributed over the domain Ω without any geometrical control condition in particular this including the case when the damping coefficient is not uniformly positive defined. So we expect to prove a weaker decay rate then given in [2]. More precisely, we prove a logarithm decay rate of the energy. Our approach is based in the frequency domain method which consist to prove an exponential loss on the resolvent estimate [7, 6, 10] where the main tool for establishing a such decay is the Carleman estimate.

The theory of Carleman estimates for scalar equations is rather well developed by now. We refer to Hörmander [14] and Lebeau and Robbiano [17, 18, 19] for the second-order elliptic and hyperbolic PDE's and to Isakov [15] second-order parabolic and Schrödinger operators. However, it turned out that Carleman estimates for systems in more than two variables are difficult to obtain and still somehow very limited: The first results to systems go back to then Carleman's original work [9] which is written for a system in two independent variables, and we refer to Calderón [8] and Kreiss [16] for more relevant systems. Recently, Eller and Toundykov [11] have established a Carleman estimate for some first-order elliptic systems. This estimate is extended to elliptic boundary value problems provided the boundary condition satisfies a Lopatinskiï-type requirement. In this paper we provide a Carleman estimate for a system of first-order which does not fit into the same framework as that of Eller and Toundykov [11]. Unlike their approach, our method is based into the Hörmander approach which is essentially based on the sub-ellipticity condition and the Gårding inequality in order to control the non-elliptical regions.

The paper is organized as follows. Section 2 summarizes some well known facts concerning the acoustic system (1.1). In section 3, we establish a new Carleman estimate needed for the stabilization problem of the system (1.1). In Section 4, we prove the logarithmic stability for the system (1.1).

2. Preliminaries

We start with a simple observation that the problem (1.1) can be viewed as a bounded (in H) perturbation of the conservative system

$$(2.8) \quad \begin{aligned} u_t + \nabla r &= 0, & \text{in } \Omega \times \mathbb{R}^+, \\ r_t + \operatorname{div} u &= 0, & \text{in } \Omega \times \mathbb{R}^+, \end{aligned}$$

which can be recast as the standard *wave equation*

$$r_{tt} - \Delta r = 0.$$

Consequently, the basic existence theory for (1.1) derives from that of (2.8). Hence A_d generates a C_0 -semigroup $(S(t))_{t \geq 0}$ in H that is even of contraction because A_d is dissipative (see (1.5)).

The first main difficulty is that the operator A_d possesses a non-trivial (and large) kernel that is left invariant by the evolution. Indeed if (u, r) belongs to $\ker A_d$, then it is solution of the "stationary" problem

$$(2.9) \quad \nabla r + bu = 0, \operatorname{div} u = 0, \text{ in } \Omega.$$

Thus multiplying the first identity of (2.9) by \bar{u} and integrating over Ω yields

$$\int_{\Omega} (\nabla r \cdot \bar{u} + b|u|^2) dx = 0.$$

By an integration by parts, using the fact that u is solenoidal and the boundary condition $u \cdot n = 0$ on Γ , we get

$$\int_{\Omega} \nabla r \cdot \bar{u} dx = 0,$$

and therefore we obtain

$$\int_{\Omega} b|u|^2 dx = 0.$$

In other words, we have

$$u = 0 \text{ on } \text{supp } b,$$

and coming back to (2.9), we find

$$\nabla r = 0.$$

Accordingly, we have shown that

$$\ker A_d = \{(u, 0) \in D(A) \mid \text{div } u = 0, u|_{\text{supp } b} = 0, u \cdot n|_{\Gamma} = 0\}.$$

For shortness set $E = \ker A_d$ and introduce also its orthogonal complement H_0 in H .

It is easy to check that

$$\langle A_d(w, s), (u, r) \rangle_H = 0 \text{ for any } (w, s) \in D(A), (u, r) \in E;$$

in particular, the semi-group associated with (1.1) leaves both E and H_0 invariant. Consequently, the decay property (1.7) may only hold for initial data emanating from the set H_0 .

The following observation can be shown by a simple density argument:

Lemma 2.1. *The solution (u, r) of (1.1) with initial datum in $D(A_d)$ satisfies*

$$(2.10) \quad E'(t) = - \int_{\Omega} b|u|^2 dx \leq 0.$$

Therefore the energy is non-increasing and (1.5) holds for all initial datum in H .

As already shown in the above, the strong stability result (1.7) may hold only if we take the initial data

$$(u^0, r^0) \in H_0 = \ker[A_d]^\perp.$$

There are several ways how to show (1.7), here we make use of the following result due to Arendt and Batty [5]:

Theorem 2.1. *Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a reflexive Banach space X . Denote by A the generator of $(T(t))$ and by $\sigma(A)$ the spectrum of A . If $\sigma(A) \cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then $\lim_{t \rightarrow +\infty} T(t)x = 0$ for all $x \in X$.*

In view of this theorem we need to identify the spectrum of A_d lying on the imaginary axis, and we have according to [2]:

- Suppose that $|\omega| > 0$. If λ is a non-zero real number, then $i\lambda$ is not an eigenvalue of A_d .
- Suppose that $|\omega| > 0$. If λ is a non-zero real number, then $i\lambda$ belongs to the resolvent set $\rho(A_d)$ of A_d .

Now, Theorem 2.1 leads to

Corollary 2.1 ([2]). *Let (u, r) be the unique semi-group solution of the problem (1.1) emanating from the initial data $(u^0, r^0) \in H$. Let P_E be the orthogonal projection onto the space $E = \ker[A_d]$ in H , and let*

$$(w, s) = P_E(u^0, r^0).$$

Then

$$\|(u, r)(t, \cdot) - (w, s)\|_H \rightarrow 0 \text{ as } t \rightarrow \infty$$

We now state the main result of this article. We begin by a proposition on an estimate of the resolvent.

Proposition 2.1. *There exist $C > 0$ such that for every $|\mu| \geq 1$, and $(f, g) \in H = (L^2(\Omega))^d \times L^2_m(\Omega)$, the solution $(u, r) \in D(A)$ of $(A_d + i\mu)(u, r) = (f, g)$ satisfied*

$$(2.11) \quad \|(u, r)\|_H \leq Ce^{C|\mu|} \|(f, g)\|_H,$$

or equivalently

$$(2.12) \quad \|(A_d + i\mu)^{-1}\|_{L(H)} \leq Ce^{C|\mu|}.$$

We recall the following result.

Theorem 2.2. *Let B a generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on H , a Hilbert space. We assume*

$$(2.13) \quad \|T(t)\|_{L(H)} \text{ is uniformly bounded with respect } t \geq 0,$$

$$(2.14) \quad B + i\mu \text{ is invertible for every } \mu \in \mathbb{R},$$

$$(2.15) \quad \text{There exists } C > 0 \text{ such that } \|(B + i\mu)^{-1}\|_{L(H)} \leq Ce^{C|\mu|}.$$

Then there exist $C_1 > 0$ such that for all $u \in D(B)$ we have

$$\|T(t)u\|_H \leq C_1 \frac{\|Bu\|_H}{\ln(3+t)}, \quad \forall t \geq 0.$$

One has also, for every $k \geq 1$ there exists $C_2 > 0$ such that if $u \in D(B^k)$, we have

$$\|T(t)u\|_H \leq C_1 \frac{\|B^k u\|_H}{\ln^k(3+t)}, \quad \forall t \geq 0.$$

A weak version of this theorem was first proven by Lebeau [17], next Burq [6] gives the precise statement. We also refer to Batty and Duyckaerts [7] for some generalizations.

On $H_0 = \ker[A_d]^+$, as seen above $A_d + i\mu$ is invertible on H_0 , in fact A_d is invertible on H_0 and $A_d + i\mu$ is invertible on H for $\mu = 0$. The semigroup is bounded as the norm on H is non-increasing by (1.5). With Proposition 2.1, we can apply Theorem 2.2. We then obtain.

Theorem 2.3. *Let (u, r) be the unique semi-group solution of the problem (1.1) emanating from the initial data $(u^0, r^0) \in \mathfrak{D} A$. Let P_E be the orthogonal projection onto the space $E = \ker[A_d]$ in H , and let*

$$(w, s) = P_E(u^0, r^0).$$

Then

$$\|(u, r)(t, \cdot) - (w, s)\|_H \leq C \frac{\|A_d(u^0, r^0)\|_H}{\ln(3+t)}, \quad \forall t \geq 0,$$

for some $C > 0$ independent of (u^0, r^0) .

Proposition 2.1 is obtained from Carleman estimates. We need two kinds of such estimates, first an estimate far away the boundary, second an estimate up to the boundary. Both estimates are proven in the next section.

3. Carleman estimates

Let Ω be an open bounded subset of \mathbb{R}^d . Let (u, r) be a solution of the resolvent problem $(A_0 + i\mu)(u, r) = (f, g) \in (L^2(\Omega))^d \times L^2_m(\Omega)$, that is

$$(3.1) \quad \begin{cases} -\nabla r + i\mu u = f & \text{in } \Omega, \\ -\operatorname{div} u + i\mu r = g & \text{in } \Omega. \end{cases}$$

Here we moreover assume that (u, r) are supported in $K \subset \Omega$ where K is a compact set. Taking the divergence of the first line and using that $\operatorname{div} u = i\mu r - g$, we obtain

$$(3.2) \quad -\Delta r - \mu^2 r = i\mu g + \operatorname{div} f \text{ in } \Omega.$$

We have to give a Carleman estimate for the solution of this type of equation. This is done in Section 3.2. to do that we need some tools on pseudo-differential operators we recall below.

3.1. Pseudo-differential operators. We start this section with some useful notations. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we introduce the following notation:

$$\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \text{and} \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

where $D_k = -i \frac{\partial}{\partial x_k} = -i \partial_{x_k}$. We denote by $C_c^\infty(V)$ the set of functions of class C^∞ compactly supported in V . For a compact subset K of \mathbb{R}^n , we note by $C_c^\infty(K)$ the set of functions in $C_c^\infty(\mathbb{R}^n)$ supported in K . The space $L^2(V)$ is equipped with the usual norm denoted by $\|\cdot\|$. For $s \in \mathbb{N}$ we set $H^s(V) = \{u \in D'(V); \partial^\alpha u \in L^2(V) \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq s\}$. The Schwartz space $S(\mathbb{R}^n)$ is the set of functions of C^∞ class with rapid decay rate. Its dual, $S'(\mathbb{R}^n)$ is the set of temperate distributions. If $u \in S(\mathbb{R}^n)$ its Fourier transform denoted by \hat{u} is defined

$$\text{by } \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy \cdot \xi} u(y) dy \quad \text{where } y \cdot \xi = \sum_{i=1}^n y_i \xi_i \text{ stands for the euclidean inner production in } \mathbb{R}^n.$$

Let f and g be two smooth functions defined in $V \times \mathbb{R}^n$, we define the Poisson bracket by $\{f, g\} = \sum_{j=1}^n (\partial_{\xi_j} f \cdot \partial_{x_j} g - \partial_{x_j} f \cdot \partial_{\xi_j} g)$. And if A and B are two operators we define there commutator by $[A, B] = A \circ B - B \circ A$.

Definition 3.1. Let $a(\cdot, \cdot, \tau) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ where $\tau \geq 1$ is a large parameter, such that for every multi-index $\alpha, \beta \in \mathbb{N}^n$ we have

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \tau)| \leq C_{\alpha, \beta} \langle \xi, \tau \rangle^{m-|\beta|}, \quad \forall x \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \tau \geq 1,$$

where we denoted by $\langle \xi, \tau \rangle = (\|\xi\|^2 + \tau^2)^{\frac{1}{2}}$. In this case we say that a is a symbol of order m and we write $a \in S_\tau^m$. We call principal symbol of $a \in S_\tau^m$ the equivalence class of a in S_τ^m / S_τ^{m-1} . We also define $S_\tau^{-\infty} = \bigcap_{r \in \mathbb{R}} S_\tau^r$ and $S_\tau^{+\infty} = \bigcup_{r \in \mathbb{R}} S_\tau^r$.

Definition 3.2. We define the pseudo-differential operator of order m by

$$a(x, D, \tau)u(x) = \operatorname{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi, \tau) \hat{u}(\xi) d\xi, \quad \forall u \in S(\mathbb{R}^n),$$

where $a \in S_\tau^m$. The set of the pseudo-differential operator of order m is denoted by Ψ_τ^m . If $A \in \Psi_\tau^m$ we denote by $\sigma_p(A)$ his principal symbol.

Remarks 3.1. Let $s \in \mathbb{R}$ for $u \in S'(\mathbb{R}^n)$ we set the following norm

$$\|u\|_{\tau, s} = \|\Lambda_\tau^s u\|_0 \quad \text{with } \Lambda_\tau^s := \operatorname{Op}(\langle \xi, \tau \rangle^s).$$

Hence we can define the corresponding space

$$H_\tau^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n); \|u\|_{\tau, s} < \infty\}.$$

Theorem 3.1. Let $s \in \mathbb{R}$ and $a(x, \xi, \tau) \in S_\tau^m$, then the operator $\operatorname{Op}(a) : H_\tau^s \rightarrow H_\tau^{s-m}$ maps continuously and uniformly for $\tau > 1$.

Lemma 3.1. Let $m \in \mathbb{R}$ and $a_j \in S_\tau^{m-j}$ with $j \in \mathbb{N}$. Then there exist $a \in S_\tau^m$ such that

$$\forall N \in \mathbb{N}, \quad a - \sum_{j=0}^N a_j \in S_\tau^{m-N-1}.$$

We then write $a \sim \sum_j a_j$. The symbol a is unique up to $S_{\tau}^{-\infty}$ in the sense that the difference of two symbols is in S_{τ}^{-M} for all $M \in \mathbb{N}$. Hence, we identify a_0 with the principal symbol of a .

Theorem 3.2. Let $a \in \mathcal{S}^m$ and $b \in \mathcal{S}^m$, then $Op(a) \circ Op(b) = Op(c)$ with $c \in S_{\tau}^{m+m'}$ which admits the following asymptotic expansion

$$c(x, \xi, \tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial^{\alpha} a(x, \xi, \tau) \partial^{\alpha} b(x, \xi, \tau).$$

Theorem 3.3. Let $a \in S_{\tau}^m$ and $b \in S_{\tau}^m$, then $[Op(a), Op(b)] = Op(c)$ with $c \in S_{\tau}^{m+m'-1}$ and principal symbol

$$\sigma(c)(x, \xi, \tau) = i \{a, b\}(x, \xi, \tau)$$

which admits the following asymptotic expansion

$$c(x, \xi, \tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial^{\alpha} a(x, \xi, \tau) \partial^{\alpha} b(x, \xi, \tau) - \alpha \partial_{\xi} b(x, \xi, \tau) \partial_x a(x, \xi, \tau).$$

Theorem 3.4. Let $a \in S_{\tau}^m$, then $Op(a)^* = Op(b)$ with $b \in S_{\tau}^m$ which admits the following asymptotic expansion

$$b(x, \xi, \tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|} \alpha!} \partial^{\alpha} \partial^{\alpha} \bar{a}(x, \xi, \tau).$$

In particular we have $\sigma_p(Op(a)^*) = \bar{a}$.

Theorem 3.5 (Gårding inequality). Let K be a compact subset of \mathbb{R}^n and $a(x, \xi, \tau) \in S_{\tau}^m$, of principal symbol a_m . We suppose that there exist $C > 0$ and $R > 0$ such that

$$\operatorname{Re} a_m(x, \xi, \tau) \geq C \langle \xi, \tau \rangle^m, \quad \forall x \in K, \xi \in \mathbb{R}^n, \tau \geq 1, \langle \xi, \tau \rangle \geq R.$$

Then for any $0 < C' < C$ there exists $\tau_* > 0$ we have

$$\operatorname{Re}(Op(a)u, u)_{L^2(\mathbb{R}^n)} \geq C' \|u\|_{L^2(\mathbb{R}^n)}^2, \quad \forall u \in C_c^{\infty}(K), \tau \geq \tau_*.$$

3.2. Local Carleman estimate away from the boundary. We set the operator

$$P(x, D) = -\mu^2 - \Delta,$$

a real values function ϕ and then we define the conjugate operator by

$$P_{\phi}(x, D) = e^{\tau\phi} P(x, D) e^{-\tau\phi},$$

where μ is a parameter that depends on τ , precisely we suppose that

$$(3.3) \quad c_0 \tau \leq |\mu| \leq c'_0 \tau \quad \forall \tau \geq 1,$$

for some constants $c'_0 > c_0 > 0$. Then we have

$$P_{\phi}(x, D)w = -\mu^2 w - \Delta w + 2\tau \nabla \phi \cdot \nabla w - \tau^2 |\nabla \phi|^2 w + \tau \Delta \phi w$$

whose symbol is given by

$$\sigma(P_{\phi}) = |\xi|^2 + 2i\tau \nabla \phi \cdot \xi - \tau^2 |\nabla \phi|^2 + \tau \Delta \phi - \mu^2$$

and with principal symbol p_{ϕ} given by

$$p_{\phi}(x, \xi, \tau) = |\xi + i\tau \nabla \phi|^2 - \mu^2 = |\xi|^2 + 2i\tau \nabla \phi \cdot \xi - \tau^2 |\nabla \phi|^2 - \mu^2.$$

We define the following self-adjoint operators

$$Q_2 = \frac{P_{\phi} + P_{\phi}^*}{2} \quad \text{and} \quad Q_1 = \frac{P_{\phi} - P_{\phi}^*}{2i}$$

with principal symbols respectively

$$q_2(x, \xi, \tau) = |\xi|^2 - \tau^2 |\nabla \phi|^2 - \mu^2 \quad \text{and} \quad q_1(x, \xi, \tau) = 2\tau \nabla \phi \cdot \xi.$$

Noting that $P_{\phi} = Q_2 + iQ_1$ and $p_{\phi} = q_2 + iq_1$.

We assume that the weight function $\phi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies the following sub-ellipticity condition in K a compact set of \mathbb{R}^d , if

$$(3.4) \quad |\nabla\phi| > 0 \text{ in } K$$

$$\forall (x, \xi, \tau) \in K \times \mathbb{R}^n \times [1, +\infty); \rho_\phi(x, \xi, \tau) = 0 \implies \{q_2, q_1\}(x, \xi, \tau) \geq C \langle \xi, \tau \rangle^3 > 0.$$

Note that the constant C does not depend on μ assuming (3.3).

Remark 3.1. *Noting that*

$$\rho_\phi(x, \xi, \tau) = 0 \iff |\xi|^2 = \tau^2 |\nabla\phi|^2 + \mu^2 \text{ and } \nabla\phi \cdot \xi = 0.$$

Lemma 3.2. *Let $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $|\nabla\psi| > 0$ in K . Then for λ large enough $\phi = e^{\lambda\psi}$ satisfies the sub-ellipticity assumption in K .*

Proof. We can assume that $\psi \geq 0$, as we can add a constant to ψ and ϕ is multiplied by a constant β . Changing τ in τ/β we can see that sub-ellipticity condition is also satisfied for a different constant C . A straightforward calculation shows that

$$\{q_2, q_1\}(x, \xi, \tau) = 4\tau \xi \psi'' \xi + \tau^2 (\nabla\phi) \phi'' \nabla\phi.$$

Using the fact that $\phi = e^{\lambda\psi}$ then we have

$$\nabla\phi = \lambda\nabla\psi\phi, \quad \phi'_j = \lambda\phi\psi'_j \text{ and } \phi''_{jk} = \lambda\phi\psi''_{jk} + \lambda^2\phi\psi'_j\psi'_k, \quad 1 \leq j, k \leq n,$$

therefore we obtain

$$\{q_2, q_1\} = 4\tau\lambda^3\phi^3 \lambda\tau^2 |\nabla\psi|^4 + \tau^2 (\nabla\psi) \psi'' \nabla\psi + |\lambda\phi|^{-2} \xi \psi'' \xi + \lambda^{-1} |\phi|^{-2} |\nabla\psi \cdot \xi|^2.$$

Now if $\rho_\phi = 0$ then $|\xi|^2 = \tau^2 |\nabla\phi|^2 + \mu^2 = \tau^2 \lambda^2 \phi^2 |\nabla\psi|^2 + \mu^2$, which gives that

$$|\lambda\phi|^{-2} \xi \psi'' \xi \geq -|\psi| \left| \tau^2 |\nabla\psi|^2 + |\lambda\phi|^{-2} \mu^2 \right| \geq -C\tau^2 |\nabla\psi|^2 + \lambda^{-2}.$$

Besides, we have

$$\tau^2 (\nabla\psi) \psi'' \nabla\psi \geq -C\tau^2 |\nabla\psi|^2.$$

Then it follows from these estimates that

$$\begin{aligned} \{q_2, q_1\} &\geq 4\tau\lambda^3\phi^3 \lambda\tau^2 |\nabla\psi|^4 + \tau^2 (\nabla\psi) \psi'' \nabla\psi + |\lambda\phi|^{-2} \xi \psi'' \xi \\ &\geq 4\tau\lambda^3\phi^3 (\lambda\tau^2 |\nabla\psi|^4 - C\tau^2 |\nabla\psi|^2 - C\tau \lambda^{-2}). \end{aligned}$$

Since $|\nabla\psi| > 0$ in the compact set K then for λ large enough we have $\{q_2, q_1\} \geq C_1 \tau^3 > 0$. As $|\xi|$ is comparable to τ on $\rho_\phi = 0$, we obtain the result. \square

Lemma 3.3. *Let f and g be two real continuous functions defined in K such that f is positive on a compact subset K of \mathbb{R}^d and verifies that*

$$\forall y \in K, \quad f(y) = 0 \implies g(y) \geq L > 0.$$

We set $h_\kappa = \kappa f + g$, then for κ sufficiently large then $h_\kappa \geq C$ for some constant $C > 0$.

Proof. Let $y_0 \in K$ to prove the result we distinguish two cases.

Case 1: We assume $f(y_0) = 0$. Then according to the assumption made in this lemma we have $h_\kappa(y_0) = g(y_0) \geq L$. Then there exists a neighborhood of y_0 , V_{y_0} such that for $y \in V_{y_0}$ and every $\kappa > 0$, $h_\kappa(y) \geq L/2$. Let $\kappa_{y_0} = 1$.

Case 2: We assume $f(y_0) > 0$. Since f and g are continuous, there exist V_{y_0} a neighborhood of y_0 and $C_1, C_2 > 0$ such that $f(y) \geq C_1$ and $|g(y)| \leq C_2$ for all $y \in V_{y_0}$. Then for all $\kappa \geq (L + C_2)/C_1$, $h_\kappa(y) \geq L$. Let $\kappa_{y_0} = (L + C_2)/C_1$.

We cover K , by compactness argument, by a finite number of such neighborhoods V_{y_1}, \dots, V_{y_p} with associated $\kappa_j = \kappa_{y_j}$. Taking $\kappa = \max_{1 \leq j \leq p} \{\kappa_j\}$ we have $h_\kappa(y) \geq L/2$ on each V_{y_j} , then on K .

This completes the proof. \square

Lemma 3.4. *Let V an open bounded subset of \mathbb{R}^d and $\kappa > 0$. We suppose that ϕ verifies the sub-ellipticity assumption on K and we set $\rho_\kappa = \kappa(q_2^2 + q_1^2) + \tau\{q_2, q_1\}$. Then for κ large enough there exists $C > 0$ such that for all $(x, \xi) \in K \times \mathbb{R}^d$ and $\tau \geq 1$ we have $\rho_\kappa(x, \xi) \geq C\langle \xi, \tau \rangle^4$.*

Proof. First we assume $|\xi|$ large with respect τ , that is $|\xi| \geq \theta\tau$ for θ sufficiently large, to be fixed below. We have

$$\begin{aligned} \rho_\kappa(x, \xi) &= \kappa(q_2^2 + q_1^2) + \tau\{q_2, q_1\} \\ &= \kappa \left(|\xi|^2 - (\tau^2 |\nabla\phi|^2 + \mu^2) \right)^2 + 4\kappa\tau^2 (\nabla\phi \cdot \xi)^2 + 4\tau^2 \xi\phi'' \xi + \tau^4 (\nabla\phi) \phi'' \nabla\phi \\ (3.5) \quad &= \kappa \langle \xi, \tau \rangle^4 \frac{(\tau^2(1 + |\nabla\phi|^2) + \mu^2)^2}{\langle \xi, \tau \rangle^2} + 4\kappa\tau^2 (\nabla\phi \cdot \xi)^2 + 4\tau^2 \xi\phi'' \xi + \tau^4 (\nabla\phi) \phi'' \nabla\phi \\ &\geq C \langle \xi, \tau \rangle^4 - C\tau^2 |\xi|^2 - C\tau^4, \end{aligned}$$

if θ is sufficiently large such that $\frac{(\tau^2(1 + |\nabla\phi|^2) + \mu^2)}{\langle \xi, \tau \rangle^2} \leq 1/2$ and for some constants $C, C' > 0$.

If θ is sufficiently large we obtain $C\tau^2 |\xi|^2 + C\tau^4 \leq C' \langle \xi, \tau \rangle^4 / 2$, from (3.5) we obtain $\rho_\kappa(x, \xi) \geq C' \langle \xi, \tau \rangle^4$, for $C' > 0$. This fixes θ .

Second we assume $|\xi| \leq \theta\tau$. As ρ_κ is homogeneous of degree 4 in (ξ, τ, μ) , we can prove the estimate on $K' = \{(x, \xi, \tau, \mu) \in K \times \mathbb{R}^d \times [0, +\infty) \times \mathbb{R}, |\xi|^2 + \tau^2 + \mu^2 = 1, |\xi| \leq \theta\tau, c_0\tau \leq |\mu| \leq c'_0\tau\}$ taking into account of (3.3). As K' is a compact set, we can apply Lemma 3.3 by taking $f = q_2^2 + q_1^2$ and $g = \tau\{q_2, q_1\}$. This completes the proof. \square

Theorem 3.6. *Let Ω be an open bounded set of \mathbb{R}^n and ϕ be a function that satisfies the sub-ellipticity assumption in K . Then there exist $\tau_* > 0$ and $C > 0$ such that*

$$(3.6) \quad \tau^3 \|e^{\tau\phi} r\|_0^2 + \tau \|e^{\tau\phi} \nabla r\|_0^2 + \tau^{-1} \sum_{\alpha=2} \|e^{\tau\phi} D_\alpha r\|_2 \leq C \|e^{\tau\phi} P r\|_2.$$

for all $r \in C_c^\infty(K)$, $\tau \geq \tau_*$ and μ satisfying (3.3).

This theorem is classical, and we can find in Hörmander [14]. Here we give a proof to be complete.

Proof. Let's take $w = e^{\tau\phi} r$ then $P r = f$ can be written as follow $P_\phi w = g = f e^{\tau\phi}$. Since $g = Q_2 w + iQ_1 w$ and Q_1 and Q_2 are symmetric then we have

$$\begin{aligned} \|g\|_0^2 &= \|Q_2 w\|_0^2 + \|Q_1 w\|_0^2 + i(Q_1 w, Q_2 w) - i(Q_2 w, Q_1 w) \\ &= (Q_2 Q_2 w, w) + (Q_1 Q_1 w, w) + i(Q_2 Q_1 w, w) - i(Q_1 Q_2 w, w) \\ (3.7) \quad &= Q_1^2 + Q_2^2 + i[Q_2, Q_1] w, w. \end{aligned}$$

We fix κ large enough such that the statement of Lemma 3.4 is fulfilled, then for τ sufficiently large satisfying $\kappa\tau^{-1} \leq 1$ then from (3.7) we obtain

$$(3.8) \quad \tau^{-1} \kappa(Q_1^2 + Q_2^2) + i\tau[Q_2, Q_1] w, w \leq \|g\|_0^2$$

Since the principal symbol of $\kappa(Q_1^2 + Q_2^2) + i\tau[Q_2, Q_1]$ is given by $\rho_\kappa(x, \xi, \tau) = \kappa(q_2^2 + q_1^2) + \tau\{q_2, q_1\}$ then from Lemma 3.4 we have $\rho_\kappa(x, \xi, \tau) \geq C\langle \xi, \tau \rangle^4$ therefore by Gårding inequality (Theorem 3.5) it follows that

$$\operatorname{Re} \kappa(Q_1^2 + Q_2^2) + i\tau[Q_2, Q_1] w, w \geq \|w\|_{\tau, 2}^2.$$

Combining this inequality with (3.8) we find

$$(3.9) \quad \tau^{-1} \|w\|_{\tau, 2}^2 \leq \|g\|_0^2$$

which reads

$$(3.10) \quad \tau^3 \|w\|_0^2 + \tau \|\nabla w\|_0^2 + \tau^{-1} \sum_{|\alpha|=2} \|D_\alpha w\|_2 \leq C \|e^{\tau\phi} f\|_2$$

Since we have

$$e^{\tau\phi}D_j r = (D_j + i\tau\partial_j\phi)w, \quad e^{\tau\phi}D_j D_k r = (D_j + i\tau\partial_j\phi)(D_k + i\tau\partial_k\phi)w,$$

then we have

$$(3.11) \quad \tau \|e^{\tau\phi}\nabla r\|_0^2 \leq C \tau^3 \|w\|_0^2 + \tau \|\nabla w\|_0^2$$

and

$$(3.12) \quad \tau^{-1} \sum_{|\alpha|=2} \|e^{\tau\phi}D_\alpha r\|_0^2 \leq C \tau^2 \|w\|_0^2 + \tau \|\nabla w\|_0^2 + \tau^{-1} \sum_{|\alpha|=2} \|D_\alpha w\|_0^2.$$

Thus, estimate (3.6) follows by replacing (3.11) and (3.12) into (3.10). This concludes the proof. \square

Theorem 3.7. *Let Ω be an open bounded set of \mathbb{R}^d , let $K \Subset \Omega$ and ϕ be a function that satisfies the sub-ellipticity assumption in K . Then there exist $\tau_* > 0$ and $C > 0$ such that*

$$\tau^3 \|e^{\tau\phi}r\|_0^2 + \tau \|e^{\tau\phi}\nabla r\|_0^2 \leq C\tau^2 (\|e^{\tau\phi}g\|_0^2 + \|e^{\tau\phi}f\|_0^2)$$

for all $r \in C_c^\infty(K)$ which satisfies (3.2), $\tau \geq \tau_*$ and μ satisfying (3.3).

From this theorem we can deduce this corollary.

Corollary 3.1. *Let Ω be an open bounded set of \mathbb{R}^d , let $K \Subset \Omega$ and ϕ be a function that satisfies the sub-ellipticity assumption in K . Then there exist $\tau_* > 0$ and $C > 0$ such that*

$$\tau^3 \|e^{\tau\phi}u\|_0^2 + \tau^3 \|e^{\tau\phi}r\|_0^2 + \tau \|e^{\tau\phi}\nabla r\|_0^2 \leq C\tau^2 (\|e^{\tau\phi}g\|_0^2 + \|e^{\tau\phi}f\|_0^2)$$

for all $r, u \in C_c^\infty(K)$ which satisfies (3.1), $\tau \geq \tau_*$ and μ satisfying (3.3).

Proof. As r satisfies (3.2), with Theorem 3.7, we only have to estimate $\tau^3 \|e^{\tau\phi}u\|_0^2$. From the first equation of (3.1) we have $\tau^3 \|e^{\tau\phi}u\|_0^2 \leq 4\tau^3 \|e^{\tau\phi}\mu^{-1}(f + \nabla r)\|_0^2$, which gives the result using (3.3). \square

Proof of Theorem 3.7. We set $P_\phi = e^{\tau\phi}P e^{-\tau\phi}$, $w = e^{\tau\phi}r$, $F = -e^{\tau\phi}f$ and $G = i\mu e^{\tau\phi}g + \tau e^{\tau\phi}\nabla\phi.f$. Then from (3.2) we have

$$P_\phi w = G + \operatorname{div}(F).$$

Let K_1 be such that $K \Subset K_1 \Subset \Omega$ and let $\chi \in C_c^\infty(K_1)$ be such that $\chi = 1$ on K . Setting $v = \chi \Lambda_\tau^{-1} w$ with $\Lambda_\tau = (\tau^2 - \Delta)^{1/2}$ and we write

$$P_\phi v = \chi \Lambda_\tau^{-1} P_\phi w + [P_\phi, \chi \Lambda_\tau^{-1}] w = \chi \Lambda_\tau^{-1} (G + \operatorname{div}(F)) + [P_\phi, \chi \Lambda_\tau^{-1}] w,$$

then we find

$$(3.13) \quad \|P_\phi v\|_0 \leq C \tau^{-1} \|G\|_0 + \|F\|_0 + \|w\|_0.$$

Applying Estimate (3.9) in the proof of Theorem 3.6 to v then we obtain

$$\tau^{-\frac{1}{2}} \|v\|_{\tau,2} \leq C \|P_\phi v\|_0$$

We have $v = \Lambda_\tau^{-1} w + [\chi, \Lambda_\tau^{-1}] w$ then $\|w\|_{\tau,1} \leq 4 \|v\|_{\tau,2} + \|w\|_0$. That together with (3.13)

$$\tau^{-\frac{1}{2}} \|w\|_{\tau,1} \leq C \tau^{-1} \|G\|_0 + \|F\|_0 + \|w\|_0.$$

Multiplying by τ which is chosen sufficiently large then we obtain

$$\tau^{\frac{1}{2}} \|w\|_{\tau,1} \leq C (\|G\|_0 + \tau \|F\|_0) \leq C\tau (\|e^{\tau\phi}g\|_0 + \|e^{\tau\phi}f\|_0).$$

As $\|w\|_{\tau,1}$ is equivalent to $\tau \|e^{\tau\phi}r\|_0$ arguing as in the proof of (3.11) we obtain the result of the theorem. \square

3.3. Local Carleman estimate at the boundary. In this section because the boundary, we use a tangential pseudo-differential calculus. This calculus is completely analogous to the one presented in Section 3.1 except that a function $a(x', x_d, \xi')$ is a symbol in (x', ξ') in the sense of Definition 3.1 where x_d is a parameter and the estimates given in Definition 3.1 are uniform with respect x_d . To avoid confusion we denote by $S_{\tau, \tau}^m$ the class of tangential symbol of order m , $\text{Op}_{\tau}(a)$ the operator associated with the symbol $a \in S_{\tau, \tau}^m$. The class of operators associated with symbols in $S_{\tau, \tau}^m$ is denoted by $\Psi_{\tau, \tau}^m$. We refer to [20] for details on these symbols and operators. We consider functions in a half space $\mathbb{R}^{d-1} \times (0, +\infty) = \mathbb{R}^d_{,+}$ and we denote by $\|\cdot\|_+ = \|\cdot\|_{L^2(\mathbb{R}^d)_+}$ the L^2 -norm and $(\cdot, \cdot)_+ = (\cdot, \cdot)_{L^2(\mathbb{R}^d)_+}$ the associated inner product. At the boundary $x_d = 0$ we denote the L^2 -norm by $|g|^2 = \int_{\mathbb{R}^{d-1}} |g(x')|^2 dx'$ and the inner product associated by $(\cdot, \cdot)_{\partial} = (\cdot, \cdot)_{L^2(\mathbb{R}^{d-1})}$. A set $W = \omega \times \Gamma$ in $\mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ is called a conic open set, if there exist ω an open set in \mathbb{R}^d , and Γ an open set in $\mathbb{R}^{d-1} \times \mathbb{R}^+$ such that for all $(\xi', \tau) \in \Gamma$ and $\lambda > 0$ then $(\lambda \xi', \lambda \tau) \in \Gamma$. For $s \in \mathbb{R}$ we denote by $\Lambda_{\tau, \tau}^s$ the tangential operator defined by $\Lambda_{\tau, \tau}^s = \text{Op}_{\tau}(\langle \xi', \tau \rangle^s)$.

We recall the following microlocal Gårding inequality obtained, for instance, by applying sharp Gårding inequality.

Theorem 3.8 (Microlocal Gårding inequality). *Let K be a compact set of \mathbb{R}^d and let W be a conic open set of $\mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ contained in $K \times \mathbb{R}^{d-1} \times \mathbb{R}^+$. Let also $\chi \in S_{\tau, \tau}^0$ be homogeneous of order 0 (for $\langle \xi', \tau \rangle \geq 1$) and be such that $\text{supp}(\chi) \subset W$.*

Let $a(x, \xi', \tau) \in S_{\tau, \tau}^m$ with principal part a_m homogeneous of order m . If there exist $C_0 > 0$ and $R > 0$ such that

$$\text{Re } a_m(x, \xi', \tau) \geq C_0 \langle \xi', \tau \rangle^m, \quad (x, \xi', \tau) \in W, \quad \tau \in [1, +\infty), \quad \langle \xi', \tau \rangle \geq R,$$

then for any $0 < C_1 < C_0$, $N \in \mathbb{N}$, there exist C_N and $\tau_ \geq 1$ such that*

$$\text{Re } \text{Op}_{\tau}(a) \text{Op}_{\tau}(\chi) u, \text{Op}_{\tau}(\chi) u \Big|_+ \geq C_1 \|\Lambda_{\tau, \tau}^{m/2} \text{Op}_{\tau}(\chi) u\|_+^2 - C_N \|\Lambda_{\tau, \tau}^{-N} u\|_+^2$$

for $u \in S(\mathbb{R}^d)$ and $\tau \geq \tau_$.*

As we want to change the variables in order to have a flat boundary which is convenient to do the computations, we use the language and usual tools of Riemannian geometry. In this framework the gradient and divergence operators keep forms we can follow after a change of variables. Our purpose is to use these tools locally and we do not use manifold tools as charts, atlas and etc. To fix the notation, let V be an open set in \mathbb{R}^d . Let $g(x) = g_{ij}(x)_{1 \leq i, j \leq d}$ be a positive symmetric matrix called the metric, we denote by $g^{-1}(x) = g^{ij}(x)_{1 \leq i, j \leq d}$ the inverse of $g(x)$. For a smooth function r , we denote by $(\nabla_g r(x))^i = \sum_{1 \leq j \leq d} g^{ij}(x) \partial_{x_j} r(x)$ the gradient of r . We have $\nabla_g r(x) \in T_x V$, this means that $\nabla_g r$ is a tangent vector field.

For $u(x) = u^1(x), \dots, u^d(x)$ a smooth tangent vector field, we define the divergence operator by

$$\text{div}_g u(x) = \det g(x)^{-1/2} \sum_{1 \leq j \leq d} \partial_{x_j} \det g(x)^{1/2} u^j(x).$$

For a smooth function r and a smooth tangent vector field u , we have

$$(3.14) \quad \text{div}_g(ru) = r \text{div}_g u + g(\nabla_g r, u), \quad \text{where } g(\nabla_g r, u) = \sum_{1 \leq i, j \leq d} g_{ij}(\nabla_g r)^i u^j.$$

For two smooth functions r_1 and r_2 we have

$$(3.15) \quad \nabla_g(r_1 r_2) = r_1 \nabla_g r_2 + r_2 \nabla_g r_1.$$

It is well-known that there exist coordinates (called *normal geodesic coordinates*) such that the boundary is defined locally by $x_d = 0$, the open set $\Omega \cap V$ is defined by $x_d > 0$, the metric g is

such that $g_{id} = g_{di} = 0$ for $i = 1, \dots, d-1$ and, $g_{dd} = 1$. We denote by $\tilde{g} = (g_{ij})_{1 \leq i, j \leq d-1}$ the metric g on x_d fixed.

We can define on each manifold $x_d = \text{const}$ the gradient and divergence operators associated with \tilde{g} and for r a smooth function and $\tilde{u} = (u^1, \dots, u^{d-1})$ a smooth vector field on $x_d = \text{const}$, we have

$$(\nabla_{\tilde{g}} r)^i = \sum_{1 \leq j \leq d-1} g^{ij} \partial_{x_j} r \text{ for } i = 1, \dots, d-1, \quad \text{div}_{\tilde{g}} \tilde{u} = (\det \tilde{g})^{-1/2} \sum_{1 \leq j \leq d-1} \partial_{x_j} (\det \tilde{g})^{1/2} u^j .$$

In such coordinates, we have $\det g = \det \tilde{g}$. The gradient and divergence operators take the following form.

$$(3.16) \quad \nabla_g r = (\nabla_{\tilde{g}} r, \partial_{x_d} r), \quad \text{div}_g u = \text{div}_{\tilde{g}} \tilde{u} + \partial_{x_d} u^d + h u^d, \quad \text{where } h = (\det \tilde{g})^{-1/2} \partial_{x_d} (\det \tilde{g})^{1/2}.$$

We recall that the equation of the resolvent problem $(A_0 + i\mu)(u, r) = (f, g)$ locally takes the form

$$(3.17) \quad \begin{cases} \cdot & \\ \cdot & -\nabla_g r + i\mu u = f \text{ in } x_d > 0, \\ \cdot & -\text{div}_g u + i\mu r = g \text{ in } x_d > 0, u^d \\ & = 0 \text{ on } x_d = 0. \end{cases}$$

We have the following theorem

Theorem 3.9. *Let $x_0 \in \mathbb{R}^{d-1} \times \{0\}$, we assume there exist a neighborhood of x_0 where ϕ satisfies (3.4) the sub-ellipticity condition and $\partial_{x_d} \phi(x_0) > 0$. Then there exist V_0 be an open set of \mathbb{R}^d such that $x_0 \in V_0$, $C > 0$, and $\tau_* > 0$ such that*

$$\tau^{1/2} |e^{\tau\phi} r|_{x_d=0} + \tau^{1/2} \|e^{\tau\phi} u\|_+ + \tau^{1/2} \|e^{\tau\phi} r\|_+ + \tau^{-1/2} \|e^{\tau\phi} \nabla_g r\|_+ \leq C \|e^{\tau\phi} f\|_+ + \|e^{\tau\phi} g\|_+,$$

for $u, r \in C^\infty(\mathbb{R}^d)$ supported on V_0 , satisfying (3.17), for every $\tau \geq \tau_*$ and μ satisfying (3.3).

Let $v = e^{\tau\phi} u$ and $w = e^{\tau\phi} r$. We have from (3.14) and (3.15)

$$\begin{aligned} \nabla_g r &= e^{-\tau\phi} \nabla_g w - \tau w \nabla_g \phi, \\ \text{div}_g u &= e^{-\tau\phi} \text{div}_g v - \tau g(\nabla_g \phi, v). \end{aligned}$$

Then System (3.17) takes the form

$$(3.18) \quad \begin{cases} \cdot & \\ \cdot & -\nabla_g w + \tau w \nabla_g \phi + i\mu v = F \text{ in } x_d > 0, \\ \cdot & -\text{div}_g v + \tau g(\nabla_g \phi, v) + i\mu w = G \text{ in } x_d > 0, v^d \\ & = 0 \text{ on } x_d = 0, \end{cases}$$

where $F = e^{\tau\phi} f$ and $G = e^{\tau\phi} g$.

In the following, we denote by $\tilde{F} = (F^1, \dots, F^{d-1})$ and by $\tilde{v} = (v^1, \dots, v^{d-1})$, then we have $F = (\tilde{F}, F^d)$ and $v = (\tilde{v}, v^d)$. Multiplying (3.18) by i , we have

$$(3.19) \quad \begin{cases} \cdot & \\ \cdot & -i\nabla_g w + i\tau w \nabla_g \phi - \mu \tilde{v} = i\tilde{F} \text{ in } x_d > 0, \\ \cdot & -i\text{div}_g v + i\tau g(\nabla_g \phi, v) + i\tau v^d \partial_{x_d} \phi - \mu w = iG \text{ in } x_d > 0, \\ \cdot & v^d = 0 \text{ on } x_d = 0. \end{cases}$$

For this system we prove the following Carleman estimate.

Proposition 3.1. *Let $x_0 \in \mathbb{R}^{d-1} \times \{0\}$, we assume there exist a neighborhood of x_0 where ϕ satisfies (3.4) the sub-ellipticity condition and $\partial_{x_d} \phi(x_0) > 0$. For $s \in \mathbb{R}$, there exist V_0 be an open set such that $x_0 \in V_0$, $C > 0$, and $\tau_* > 0$ such that*

$$|\Lambda_{\tau, \tau}^{s+1/2} w|_{x_d=0} + \tau^{1/2} \|\Lambda_{\tau, \tau}^s v\|_+ + \tau^{-1/2} \|\Lambda_{\tau, \tau}^{s+1} w\|_+ \leq C \|\Lambda_{\tau, \tau}^s F\|_+ + \|\Lambda_{\tau, \tau}^s G\|_+,$$

for $v, w \in S(\mathbb{R}^d)$ satisfying (3.19), supported in V_0 for every $\tau \geq \tau_*$ and μ satisfying (3.3).

From this proposition we deduce Theorem 3.9 taking $s = 0$. Indeed we have $\tau^{1/2}|w|_{x_d \neq 0} \leq 4|\Lambda_{\tau, \tau}^{1/2} w|_{x_d=0}$ and from (3.17) we have

$$\tau^{-1/2} \|e^{\tau\phi} \nabla_g r\|_+ \leq 4 \|e^{\tau\phi} \nabla r - i\mu u\|_+ + |\mu| \tau^{-1/2} \|e^{\tau\phi} u\|_+ \leq 4 \|e^{\tau\phi} f\|_+ + \tau^{1/2} \|e^{\tau\phi} u\|_+,$$

from (3.3).

We begin by reducing the system in a 2×2 system. We denote by $\zeta \in S_{\tau, \tau}^1$ the tangential symbol of the operator $-i\nabla_g + i\tau\nabla_g \phi$. We have

$$\zeta_i = \sum_{1 \leq j \leq d-1} g^{ij} (\xi_j + i\tau \partial_{x_j} \phi) \text{ for } i = 1, \dots, d-1.$$

Let $\text{Op}_T(\delta) := -i \text{div}_{\tilde{g}} + i\tau \tilde{g}(\nabla_{\tilde{g}} \phi, \cdot)$ where $\delta \in S_{\tau, \tau}^1$. The principal symbol of the operator δ is $(\xi_1 + i\tau \partial_{x_1} \phi, \dots, \xi_{d-1} + i\tau \partial_{x_{d-1}} \phi)$ modulo symbol in $S_{\tau, \tau}^0$. The first equation of (3.19) reads $\text{Op}_T(\zeta') w - \mu \tilde{v} = i\tilde{F}$. Applying in both side of this equation the operator $\text{Op}_T(\delta)$, we obtain

$$(3.20) \quad \text{Op}_T(\delta) \tilde{v} = -i\mu^{-1} \text{Op}_T(\delta) \tilde{F} + \mu^{-1} \text{Op}_T(\delta) \text{Op}_T(\zeta') w.$$

From (3.16) we have

$$\begin{aligned} -i \text{div}_g v + i\tau \tilde{g}(\nabla_{\tilde{g}} \phi, \tilde{v}) &= \text{Op}_T(\delta) \tilde{v} - i\partial_{x_d} v - i h v^d \\ &= D_{x_d} v^d - i\mu^{-1} \text{Op}_T(\delta) \tilde{F} + \mu^{-1} \text{Op}_T(\delta) \text{Op}_T(\zeta') w - i h v^d. \end{aligned}$$

From this equation and second and third of (3.19) we obtain two equations on w and v^d , that is

$$(3.21) \quad \begin{aligned} & D_x w + i\tau w \partial_x \phi - \mu v^d = iF^d \text{ in } x_d > 0, \\ & D_x v^d + \mu^{-1} \text{Op}_T(\delta) \text{Op}_T(\zeta') w - \mu w + i\tau v^d \partial_x \phi - i h v^d = iG + i\mu^{-1} \text{Op}_T(\delta) \tilde{F} \text{ in } x_d > 0, \\ & v^d = 0 \text{ on } x_d = 0. \end{aligned}$$

Let $U = (w, v^d)$, the system (3.21) has the form

$$D_{x_d} U + BU = H, \text{ where } H = (iF^d, iG + \mu^{-1} \text{Op}_T(\delta) \tilde{F}),$$

and B is a tangential matrix operators with principal symbol

$$b = \begin{pmatrix} i\tau \partial_{x_d} \phi & -\mu \\ \mu^{-1} q(x, \xi) - \mu & i\tau \partial_x \phi \end{pmatrix},$$

modulo $\mu^{-1} S_{\tau, \tau}^1$, where $q(x, \xi) = \sum_{1 \leq i, j \leq d-1} g^{ij}(x) \xi_i + i\tau \partial_{x_i} \phi(x) \xi_j + i\tau \partial_{x_j} \phi(x)$. The char-

acteristic polynomial of b is given by $P(\lambda) = (\lambda - i\tau \partial_{x_d} \phi)^2 + q - \mu^2$. Let $\alpha \in \mathbb{C}$ such that $\alpha^2 = q - \mu^2$ with $\text{Re } \alpha \geq 0$. The definition of α is ambiguous when $q - \mu^2 \leq 0$ but in this case if $q - \mu^2 < 0$ the root are simple and the analysis below is independent of the choice of root. In particular the roots are smooth, or if $q - \mu^2 = 0$ the root is double and below, we give a specific analysis in this case. The root of $P(\lambda)$ are $i\tau \partial_{x_d} \phi \pm i\alpha$ and the analysis in what follows depends on the location of roots in complex plane. We have the following result, denoting $s = t^2$ where $t, s \in \mathbb{C}$ we have for $r_0 > 0$,

$$(3.22) \quad |\text{Re } t| \leq r_0 \iff 4r_0^2 \text{Re } s - 4r_0^4 + (\text{Im } s)^2 \leq 0.$$

Indeed, let $t = x + iy$, we have $\text{Re } s = x^2 - y^2$ and $\text{Im } s = 2xy$, we obtain $4r_0^2 \text{Re } s - 4r_0^4 + (\text{Im } s)^2 = 4(r_0^2 + y^2)(x^2 - r_0^2)$ which gives the result.

From (3.22), we obtain that $|\text{Re } \alpha| \leq \tau |\partial_{x_d} \phi|$ is equivalent to

$$(3.23) \quad 4\tau^2 (\partial_{x_d} \phi)^2 (\text{Re } q - \mu^2) - 4\tau^4 (\partial_{x_d} \phi)^4 + (\text{Im } q)^2 \leq 0,$$

where, from the definition of q , we have

$$\begin{aligned} \operatorname{Re} q(x, \xi) &= \sum_{1 \leq i, j \leq d-1} g^{ij}(x) \xi_i \xi_j - \tau^2 \partial_x \phi(x) \partial_x \phi(x), \\ \operatorname{Im} q(x, \xi) &= \tau \sum_{1 \leq i, j \leq d-1} g^{ij}(x) \xi_j \partial_{x_i} \phi(x). \end{aligned}$$

We prove a microlocal Carleman estimate.

Lemma 3.5. *Let $x_0 \in \mathbb{R}^{d-1} \times \{0\}$, we assume there exist a neighborhood of x_0 where ϕ satisfies (3.4) the sub-ellipticity condition and $\partial_x \phi(x_0) > 0$. Let $(\xi'_0, \tau_0) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ be such that $|\xi'_0|^2 + \tau_0^2 = 1$. There exist W be an open conic set of (x_0, ξ', τ_0) , $\chi_1 \in S^0_{\tau, \tau}$ be an homogenous symbol of order 0 for $\langle \xi', \tau \rangle \geq 1$ supported in W and $\chi_1 = 1$ in a conic neighborhood of (x_0, ξ', τ_0) . For $s \in \mathbb{R}$, there exist $C > 0$, and $\tau_* > 0$ such that*

$$\begin{aligned} \|\Lambda_{\tau, \tau}^{s+1/2} \operatorname{Op}_{\tau}(\chi_1) w|_{x_d=0}\| + \tau^{1/2} \|\Lambda_{\tau, \tau}^s \operatorname{Op}_{\tau}(\chi_1) v\|_+ + \tau^{-1/2} \|\Lambda_{\tau, \tau}^{s+1} \operatorname{Op}_{\tau}(\chi_1) w\|_+ \\ \leq C \|\Lambda_{\tau, \tau}^s F\|_+ + \|\Lambda_{\tau, \tau}^s G\|_+ + \|\Lambda_{\tau, \tau}^s w\|_+ + \|\Lambda_{\tau, \tau}^s v\|_+, \end{aligned}$$

for $v, w \in S(\mathbb{R}^d)$ satisfying (3.19), for every $\tau \geq \tau_*$ and μ satisfying (3.3).

This lemma implies Proposition 3.1 as we can cover $\xi', \tau = 1$ by a finite number of open sets given by the statement of Lemma 3.5.

For the proof of Lemma 3.5, we distinguish two cases, $\alpha \neq 0$ and $\alpha = 0$.

Assume that $\alpha(x_0, \xi'_0, \tau_0) \neq 0$. By continuity and homogeneity in (ξ', τ) , $\alpha \neq 0$ in a conic neighborhood W of (x_0, ξ'_0, τ_0) . Let $\chi_0 \in S^0_{\tau, \tau}$ be an homogenous symbol of order 0 for $\langle \xi', \tau \rangle \geq 1$ such that $\chi_0 = 1$ in a conic neighborhood of (x_0, ξ'_0, τ_0) , supported in W and χ_1 supported on $\chi_0 = 1$. Writing

$$b = \begin{pmatrix} i\tau \partial_{x_d} \phi & -\mu \\ \mu^{-1} \alpha^2 & i\tau \partial_{x_d} \phi \end{pmatrix},$$

the left eigenvector associated with $i\tau \partial_{x_d} \phi + i\alpha$ (resp. $i\tau \partial_{x_d} \phi - i\alpha$) is $-i\alpha \mu$ (resp. $i\alpha \mu$).

Let $\tilde{\alpha} = \chi_0 \alpha$, as α is a smooth homogenous function of order 1 in W , $\tilde{\alpha} \in S^1_{\tau, \tau}$. Recall the notation $\Lambda_{\tau, \tau}^s = \operatorname{Op}_{\tau}(\langle \xi', \tau \rangle^s)$, according with the above algebraic computations and with the left vector found, we define

$$(3.24) \quad \begin{aligned} z_1 &= -i\Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\tilde{\alpha}) \operatorname{Op}_{\tau}(\chi_1) w + \mu \Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\chi_1) v^d \\ z_2 &= i\Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\tilde{\alpha}) \operatorname{Op}_{\tau}(\chi_1) w + \mu \Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\chi_1) v. \end{aligned}$$

As $v^d = 0$ on $x_d = 0$ we obtain $z_1 + z_2 = 0$ on $x_d = 0$. Applying $\pm i\Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\tilde{\alpha}) \operatorname{Op}_{\tau}(\chi_1)$ to the first equation (3.21), $\mu \Lambda_{\tau, \tau}^{-1} \operatorname{Op}_{\tau}(\chi_1)$ to the second equation and summing up, we obtain

$$(3.25) \quad D_{x_d} z_j + \operatorname{Op}_{\tau}(i\tau \partial_{x_d} \phi + (-1)^j i\tilde{\alpha}) z_j = H_j \text{ where} \\ \|\Lambda_{\tau, \tau}^s H_j\|_+ \leq 4 \|\Lambda_{\tau, \tau}^s F^d\|_+ + \|\Lambda_{\tau, \tau}^s G\|_+ + \|\Lambda_{\tau, \tau}^s \tilde{F}\|_+ + \|\Lambda_{\tau, \tau}^s w\|_+ + \|\Lambda_{\tau, \tau}^s v^d\|_+.$$

We compute

$$(3.26) \quad 2 \operatorname{Re}(H_j, i\Lambda_{\tau, \tau}^{2s+1} z_j)_+ = 2 \operatorname{Re}(D_{x_d} z_j + \operatorname{Op}_{\tau}(i\tau \partial_{x_d} \phi + (-1)^j i\tilde{\alpha}) z_j, i\Lambda_{\tau, \tau}^{2s+1} z_j)_+ \\ = |\Lambda_{\tau, \tau}^{s+1/2} z_j|_{x_d=0}|^2 + 2 \operatorname{Re}(\Lambda_{\tau, \tau}^{2s+1} \operatorname{Op}_{\tau}(i\tau \partial_{x_d} \phi + (-1)^j i\tilde{\alpha}) z_j, z_j)_+,$$

using that

$$(3.27) \quad 2 \operatorname{Re}(D_{x_d} h, i\Lambda_{\tau, \tau}^{2m} h)_+ = |\Lambda_{\tau, \tau}^m h|_{x_d=0}|^2,$$

for $h \in S(\mathbb{R}^d)$.

If $j = 2$, we have $i\tau \partial_{x_d} \phi + \operatorname{Re} \alpha \leq 4\tau + |\xi'|$ in W . Let $\chi_2 \in S^0_{\tau, \tau}$ supported in $\chi_0 = 1$ and $\chi_2 = 1$ on the support of χ_1 . From symbolic calculus we have

$$(3.28) \quad \|\Lambda_{\tau, \tau}^s z_2 - \operatorname{Op}_{\tau}(\chi_2) z_2\|_+ \leq 4 \|\Lambda_{\tau, \tau}^s w\|_+ + \|\Lambda_{\tau, \tau}^s v^d\|_+.$$

Then the tangential Gårding inequality of Theorem 3.8 applies and we have

$$2 \operatorname{Re}(\Lambda_{\tau, \tau}^{2s+1} \operatorname{Op}_{\tau} \tau \partial_{x_d} \phi + \tilde{\alpha} z_2, z_2)_+ \geq C_1 \|\Lambda_{\tau, \tau}^{s+1} z_2\|_+^2 - C_N \|\Lambda_{\tau, \tau}^{-N} w\|_+^2 + \|\Lambda_{\tau, \tau}^{-N-d} v\|_+^2.$$

From (3.26), we then deduce

$$(3.29) \quad 2 \operatorname{Re}(H_2, i \Lambda_{\tau, \tau}^{2s+1} z_2)_+ \geq C_1 \|\Lambda_{\tau, \tau}^{s+1/2} (z_2)|_{x_d=0}\|_+^2 + \|\Lambda_{\tau, \tau}^{s+1} z_2\|_+^2 - C_N \|\Lambda_{\tau, \tau}^{-N} w\|_+^2 + \|\Lambda_{\tau, \tau}^{-N-d} v\|_+^2,$$

for $C_1 > 0$, for every $N > 0$ and $C_N > 0$, uniformly with respect to τ chosen sufficiently large. This implies

$$(3.30) \quad \|\Lambda_{\tau, \tau}^{s+1/2} (z_2)|_{x_d=0}\|_+ + \|\Lambda_{\tau, \tau}^{s+1} z_2\|_+ \leq 4 \|\Lambda_{\tau, \tau}^s H_2\|_+ + \|\Lambda_{\tau, \tau}^{-N} w\|_+ + \|\Lambda_{\tau, \tau}^{-N-d} v^d\|_+.$$

Lemma 3.6. Assume that $\alpha \neq 0$ in W .

If $\operatorname{Re} \alpha - \partial_{x_d} \phi = 0$ on W , we have

$$(3.31) \quad \|\Lambda_{\tau, \tau}^{s+1} z\|_+ \leq C \|\Lambda_{\tau, \tau}^s H\|_+ + \|\Lambda_{\tau, \tau}^{s+1/2} (z)|_{x_d=0}\|_+ + \|\Lambda_{\tau, \tau}^{-N} w\|_+ + \|\Lambda_{\tau, \tau}^{-N-d} v^d\|_+$$

for some $C > 0$.

If $\operatorname{Re} \alpha - \tau \partial_x \phi = 0$ at (x_0, ξ', τ_0) , we have

$$(3.32) \quad \|\Lambda_{\tau, \tau}^{s+1/2} z_1\|_+ \leq C \|\Lambda_{\tau, \tau}^s H\|_+ + \|\Lambda_{\tau, \tau}^{s+1/2} (z)|_{x_d=0}\|_+ + \|\Lambda_{\tau, \tau}^{-N} w\|_+ + \|\Lambda_{\tau, \tau}^{-N-d} v^d\|_+$$

for some $C > 0$.

Proof. We have to distinguish three cases, that is $|\operatorname{Re} \alpha| \leq \tau |\partial_{x_d} \phi|$ at (x_0, ξ', τ_0) .

- If $|\operatorname{Re} \alpha| < \tau |\partial_{x_d} \phi|$, from (3.23) this is equivalent to

$$4\tau^2 (\partial_{x_d} \phi)^2 (\operatorname{Re} q - \mu^2) - 4\tau^4 (\partial_x \phi)^4 + (\operatorname{Im} q)^2 < 0.$$

We have $\tau \partial_x \phi + \operatorname{Re} \alpha \leq 4\tau + \xi'$ in W . Then we have the same computations as in (3.29) and (3.30), and we have

$$\|\Lambda_{\tau, \tau}^{s+1/2} (z_1)|_{x_d=0}\|_+ + \|\Lambda_{\tau, \tau}^{s+1} z_1\|_+ \leq 4 \|\Lambda_{\tau, \tau}^s H_1\|_+ + \|\Lambda_{\tau, \tau}^{-N} w\|_+ + \|\Lambda_{\tau, \tau}^{-N-d} v\|_+$$

which is a better estimate than (3.31).

- If $|\operatorname{Re} \alpha| > \tau |\partial_{x_d} \phi|$, from (3.22) this is equivalent to

$$4\tau^2 (\partial_{x_d} \phi)^2 (\operatorname{Re} q - \mu^2) - 4\tau^4 (\partial_x \phi)^4 + (\operatorname{Im} q)^2 > 0.$$

Observe that this case contains the case where $\tau_0 = 0$ as $|\xi'_0| = 1$, and in W we have $q(x, \xi, \tau) \geq c|\xi'|^2$ and $|\xi'| \gg \tau$.

As $-\tau \partial_x \phi + \operatorname{Re} \alpha \leq 4\tau + \xi'$ in W , from (3.26), we can introduce a cutoff as in (3.28) to apply the tangential Gårding inequality of Theorem 3.8, we deduce

$$-2 \operatorname{Re}(H_1, i \Lambda_{\tau, \tau}^{2s+1} z_1)_+ + \|\Lambda_{\tau, \tau}^{s+1/2} (z_1)|_{x_d=0}\|_+^2 \leq 4 \|\Lambda_{\tau, \tau}^{s+1} z_1\|_+^2 - C_N \|\Lambda_{\tau, \tau}^{-N} w\|_+^2 + \|\Lambda_{\tau, \tau}^{-N-d} v\|_+^2,$$

and then

$$\|\Lambda_{\tau, \tau}^{s+1} z_1\|_+ \leq 4 \|\Lambda_{\tau, \tau}^s H_1\|_+ + \|\Lambda_{\tau, \tau}^{s+1/2} (z_1)|_{x_d=0}\|_+ + \|\Lambda_{\tau, \tau}^{-N} w\|_+ + \|\Lambda_{\tau, \tau}^{-N-d} v\|_+,$$

which implies (3.31).

- If $\operatorname{Re} \alpha = \tau \partial_{x_d} \phi$ at (x_0, ξ'_0, τ_0) , from (3.22) and as $\operatorname{Re} \alpha$ and $\partial_{x_d} \phi$ are positive, this is equivalent to

$$4\tau^2 (\partial_{x_d} \phi)^2 (\operatorname{Re} q - \mu^2) - 4\tau^4 (\partial_x \phi)^4 + (\operatorname{Im} q)^2 = 0, \text{ at } (x_0, \xi', \tau_0).$$

We use Carleman technics to obtain an estimate. Before doing that we must translate sub-ellipticity assumption (3.4) on p_ϕ on analogous condition on α . First observe that

$$p_\phi(x, \xi, \tau) = (\xi_d + i\tau\partial_{x_d}\phi)^2 + \alpha^2 = (\xi_d + i\tau\partial_{x_d}\phi + i\alpha)(\xi_d + i\tau\partial_{x_d}\phi - i\alpha).$$

As $i\tau_0\partial_{x_d}\phi(x_0) - i\alpha(x_0, \xi', \tau_0) \in \mathbb{R}$, $p_\phi = 0$ is equivalent to $\xi_d + i\tau\partial_{x_d}\phi - i\alpha = 0$. Noting that $i\tau\partial_{x_d}\phi + i\alpha \notin \mathbb{R}$ thus $\xi_d + i\tau\partial_{x_d}\phi + i\alpha = 0$ in W . Second, for a smooth function $q = q_r + iq_i$ where q_r, q_i are real valued, we have $\{q, \bar{q}\} = 2i\{q_i, q_r\}$. Thus on $p_\phi = 0$ we have

$$\begin{aligned} \{p_\phi, \bar{p}_\phi\} &= |\xi_d + i\tau\partial_{x_d}\phi + i\alpha|^2 \{ \xi_d + i\tau\partial_{x_d}\phi - i\alpha, \xi_d - i\tau\partial_{x_d}\phi + i\alpha \} \\ &= 2i|\xi_d + i\tau\partial_{x_d}\phi + i\alpha|^2 \{ \tau\partial_{x_d}\phi - \operatorname{Re} \alpha, \xi_d + \operatorname{Im} \alpha \}. \end{aligned}$$

Thus sub-ellipticity condition reads in W , there exists $C > 0$ such that

$$(3.33) \quad \xi_d + i\tau\partial_{x_d}\phi - i\alpha = 0 \implies \{ \xi_d + \operatorname{Im} \alpha, \tau\partial_{x_d}\phi - \operatorname{Re} \alpha \} \geq C \langle \xi', \tau \rangle.$$

At (x_0, ξ', τ_0) , observe that we can choose ξ_d such that $\xi_d + \operatorname{Im} \alpha = 0$ and as $\tau_0\partial_{x_d}\phi - \operatorname{Re} \alpha = 0$, the condition (3.33) means, by continuity and homogeneity, there exists $C > 0$ such that

$$(3.34) \quad \{ \xi_d + \operatorname{Im} \alpha, \tau\partial_{x_d}\phi - \operatorname{Re} \alpha \} \geq C \langle \xi', \tau \rangle \text{ in } W,$$

eventually shrinking W .

Let

$$\begin{aligned} A &= \frac{1}{2} \operatorname{Op}_T(i\tau\partial_{x_d}\phi - i\tilde{\alpha}) + \operatorname{Op}_T(i\tau\partial_{x_d}\phi - i\tilde{\alpha})^*, \\ B &= \frac{1}{2i} \operatorname{Op}_T(i\tau\partial_{x_d}\phi - i\tilde{\alpha}) - \operatorname{Op}_T(i\tau\partial_{x_d}\phi - i\tilde{\alpha})^*. \end{aligned}$$

We have $A = A^*$, $B = B^*$, $\operatorname{Op}_T(i\tau\partial_{x_d}\phi - i\tilde{\alpha}) = A + iB$, and principal symbol of A is $\operatorname{Im} \tilde{\alpha}$ and principal symbol of B is $\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha}$.

$$(3.35) \quad \begin{aligned} \left\| \begin{matrix} D_{x_d} \\ \tau \end{matrix} + \operatorname{Op} \begin{pmatrix} i\tau\partial_{x_d}\phi - \\ i\tilde{\alpha} \end{pmatrix} z \right\|_+^2 &= \left\| \begin{matrix} D_{x_d} + A \\ z \end{matrix} \right\|_+^2 + \|Bz\|_+^2 \\ &\quad + 2 \operatorname{Re} \begin{pmatrix} D_{x_d} + A \\ z \end{pmatrix}, iBz \Big|_+. \end{aligned}$$

We have

$$2 \operatorname{Re} \begin{pmatrix} D_{x_d} z, iBz \end{pmatrix}_+ = [D_{x_d}, iB]z, z \Big|_+ + (Bz|_{x_d=0}, z|_{x_d=0})_\partial.$$

As the principal symbol of B is $\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha}$, we obtain

$$(3.36) \quad \begin{aligned} 2 \operatorname{Re} \begin{pmatrix} D_{x_d} z, iBz \end{pmatrix}_+ &\geq \operatorname{Re} i[D_{x_d}, \operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})]z, z \Big|_+ \\ &\quad + \operatorname{Re}(\operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})z|_{x_d=0}, z|_{x_d=0})_\partial - C\|z\|_+^2 - C|z|_{x_d=0}|_+^2, \end{aligned}$$

for some constant $C > 0$. We also have

$$2 \operatorname{Re} \begin{pmatrix} Az, iBz \end{pmatrix}_+ = i[A, B]z, z \Big|_+ \geq \operatorname{Re} i[\operatorname{Op}_T(\operatorname{Im} \tilde{\alpha}), \operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})]z \Big|_+ - C\|z\|_+^2.$$

Then from this estimate and (3.36), we obtain

$$(3.37) \quad \begin{aligned} 2 \operatorname{Re} \begin{pmatrix} D_{x_d} + A \\ z \end{pmatrix}, iBz \Big|_+ &\geq \operatorname{Re} i[D_{x_d} + \operatorname{Op}_T(\operatorname{Im} \tilde{\alpha}), \operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})]z, z \Big|_+ \\ &\quad + \operatorname{Re}(\operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})z|_{x_d \neq 0}, z|_{x_d \neq 0})_\partial - C\|z\|_+^2 - C|z|_{x_d=0}|_+^2. \end{aligned}$$

The principal symbol of $i[D_{x_d} + \operatorname{Op}_T(\operatorname{Im} \tilde{\alpha}), \operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})]$ is $\{ \xi_d + \operatorname{Im} \alpha, \tau\partial_{x_d}\phi - \operatorname{Re} \alpha \}$, then from (3.34) and microlocal Gårding inequality of Theorem 3.8, we have

$$(3.38) \quad \operatorname{Re} i \begin{pmatrix} D_{x_d} \\ \tau \end{pmatrix} + \operatorname{Op} \begin{pmatrix} \operatorname{Im} \tilde{\alpha} \\ \tau \end{pmatrix}, \operatorname{Op} \begin{pmatrix} \tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha} \\ \tau \end{pmatrix} z, z \Big|_+ \geq C \|\Lambda_{\tau, \tau}^{1/2} z\|_+^2 - C \|\Lambda_{\tau, \tau}^{-N} z\|_+^2.$$

We have

$$\left| (\operatorname{Op}_T(\tau\partial_{x_d}\phi - \operatorname{Re} \tilde{\alpha})z|_{x_d=0}, z|_{x_d=0})_\partial \right| \leq C \|\Lambda_{\tau, \tau}^{1/2} z|_{x_d=0}\|_+^2,$$

then from (3.35), (3.37) and (3.38) we obtain

$$(3.39) \quad \|\Lambda_{T,\tau} z\|_+ \leq 4 \left\| D_{x_d} + \text{Op}_T(i\tau \partial_{x_d} \phi - i\tilde{\alpha}) z \right\|_+ + |\Lambda_{T,\tau} z|_{x_d=0} + \|\Lambda_{T,\tau} z_1\|_+,$$

as we can absorb the remainder term $\|z\|_+^2$ by the left hand side. Recalling the definition of z_1 given by formula (3.24), the symbolic calculus yields

$$\|\Lambda_{T,\tau} \text{Op}_T(\chi_0) \Lambda_{T,\tau} z_1 - \Lambda_{T,\tau} z_1\|_+ \leq 4 \|\Lambda_{T,\tau} w\|_+ + \|\Lambda_{T,\tau} v^d\|_+.$$

From $z = \text{Op}_T(\chi_0) \Lambda_{T,\tau} z_1$, we deduce

$$(3.40) \quad \|\Lambda_{T,\tau}^{s+1/2} z\|_+ \leq 4 \|\Lambda_{T,\tau}^{1/2} z\|_+ + \|\Lambda_{T,\tau}^{s-1/2} w\|_+ + \|\Lambda_{T,\tau}^{s-1/2} v^d\|_+.$$

Symbolic calculus also gives

$$(3.41) \quad \left\| D_{x_d} + \text{Op}_T(i\tau \partial_{x_d} \phi - i\tilde{\alpha}) z \right\|_+ \leq 4 \|\Lambda_{T,\tau} H_1\|_+ + \|\Lambda_{T,\tau} w\|_+ + \|\Lambda_{T,\tau} v^d\|_+$$

and

$$(3.42) \quad |\Lambda_{T,\tau}^{1/2} z|_{x_d=0} \leq 4 |\Lambda_{T,\tau}^{s+1/2} (z_1)|_{x_d=0}.$$

Then from (3.39)–(3.42) we obtain

$$(3.43) \quad \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ \leq 4 \|\Lambda_{T,\tau}^s H_1\|_+ + |\Lambda_{T,\tau}^{s+1/2} (z_1)|_{x_d=0} + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+,$$

which is (3.32). This achieves the proof of Lemma 3.6 as we have treated the three cases. \square

We can prove Lemma 3.5 in the case $\alpha = 0$.

If $\text{Re } \alpha - \partial_{x_d} \phi \neq 0$ on W , from (3.30), Lemma 3.6, and as $z_1 + z_2 = 0$ on $x_d = 0$, we deduce

$$\begin{aligned} |\Lambda_{T,\tau}^{s+1/2} (z_2)|_{x_d=0} + \|\Lambda_{T,\tau}^{s+1} z_2\|_+ + \|\Lambda_{T,\tau}^{s+1} z_1\|_+ \\ \leq 4 \|\Lambda_{T,\tau}^s H_2\|_+ + \|\Lambda_{T,\tau}^s H_1\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+. \end{aligned}$$

From (3.25) we deduce

$$(3.44) \quad |\Lambda_{T,\tau}^{s+1/2} (z_2)|_{x_d=0} + \|\Lambda_{T,\tau}^{s+1} z_2\|_+ + \|\Lambda_{T,\tau}^{s+1} z_1\|_+ \leq 4 \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ \\ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s w\|_+.$$

We have from (3.24), $z_1 + z_2 = 2\mu \Lambda_{T,\tau}^{-1} \text{Op}_T(\chi_1) v^d$ and from (3.3) we deduce

$$(3.45) \quad \tau \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) v^d\|_+ \leq 4 |\mu| \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) v^d\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1} z_2\|_+.$$

We have $\text{Op}_T(\tilde{\alpha}) * \Lambda_{T,\tau}^s \Lambda_{T,\tau}^s \text{Op}_T(\tilde{\alpha}) = \text{Op}_T(\langle \xi', \tau \rangle^{2s} \tilde{\alpha}^2)$ modulo an operator of order $2s + 1$. As $\tilde{\alpha}$ is not 0 on the support of χ_1 , the tangential Gårding inequality of Theorem 3.8 yields

$$\|\Lambda_{T,\tau}^s \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^N w\|_+ \leq 4 \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) w\|_+,$$

for every $N > 0$. From this and as $z_2 - z_1 = 2i \Lambda_{T,\tau}^{-1} \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w$ from (3.24), we deduce

$$(3.46) \quad \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1} z_2\|_+ + \|\Lambda_{T,\tau}^N w\|_+.$$

From the first equation of (3.19) and from (3.3) we have

$$(3.47) \quad \tau \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) \tilde{v}\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s w\|_+ \\ + 4 \|\Lambda_{T,\tau}^{s+1} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1} z_2\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s w\|_+.$$

From (3.24), $(z_2)|_{x_d=0} = i \Lambda_{T,\tau}^{-1} \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w|_{x_d=0}$, arguing as from above and using the Gårding estimate of Theorem 3.5, we have

$$(3.48) \quad |\Lambda_{T,\tau}^{s+1/2} \text{Op}_T(\chi_1) w|_{x_d=0} \leq 4 |\Lambda_{T,\tau}^{s+1/2} (z_2)|_{x_d=0}.$$

From (3.44)–(3.48) we obtain Lemma 3.5.

If $\text{Re } \alpha - \tau \partial_x \phi = 0$ at (x_0, ξ', τ_0) , adding (3.30) to $\varepsilon(3.32)$ for $\varepsilon > 0$, we deduce

$$\begin{aligned} & |\Lambda_{T,\tau}^{s+1/2}(z_2)|_{x_d=0}| + \|\Lambda_{T,\tau}^{s+1/2} z_2\|_+ + \varepsilon \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ \\ & 4 \|\Lambda_{T,\tau}^s H_2\|_+ + \varepsilon \|\Lambda_{T,\tau}^s H_1\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \varepsilon |\Lambda_{T,\tau}^{s+1/2}(z_1)|_{x_d=0}|. \end{aligned}$$

From (3.25) and as $z_1 + z_2 = 0$ on $x_d = 0$, we deduce for ε small enough that

$$(3.49) \quad |\Lambda_{T,\tau}^{s+1/2}(z_2)|_{x_d=0}| + \|\Lambda_{T,\tau}^{s+1/2} z_2\|_+ + \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ \\ 4 \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s w\|_+.$$

We have from (3.24), $z_1 + z_2 = 2\mu \Lambda_{T,\tau}^{-1} \text{Op}_T(\chi_1) v^d$. Let $\chi_2 \in S_{T,\tau}^0$ supported in $\chi_0 = 1$ and $\chi_2 = 1$ on the support of χ_1 . From symbolic calculus we have

$$\|(\text{Op}_T(\chi_2) \Lambda_{T,\tau}^{s+3/2} \mu^{-1})(\mu \Lambda_{T,\tau}^{-1} \text{Op}_T(\chi_1) v^d) - \Lambda_{T,\tau}^{s+1/2} \text{Op}_T(\chi_1) v^d\|_+ \leq 4 \|\Lambda_{T,\tau}^N v^d\|_+.$$

As $\text{Op}_T(\chi_2) \Lambda_{T,\tau}^{s+3/2} \mu^{-1}$ is an operator of order $s + 1/2$ as $|\mu|$ and $|\xi'|$ are comparable on the support of χ_2 , we deduce

$$(3.50) \quad \tau^{1/2} \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) v^d\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1/2} \text{Op}_T(\chi_1) v^d\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1/2} z_2\|_+ + \|\Lambda_{T,\tau}^N v^d\|_+.$$

We have $\tau^{-1} \text{Op}_T(\tilde{\alpha}) * \Lambda_{T,\tau}^s \Lambda_{T,\tau}^s \text{Op}_T(\tilde{\alpha}) = \tau^{-1} \text{Op}_T(\langle \xi', \tau \rangle^{2s} \tilde{\alpha}^2)$ modulo an operator of order $2s$. As $\tilde{\alpha}$ is not 0 on the support of χ_1 , the tangential Gårding inequality of Theorem 3.8 yields

$$\tau^{-1/2} \|\Lambda_{T,\tau}^s \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^N w\|_+ \leq 4 \tau^{-1/2} \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+,$$

for every $N > 0$. From this and as $z_2 - z_1 = 2i \Lambda_{T,\tau}^{-1} \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w$ from (3.24), we deduce,

$$(3.51) \quad \tau^{-1/2} \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+ \leq 4 \tau^{-1/2} \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_2) \Lambda_{T,\tau}^{-1} \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^N w\|_+ \\ 4 \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1/2} z_2\|_+ + \|\Lambda_{T,\tau}^N w\|_+,$$

as $\tau^{-1/2} \Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_2)$ is an operator of order $s + 1/2$.

From the first equation of (3.19) and (3.3), we have

$$(3.52) \quad \tau^{1/2} \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) \tilde{v}\|_+ \leq 4 \|\Lambda_{T,\tau}^{s+1} \mu^{-1} \tau^{1/2} \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s w\|_+ \\ 4 \|\Lambda_{T,\tau}^{s+1/2} z_1\|_+ + \|\Lambda_{T,\tau}^{s+1/2} z_2\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+,$$

from (3.51).

From (3.24), $(z_2)|_{x_d=0} = i \Lambda_{T,\tau}^{-1} \text{Op}_T(\tilde{\alpha}) \text{Op}_T(\chi_1) w|_{x_d=0}$, arguing as from above and using the Gårding estimate of Theorem 3.5, we have

$$(3.53) \quad |\Lambda_{T,\tau}^{s+1/2} \text{Op}_T(\chi_1) w|_{x_d=0}| \leq 4 |\Lambda_{T,\tau}^{s+1/2}(z_2)|_{x_d=0}|.$$

From (3.49)–(3.53) we obtain Lemma 3.5.

Now we consider the case $q - \mu^2 = 0$. Let $\varepsilon > 0$, we can shrink W such that $|q - \mu^2| \leq \varepsilon \langle \xi', \tau \rangle^2$ in W . Note that $|\mu| \sim \tau \sim |\xi'|$ on W . Let χ_1 be the cutoff defined previously supported on W and χ_0 supported on W and $\chi_0 = 1$ on the support of χ_1 . By symbolic calculus we have

$$(3.54) \quad \text{Op}_T(\chi_1) \text{Op}_T(\delta) \text{Op}_T(\zeta') = \text{Op}_T(\delta) \text{Op}_T(\zeta') \text{Op}_T(\chi_1) + [\text{Op}_T(\chi_1), \text{Op}_T(\delta) \text{Op}_T(\zeta')] \\ = \text{Op}_T(q) \text{Op}_T(\chi_1) + \text{Op}_T(r_1) \text{Op}_T(\chi_1) \\ + \text{Op}_T(\chi_0) [\text{Op}_T(\chi_1), \text{Op}_T(\delta) \text{Op}_T(\zeta')] + \text{Op}_T(r_{-N}),$$

where $r_1 \in S_{T,\tau}^1$ and $r_{-N} \in S_{T,\tau}^N$. Observe that $\mu^{-1} \chi_j \in S_{T,\tau}^1$ for $j = 1, 2$.

From (3.21), (3.54), and by symbolic calculus we have

$$\begin{aligned} & D_x \text{Op}_T(\chi_1) w + i\tau(\partial_x \phi) \text{Op}_T(\chi_1) w - \mu \text{Op}_T(\chi_1) v^d = H_1 && \text{in } x_d > 0, \\ & D_x \text{Op}_T(\chi_1) v^d + \mu^{-1} \text{Op}_T(q - \mu^2) \text{Op}_T(\chi_1) w + i\tau(\partial_x \phi) \text{Op}_T(\chi_1) v^d = H_2 && \text{in } x_d > 0, \\ & v^d = 0 && \text{on } x_d = 0, \end{aligned}$$

where

$$(3.55) \quad \|\Lambda_{T,\tau}^s H_j\|_+ \leq C_\varepsilon \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+,$$

for $j = 1, 2$ with C_ε depends on ε . We compute

$$(3.56) \quad 2 \operatorname{Re}(H_{1,\tau} i \Lambda_{T,\tau}^{2s+1} \operatorname{Op}_T(\chi_1) w)_+ \\ = 2 \operatorname{Re}(D_{x_d} \operatorname{Op}_T(\chi_1) w + i\tau(\partial_{x_d} \phi) \operatorname{Op}_T(\chi_1) w - \mu \operatorname{Op}_T(\chi_1) v^d, i \Lambda_{T,\tau}^{2s+1} \operatorname{Op}_T(\chi_1) w)_+.$$

By microlocal Gårding inequality of Theorem 3.8 we have, using $\tau \sim |\xi|$ on W

$$(3.57) \quad 2 \operatorname{Re}(i\tau(\partial_{x_d} \phi) \operatorname{Op}_T(\chi_1) w, i \Lambda_{T,\tau}^{2s+1} \operatorname{Op}_T(\chi_1) w)_+ \geq C_0 \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_T(\chi_1) w\|_+^2 - C_N \|\Lambda_{T,\tau}^{-N} w\|_+^2$$

for $C_0 > 0$, for all $N > 0$ and $C_N > 0$.

From this, (3.27) and (3.56) we obtain

$$(3.58) \quad 2 \operatorname{Re}(H_{1,\tau} i \Lambda_{T,\tau}^{2s+1} \operatorname{Op}_T(\chi_1) w)_+ \\ \geq |\Lambda_{T,\tau}^{s+1/2} \operatorname{Op}_T(\chi_1) w|_{x_d=0}^2 + C_1 \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_T(\chi_1) w\|_+^2 - \mu^2 C_2 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+^2 - C_N \|\Lambda_{T,\tau}^{-N} w\|_+^2,$$

for $C_1, C_2 > 0$, for all $N > 0$ and $C_N > 0$.

We then obtain

$$(3.59) \quad |\Lambda_{T,\tau}^{s+1/2} \operatorname{Op}_T(\chi_1) w|_{x_d=0}^2 + \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_T(\chi_1) w\|_+^2 \leq \mu^2 C_3 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+^2 \\ + C_\varepsilon \|\Lambda_{T,\tau}^s F\|_+^2 + \|\Lambda_{T,\tau}^s G\|_+^2 + \|\Lambda_{T,\tau}^s w\|_+^2 + \|\Lambda_{T,\tau}^s v^d\|_+^2 + \|\Lambda_{T,\tau}^s \tilde{F}\|_+^2,$$

for $C_3 > 0$, for all $N > 0$ and $C_N, C_\varepsilon > 0$.

Now we compute

$$(3.60) \quad 2 \operatorname{Re}(H_{2,\tau} i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+ = 2 \operatorname{Re}(D_{x_d} \operatorname{Op}_T(\chi_1) v^d, i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+ \\ + 2 \operatorname{Re}(\mu^{-1} \operatorname{Op}_T(q) \operatorname{Op}_T(\chi_1) w - \mu \operatorname{Op}_T(\chi_1) w + i\tau(\partial_{x_d} \phi) \operatorname{Op}_T(\chi_1) v^d, i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+.$$

From (3.27) we have $2 \operatorname{Re}(D_{x_d} \operatorname{Op}_T(\chi_1) v^d, i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+ = 0$ as $v^d = 0$ on $x_d = 0$.

As $C\varepsilon^2 \langle \xi', \tau \rangle^2 - \mu^{-2}(q - \mu^2)^2 \geq \varepsilon^2 \langle \xi', \tau \rangle^2$, on W with $C > 0$, using that $\tau \sim |\mu| \sim |\xi|$, we have by microlocal Gårding inequality of Theorem 3.8

$$2 \operatorname{Re}(\mu^{-1} \operatorname{Op}_T(q - \mu^2) \operatorname{Op}_T(\chi_1) w, \Lambda_{T,\tau}^{2s} \mu^{-1} \operatorname{Op}_T(q - \mu^2) \operatorname{Op}_T(\chi_1) w)_+ \\ \leq C_4 \varepsilon \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_T(\chi_1) w\|_+^2 + C_{N,\varepsilon} \|\Lambda_{T,\tau}^{-N} w\|_+^2.$$

Then we have

$$(3.61) \quad 2 \operatorname{Re}(\mu^{-1} \operatorname{Op}_T(q) \operatorname{Op}_T(\chi_1) w - \mu \operatorname{Op}_T(\chi_1) w, i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+ \\ \leq \varepsilon |\mu| C_5 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+^2 + \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) w\|_+^2 + C_{N,\varepsilon} \|\Lambda_{T,\tau}^{-N} w\|_+^2,$$

for $C_5 > 0$.

From microlocal Gårding inequality of Theorem 3.8 and as $\partial_{x_d} \phi(x_0) > 0$, we have

$$(3.62) \quad 2 \operatorname{Re}(i\tau(\partial_{x_d} \phi) \operatorname{Op}_T(\chi_1) v^d, i \mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_T(\chi_1) v^d)_+ \geq \mu^2 C_6 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+^2 - C_N \|\Lambda_{T,\tau}^{-N} v^d\|_+^2,$$

where $C_6 > 0$ is independent of ε , for all $N > 0$, $C_N > 0$.

From (3.55) and (3.60)–(3.62) we obtain

$$\mu^2 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+^2 \leq |\mu| \varepsilon C_2 \|\Lambda_{T,\tau}^s \operatorname{Op}_T(\chi_1) v^d\|_+ \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_T(\chi_1) w\|_+ + C_{N,\varepsilon} \|\Lambda_{T,\tau}^{-N} w\|_+^2 \\ + C_\varepsilon \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+^2.$$

We deduce

$$(3.63) \quad \begin{aligned} \mu^2 \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) v^d\|_+^2 &\leq \varepsilon^2 C_7 \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+^2 \\ &+ C_\varepsilon \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+^2. \end{aligned}$$

By the linear combination (3.63)+ ε (3.59) and fixing ε sufficiently small, from (3.3) and τ sufficiently large, we deduce

$$(3.64) \quad \begin{aligned} \tau \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) v^d\|_+ + |\Lambda_{T,\tau}^{s+1/2} \text{Op}_T(\chi_1) w|_{x^d=0} + \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+ \\ 4 \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+. \end{aligned}$$

From the first equation of (3.19) and from (3.3) we have

$$(3.65) \quad \begin{aligned} \tau \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1) \tilde{v}\|_+ + 4 \|\Lambda_{T,\tau}^{s+1} \text{Op}_T(\chi_1) w\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s w\|_+ \\ 4 \|\Lambda_{T,\tau}^s F^d\|_+ + \|\Lambda_{T,\tau}^s G\|_+ + \|\Lambda_{T,\tau}^s \tilde{F}\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+, \end{aligned}$$

from (3.64). From (3.64) and (3.65) we obtain Lemma 3.5 in the case $\alpha = 0$.

4. Logarithmic stability

The exponential estimate of Proposition 2.1 is the consequence of the two following results. First a global Carleman estimate with an observability term and second an estimate of the observability term coming from the dissipation.

Let ω_0 and ω_1 be open sets such that $\omega_1 \Subset \omega_0 \Subset \omega$, and, from (1.2), we have $b(x) \geq b_- > 0$ for $x \in \omega$. In what follows we denote by $\|\cdot\|_0 := \|\cdot\|_{L^2(\Omega)}$.

Theorem 4.1. *Let Ω be an open bounded set of \mathbb{R}^d with smooth boundary. Let $\phi \in C(\overline{\Omega})$ be a function that satisfies the sub-ellipticity assumption in $\overline{\Omega} \setminus \omega_1$. Then there exist $\tau_* > 0$ and $C > 0$ such that*

$$\tau^{3/2} \|e^{\tau\phi} r\|_0 + \tau^{3/2} \|e^{\tau\phi} u\|_0 \leq C \left(\tau \|e^{\tau\phi} g\|_0 + \tau \|e^{\tau\phi} f\|_0 + \tau^{3/2} \|e^{\tau\phi} r\|_{L^2(\omega_0)} + \tau^{3/2} \|e^{\tau\phi} u\|_{L^2(\omega_0)} \right),$$

for all $u, r \in C_c^\infty(\overline{\Omega})$ which satisfies (3.1), $u \cdot n|_\Gamma = 0$, $\tau \geq \tau_*$, and μ satisfying (3.3).

Remarks 4.1. *It is classical that there exist ψ such that $\partial_n \psi(x) < 0$ for $x \in \partial\Omega$ and $\nabla \psi \neq 0$ for $x \in \overline{\Omega} \setminus \omega_1$ (see Fursikov-Imanuvilov [12]). From Lemma 3.2, $\phi = e^{\lambda\psi}$ satisfies sub-ellipticity condition in $\Omega \setminus \omega_1$ for λ sufficiently large. In what follows we fix such a function ϕ .*

Proposition 4.1. *Let $(u, r) \in D(A_-)$ solution of $(A_d + i\mu)(u, r) = (f, g) \in H$. Then we have*

$$(4.1) \quad \begin{aligned} |\mu| \| \sqrt{b} u \|_0^2 &\leq C \| (u, r) \|_H \| (f, g) \|_H \\ |\mu| \| r \|_{L^2(\omega_0)}^2 &\leq C \| (u, r) \|_H \| (f, g) \|_H, \end{aligned}$$

for some constant $C > 0$.

From these two results we are able to prove Proposition 2.1.

Proof of Proposition 2.1. Noting that the resolvent problem $(A - i\mu)(u, r) = (f, g)$ is written as follow

$$\begin{aligned} &\cdot \\ &\cdot \nabla r + i\mu u = f - bu \quad \text{in } \Omega \\ &\cdot \text{div}(u) + i\mu r = g \quad \text{in } \Omega \\ &\cdot u \cdot n = 0 \quad \text{on } \Gamma. \end{aligned}$$

This allows us to apply Theorem 4.1. So let $C_2 = \max_{x \in \Omega} \phi(x)$ and $C_1 = \min_{x \in \Omega} \phi(x)$ we deduce from the Carleman estimate of Theorem 4.1 that

$$(4.2) \quad \|r\|_0 + \|u\|_0 \leq 4 e^{(C_2 - C_1)\tau} \left(\|g\|_0 + \|f + bu\|_0 + \|r\|_{L^2(\omega_0)} + \|u\|_{L^2(\omega_0)} \right).$$

Taking $\tau = |\mu|/c_0$ accordingly with (3.3), by the estimates of Proposition 4.1 and as $\|bu\|_0 \leq 4 \|bu\|_0$, we have

$$(4.3) \quad \|(u, r)\|_H \leq 4 C e^{K|\mu|} \| (f, g) \|_H + \|(u, r)\|_H^{1/2} \| (f, g) \|_H^{1/2},$$

which yields $\|(u, r)\|_H \leq 4 e^{K|\mu|} \| (f, g) \|_H$. This is the sought result. \square

Proof of Proposition 4.1. From equation, we have $-\nabla r + i\mu u - bu = f$ taking the inner product with u , we obtain $-(\nabla r, u) + i\mu \|u\|^2 - (bu, u) = (f, u)$. Integrating by parts, we have $-(\nabla r, u) = (r, \operatorname{div} u)$ as $u \cdot n = 0$ on $\partial\Omega$. Using the second equation $-\operatorname{div} u + i\mu r = g$, we have $-(\nabla r, u) = (r, i\mu r - g)$. We thus obtain

$$-i\mu \|r\|^2 - (r, g) + i\mu \|u\|^2 - \sqrt{b} \|u\|^2 = (f, u).$$

Taking the real part of this equation we have $|\mu| \|u\|^2 \leq |(f, u)| + |(r, g)|$. This implies the first estimate of (4.1).

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ for $x \in \omega_0$ and χ supported in ω . Taking the inner product between $-\operatorname{div} u + i\mu r = g$ and $\chi^2 r$, we obtain $(-\operatorname{div} u, \chi^2 r) + i\mu \|\chi r\|^2 = (g, \chi^2 r)$. Integrating by parts we have $(-\operatorname{div} u, \chi^2 r) = (u, \chi^2 \nabla r) + (u, 2\chi r \nabla \chi)$ and by equation $-\nabla r + i\mu u - bu = f$ we have

$$-i\mu \|\chi u\|^2 - (u, \chi^2 bu) - (u, \chi^2 f) + (u, 2\chi r \nabla \chi) + i\mu \|\chi r\|^2 = (g, \chi^2 r).$$

Taking account that $b \geq b_-$ in ω , thus on the support of χ , we have

$$|\mu| \|\chi r\|^2 \leq 4 \| (u, r) \|_H \| (f, g) \|_H + \|u \nabla \chi\| \|\chi r\| + |\mu| \|\chi u\|^2 + \sqrt{b_-} \|bu\|^2.$$

We can estimate $\|u \nabla \chi\|$ and $\|\chi u\|$ by $\sqrt{b_-}$ and by the first estimate of Proposition 4.1 we obtain the second estimate. \square

Proof of Theorem 4.1. Let $x_0 \in \overline{\Omega} \setminus \omega_1$, from Corollary 3.1 if $x_0 \in \Omega$ or from Theorem 3.9 if $x_0 \in \partial\Omega$ we obtain, in both cases, an open neighborhood (in \mathbb{R}^d) of x_0 , V such that

$$(4.4) \quad \tau^{3/2} \|e^{\tau\phi} r\|_0 + \tau^{3/2} \|e^{\tau\phi} u\|_0 \leq C (\tau \|e^{\tau\phi} g\|_0 + \tau \|e^{\tau\phi} f\|_0),$$

for $u, r \in C_c^\infty(V)$. By compactness of $\overline{\Omega} \setminus \omega_1$ we can find a finite covering $(V_j)_{j \in J}$ of $\overline{\Omega} \setminus \omega_1$. Let $(\chi_j)_{j \in J}$ be a partition of unity subordinated to $(V_j)_{j \in J}$ such that $\sum_{j \in J} \chi_j(x) = 1$ for $x \in \overline{\Omega} \setminus \omega_1$. Let $u_j = \chi_j u$ and $r_j = \chi_j r$ where (u, r) solution to (3.1), $u \cdot n|_\Gamma = 0$. We have

$$\begin{aligned} -\nabla r_j + i\mu u_j &= \chi_j f - r \nabla \chi_j \\ -\operatorname{div} u_j + i\mu r_j &= \chi_j g - u \cdot \nabla \chi_j. \end{aligned}$$

We can apply the Carleman estimate (4.4) in each V_j and we obtain

$$\begin{aligned} \tau^{3/2} \|e^{\tau\phi} r_j\|_0 + \tau^{3/2} \|e^{\tau\phi} u_j\|_0 &\leq 4 \tau \|e^{\tau\phi} (\chi_j g - u \cdot \nabla \chi_j)\|_0 + \tau \|e^{\tau\phi} (\chi_j f - r \nabla \chi_j)\|_0 \\ &\leq 4 \tau \|e^{\tau\phi} g\|_0 + \tau \|e^{\tau\phi} u\|_0 + \tau \|e^{\tau\phi} f\|_0 + \tau \|e^{\tau\phi} r\|_0. \end{aligned}$$

We have

$$\begin{aligned} \tau^{3/2} \|e^{\tau\phi} r\|_0 + \tau^{3/2} \|e^{\tau\phi} u\|_0 &\leq 4 \tau^{3/2} \sum_{j \in J} (\|e^{\tau\phi} r_j\|_0 + \|e^{\tau\phi} u_j\|_0) + \tau^{3/2} \|e^{\tau\phi} r\|_{L^2(\omega_0)} + \tau^{3/2} \|e^{\tau\phi} u\|_{L^2(\omega_0)} \\ &\leq 4 \tau \|e^{\tau\phi} g\|_0 + \tau \|e^{\tau\phi} u\|_0 + \tau \|e^{\tau\phi} f\|_0 + \tau \|e^{\tau\phi} r\|_0 + \tau^{3/2} \|e^{\tau\phi} r\|_{L^2(\omega_0)} + \tau^{3/2} \|e^{\tau\phi} u\|_{L^2(\omega_0)}. \end{aligned}$$

This gives the sought result as we can absorb the term $\tau \|e^{\tau\phi} u\|_0 + \tau \|e^{\tau\phi} r\|_0$ with the left hand side. \square

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