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# Treelength of series-parallel graphs<sup>\*</sup>

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## Abstract

The *length* of a *tree-decomposition* of a graph is the maximum distance (in the graph) between two vertices of a same bag of the decomposition. The *treelength* of a graph is the minimum length among its tree-decompositions. Treelength of graphs has been studied for its algorithmic applications in classical metric problems such as Traveling Salesman Problem or metric dimension of graphs and also, in compact routing in the context of distributed computing. Deciding whether the treelength of a general graph is at most 2 is NP-complete (graphs of treelength one are precisely the chordal graphs), and it is known that the treelength of a graph cannot be approximated up to a factor less than  $\frac{3}{2}$  (the best known approximation algorithm for treelength has an approximation ratio of 3). However, nothing is known on the computational complexity of treelength in planar graphs, except that the treelength of any outerplanar graph is equal to the third of the size of a largest isometric cycle. This work initiates the study of treelength in planar graphs by considering the next natural subclass of planar graphs, namely the one of series-parallel graphs.

We first fully describe the treelength of melon graphs (set of pairwise internally disjoint paths linking two vertices), showing that, even in such a restricted graph class, the expression of the treelength is not trivial. Then, we show that treelength can be approximated up to a factor  $\frac{3}{2}$  in series-parallel graphs. Our main result is a quadratic-time algorithm for deciding whether a series-parallel graph has treelength at most 2. Our latter result relies on a characterization of series-parallel graphs with treelength 2 in terms of an infinite family of forbidden isometric subgraphs.

**Keywords:** Tree-decomposition, Treelength, Series-parallel graphs, Isometric subgraphs

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## 1. Introduction

**Treewidth.** *Tree-decompositions* of graphs have been initially introduced by Halin [2] and then rediscovered as part of the Graph Minor Theory by Robertson and Seymour [3]. Roughly speaking, a *tree-decomposition* of a graph describes it using a set of subsets of its vertices (called *bags*) that are organized in a tree-like fashion. The classical measure of a tree-decomposition is its *width*, i.e., the maximum size (minus one) of its bags, and the *treewidth* of a graph  $G$ , denoted by  $tw(G)$ , is the minimum width of its tree-decompositions. Tree-decompositions have been extensively studied due to their various algorithmic applications. For instance, numerous NP-hard problems can be solved in linear time in bounded treewidth graphs [4, 5]; tree-decompositions are used as part of many efficient parameterized algorithms [6]; they play a crucial role in the design of sub-exponential algorithms in the context of bi-dimensionality [7], *etc.* (see [8, 9] for more details).

To make the most of previous results, being able to compute tree-decompositions with small width is an important pre-requisite. Unfortunately, computing the treewidth of an  $n$ -vertex graph  $G$  is NP-hard [10] and

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the best known approximation algorithm has approximation-ratio  $O(\sqrt{\log tw(G)})$  [11]. While computing the treewidth is *FPT*, i.e., deciding whether  $tw(G) \leq k$  can be solved in time  $O(2^{k^3} n)$  [12], the latter algorithm cannot be used in practice since it is super-exponential in  $k$  and due to the large constant hidden in the “big  $O$ ”. On the positive side, an integer  $k$  being fixed, there exists an algorithm that, given an input  $n$ -node graph  $G$ , decides if  $tw(G) > k$  or computes a tree-decomposition of  $G$  with width at most  $2k + 1$  in time  $2^{O(k)} n$  [13]. The case of planar graphs is particularly interesting since, while approximation algorithms exist [14, 15], the status of the computational complexity of treewidth in planar graphs has been open for almost 40 years.

**Treelength.** Apart from its width, other parameters have been proposed as “measures” of a tree-decomposition. In particular, the *length* (resp., *breadth*) of a tree-decomposition is the maximum *diameter* (resp., *radius*) of its bags (where the distances are considered in the whole graph). The *treelength* of a graph  $G$ , denoted by  $tl(G)$ , is then the minimum length of its tree-decompositions [16] and the *treebreadth* is defined accordingly [17]. Both treelength and treebreadth also have algorithmic interests. For instance, the *Traveling Salesman Problem* admits an FPTAS in bounded treelength graphs [18]; computing the *metric dimension* is FPT in the treelength plus the maximum degree [19]; efficient compact routing schemes and sparse additive spanners can be built in the class of bounded treelength or bounded treebreadth graphs [16, 17, 20], *etc.* Unfortunately, both these parameters are not even FPT since deciding if a graph has treelength at most two (resp., has treebreadth at most one) is NP-complete [21, 22]. On the positive side, both parameters can be efficiently approximated: treelength can be approximated up to a factor 3 using a BFS-like algorithm [16] (the approximation for treebreadth follows since the treelength of a graph is at most twice its treebreadth). Concerning treelength and treebreadth of planar graphs, very few is known. In [21], it was shown that deciding whether the treebreadth is at most one can be solved in polynomial-time in the class of  $K_{3,3}$ -minor-free graphs. The treelength of an *outerplanar* graph equals the third of its largest *isometric* cycle [16], but it is not even known whether deciding if the treelength of a planar graph is at most two can be solved in polynomial time.

This paper initiates the study of the computational complexity of treelength in planar graphs by considering the next natural superclass of 2-connected outerplanar graphs, namely the *series-parallel* graphs [23, 24].

**Relationship between treewidth and treelength.** Another motivation for this work is that achieving exact (or better approximation) algorithms for computing the treelength may lead toward better (more efficient or with better approximation ratio) approximation algorithms for computing the treewidth in large graph classes. In general, treewidth and treelength are not comparable. The treewidth of any  $n$ -node cycle equals 2 while its treelength equals  $\lceil \frac{n}{3} \rceil$  [16]. On the other hand, the treewidth of the complete graph with  $n$  vertices equals  $n - 1$  while its treelength equals one. However, these graph classes (cycles and complete graphs) are somehow the extreme cases since it has been proved that  $tw(G) = \Theta(tl(G))$  in the class of graphs  $G$  excluding an *apex* graph as minor (including planar graphs) and with bounded largest isometric cycle [25]. More specifically, [26] presents a polynomial-time algorithm that, given a tree-decomposition of length  $\ell$  of a planar graph  $G$ , computes a tree-decomposition of width at most  $9\ell$  for  $G$ . Therefore, computing tree-decompositions with “small” length would imply “good” approximation algorithms for the treewidth of planar graphs.

**Our contributions.** We focus on the computation of the treelength in series-parallel graphs. Section 2 is devoted to the formal definitions of the main concepts used throughout the paper. In Section 3, we consider *melon* graphs, i.e., series-parallel graphs  $G$  obtained by identifying the endpoints of pairwise internally disjoint paths  $(P_i)_{i \leq p}$  of respective length  $\ell_i$  (with  $\ell_1 \geq \dots \geq \ell_p$ ). We show that, in any melon graph  $G$ ,  $tl(G) = \min\{\lceil \frac{lc(G)}{3} \rceil, \max\{\lceil \frac{is(G)}{3} \rceil, \ell_p\}\}$  where  $is(G)$  (resp.,  $lc(G)$ ) is the size of a largest isometric (resp., of a largest) cycle in  $G$ . Moreover, we exhibit an example of series-parallel graphs for which it seems harder to link the treelength to the size of its largest (isometric) cycles. In our next results, we make use of the *nested ear-decompositions* [23] of series-parallel graphs. In Section 4, we design a  $\frac{3}{2}$ -approximation algorithm for computing the treelength of series-parallel graphs. In Section 5, our main result is that a series-parallel graph  $G$  has treelength at most two if and only if its largest isometric cycle has length at most 6 and  $G$  has

no *Dumbo graph* (see definition below) as an isometric subgraph. This characterization leads to a quadratic-time algorithm that decides if a series-parallel graph has treelength at most two. Finally, we conclude in Section 6 by discussing how our results may be generalized to compute treelength of series-parallel graphs.

## 2. Preliminaries

In this paper, we consider only undirected unweighted simple (without loops or parallel edges) graphs. A graph  $G = (V, E)$  is *connected* if, for every  $u, v \in V$ , there exists a path between  $u$  and  $v$  in  $G$ . We now only consider connected graphs. For any  $v \in V$ , let  $N_G(v)$  be the neighbors of  $v$  in  $G$  (i.e.,  $N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$ ) and let  $N_G(S)$  be the set of vertices in  $G$  adjacent to a vertex in  $S$  (i.e.,  $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$ ). The *distance*  $\text{dist}_G(u, v)$  in  $G = (V, E)$  between two vertices  $u, v \in V$  equals the minimum *length* (number of edges) of a path linking  $u$  and  $v$  in  $G$  (the subscript  $G$  is omitted when there is no ambiguity), and  $P_G(u, v)$  denotes any shortest  $u, v$ -path. The *diameter* of  $G$  is the maximum distance between its vertices, i.e.,  $\max_{u, v \in V} \text{dist}_G(u, v)$ . A *subgraph*  $H = (V(H) \subseteq V, E(H) \subseteq E \cap (V(H) \times V(H)))$  of  $G$  is *isometric* if  $\text{dist}_H(u, v) = \text{dist}_G(u, v)$  for every  $u, v \in V(H)$ , i.e., if  $H$  *preserves* the distances of  $G$ . Let  $\text{is}(G)$  be the largest size of an isometric cycle in  $G$ .

**Tree-decompositions.** A *tree-decomposition* of a graph  $G = (V, E)$  is a pair  $(T, \mathcal{X} = \{X_t \mid t \in V(T)\})$  such that  $T$  is a tree, and  $\mathcal{X}$  is a set of subsets (called *bags*) of vertices of  $G$ , indexing the nodes of  $T$  such that:

- $\bigcup_{t \in V(T)} X_t = V(G)$ ;
- for every  $\{u, v\} \in E(G)$ , there exists  $t \in V(T)$  such that  $u, v \in X_t$ ;
- for every  $v \in V(G)$ , the set  $\{t \in V(T) \mid v \in X_t\}$  induces a subtree of  $T$ .

We may further assume that  $(T, \mathcal{X})$  is *reduced*, i.e., no bag is included in another one. The *width* of  $(T, \mathcal{X})$  equals  $\max_{t \in V(T)} |X_t| - 1$ , i.e., the largest size of the bags of  $(T, \mathcal{X})$  minus one. The *treewidth*  $\text{tw}(G)$  of  $G$  is the minimum width of the (reduced) tree-decompositions of  $G$ . The *length*  $\ell(T, \mathcal{X})$  of  $(T, \mathcal{X})$  equals  $\max_{t \in V(T)} \ell(X_t) = \max_{t \in V(T)} \max_{u, v \in X_t} \text{dist}_G(u, v)$ , i.e., the maximum diameter (in  $G$ ) of its bags. The *treelength*  $\text{tl}(G)$  of  $G$  is the minimum length of the (reduced) tree-decompositions of  $G$ .

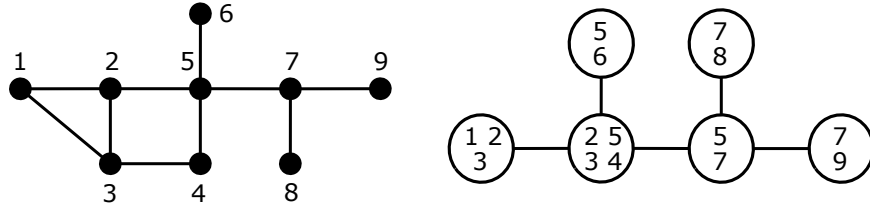


Figure 1: Example of a tree-decomposition  $(T, \mathcal{X})$  (right) of minimum length ( $\text{tl}(G) = 2$ ) of the graph  $G$  (left) where the bag  $\{2, 3, 4, 5\}$  has length 2 and all other bags have length 1.

In what follows, we will use the following lemma that follows from the fact that  $\text{tl}(C_n) = \lceil \frac{n}{3} \rceil$  for any  $n$ -vertex cycle  $C_n$  [16].

**Lemma 1.** [16] *Let  $G$  be any graph and  $H$  be any isometric subgraph of  $G$ . Then,  $\text{tl}(H) \leq \text{tl}(G)$ . In particular,  $\text{tl}(G) \geq \lceil \frac{\text{is}(G)}{3} \rceil$ .*

Given a connected graph  $G = (V, E)$ , a set  $S \subset V$  is a *separator* if  $G - S$  (obtained from  $G$  by removing the vertices of  $S$ ) is not connected. It is well known that in any reduced tree-decomposition, the intersection between two adjacent bags is a separator of the graph. The set  $S$  is a *clique-separator* of  $G$  if moreover the subgraph  $G[S]$  induced by  $S$  in  $G$  is a complete graph. It is easy to show that, for any graph  $G$  with a

clique-separator  $S$  and  $\mathcal{C}$  being the set of connected components of  $G \setminus S$ , then  $tw(G) = \max_{C \in \mathcal{C}} tw(G[C \cup S])$  and  $tl(G) = \max_{C \in \mathcal{C}} tl(G[C \cup S])$ . A graph  $G$  is called *prime* if it does not admit any clique-separator. Therefore, from now on, we will only consider prime graphs. In particular, we only consider 2-connected graphs, i.e., graphs with no separator of size one (Lemma 7 even considers graphs without clique-separators of size 2, i.e., without edge-separators). Note that computing the prime components of any planar graph can be done in linear-time [27].

**Series-parallel graphs.** An  $(s, t)$ -series-parallel graph is any graph (with two distinguished vertices  $s$  and  $t$ ) recursively defined as follows. An edge  $st$  is an  $(s, t)$ -series-parallel graph. Moreover, given an  $(s_1, t_1)$ -series-parallel graph  $G_1$  and an  $(s_2, t_2)$ -series-parallel graph  $G_2$ , an  $(s, t)$ -series-parallel graph  $G$  can be obtained from the disjoint union of  $G_1$  and  $G_2$  by identifying either:

**series composition:**  $t_1$  and  $s_2$  (in which case  $s = s_1$ , and  $t = t_2$ ) or,

**parallel composition:**  $s_1$  and  $s_2$  on the one hand, and  $t_1$  and  $t_2$  on the other hand (in which case  $s = s_1 = s_2$ , and  $t = t_1 = t_2$ ).

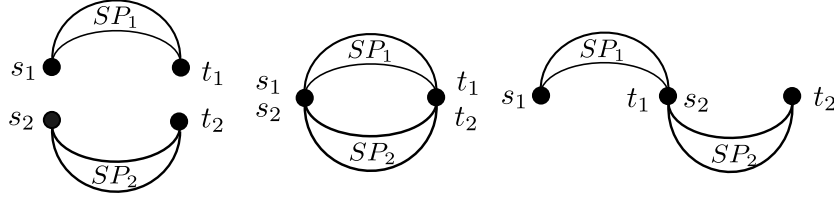


Figure 2: Series (right) and parallel (middle) composition of two series-parallel graphs (left)  $SP_1$  and  $SP_2$ .

A graph  $G = (V, E)$  is *series-parallel* if there are two vertices  $s, t \in V$  such that  $G$  is an  $(s, t)$ -series-parallel graph. It is well known that a graph is  $K_4$ -minor free, or equivalently has treewidth at most 2, if and only if its 2-connected components are series-parallel. Note that outerplanar graphs are precisely  $(K_4, K_{2,3})$ -minor free graphs and so 2-connected outerplanar graphs are included in the class of series-parallel graphs.

Note that, in any series-parallel graph  $G$ , a largest isometric cycle (and so  $is(G)$ ) can be computed in linear time by a simple dynamic programming algorithm (using a recursive sequence of compositions that can be obtained in linear time [24]).

**Ear-decompositions.** An *ear-decomposition* of a graph  $G = (V, E)$  is a partition  $(E_0, \dots, E_p)$  of  $E$  such that  $E_0$  induces a cycle in  $G$  and, for every  $1 \leq i \leq p$ ,  $E_i$  induces a path between two vertices  $a_i$  and  $b_i$  in  $G$ . Moreover,  $V(E_i) \cap V(G_{i-1}) = \{a_i, b_i\}$  where  $G_{i-1}$  is the subgraph induced by  $\bigcup_{j \leq i-1} V(E_j)$  (that is, the path induced by  $E_i$  is internally disjoint from  $V(E_0), \dots, V(E_{i-1})$ ). We say that  $a_i$  and  $b_i$  are the *attachment vertices* of  $E_i$  in  $G_{i-1}$  (note that  $\{a_i, b_i\}$  is a separator of  $G_i$ ). It is well known that a graph admits an ear decomposition if and only if it is 2-connected [9].

An ear decomposition is *nested* if moreover, for every  $1 \leq i \leq i' \leq p$ :

- the attachment vertices  $a_i$  and  $b_i$  of  $E_i$  appear in a previous ear  $E_j$ , with  $j < i$ , i.e., there exists  $j < i$  such that  $a_i, b_i \in V(E_j)$ , in which case we say that  $E_i$  is *attached* to  $E_j$ . Let  $j_i$  be the smallest index  $0 \leq j < i$  such that  $E_i$  is attached to  $E_j$ , and
- if two ears  $E_i$  and  $E_{i'}$  are both attached to some ear  $E_j$ , then either the path  $P_{E_j}(a_i, b_i)$  between  $a_i$  and  $b_i$  in  $E_j$  contains (not necessarily properly)  $P_{E_j}(a_{i'}, b_{i'})$ , or vice versa, or  $P_{E_j}(a_i, b_i)$  and  $P_{E_j}(a_{i'}, b_{i'})$  are internally vertex-disjoint. That is, two ears “do not cross” each other.

A graph is a 2-connected series-parallel graph if and only if it admits a nested ear decomposition [23]. It is easy to prove that we may further assume that  $E_0$  is a largest isometric cycle of  $G$  and that, for every  $1 \leq i \leq p$ ,  $|E(E_i)| \geq |E(P_{E_{j_i}}(a_i, b_i))|$ , i.e., equivalently that  $G_i$  is an isometric subgraph of  $G$  for every  $0 \leq i \leq p$ . A nested ear-decomposition satisfying the latter condition is called *isometric*.

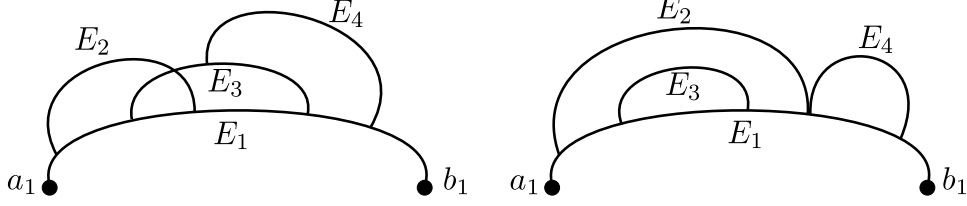


Figure 3: Example of not nested (left) and nested (right) ears.

**Lemma 2.** *For any 2-connected series-parallel graph  $G$ , an isometric nested ear decomposition starting from a largest isometric cycle of  $G$  can be computed in quadratic time.*

*Proof.* First, note that, by a remark above, a largest isometric cycle  $C$  of  $G$  (and its length) can be computed in linear time. Then an isometric nested ear decomposition of  $G$  can be computed in quadratic time as follows:

- Step  $E_0$ :  $G_0 = G[V(C)]$  and let  $E_0 = G_0$ .
- Step  $E_i$ , for  $1 \leq i \leq p$ : Let  $C_1, \dots, C_k$  be the  $k$  connected components of  $G - G_{i-1}$ . For any  $1 \leq j \leq k$ , consider the subgraph  $C^*$  induced by  $V(C_j) \cup V(N_{G_{i-1} \cup C_j}(C_j))$ . Note that  $|V(N_{G_{i-1} \cup C_j}(C_j))| = 2$  (otherwise there is a  $K_4$ -minor) and let  $V(N_{G_{i-1} \cup C_j}(C_j)) = \{a_i, b_i\}$ . Let  $E_i$  be a shortest path between  $a_i$  and  $b_i$  in  $C^*$  and  $G_i = G[V(G_{i-1}) \cup V(E_i)]$ .

Note that since we define  $E_i$  as a shortest path in  $C^*$ , it is impossible that an ear  $E_{i'}$  is attached to  $E_i$  such that  $|E(E_{i'})| < |E(P_{E_i}(a_{i'}, b_{i'}))|$ .  $\square$

### 3. Simple series-parallel graphs

This section is devoted to the simplest (including the cycles) subclass of 2-connected series-parallel graphs that we call the *melon* graphs. A *melon* graph is any graph  $G = (P_1, \dots, P_p)$  obtained from two vertices  $x$  and  $y$  by adding  $p \geq 2$  internally vertex-disjoint paths  $P_1, \dots, P_p$  between  $x$  and  $y$ . In what follows, let  $\ell_i = |E(P_i)|$  be the length of  $P_i$  for every  $1 \leq i \leq p$  and, w.l.o.g., let us assume that  $\ell_1 \geq \dots \geq \ell_p > 0$ . Note that a largest isometric cycle of  $G$  consists of  $P_1$  and  $P_p$  and so  $is(G) = \ell_1 + \ell_p$  and that a largest cycle consists of  $P_1$  and  $P_2$  and has size  $lc(G) = \ell_1 + \ell_2 \geq is(G)$ .

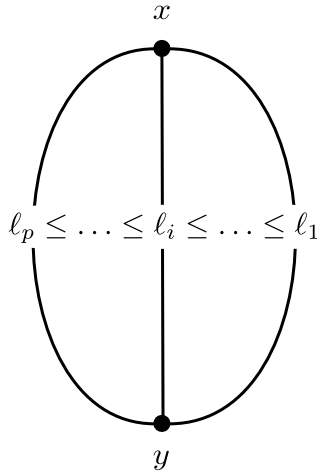


Figure 4: A melon graph.

**Theorem 1.** For any melon graph  $G = (P_1, \dots, P_p)$ ,  $tl(G) = \min\{\lceil \frac{lc(G)}{3} \rceil; \max\{\lceil \frac{is(G)}{3} \rceil, \ell_p\}\}$ .

*Proof.* Let us first show the upper bounds in each of the three cases:  $\ell_p = |E(P_p)| \leq \lceil \frac{is(G)}{3} \rceil$  (in which case we aim at proving that  $tl(G) = \lceil \frac{is(G)}{3} \rceil$ );  $\lceil \frac{is(G)}{3} \rceil \leq \ell_p \leq \lceil \frac{lc(G)}{3} \rceil$  (in which case  $tl(G) = \ell_p$ ); and  $\lceil \frac{lc(G)}{3} \rceil \leq \ell_p$  (in which case  $tl(G) = \lceil \frac{lc(G)}{3} \rceil$ ).

- First, let us assume that  $\ell_p \leq \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ . Let  $I_1 = \{1 \leq i < p \mid \ell_i > \lceil \frac{\ell_1 + \ell_p}{3} \rceil - \ell_p\}$  and let  $I_2 = \{1, \dots, p-1\} \setminus I_1$ . Note that, for any  $i \in I_2$ ,  $\ell_i = |E(P_i)| \leq \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ .

For any  $i \in I_1$ , let  $z_i$  be the vertex of  $P_i$  such that the subpath  $P'_i$  of  $P_i$  from  $x$  to  $z_i$  has length  $\lceil \frac{\ell_1 + \ell_p}{3} \rceil - \ell_p$  and does not pass through  $y$  (possibly  $z_i = x$ ). The path  $P_p \cup P'_i$  going from  $y$  to  $z_i$  and passing through  $x$  has length  $\lceil \frac{is(G)}{3} \rceil$ . For any  $i \in I_1$ , let  $P''_i = (P_i \setminus P'_i) \cup z_i$  and let  $\gamma_i$  be the central vertex of  $P''_i$ , i.e., such that  $dist_{P''_i}(\gamma_i, z_i) = \lfloor \frac{|E(P''_i)|}{2} \rfloor$ . Let  $Q_i$  (resp.  $Q'_i$ ) be the subpath of  $P''_i$  going from  $\gamma_i$  to  $z_i$  (resp., to  $y$ ). Note that  $|E(Q_i)| \leq |E(Q'_i)| = \lceil \frac{|E(P_i)| - |E(P'_i)|}{2} \rceil = \lceil \frac{\ell_i - (\lceil \frac{\ell_1 + \ell_p}{3} \rceil - \ell_p)}{2} \rceil = \lceil \frac{\ell_p + \ell_i - \lceil \frac{\ell_1 + \ell_p}{3} \rceil}{2} \rceil \leq \lceil \frac{\ell_p + \ell_1 - \lceil \frac{\ell_1 + \ell_p}{3} \rceil}{2} \rceil = \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ .

For any  $i \in I_2$ , let  $z_i = x$  and  $V(P'_i) = \{z_i\}$  and  $P''_i = P_i$ . Then  $Q_i$  and  $Q'_i$  are defined similarly as above. By definition of  $I_2$ ,  $|E(Q_i)| \leq |E(Q'_i)| \leq |E(P_i)| = \ell_i \leq \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ .

Let us build a tree-decomposition as follows. Start with a bag  $X_0 = V(P_p)$ . For every  $1 \leq i < p$ , add a bag  $X_i^1 = X_0 \cup V(P'_i)$  adjacent to  $X_0$ , then a bag  $C_i = \{z_i, \gamma_i, y\}$  adjacent to  $X_i^1$  and two bags  $X_i^2 = V(Q_i)$  and  $X_i^3 = V(Q'_i)$  both adjacent to  $C_i$ . By the previous paragraph, this is a tree-decomposition with length  $\lceil \frac{\ell_1 + \ell_p}{3} \rceil$  (see Figure 5).

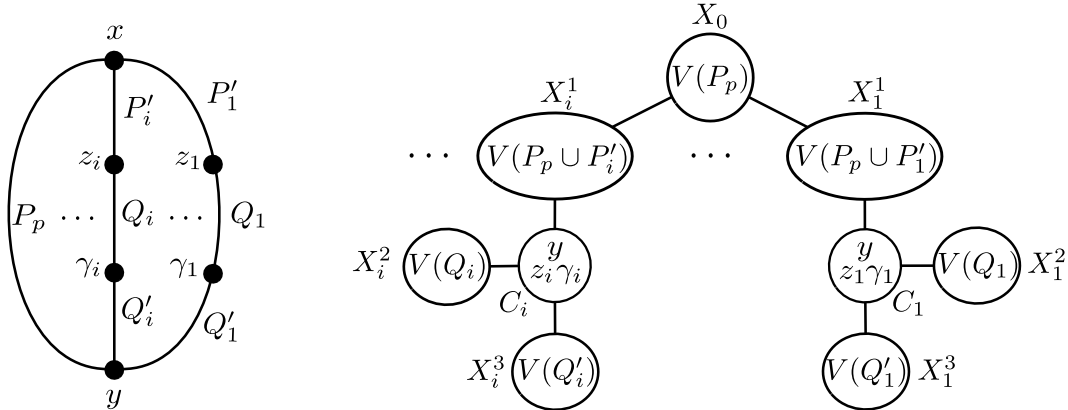


Figure 5: The graph  $G = (P_1, \dots, P_p)$  (left) and a tree-decomposition of length  $\lceil \frac{\ell_1 + \ell_p}{3} \rceil$  (right) when  $\ell_p \leq \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ .

- Let us assume that  $\lceil \frac{\ell_1 + \ell_p}{3} \rceil \leq \ell_p \leq \lceil \frac{\ell_1 + \ell_2}{3} \rceil$ . The tree-decomposition is obtained as in the previous case with the only difference that  $z_i = x$  for every  $1 \leq i \leq p$  (i.e.,  $P'_i = \{z_i\}$  for every  $i$ ). This is a tree-decomposition with length  $\ell_p$  (see Figure 6).
- Finally, let us consider the case when  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil < \ell_p$ . For every  $1 \leq i \leq p$ , let  $\gamma_i$  be the vertex of  $P_i$  at distance  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil < \ell_p$  from  $x$  and let  $P'_i$  be the subpath of  $P_i$  which is a shortest path from  $x$  to  $\gamma_i$ . For every  $1 \leq i, j \leq p$ ,  $dist_G(\gamma_i, \gamma_j) \leq \lceil \frac{\ell_1 + \ell_2}{3} \rceil$  (via the shortest path going through  $y$ ). Let  $Q$  be the subtree induced by  $\{\gamma_1, \dots, \gamma_p\}$  and the connected component of  $G - \{\gamma_1, \dots, \gamma_p\}$  that contains  $y$ . Let us build a tree-decomposition as follows. Start with a bag  $X_0 = \{x, \gamma_1, \dots, \gamma_p\}$ . For every  $1 \leq i \leq p$ , add a bag  $X_i = V(P'_i)$  adjacent to  $X_0$ . Finally, add a bag  $X_{p+1} = V(Q)$  adjacent to  $X_0$ . This is a tree-decomposition with length  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil$  (see Figure 7).

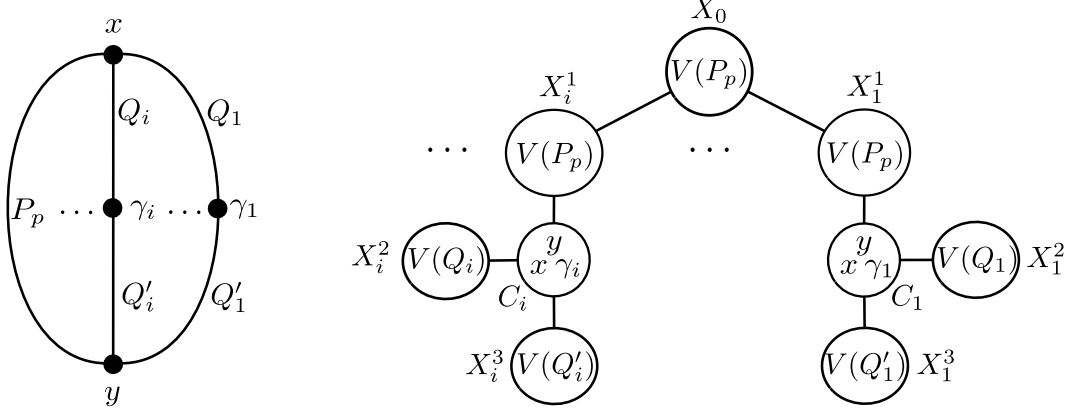


Figure 6: The graph  $G = (P_1, \dots, P_p)$  (left) and a tree-decomposition of length  $\ell_p$  (right) when  $\lceil \frac{\ell_1 + \ell_p}{3} \rceil \leq \ell_p \leq \lceil \frac{\ell_1 + \ell_2}{3} \rceil$ .

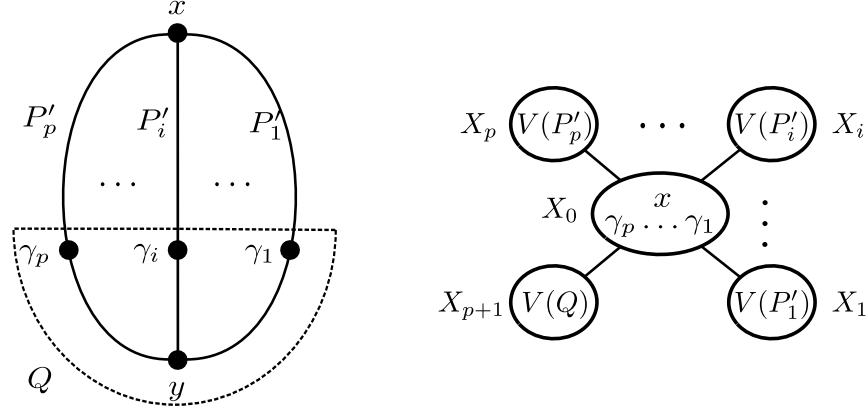


Figure 7: The graph  $G = (P_1, \dots, P_p)$  (left) and a tree-decomposition of length  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil$  (right) when  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil < \ell_p$ .

Now, let us prove the lower bounds. By Lemma 1, in all cases,  $tl(G) \geq \lceil \frac{is(G)}{3} \rceil = \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ . Thus, if  $\ell_p \leq \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ ,  $tl(G) = \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ . We now prove that, if  $\ell_p > \lceil \frac{\ell_1 + \ell_p}{3} \rceil$ , then  $tl(G) \geq \min\{\ell_p, \lceil \frac{\ell_1 + \ell_2}{3} \rceil\}$ . For the purpose of contradiction, let us assume that  $tl(G) < k$  for some  $k \leq \min\{\ell_p, \lceil \frac{\ell_1 + \ell_2}{3} \rceil\}$  and consider a tree-decomposition  $(T, \mathcal{B})$  of  $G$  with minimum length. Let  $\alpha$  (resp.  $\beta$ ) be the vertex at distance  $k$  from  $x$  on  $P_1 - y$  (resp. on  $P_2 - y$ ). Note that  $\alpha$  and  $\beta$  are well defined since either  $\ell_1 \geq \ell_2 \geq \ell_p > \lceil \frac{\ell_1 + \ell_2}{3} \rceil \geq k$  or  $\lceil \frac{\ell_1 + \ell_2}{3} \rceil \geq \ell_p > \lceil \frac{\ell_1 + \ell_p}{3} \rceil$  and so  $\ell_1 \geq \ell_2 > \ell_p \geq k$ . Since  $k \leq \min\{\ell_p, \lceil \frac{\ell_1 + \ell_2}{3} \rceil\}$ ,  $dist_G(\alpha, \beta) \geq k$  and, therefore, no bag of  $(T, \mathcal{B})$  can contain at least two of  $x, \alpha$  and  $\beta$ . Let  $B_x, B_\alpha$  and  $B_\beta$  be three bags containing  $x, \alpha$  and  $\beta$  respectively. There are several cases to be considered.

- First, let us assume that  $B_x$  is on the path of  $T$  between  $B_\alpha$  and  $B_\beta$ . Therefore,  $\alpha$  and  $\beta$  must be in different connected components of  $G - B_x$ . Hence,  $B_x$  must contain a vertex of the path from  $\alpha$  to  $\beta$  going through  $y$  (and not through  $x$ ). Every vertex of this path is at distance at least  $k$  from  $x$ , a contradiction.
- Second, assume that  $B_\alpha$  is on the path of  $T$  between  $B_x$  and  $B_\beta$ . Therefore,  $x$  and  $\beta$  must be in different connected components of  $G - B_\alpha$ . Hence,  $B_\alpha$  must contain a vertex of the path from  $x$  to  $\beta$  not going through  $y$ . Every vertex of this path is at distance at least  $k$  from  $\alpha$ , a contradiction. Note that the same statement holds if  $B_\beta$  is between  $B_x$  and  $B_\alpha$ .



- Finally, assume that there is a bag  $B$  such that  $B_x$ ,  $B_\alpha$  and  $B_\beta$  are each in distinct connected component of  $T - B$ . The set  $B$  must separate  $x$ ,  $\alpha$  and  $\beta$ . Therefore,  $B$  must contain a vertex in each of the three paths from  $x$  to  $\alpha$  (not going through  $y$ ), from  $x$  to  $\beta$  (not going through  $y$ ) and from  $\alpha$  to  $\beta$  going through  $y$  (and not through  $x$ ). Since the cycle  $P_1 \cup P_2$  containing these three vertices has length at least  $3k \leq \ell_1 + \ell_2$  and  $k \leq \ell_p$ , at least two of these three vertices are at distance at least  $k$ , a contradiction.

□

The above result (involving 2-connected series-parallel graphs with at most 2 vertices with degree larger than 2) and the following family of series-parallel graphs (with only four vertices with degree larger than 2) give us the impression that the treelength of series-parallel graphs cannot be expressed by a “nice” formula (such as in the case of outerplanar graphs). Let  $p \in \mathbb{N}^*$ . Let  $G_p$  be the graph obtained from a cycle of length  $12p$  and let  $a, b, c, d$  be four distinct vertices of it such that  $\text{dist}(a, b) = \text{dist}(c, d) = 4p$  and  $\text{dist}(a, d) = \text{dist}(b, c) = 2p$ . Then, add one path of length  $8p$  from  $a$  to  $b$  and one path of length  $8p$  from  $c$  to  $d$ . Note that  $\text{is}(G_p) = 12p$ , that its largest cycle (not isometric) has length  $20p$ , all other cycles (not isometric) have length  $16p$  and that its maximal paths with internal vertices of degree 2 have length  $2p, 4p$  or  $8p$ . By similar arguments as in the previous proof, it can be shown that  $\text{tl}(G_p) = 5p$  which seems not directly related to the invariants previously mentioned.

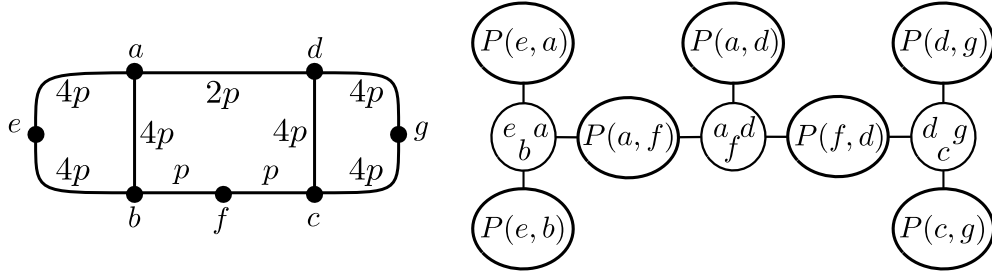


Figure 8: The graph  $G_p$  (left) and a tree-decomposition of length  $5p$  (right).

**Lemma 3.** For any  $p \in \mathbb{N}^*$ ,  $\text{tl}(G_p) = 5p$ .

*Proof.* Let us build a tree-decomposition as follows (see Figure 8). Start with a bag  $X_0 = \{a, b, e\}$ . Add two bags adjacent to  $X_0$  containing respectively the shortest path between  $a$  and  $e$ , and the shortest path between  $b$  and  $e$ . Then, add a bag  $X_1$ , adjacent to  $X_0$ , containing the shortest path between  $a$  and  $b$ , and the shortest path between  $b$  and  $f$  (the diameter of this bag is  $5p$ ). Add the bag  $X_2 = \{a, d, f\}$  adjacent to  $X_1$ . Add a bag, adjacent to  $X_2$ , containing the shortest path from  $a$  to  $d$ . Add the bag  $X_3$ , adjacent to  $X_2$ , containing the shortest path from  $d$  to  $f$  (this bag has also diameter  $5p$ ). Then, add the bag  $X_4 = \{d, c, g\}$  adjacent to  $X_3$ . Finally, add the bags, adjacent to  $X_4$ , containing respectively the shortest path between  $d$  and  $g$ , and between  $c$  and  $g$ . This tree-decomposition has length  $5p$ .

To prove the lower bound, let us consider the vertices  $a$ ,  $f$  and  $g$ . In any tree-decomposition of length  $< 5p$ , no bag can contain at least two of these vertices. Let  $B_a$ ,  $B_f$  and  $B_g$  be some bags containing respectively  $a$ ,  $f$  and  $g$  in such a decomposition (that we suppose to exist for the purpose of contradiction). There are several cases to be considered.

- First, let us assume that  $B_f$  is on the path of  $T$  between  $B_a$  and  $B_g$ . Therefore,  $a$  and  $g$  must be in different connected components of  $G_p - B_f$ . Hence,  $B_f$  must contain a vertex of the shortest path from  $a$  to  $g$  going through  $d$ . Every vertex of this path is at distance at least  $5p$  from  $f$ , a contradiction.
- Second, assume that  $B_a$  is on the path of  $T$  between  $B_f$  and  $B_g$ . Therefore,  $f$  and  $g$  must be in different connected components of  $G_p - B_a$ . Hence,  $B_a$  must contain a vertex of the shortest path from  $g$  to  $f$  going through  $c$ . Every vertex of this path is at distance at least  $5p$  from  $a$ , a contradiction.

- Then, assume that  $B_g$  is on the path of  $T$  between  $B_f$  and  $B_a$ . Therefore,  $f$  and  $a$  must be in different connected components of  $G_p - B_g$ . Hence,  $B_g$  must contain a vertex of the shortest path from  $a$  to  $f$  going through  $b$ . Every vertex of this path is at distance at least  $5p$  from  $g$ , a contradiction.
- Finally, assume that there is a bag  $B_0$  such that  $B_a$ ,  $B_g$  and  $B_f$  are each in distinct connected components of  $T - B_0$ . The set  $B_0$  must separate  $a$ ,  $f$  and  $g$ . There are several cases to be considered.
  - Assume first that  $B_0$  contains a vertex  $v$  of the shortest path between  $b$  and  $f$  (to separate  $a$  from  $f$ ). The bag  $B_0$  must also contain a vertex  $u$  of the shortest path from  $g$  to  $a$  (containing  $d$ ). If  $\text{dist}_{G_p}(v, u) < 5p$ , then  $\text{dist}_{G_p}(a, u) \leq p$ . Finally,  $B_0$  must contain a vertex on the shortest path from  $g$  to  $f$  (going through  $c$ ) which are all at distance at least  $5p$  from  $u$ , a contradiction.
  - Otherwise,  $B_0$  must contain a vertex  $v$  of the path from  $b$  to  $a$  in  $G_p$  (containing  $e$ ). The bag  $B_0$  must also contain a vertex  $u$  of the path between  $a$  and  $g$  (containing  $d$ ) and a vertex  $w$  of the path from  $f$  to  $g$  (through  $c$ ). Note that if  $\text{dist}_{G_p}(u, w) < 5p$ , then  $\text{dist}_{G_p}(a, u) > p$ . Hence,  $v$  has to be in the path between  $a$  and  $e$  (otherwise  $\text{dist}_{G_p}(u, v) > 5p$ ). Therefore,  $\text{dist}_{G_p}(v, w) > 5p$ , a contradiction.

□

#### 4. Approximation algorithm

This section shows that, even if it is still unknown whether computing the treelength of series-parallel graphs can be done in polynomial time, there exists an efficient approximation algorithm using ear-decompositions.

**Theorem 2.** *For any series-parallel graph  $G$ , a tree-decomposition of  $G$  with length at most  $\frac{3}{2} \cdot \text{tl}(G)$  can be computed in quadratic time.*

*Proof.* By a remark in Section 2, it is sufficient to consider 2-connected graphs. Let  $G$  be a 2-connected series-parallel graph. It follows from Lemma 1 that  $\text{tl}(G) \geq \lceil \frac{\text{is}(G)}{3} \rceil$ . Let us show how to compute a tree-decomposition of length at most  $\lceil \frac{\text{is}(G)}{2} \rceil$ . Intuitively, every bag will consist of a subgraph of an isometric cycle, and so, for every  $x$  and  $y$  in a bag, they will belong to an isometric cycle  $C$  and  $\text{dist}_G(x, y) = \text{dist}_C(x, y) \leq \lfloor \frac{|E(C)|}{2} \rfloor \leq \lfloor \frac{\text{is}(G)}{2} \rfloor$ . Let us consider an isometric nested ear decomposition  $\mathcal{E} = (E_0, \dots, E_p)$  starting with a largest isometric cycle  $E_0$  for  $G$  (it exists and can be computed in quadratic time by Lemma 2). For any  $1 \leq i \leq p$ , let  $a_i$  and  $b_i$  be the endpoints of  $E_i$  as defined in the proof of Lemma 2. Let us build the tree-decomposition as follows. Start with a bag containing  $V(E_0)$ . Then, for every  $1 \leq i \leq p$ , let us add a bag consisting of  $V(E_i)$  adjacent to the bag containing  $V(E_{j_i})$  where  $0 \leq j_i < i$  is the minimum index such that  $E_{j_i}$  contains  $a_i$  and  $b_i$ , the two endpoints of  $E_i$ . Recall that, for any  $0 \leq j \leq p$ ,  $G_j$  is the subgraph induced by  $E_0, \dots, E_j$ . Since  $\mathcal{E}$  is an isometric nested ear decomposition, i.e.,  $G_{i-1}$  is an isometric subgraph of  $G$ , for any  $x, y \in G_{i-1}$ ,  $d_{G_{i-1}}(x, y) \leq d_G(x, y)$ . Therefore, the cycle  $C'$  that consists of  $E_i$  and a shortest path between  $a_i$  and  $b_i$  in  $G_{i-1}$  is isometric in  $G_i$ . Since, moreover,  $G_i$  is an isometric subgraph of  $G$ ,  $C'$  is an isometric cycle of  $G$ . Therefore, the length of the tree-decomposition is at most  $\lfloor \frac{\text{is}(G)}{2} \rfloor \leq \lceil \frac{\text{is}(G)}{3} \rceil \cdot \frac{3}{2} \leq \frac{3}{2} \cdot \text{tl}(G)$ . □

#### 5. Characterization of series-parallel graphs with treelength 2

Before stating our main theorem, a last ingredient is required, namely the *Dumbo* graphs. A *Dumbo* graph is any graph built as follows (see Figure 9). Start with a cycle  $C_0 = (v_0, \dots, v_5)$  of order 6, and add a path  $R$  of length (number of edges) at least 3 and at most 4 between  $v_0$  and  $v_2$  and a path  $L$  of length at least 3 and at most 4 between  $v_3$  and  $v_5$ . Note that a Dumbo graph is series-parallel.

This section is devoted to prove the following theorem which highly relies on the ear-decompositions of series-parallel graphs.

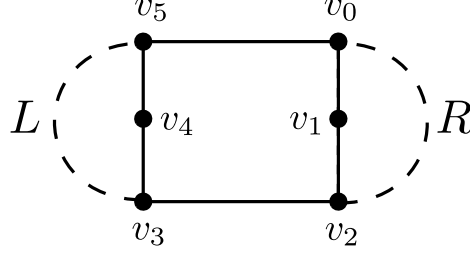


Figure 9: A Dumbo graph of treelength 3 with two ears  $L$  and  $R$  such that  $3 \leq |E(L)| \leq 4$  and  $3 \leq |E(R)| \leq 4$ .

**Theorem 3.** *For any series-parallel graph  $G$ ,  $tl(G) \leq 2$  if and only if  $is(G) \leq 6$  and  $G$  does not contain a Dumbo graph as an isometric subgraph.*

*Moreover, there is a quadratic-time algorithm that either computes a tree-decomposition of length at most 2 of  $G$  or exhibits a certificate that  $tl(G) > 2$  (a large isometric cycle or an isometric Dumbo subgraph).*

The “only if” part of Theorem 3 follows from Lemma 1 ( $tl(G) \geq \lceil \frac{is(G)}{3} \rceil$ ) and from Lemma 5 whose proof uses Lemma 4.

**Lemma 4.** *Let  $G$  be a graph and  $C$  be any isometric cycle of length  $\ell$  in  $G$ . In any tree-decomposition  $(T, \mathcal{X})$  of  $G$  with length at most  $\lceil \frac{\ell}{3} \rceil$ , there exists a bag  $X \in \mathcal{X}$  containing three vertices  $a, b, c \in V(C)$  such that  $\lceil \frac{\ell}{3} \rceil = dist(a, b) \geq dist(a, c) \geq \lfloor \frac{\ell}{3} \rfloor$  and  $dist(a, c) \geq dist(c, b) \geq \lfloor \frac{\ell}{3} \rfloor - 1$ .*

*Proof.* Let  $(T, \mathcal{X})$  be any tree-decomposition of  $G$  of length at most  $\lceil \frac{\ell}{3} \rceil$ . Note that, by Lemma 1,  $(T, \mathcal{X})$  has length exactly  $\lceil \frac{\ell}{3} \rceil$ . Since every edge must appear in some bag, there must be bags containing at least two vertices of  $C$ . For every  $X \in \mathcal{X}$  with  $|X \cap V(C)| \geq 2$ , let  $d(X) = \max_{u, v \in X \cap V(C)} dist(u, v)$ . Let  $X$  be a bag maximizing  $d(X)$  and  $a, b \in X \cap V(C)$  with  $dist(a, b) = d(X)$ . Since  $d(X) \leq \ell(X) \leq \ell(T, \mathcal{X})$ , then  $dist(a, b) \leq \lceil \frac{\ell}{3} \rceil$ . Moreover, since the restriction  $(T, \mathcal{X} \cap V(C))$  of  $(T, \mathcal{X})$  to  $C$  is a tree-decomposition of  $C$ , the length of  $(T, \mathcal{X} \cap V(C))$  is at least  $\lceil \frac{\ell}{3} \rceil$ . This implies that  $d(X) \geq \lceil \frac{\ell}{3} \rceil$ . Therefore,  $dist(a, b) = \lceil \frac{\ell}{3} \rceil$ .

Let  $P$  be the path in  $C$  between  $a$  and  $b$  of length  $\ell - dist(a, b)$ , and let  $c \in V(P)$  be such that  $0 \leq dist(a, c) - dist(b, c) \leq 1$ . Since  $dist(a, b) = \lceil \frac{\ell}{3} \rceil$ , we get that  $|E(P)| = \ell - \lceil \frac{\ell}{3} \rceil$  and since  $dist(b, c) = \lfloor \frac{|E(P)|}{2} \rfloor$  and  $dist(a, c) = \lceil \frac{|E(P)|}{2} \rceil$ , it follows that  $\lceil \frac{\ell}{3} \rceil \geq dist(a, c) \geq dist(b, c) = \lfloor \frac{\ell}{3} \rfloor$ .

Hence, if  $c \in X$ ,  $a, b, c$  and  $X$  satisfy the statement.

Thus, let us assume that no bag contains  $a, b$  and  $c$ . Let  $Y$  be a bag containing  $c$  (which exists by the properties of a tree-decomposition) that is closest to  $X$  in  $T$  and let  $X'$  be a bag containing  $a$  and  $b$  that is closest to  $Y$ . Let  $X''$  be the bag adjacent to  $X'$  on the path between  $X'$  and  $Y$  in  $T$  (possibly  $X'' = Y$ ) and let  $Z = X' \cap X''$ . Note that  $c \notin Z$  and at least one of  $a$  and  $b$  is not in  $Z$  (otherwise, it would contradict either the fact that  $X'$  is closest to  $Y$  or that no bag contains  $\{a, b, c\}$ ).

Let us first assume that  $b \notin Z$ . Since  $Z$  is the intersection between two adjacent bags,  $Z$  must separate  $b$  and  $c$ . Hence, there is a vertex  $u$  between  $b$  and  $c$  in  $P$  that belongs to  $Z$ . Note that  $dist(a, u) = dist(a, c) + dist(c, u)$  since  $dist(a, u) \leq dist(a, b) = d(X) = d(X')$  by the maximality of  $d(X)$ . Therefore  $dist(a, c) < \lceil \frac{\ell}{3} \rceil$ , otherwise,  $d(a, u) \geq \lceil \frac{\ell}{3} \rceil + 1$ , a contradiction to the length of  $(T, \mathcal{X})$ . Hence,  $dist(a, c) = dist(b, c) = \lfloor \frac{\ell}{3} \rfloor$ . It follows that  $\ell \equiv 1 \pmod{3}$ . Therefore,  $dist(a, u) = \lfloor \frac{\ell}{3} \rfloor + 1 = \lceil \frac{\ell}{3} \rceil$  and  $dist(b, u) = \lfloor \frac{\ell}{3} \rfloor - 1$ . Hence,  $a, b, u$  and  $X'$  satisfy the statement.

Let us assume now that  $a \notin Z$ . Since  $Z$  is the intersection between two adjacent bags,  $Z$  must separate  $a$  and  $c$ . Hence, there is a vertex  $u$  between  $a$  and  $c$  in  $P$  that belongs to  $Z$ . We claim that  $a, b$  and  $u$  are the required vertices. Indeed, by the maximality of  $d(X)$ ,  $d(X) \geq d(X')$ . Thus  $dist(a, b) \geq dist(u, b)$  and so the shortest path between  $u$  and  $b$  in  $C$  goes through  $c$ . Hence  $dist(u, b) > dist(b, c) = \lfloor \frac{\ell}{3} \rfloor$  and so  $dist(u, b) = \lceil \frac{\ell}{3} \rceil$  (because  $dist(u, b) \leq \ell(X') \leq \lceil \frac{\ell}{3} \rceil$ ). So,  $dist(u, b) = dist(b, c) + 1$  and  $dist(a, u) = dist(a, c) - 1$ . Since

$\lceil \frac{\ell}{3} \rceil \geq \text{dist}(a, c) \geq \lfloor \frac{\ell}{3} \rfloor$ , then  $\lceil \frac{\ell}{3} \rceil - 1 \geq \text{dist}(a, u) \geq \lfloor \frac{\ell}{3} \rfloor - 1$ . This implies that  $a, b, u$  and  $X'$  satisfy the statement.  $\square$

**Lemma 5.** *If a series-parallel graph  $G$  contains a Dumbo graph as an isometric subgraph, then  $\text{tl}(G) > 2$ .*

*Proof.* Let  $G = (V, E)$  be any series-parallel graph containing a Dumbo graph  $D = (C_0, R, L)$  as an isometric subgraph. For the purpose of contradiction, let us assume that  $G$  admits a tree-decomposition  $(T, \mathcal{X})$  of length at most 2. By Lemma 4, there must be a bag  $X \in \mathcal{X}$  containing  $\{v_0, v_2, v_4\}$  or  $\{v_1, v_3, v_5\}$ . By symmetry, let us assume that  $\{v_0, v_2, v_4\} \subseteq X$ . Let  $z$  be a vertex of  $L - \{v_5, v_3\}$  such that  $|\text{dist}(z, v_5) - \text{dist}(z, v_3)| \leq 1$ . Note that  $\text{dist}(z, v_5), \text{dist}(z, v_3) \geq 1$  and  $\max\{\text{dist}(z, v_5), \text{dist}(z, v_3)\} \geq 2$ . Moreover, because  $G$  is series-parallel, every path from  $z$  to  $v_0, v_2$  or  $v_4$  goes through  $v_3$  or  $v_5$  (otherwise, there would be a  $K_4$  minor). Note also that no bag contains  $\{v_0, v_2, v_4, z\}$  since  $z$  is at distance at least 3 from some of  $v_0, v_2, v_4$ .

Let  $Y$  be the bag containing  $z$  that is closest to  $X$ , and let  $X'$  be the bag containing  $v_0, v_2, v_4$  that is closest to  $Y$ . If  $X'Y \in E$ , let  $Z = X' \cap Y$ , otherwise, let  $Z'$  be the neighbor of  $X'$  on the path between  $X'$  and  $Y$  in  $T$  and let  $Z = Z' \cap X'$ . Note that  $z \notin Z$  and at least one of  $v_0, v_2$  and  $v_4$  is not in  $Z$  (otherwise, it would contradict either the fact that  $X'$  is closest to  $Y$ , that  $Y$  is closest to  $X$  or that no bag contains all  $v_0, v_2, v_4$  and  $z$ ). Let  $W = \{v_0, v_2, v_4\} \setminus Z$ . Since  $Z$  is the intersection between two adjacent bags,  $Z$  must separate every  $w \in W$  from  $z$ . There are several cases to be considered depending on which vertices among  $v_0, v_2$  and  $v_4$  are not in  $Z$ :

- If  $v_2$  belongs to  $Z$ , then  $W \subseteq \{v_0, v_4\}$ . Hence, there must be  $u$  in the  $z$ - $v_5$  subpath of  $L$  that is in  $Z$  if  $\{v_0, v_4\} \cap W \neq \emptyset$  (which is the case by the previous assumptions). Since  $z \notin Z$ ,  $u \neq z$  and  $\text{dist}(u, v_3) \geq 2$  and then  $\text{dist}(u, v_2) \geq 3$ . Therefore, there is no tree-decomposition of length 2 with  $v_2$  in  $Z$ .
- If  $v_0$  belongs to  $Z$ , then  $W \subseteq \{v_2, v_4\}$ . Hence, there must be  $v$  in the  $z$ - $v_3$  subpath of  $L$  that is in  $Z$  if  $\{v_2, v_4\} \cap W \neq \emptyset$  (which is the case by the previous assumptions). Since  $z \notin Z$ ,  $v \neq z$  and  $\text{dist}(v, v_5) \geq 2$  and then  $\text{dist}(v, v_0) \geq 3$ . Therefore, there is no tree-decomposition of length 2 with  $v_0$  in  $Z$ .
- Finally, since  $v_0, v_2 \notin Z$ , we have that  $v_0, v_2 \in W$ . Hence, there must be  $u$  in the  $z$ - $v_5$  subpath of  $L$  that is in  $Z$  and there must be  $v$  in the  $z$ - $v_3$  subpath of  $L$  that is in  $Z$ . Since  $z \notin Z$ ,  $v \neq z$ ,  $u \neq z$ ,  $\text{dist}(u, v_3) \geq 2$  and  $\text{dist}(v, v_5) \geq 2$  and then  $\text{dist}(v, v_0) \geq 3$  and  $\text{dist}(u, v_2) \geq 3$ . Therefore, there is no tree-decomposition of length 2 with  $v_0, v_2$  in  $W$ .

$\square$

Note that the previous lemma implies that  $\text{tl}(D) \geq 3$  when  $D = (C_0, L, R)$  is a dumbo graph but it is easy to show that  $\text{tl}(D) \leq 3$  since it admits the decomposition with the three bags  $V(L), V(C_0)$  and  $V(R)$ . It follows:

**Corollary 1.** *Let  $D$  be a Dumbo graph. Then,  $\text{tl}(D) = 3$ .*

The “if” part of Theorem 3 follows from Lemma 7 whose proof describes the algorithm. Lemma 6 will be used in the proof of Lemma 7 to deal with the case of ears of length 2.

Let us first give an intuition of the algorithm. It takes an isometric ear decomposition  $\mathcal{E} = (E_0, \dots, E_p)$  of  $G$  as input. Let  $G_i$  be the subgraph induced by the first  $i$  ears. The algorithm first checks that  $\text{is}(G) \leq 6$  (otherwise  $\text{tl}(G) > 2$  by Lemma 1). Then, it proceeds by induction on the number of ears and, in polynomial time, either computes a tree-decomposition of length 2, or exhibits a Dumbo graph as an isometric subgraph. Informally, let us assume by induction on  $i$  that, for  $0 \leq i < p$ , our algorithm has computed a tree-decomposition  $(T^i, \mathcal{X}^i)$  of  $G_i$  of length at most 2 and such that, for every ear  $E_j$ ,  $j > i$ , of length at least 3 and whose attachment vertices are in  $G_i$ , these attachment vertices are in some bag of  $(T^i, \mathcal{X}^i)$ . If  $E_{i+1}$  is of length 2, we extend the tree-decomposition  $(T^i, \mathcal{X}^i)$  “directly” to a tree-decomposition  $(T^{i+1}, \mathcal{X}^{i+1})$  (with the desired properties) of  $G_{i+1}$  (using Lemma 6). Otherwise, by a case analysis on the length  $|E_{i+1}|$  of  $E_{i+1}$  and  $\text{is}(G)$  (the proof of Lemma 7 is mainly dedicated to that part), we show that either  $(T^i, \mathcal{X}^i)$  can be

extended to a decomposition  $(T^{i+1}, \mathcal{X}^{i+1})$  of  $G_{i+1}$  (with the desired properties), or a Dumbo graph can be identified. The main difficulty is that, in the latter case (when  $|E_{i+1}| > 2$ ), we are sometimes forced to deal simultaneously with a subset of new ears.

Recall that, given a nested ear decomposition  $\mathcal{E} = (E_i)_{0 \leq i \leq p}$  of a graph  $G$ ,  $j_i$  denotes the smallest index such that  $E_{j_i}$  contains both endpoints of  $E_i$  for every  $0 \leq i \leq p$ .

**Lemma 6.** *Let  $G$  be any 2-connected series-parallel graph without clique-separators, with an isometric nested ear decomposition  $\mathcal{E} = (E_i)_{0 \leq i \leq p}$ . Let  $(T', \mathcal{X}')$  be a tree-decomposition, with length at least 2, of the subgraph  $G_j$  of  $G$  induced by  $E_0, \dots, E_j$  and let  $E_i$  be such that  $1 \leq j_i \leq j < i \leq p$  and  $|E_i| = 2$ , i.e.,  $E_i$  is an ear of length 2 not in  $G_j$  but both its endpoints are in  $G_j$ . Then, there exists a tree-decomposition  $(T, \mathcal{X})$  of  $G_j \cup E_i$  with the same length and such that, for every  $B' \in \mathcal{X}'$ , there exists  $B \in \mathcal{X}$  such that  $B' \subseteq B$ .*

*Proof.* Note that, by hypothesis, both endpoints of  $E_i$  belong to  $G_j$  since they belong to  $E_{j_i}$ . Let us first suppose that the endpoints of  $E_i$  are in a same bag  $B$  of  $(T', \mathcal{X}')$ . Then, the tree-decomposition obtained from  $(T', \mathcal{X}')$  by adding a bag  $V(E_i)$  adjacent to  $B$  satisfies the statement of the lemma.

Let us now consider the case where no bag of  $(T', \mathcal{X}')$  contains both the endpoints  $a_i$  and  $b_i$  of  $E_i$ . Let  $X \in \mathcal{X}'$  and  $Y \in \mathcal{X}'$  be such that  $a_i \in X$ ,  $b_i \in Y$  and the distance in  $T$  between two such bags is minimum.

Note that, because  $G$  has no edge-separator and because the ears are added in isometric order (i.e.,  $2 \leq |E(P_{E_{j_i}}(a_i, b_i))| \leq |E(E_i)| = 2$ ),  $a_i$  and  $b_i$  must have common neighbors in  $G_j$ . Note also that, because  $G$  is series-parallel (in particular, the ears are nested) without clique-separators, then every common neighbor  $w$  of  $a_i$  and  $b_i$  satisfies  $N(w) = \{a_i, b_i\}$ . Indeed, if  $w$  admits a neighbor  $u$  different from  $a_i$  and  $b_i$ , the edge  $\{u, w\}$  must belong to an ear whose attachment vertices are  $w$  and either  $a_i$  or  $b_i$  (because the decomposition is nested), which would imply that either  $\{a_i, w\}$  or  $\{w, b_i\}$  is an edge-separator, a contradiction.

Since  $(T', \mathcal{X}')$  is a tree-decomposition, every bag  $W$  on the  $X$ - $Y$  path in  $T'$  must separate  $X \setminus Y$  from  $Y \setminus X$ . In particular,  $N_{G_j}(a_i) \cap N_{G_j}(b_i) \subseteq W$ . Similarly,  $N_{G_j}(a_i) \cap N_{G_j}(b_i) \subseteq X$  and  $N_{G_j}(a_i) \cap N_{G_j}(b_i) \subseteq Y$ . Let  $v$  be the common neighbor of  $a_i$  and  $b_i$  in  $E_i$ . Then, adding  $v$  to every bag  $W$  on the  $X$ - $Y$  path in  $T'$  (including  $X$  and  $Y$ ) gives the desired decomposition. This is clearly a tree-decomposition  $(T, \mathcal{X})$  of  $G_j \cup E_i$  and, for every  $B' \in \mathcal{X}'$ , there exists  $B \in \mathcal{X}$  such that  $B' \subseteq B$ . We only need to prove that its length is at most  $\ell(T', \mathcal{X}')$ . Let  $W'$  be any bag on the  $X$ - $Y$  path in  $T'$  (including  $X$  and  $Y$ ) and let  $W = W' \cup \{v\}$ . Let  $v' \in W'$  and let  $w \in N(a_i) \cap N(b_i) \cap V(E_{j_i})$ . Since  $w, v' \in W' \subseteq W$ , then  $\text{dist}(w, v') \leq \ell(W')$ . Moreover,  $\text{dist}(w, v') = \min\{\text{dist}(w, a_i) + \text{dist}(a_i, v'), \text{dist}(w, b_i) + \text{dist}(b_i, v')\} = \min\{\text{dist}(v, a_i) + \text{dist}(a_i, v'), \text{dist}(v, b_i) + \text{dist}(b_i, v')\} = \text{dist}(v, v')$  (since,  $N(w) = N(v) = \{a_i, b_i\}$ ). Hence,  $\ell(W) \leq \ell(T', \mathcal{X}')$  for every such bag  $W$ .  $\square$

Some notations are still needed. Let  $G$  be a 2-connected series-parallel graph with an isometric nested ear decomposition  $\mathcal{E} = (E_i)_{0 \leq i \leq p}$  such that  $E_0$  is a largest isometric cycle of  $G$ . Recall that  $a_i$  and  $b_i$  denote the endpoints of  $E_i$  ( $a_i, b_i \in V(G_{i-1})$ ). Let  $\ell_i = |E(E_i)|$  and  $d_i = \text{dist}_{G_{i-1}}(a_i, b_i)$ . Since  $\mathcal{E}$  is isometric,  $d_i \leq \ell_i$  for all  $1 \leq i \leq p$ . Finally, for any subgraph  $H$  of  $G$  induced by  $\bigcup_{i' \leq j \leq i} V(E_{j'})$ , let  $\text{Att}(H) \subseteq V(H)$  be the set of vertices of  $H$  that are the attachment vertices ( $a_k$  and  $b_k$ ) of some ear  $E_k$  with  $k > i$ .

Recall that a graph is prime if it has no clique-separators. Moreover, an edge-separator is a separator that consists of two adjacent vertices.

**Lemma 7.** *Let  $G$  be any (simple) prime series-parallel graph with  $\text{is}(G) \leq 6$ . If  $G$  does not contain a Dumbo graph as an isometric subgraph, then  $\text{tl}(G) \leq 2$ .*

*Proof.* Let us assume that  $G$  is not a chordal graph in which case the result is trivial (recall that  $\text{tl}(G) = 1$  if and only if  $G$  is chordal, which can be decided in linear time). Hence, we may assume that  $\text{tl}(G) \geq 2$ .

Let  $G$  be any prime series-parallel graph with  $\text{is}(G) \leq 6$ , and with no Dumbo graph as an isometric subgraph. Let  $\mathcal{E} = (E_i)_{0 \leq i \leq p}$  be an isometric nested ear-decomposition of  $G$  with  $E_0$  being a largest isometric cycle. Note that  $\mathcal{E}$  contains no ear of length one since  $G$  is simple, series-parallel, and prime (an ear of length one would be an edge-separator).

We will build a sequence  $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_{p'} = \mathcal{E}$  such that  $E_0 \in \mathcal{E}_1$  and, for every  $1 \leq i \leq p'$ ,

1.  $G_i = G[\bigcup_{E' \in \mathcal{E}_i} V(E')]$  is an isometric series-parallel subgraph of  $G$  with  $\mathcal{E}_i$  as an ear-decomposition;

2. There are no ears of length two attached to  $G_i$ , i.e., every ear of  $\mathcal{E}$  not yet in  $G_i$  with both endpoints in  $G_i$  has length at least 3;
3.  $G_i$  admits a tree-decomposition  $(T^i, \mathcal{X}^i)$  of length 2;
4. For every ear  $E_j \in \mathcal{E} \setminus \mathcal{E}_i$  attached to  $G_i$ , there exists  $t \in V(T^i)$  such that  $\{a_j, b_j\} \subseteq X_t^i \in \mathcal{X}^i$ , i.e., every ear not yet in  $G_i$  with both endpoints in  $G_i$  (so with length at least 3) has both its endpoints in some bag of  $(T^i, \mathcal{X}^i)$ .

The proof is by induction on  $1 \leq i \leq p'$ . The base case consists in building  $\mathcal{E}_1$ . There are several cases depending on the size  $\ell_0$  of  $E_0$ . Note that  $\ell_0 > 3$  since otherwise,  $G$  would be chordal or not prime, and so  $4 \leq \ell_0 \leq 6$ .

- If  $E_0 = (a, b, c, d)$  has length 4, recall that since  $G$  is prime, for any  $E_j \in \mathcal{E}$ ,  $d_j > 1$  and  $\ell_j > 1$ . Moreover, since  $\mathcal{E}$  is nested, for any two ears  $E_q$  and  $E_{q'}$  in  $\mathcal{E}$ , either  $P_{E_{j_q}}(a_q, b_q) \subseteq P_{E_{j_{q'}}}(a_{q'}, b_{q'})$  or  $P_{E_{j_{q'}}}(a_{q'}, b_{q'}) \subseteq P_{E_{j_q}}(a_q, b_q)$  or they are disjoint. Therefore it is not possible that  $G$  contains an ear attached to  $a$  and  $c$  and another ear attached to  $b$  and  $d$ . Therefore, up to symmetries,  $\text{Att}(E_0) = \{a, c\}$  (if  $\text{Att}(E_0) = \emptyset$ , then  $G = E_0$  and the result is trivial). Let  $\mathcal{E}_1$  consist of  $E_0$  and the set of all ears

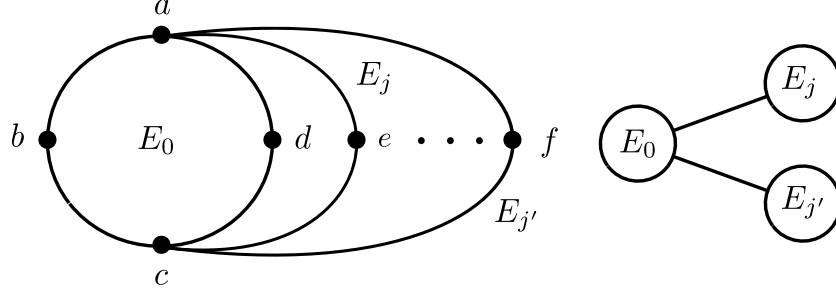


Figure 10: Graph (left) and tree-decomposition (right) with  $\ell_0 = 4$ .

of length two attached to  $a$  and  $c$ . Then,  $(T^1, \mathcal{X}^1)$  is the tree-decomposition with one “central” bag  $\{a, b, c, d\}$ , with one neighboring bag  $E_j$  for every ear  $E_j \in \mathcal{E}_1 \setminus \{E_0\}$  (see Figure 10). Clearly,  $(T^1, \mathcal{X}^1)$  is a tree-decomposition of  $G_1$  with length 2. Finally, because the ears are nested and there are no clique-separators, every ear in  $\mathcal{E} \setminus \mathcal{E}_1$  with attachment vertices in  $G_1$  must have  $a$  and  $c$  as attachment vertices. If such an ear in  $\mathcal{E} \setminus \mathcal{E}_1$  exists, it must have length at least 3 which would contradict the fact that  $E_0$  is a largest isometric cycle. Hence, no such ear exists and  $G_1 = G$ .

- If  $E_0 = (a, b, c, d, e)$  has length 5 then, up to symmetries,  $\text{Att}(E_0) \subseteq \{a, c, d\}$  (see Fig. 11). Indeed, if  $\text{Att}(E_0) = \emptyset$ , then  $G = E_0$  and the result is trivial. Otherwise, since  $G$  has no clique-separator, no ear can be attached to two adjacent vertices. Moreover, since  $\mathcal{E}$  is nested, up to symmetries, the ears can only be attached to  $a$  and  $c$  or to  $a$  and  $d$ . Moreover, all ears have length two or three, otherwise, there would be an isometric cycle of length larger than 5.

Let  $\mathcal{E}_1$  consist of  $E_0$  and the set of all ears of length two attached to  $E_0$ , and let  $G_1$  be the graph induced by these ears. Then,  $(T^1, \mathcal{X}^1)$  is the tree-decomposition with one “central” bag  $\{a, b, c, d, e\}$ , with one neighboring bag  $E_k$  for every ear  $E_k \in \mathcal{E}_1 \setminus \{E_0\}$  (see Figure 11). Clearly,  $(T^1, \mathcal{X}^1)$  is a tree-decomposition of  $G_1$  with length 2. Finally, every ear in  $\mathcal{E} \setminus \mathcal{E}_1$  attached to  $G_1$  has its attachment vertices in  $E_0$  because  $\mathcal{E}$  is nested and  $G$  has no clique-separator. More precisely, otherwise, since an ear cannot have adjacent attachment vertices (no clique-separator), there would be an ear  $E_j \in \mathcal{E} \setminus \mathcal{E}_1$  and an ear  $E_k \in \mathcal{E}_1 \setminus \{E_0\}$  (w.l.o.g., say with attachment vertices  $a$  and  $c$ ) with  $a_j \in E_k \setminus \{a, c\}$  and  $b_j \notin \{a, c\}$ . This would imply that  $G$  contains a  $K_4$  as minor, a contradiction. Thus, the vertices of  $G_1$  that are attachment vertices of ears in  $\mathcal{E} \setminus \mathcal{E}_1$  all belong to  $V(E_0)$  (which is a bag of  $(T^1, \mathcal{X}^1)$ ), and moreover, the ears of  $\mathcal{E} \setminus \mathcal{E}_1$  with attachment vertices in  $G_1$  (so in  $V(E_0)$ ) have length at least 3, by definition of  $\mathcal{E}_1$ . Hence, the induction hypotheses are satisfied.

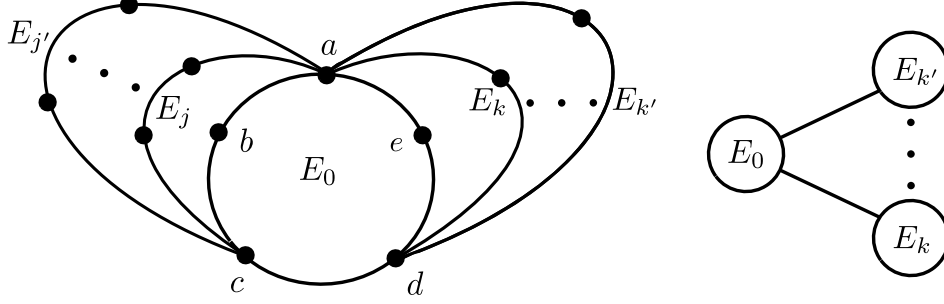


Figure 11: Graph (left) and tree-decomposition (right) with  $\ell_0 = 5$  ( $E_k$  and  $E_{k'}$  are contained in a bag since they have length 2.  $E_j$  and  $E_{j'}$  are not contained in a bag since they have length 3).

- Then, let us consider the case when  $E_0 = (a, b, c, d, e, f)$  has length 6. If there is an ear attached to two vertices at distance 3, note that every such ear has length exactly 3 since  $E_0$  is a largest isometric cycle. Moreover, all such ears have the same attachment vertices since the ears are nested (otherwise, there would be a  $K_4$  minor). W.l.o.g., let  $a$  and  $d$  be the attachment vertices of all (if any) ears attached to

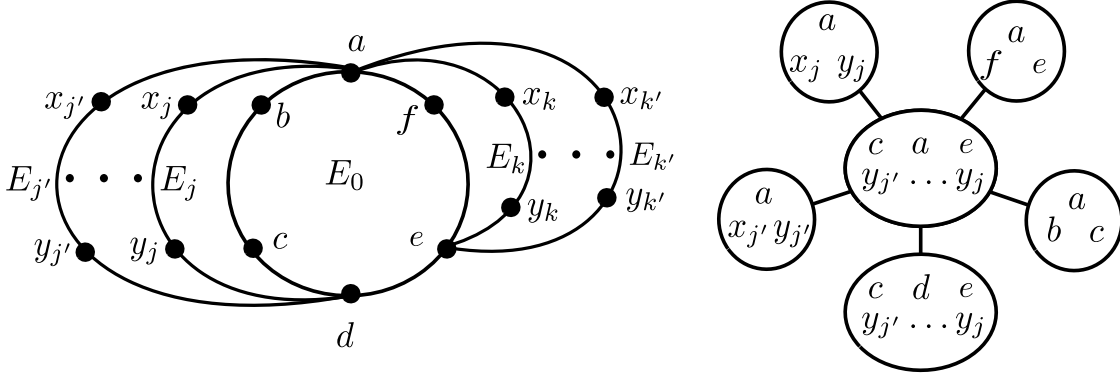


Figure 12: Graph (left) and tree-decomposition (right) with  $\ell_0 = 6$ .

vertices at distance 3 in  $E_0$ . Let  $\mathcal{E}'_1$  consist of  $E_0$  and all ears  $E_j = (a_j = a, x_j, y_j, b_j = d)$  attached to  $a$  and  $d$ .

Let  $E$  be an ear of  $\mathcal{E} \setminus \mathcal{E}'_1$  of length at least 3 and with both its attachment vertices  $x$  and  $y$  in  $G' = G[\bigcup_{E' \in \mathcal{E}'_1} V(E')]$ . Recall that  $x$  and  $y$  cannot be adjacent since  $G$  has no edge-separator. Moreover, if  $\mathcal{E}'_1 \neq \{E_0\}$ , we must have  $\{x, y\} \cap \{a, d\} \neq \emptyset$  since  $\mathcal{E}$  is nested (if  $\mathcal{E}'_1 = E_0$ , up to symmetries, we may assume that  $\{x, y\} \cap \{a, d\} \neq \emptyset$ ). Up to symmetry, let us assume that  $x = a$ . Then,  $y \neq d$  (since otherwise  $E \in \mathcal{E}'_1$ ). Hence, since  $G$  has no edge-separator, we must have  $y \in \{c, e\} \cup \bigcup_{E_j \in \mathcal{E}'_1} \{y_j\}$ . Finally, let  $E'$  be another (if any) ear of  $\mathcal{E} \setminus \mathcal{E}'_1$  of length at least 3 and with both its attachment vertices  $x'$  and  $y'$  in  $G' = G[\bigcup_{E'' \in \mathcal{E}'_1} V(E'')]$ . Let us assume for the purpose of contradiction that  $x' = d$ , by similar arguments,  $y' \in \{b, f\} \cup \bigcup_{E_j \in \mathcal{E}'_1} \{x_j\}$ . In that case, either  $E$  and  $E'$  are not nested (if they are attached to the same ear  $E_j \in \mathcal{E}'_1$ ), or  $E$ ,  $E'$  and the cycle containing  $x, y, x'$  and  $y'$  would be an isometric Dumbo graph, a contradiction. Therefore, up to symmetries, all ears of length at least 3 that are attached to  $G'$  have  $a$  and some vertex in  $B = \{c, e\} \cup \bigcup_{E_j \in \mathcal{E}'_1} \{y_j\}$  as attachment vertices (see Figure 12).

Let  $(T', \mathcal{X}')$  be the tree-decomposition with one “central” bag  $C = B \cup \{a\}$  with one neighboring bag  $\{a, x_j, y_j\}$  for every ear  $E_j \in \mathcal{E}'_1 \setminus \{E_0\}$ , one neighboring bag  $\{a, b, c\}$ , one neighboring bag  $\{a, f, e\}$ ,

and one neighboring bag  $\{d\} \cup B$ . Then,  $(T', \mathcal{X}')$  is clearly a tree-decomposition of  $G'$  of length 2 such that all ears of length at least 3 attached to  $G'$  have their attachment vertices in  $C$ . Finally, let  $F$  be the set of all ears of length 2 attached to  $G'$ . Let  $\mathcal{E}_1 = \mathcal{E}'_1 \cup F$ . By Lemma 6, from  $(T', \mathcal{X}')$ , we can obtain a tree-decomposition  $(T^1, \mathcal{X}^1)$  of  $G_1$  of length 2 such that every bag in  $\mathcal{X}'$  is contained in some bag of  $\mathcal{X}^1$  (see Figure 12).

Finally, since  $G$  has no clique-separator and is series-parallel (in particular the ears are nested), every ear attached to  $G^1$  must have both its attachment vertices in the same bag of  $(T^1, \mathcal{X}^1)$ , and must have length at least 3 (since otherwise it would have been included in  $\mathcal{E}_1$ ).

Now, let us prove by induction on  $1 \leq i < p'$  that we can build an ear decomposition  $\mathcal{E}_{i+1}$  from  $\mathcal{E}_i$  with all the desired properties. Let  $E_j$  be any shortest ear not in  $\mathcal{E}_i$  with attachment vertices  $\{a_j, b_j\} \in V(G_i)$ . Because  $G$  has no clique-separator and, by the induction hypothesis,  $G_i$  has a tree-decomposition  $(T^i, \mathcal{X}^i)$  of length 2 with a bag containing  $a_j$  and  $b_j$ , note that  $d_j = \text{dist}_G(a_j, b_j) = \text{dist}_{G_i}(a_j, b_j) = 2$ . Moreover, because  $is(G) = 6$  and there is no ear of length 2 attached to  $G_i$ , the length  $\ell_j$  of  $E_j$  is such that  $3 \leq \ell_j \leq 4$ . There are two cases depending on the length of  $E_j$ .

- If  $E_j = (a_j, x, y, b_j)$  has length 3, then up to symmetries  $\text{Att}(G_i \cup E_j) \cap V(E_j) \subseteq \{a_j, y, b_j\}$ . Indeed, since  $G$  has no clique-separator, no ear can be attached to two adjacent vertices. Moreover, since all ears of  $\mathcal{E}$  are nested, first, there are no two ears, one attached to  $a_j$  and  $y$  and the other one attached to  $x$  and  $b_j$ , and, second, there is no ear attached to a vertex of  $V(E_j) \setminus \{a_j, b_j\}$  and to a vertex of  $V(G_i) \setminus \{a_j, b_j\}$  (see Figure 13). Let  $\mathcal{E}'_{i+1}$  consist of  $\mathcal{E}_i$  and  $E_j$ . Let  $G' = G[\bigcup_{E' \in \mathcal{E}'_{i+1}} V(E')]$  and  $(T', \mathcal{X}')$  be the tree-decomposition build from  $(T^i, \mathcal{X}^i)$  with a bag  $B = \{a_j, x, y, b_j\}$  connected to a bag of  $(T^i, \mathcal{X}^i)$  containing  $a_j$  and  $b_j$ . Then,  $(T', \mathcal{X}')$  is clearly a tree-decomposition of  $G'$  of length 2. Finally, let  $F$  be the set of all ears of length 2 attached to  $G'$  (note that, because of the induction hypothesis and the fact that the initial ear decomposition is isometric, all such ears are attached to  $a_j$  and  $y$ ). Let  $\mathcal{E}_{i+1} = \mathcal{E}'_{i+1} \cup F$ . By Lemma 6, from  $(T', \mathcal{X}')$ , we can obtain a tree-decomposition  $(T^{i+1}, \mathcal{X}^{i+1})$  of  $G_{i+1}$  of length 2 such that every bag of  $\mathcal{X}'$  is contained in some bag of  $\mathcal{X}^{i+1}$  (see Figure 13). Clearly if there is an ear attached to the only middle vertex of an ear  $E_f$  of  $F$  then by definition of a nested ear decomposition, its second endpoint is a vertex in  $E_f$  which contradicts the fact that  $G$  has no clique-separator. We can deduce that for every  $E_m$  attached to  $G_{i+1}$  there exists  $t \in V(T^{i+1})$  such that  $\{a_m, b_m\} \subseteq X_t^{i+1}$ .

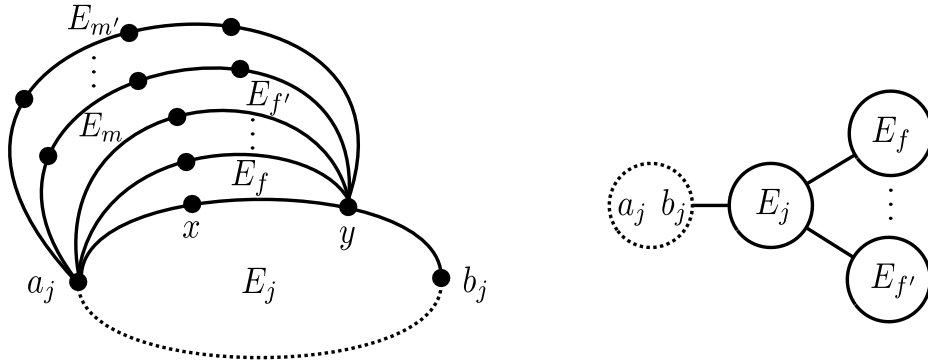


Figure 13: Graph (left) and tree-decomposition (right) of  $G_{i+1}$  with  $\ell_j = 3$ .

- Now, let us assume that  $E_j = (a_j, x, y, z, b_j)$  has length 4. There are several cases depending on the vertices of  $E_j$  that are attachment vertices for other ears  $E_l$  in  $\mathcal{E} \setminus (\mathcal{E}_i \cup \{E_j\})$  attached to  $E_j$ . Because  $G$  has no clique-separator and  $\mathcal{E}$  is an isometric nested ear decomposition, we have the following possibilities up to symmetries.



- If  $\text{Att}(E_j) \subseteq \{a_j, y, b_j\}$  (see Figure 14), then let  $\mathcal{E}'_{i+1}$  consist of  $\mathcal{E}_i$  and  $E_j$ . Let  $(T', \mathcal{X}')$  be the tree-decomposition of  $G' = G[\bigcup_{E \in \mathcal{E}'_{i+1}} V(E)]$  built from  $(T^i, \mathcal{X}^i)$  as follows. Let  $B$  be any bag of  $(T^i, \mathcal{X}^i)$  containing both  $a_j$  and  $b_j$  (which exists by the induction hypothesis). Let us add the bag  $\{a_j, y, b_j\}$  adjacent to  $B$  and to the bags  $\{a_j, x, y\}$  and  $\{y, z, b_j\}$ . Since  $(T^i, \mathcal{X}^i)$  is a tree-decomposition of  $G_i$  of length 2, then  $(T', \mathcal{X}')$  is also a tree-decomposition of  $G'$  of length 2. Let  $F$  be the set of ears of length 2 attached to  $E_j$  and let  $\mathcal{E}_{i+1}$  consist of  $\mathcal{E}'_{i+1}$  and  $F$ . By Lemma 6, we can obtain from  $(T', \mathcal{X}')$  a tree-decomposition  $(T^{i+1}, \mathcal{X}^{i+1})$  of length 2 of  $G_{i+1}$ . Finally,  $(T^{i+1}, \mathcal{X}^{i+1})$  satisfies the desired properties (in particular because  $G$  has no edge separator, every ear attached to  $G_{i+1}$  has its attachment vertices in a bag of  $(T^{i+1}, \mathcal{X}^{i+1})$ ).

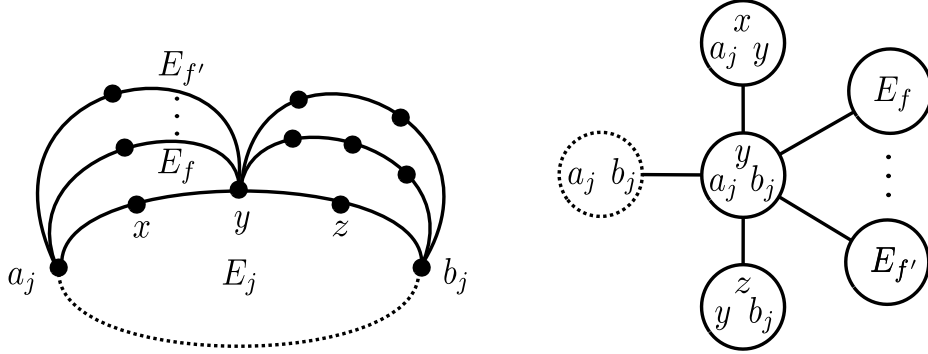


Figure 14: Graph (left) and tree-decomposition (right) of  $G_{i+1}$  with  $\ell_j = 4$  and  $\text{Att}(E_j) \subseteq \{a_j, y, b_j\}$ .

- Now, let us assume, up to symmetry, that there exists an ear  $E'$  attached to  $a_j$  and  $z$ . Note that such an ear has length exactly 3 since  $\mathcal{E}$  is an isometric nested ear decomposition and no isometric cycle has length more than 6. Let  $\mathcal{E}'$  be the set of all ears  $E_{j'} = (a_j = a_{j'}, x_{j'}, y_{j'}, b_{j'} = z) \notin \mathcal{E}_i$  of length 3 attached to  $a_j$  and  $z$  (in particular,  $E'$  is such an ear), and let  $\mathcal{E}'_{i+1}$  consist of  $\mathcal{E}_i \cup E_j \cup \mathcal{E}'$  (see Figure 15).

Let us first show that no ear  $E_q \in \mathcal{E} \setminus \mathcal{E}'_{i+1}$  of length at least 3 is attached to  $x_{j'}$  and  $z = b_{j'}$  for some  $j'$  such that  $E_{j'} \in \mathcal{E}'$  (resp. to  $x$  and  $z$ ). For the purpose of contradiction, let us assume that such an ear  $E_q$  exists. Recall that, by the induction hypothesis,  $a_j$  and  $b_j$  must belong to a same bag of  $(T^i, \mathcal{X}^i)$  of length 2 and that, because there is no clique-separator,  $\{a_j, b_j\} \notin E(G)$ . Hence,  $\text{dist}_G(a_j, b_j) = \text{dist}_{G_i}(a_j, b_j) = 2$ . Let  $E_\ell$  be the first (i.e., with minimum  $\ell$ ) ear of  $G_i$  containing both  $a_j$  and  $b_j$  (such an ear must exist since  $E_j$  can only be attached to the vertices of some previous ear).

- \* If  $E_\ell = E_0$ , then the subgraph induced in  $G$  by  $V(E_0) \cup V(E_{j'}) \cup V(E_q)$  (resp.  $V(E_0) \cup V(E_j) \cup V(E_q)$ ) is an isometric Dumbo graph, a contradiction.
- \* Otherwise (if  $\ell \neq 0$ ), let  $a_\ell$  and  $b_\ell$  be the endpoints of  $E_\ell$ , and let  $G^*$  be the subgraph induced by the vertices of the ears in  $\{E_m \in \mathcal{E}^i \mid m < \ell\}$ . Note that  $G^*$  is an isometric subgraph of  $G_i$ . W.l.o.g.,  $a_\ell \notin \{a_j, b_j\}$  (otherwise this would contradict that  $E_\ell$  is the first ear in which both  $a_j$  and  $b_j$  appear). Let  $P$  be any shortest  $a_\ell$ - $b_\ell$  path in  $G^*$ . Since  $a_\ell$  and  $b_\ell$  are not adjacent (otherwise there would be an edge separator in  $G$ ),  $P$  has length at least 2. Then, the subgraph induced by  $V(P) \cup V(E_\ell) \cup V(E_{j'}) \cup V(E_q)$  (resp.  $V(P) \cup V(E_\ell) \cup V(E_j) \cup V(E_q)$ ) is an isometric Dumbo graph, a contradiction.

Let  $B$  be any bag of  $(T^i, \mathcal{X}^i)$  containing both  $a_j$  and  $b_j$  (which exists by the induction hypothesis). Let  $B' = \{a_j, b_j, y\} \bigcup_{j', E_{j'} \in \mathcal{E}'} \{y_{j'}\}$ , let  $B_{j'} = \{a_j, x_{j'}, y_{j'}\}$  for all  $j'$  such that  $E_{j'} \in \mathcal{E}'$ , let  $B'' = \{b_j, z, y\} \bigcup_{j', E_{j'} \in \mathcal{E}'} \{y_{j'}\}$ , and let  $B_j = \{a_j, x, y\}$ .

Let  $(T', \mathcal{X}')$  be the tree-decomposition of  $G' = G[\bigcup_{E' \in \mathcal{E}'_{i+1}} V(E')]$  built from  $(T^i, \mathcal{X}^i)$  by adding the bag  $B'$  adjacent to  $B$ , to  $B''$ , to  $B_j$ , and to  $B_{j'}$  for all  $j'$  such that  $E_{j'} \in \mathcal{E}'$ . It can be shown

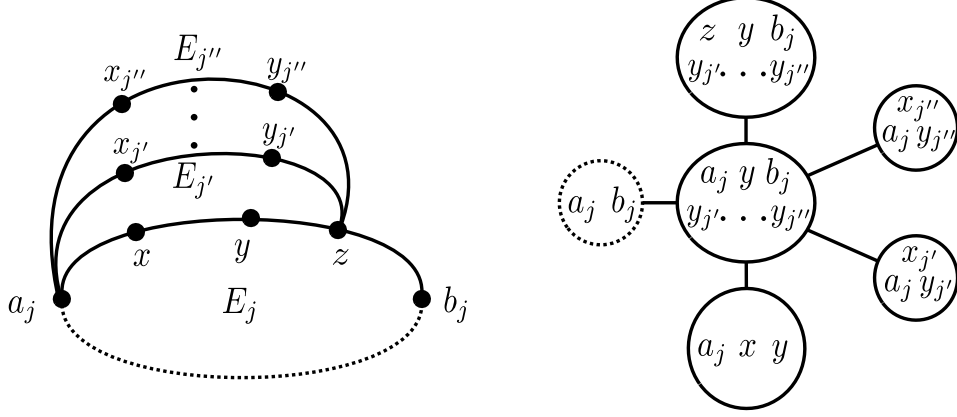


Figure 15: Case where  $E_j$  has length 4 and there is at least one ear attached to  $a_j$  and  $z$ .

that  $(T', \mathcal{X}')$  is a tree-decomposition of  $G'$ , with length 2 and such that every ear of length at least 3 attached to  $G'$  has both its attachment vertices in some bag of  $(T', \mathcal{X}')$ . Let  $F$  be the set of ears of length 2 attached to some ear in  $\mathcal{E}' \cup E_j$  and let  $\mathcal{E}_{i+1}$  consist of  $\mathcal{E}'_{i+1} \cup F$ . By Lemma 6, we can obtain from  $(T', \mathcal{X}')$  a tree-decomposition  $(T^{i+1}, \mathcal{X}^{i+1})$  of length 2 of  $G_{i+1}$ .

Finally,  $(T^{i+1}, \mathcal{X}^{i+1})$  satisfies the desired properties (in particular because  $G$  has no edge separator, every ear attached to  $G_{i+1}$  has its attachment vertices in a bag of  $(T^{i+1}, \mathcal{X}^{i+1})$ ).

This concludes the proof. Note that this proof is constructive and provides a quadratic-time algorithm that takes a series-parallel graph  $G$  as input and either returns a certificate that  $tl(G) > 2$  (an isometric Dumbo graph, or an isometric cycle of size at least 7) or a tree-decomposition of length 2 of  $G$ . The algorithm first checks whether  $G$  is chordal or not in linear time [28]. If  $G$  is chordal, then the algorithm computes a tree-decomposition of length 1. Otherwise, it computes, in linear time, the size of a largest isometric cycle in  $G$  (using a decomposition computed in linear time [24], see section 2). If there is an isometric cycle of size at least 7, this cycle is returned. Otherwise, the algorithm computes, in quadratic time (by Lemma 2), an isometric nested ear decomposition of  $G$ . Finally, it considers sequentially each ear (sometimes several ears simultaneously) and adds it (them) to the current tree-decomposition in constant time (it looks for the bags containing the attachment vertices for instance using a dictionary, and it considers a constant number of cases). Overall, the time complexity is quadratic in the size of the graph.  $\square$

## 6. Further work

This work presents the first characterization of the treelength of a class of graphs in terms of forbidden isometric subgraphs. In particular, we show that deciding if the treelength of a series-parallel graph is at most 2 can be done in polynomial time while this problem is NP-complete in general graphs. Our approach seems difficult to generalize to larger values of the treelength. Indeed, for treelength 3, we have already identified about 20 families of forbidden isometric subgraphs. All these families are slight variations of the Dumbo graphs but we still do not know how to describe them in a synthetic way. The next step is then to find a polynomial-time algorithm that computes the treelength of series-parallel graphs (or to prove that it is an NP-hard problem). The main goal is to further investigate the computational complexity of computing the treelength (or even the treewidth) of planar graphs. Designing better approximation algorithms for general or planar or series-parallel graphs is also an interesting open problem.

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