



**HAL**  
open science

## State prediction using a high-gain distributed scheme

Mathieu Bajodek, Fernando Castaños, Sabine Mondié

► **To cite this version:**

Mathieu Bajodek, Fernando Castaños, Sabine Mondié. State prediction using a high-gain distributed scheme. 2023. hal-04267738

**HAL Id: hal-04267738**

**<https://hal.science/hal-04267738>**

Preprint submitted on 2 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# State prediction using a high-gain distributed scheme

Mathieu Bajodek <sup>a</sup>, Fernando Castaños <sup>b,\*</sup> and Sabine Mondié <sup>b</sup>

<sup>a</sup>Laboratoire d'Automatique, de Génie des Procédés et de Génie Pharmaceutique (LAGEPP), CPE Lyon, Villeurbanne, France

<sup>b</sup>Departamento de Control Automático, Cinvestav-IPN, México

---

## Abstract

Time delays pose unique challenges in control engineering, as the inherent delays can introduce instabilities and compromise system performance. To overcome these complexities, the concept of predictors or compensators has been instrumental in stabilising closed-loop systems. Still, it requires an accurate delayed state estimation through the design and implementation of predictors. In this paper, we delve into the design of novel prediction architectures tailored to nonlinear systems with input and output delays that not only ensure stability but also bolster robustness, emphasising their ability to preserve stability despite measurement noises or external perturbations.

*Key words:* Time-delay systems; Nonlinear systems; Observer-predictor; Stability analysis.

---

## 1 Introduction

In applications across diverse engineering domains, the influence of delays on closed-loop stability is critical [28]. Feedback and state predictors have been introduced to address this challenge, allowing the application of control strategies originally designed for delay-free systems.

Two approaches stand out among strategies aiming at determining the future state. The first is to compute the predictor mapping by integrating the system dynamics, i.e., by the Cauchy Formula. This approach, initiated by Manitius [21] for linear input delay linear systems, has been extended to systems with state and input delays [16], stochastic input delays [18] and others. For nonlinear systems, determining the integral predictor mapping is seldom possible. Thus, an approximation is suggested [14]. The integral nature of the control law makes the robust closed-loop stability analysis involved, an issue that is resolved by filtering the control law [26,17]. For the case of partial state information, a two-step approach consisting of an observer of the state at the present time followed by the integral prediction is possible [14].

The second approach is an observer-based prediction method, called *observer-prediction* for short, and simply *prediction* when the context is clear. It merges the observation and prediction tasks, an idea that was initially introduced in the framework of output delay sys-

tems [10]. In the case of linear input delay systems, it consists of estimating the future state a period ahead of time by feeding a Luenberger-type observer with the control variable at the present time [27]. It is straightforwardly generalised to other classes of systems such as sampled, time-varying [23], state delay [30], and nonlinear [22,7], as the observer amounts to a copy of the system amended by a correction term depending on the prediction error. The prediction-observation error is now governed by a delay system whose stability margin is limited by the delay, an issue that is solved by introducing several sub-predictors [10,27]. Thus, within this approach, the observer-predictor construction is straightforward, but the choice of the control gain and number of sub-predictors requires special attention.

Nonlinear terms cannot be swept under the rug either. High-gain observers [2,4,5] provide an efficient tool for the design of observers in the presence of nonlinearities. Known for their ease of tuning, they ensure robustness and adaptability in state estimation. Global stability results can be obtained under Lipschitz conditions and particular system structures. This turnkey tool also proves its suitability for time-delay systems [13]. However, the limitations of the high-gain strategy for long input delays are well established [15]. While using sub-predictors is a natural solution [1], the number of required sub-predictors grows exponentially with the system's dimension [29], a fact that suggests considering continuously distributed sub-predictors, rather than a finite number of them.

---

\* Corresponding author. Address: castanos@ieee.org

At the heart of this research, a new observer-predictor is designed for nonlinear systems with input and output delays using synergistic integration of high-gain observers and continuously distributed sub-predictors. The main objective is to ensure stability and robustness against measurement noises and external perturbations.

This new predictor is implemented as a partial differential equation specified by three distinct components:

- Similarly to the delay-free case, a high-gain observer is tuned with a sufficiently large scalar gain [12]. The delayed state is estimated and used to define a boundary condition.
- The differential equation itself, relating the temporal and spatial derivatives. The equation results from taking a chain of sub-predictors and letting their number grow to infinity.
- The predictor state at one of the domain boundaries is fed to a filter to enhance robustness with respect to measurement noises.

This structure enables handling systems with long dead times, hard nonlinearities, and potentially high-dimensional state spaces. The most innovative part concerns the distributed predictor, which can be written as a partial differential equation, the solutions of which are easy to analyse with infinite-dimensional tools [24,25]. Through Lyapunov theory and rigorous mathematical analysis, we provide stability guarantees in the presence and absence of uncertainty and provide upper bounds on the estimation errors.

We assess the predictor's performance using a bioreactor modeled by a Contois model [3,6]. The simulations showcase our proposed predictor's efficacy, affirming its potential as a robust and reliable solution for state prediction.

This paper is organised as follows. In Section 2, the predictor design strategy is presented. It consists of three cascaded systems: a high-gain observer, a distributed predictor, and a filter. In Section 3, the stability analysis of the error system is regarded. Through Lyapunov arguments, we show exponential stability and input-to-state stability properties with respect to perturbations and measurement noises. In Section 4, the results are illustrated by the numerical simulation of a bioreactor.

**Notation.** Throughout the paper,  $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}^n$  the  $n$ -dimensional Euclidean space, and  $\mathbb{R}^{n \times m}$  the set of  $n \times m$  real matrices. Moreover, for any square matrix  $A \in \mathbb{R}^{n \times n}$ ,  $A^\top$  represents its transpose,  $\mathcal{H}(A) = A + A^\top$ , and  $\underline{A}, \overline{A}$  are the square roots of, respectively, the minimal and maximal eigenvalues of the symmetric matrix  $A^\top A$ . Lastly, the real-valued function  $|\cdot|$  is the Euclidean norm and  $\|\cdot\|$  is the  $\mathcal{L}^2(-h_1, h_2)$  norm.

## 2 Structure of the observer-predictor

In this section, we present the class of systems for which we can predict the state and then propose the general structure of the predictor. The following section is devoted to choosing appropriate gains and the corresponding stability analysis.

### 2.1 Problem statement

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \phi(x(t), u(t - h_2)) \\ y(t) &= Cx(t - h_1) \end{aligned}, \quad (1)$$

where the function  $\phi : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$  is triangular in  $x$ ,

$$\phi(x, u) = \begin{bmatrix} \phi(x_1, u) & \phi(x_1, x_2, u) & \cdots & \phi(x_1, \dots, x_n, u) \end{bmatrix}, \quad (2)$$

and where the matrix pair  $(C, A) \in \mathbb{R}^{1 \times n} \times \mathbb{R}^{n \times n}$  is in the observability Brunowski normal form, that is,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \text{and} \quad C = [1 \ 0 \ \cdots \ 0]. \quad (3)$$

**Remark 1** A necessary and sufficient condition for a (possibly nonlinear) system to be observable, uniformly with respect to the inputs, is the existence of a change of coordinates putting it in the form (1) with (2) and (3). See, e.g., [5] for details.  $\lrcorner$

As we will show later, the following assumption establishes the possibility of stabilising the nonlinear dynamics of the prediction error using linear output injection.

**Assumption 1** *The function  $\phi$  is globally Lipschitz in  $x$ , uniformly in  $\mathcal{U}$ . In other words, there exists a positive scalar  $\gamma_\phi$  such that*

$$|\phi(x, u) - \phi(\hat{x}, u)| \leq \gamma_\phi |x - \hat{x}|, \quad \forall (x, u) \in \mathbb{R}^n \times \mathcal{U}. \quad (4)$$

We seek an observer-predictor that reconstructs the state  $x(t + \theta)$  for  $\theta \in [-h_1, h_2]$ . In particular, the reconstruction of the state  $x(t + h_2)$  eases the control of (1), bringing the design back to the case without delays. The two main difficulties are:

- the nonlinearity  $\phi$ , which is tackled with a high-gain observer, and
- the delays  $h_1$  and  $h_2$ , tackled with a distributed predictor scheme.

It is worth mentioning that the classical high-gain strategy cannot be freely applied in the presence of delays, as the delay effectively imposes a constraint on how much the high-gain can be increased before losing stability. This limitation has been overcome in, e.g., [1,29] by cascading several sub-predictors. This paper addresses the limitation by endowing the observer-predictor with a continuously distributed architecture, recovering full freedom in the choice of the high gain.

## 2.2 High-gain observer

The observability of the linear structure (3) implies the stabilisability of the linear error dynamics.

**Lemma 1** *Let  $A$  and  $C$  be as in (3) and let  $\alpha > 0$  and  $Q \succ 0$ . There exist matrices  $L \in \mathbb{R}^{n \times 1}$  and  $P \succ 0$  such that*

$$P(A + LC) + (A + LC)^\top P = -\alpha P - Q. \quad (5)$$

As in [29], we discuss high-gain observation through the lens of homogeneity. We begin by recalling the scaling or *dilation* matrix

$$\Lambda(\lambda) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda^n \end{bmatrix}$$

and proceed to construct a Luenberger-type observer at the spatial boundary  $\theta = -h_1$ ,

$$\begin{aligned} \dot{\hat{z}}(t, -h_1) &= A\hat{z}(t, -h_1) + \Lambda(\lambda)L(C\hat{z}(t, -h_1) - y(t)) \\ &\quad + \phi(\hat{z}(t, -h_1), u(t - h_1 - h_2)), \end{aligned} \quad (6)$$

where  $\lambda \geq 1$  is the *high-gain parameter* and  $L \in \mathbb{R}^{n \times 1}$  is a vector gain. Consider the time and state scalings

$$\xi(t, \theta) = \Lambda(\lambda^{-1})x(\lambda^{-1}t - \theta), \quad \zeta(t, \theta) = \Lambda(\lambda^{-1})\hat{z}(\lambda^{-1}t, \theta),$$

and define the error

$$e(t) = \xi(t, -h_1) - \zeta(t, -h_1). \quad (7)$$

Its dynamics are given by

$$\begin{aligned} \dot{e}(t) &= \lambda^{-1}\Lambda(\lambda^{-1})A\Lambda(\lambda)e(t) + \lambda^{-1}LC\Lambda(\lambda)e(t) \\ &\quad + \lambda^{-1}\tilde{\Phi}_\lambda(e(t), \xi(t, -h_1), u(\lambda^{-1}t - h_1 - h_2)), \end{aligned}$$

where

$$\tilde{\Phi}_\lambda(e, \xi, u) = \Lambda(\lambda^{-1})[\phi(\Lambda(\lambda)\xi, u) - \phi(\Lambda(\lambda)(\xi - e), u)].$$

By the homogeneity of the vector field  $Ax$  and of the function  $Cx$ , we have  $A = \lambda^{-1}\Lambda(\lambda^{-1})A\Lambda(\lambda)$  and  $C\Lambda(\lambda) = \lambda C$  [29, Lemma 1], so the error dynamics simplify to

$$\begin{aligned} \dot{e}(t) &= (A + LC)e(t) + \\ &\quad \lambda^{-1}\tilde{\Phi}_\lambda(e(t), \xi(t, -h_1), u(\lambda^{-1}t - h_1 - h_2)). \end{aligned}$$

Assumption 1 implies the existence of a positive scalar  $\gamma$  such that

$$|\tilde{\Phi}_\lambda(e, \xi, u)| \leq \gamma|e| \quad (8)$$

for all  $(e, \xi, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{U}$  and all  $\lambda \geq 1$  [29, Lemma 2]. The standard high-gain approach suggests the observation problem be solved by first fixing  $L$  such that  $A + LC$  is Hurwitz, and then making  $\lambda$  sufficiently large so that the term  $\lambda^{-1}\tilde{\Phi}_\lambda$  does not alter the stability properties of the system.

## 2.3 Distributed predictor

To motivate the distributed nature of the proposed observer-predictor, consider a cascade of  $r \in \mathbb{N}$  sub-predictors given by

$$\begin{aligned} \dot{\hat{z}}(t, \theta_i) &= A\hat{z}(t, \theta_i) + \phi(\hat{z}(t, \theta_i), u(t + \theta_i - h_2)) \\ &\quad + \lambda_r \cdot (\hat{z}(t, \theta_{i+1}) - \hat{z}(t - \frac{h_1+h_2}{r}, \theta_i)). \end{aligned} \quad (9)$$

Here,  $\theta_i = h_2 - i\frac{h_1+h_2}{r}$  for  $i \in \{0, \dots, r-1\}$  and  $\lambda_r \in \mathbb{R}$  is the gain for each sub-predictor. The last sub-system has index  $i = r$ , state  $\hat{z}(t, -h_1)$ , and evolves according to (6). Our interest in the cascade (9) is that the delay that appears in the resulting error dynamics for each sub-predictor is reduced from  $h_1 + h_2$  to  $(h_1 + h_2)/r$ .

The new predictor is then obtained by letting the number of sub-predictors grow to infinity. In doing so, the discrete index  $\theta_i$  becomes a real independent variable  $\theta$ . Notice that the last term on the right-hand side of (9) can be expanded as

$$\begin{aligned} \hat{z}(t, \theta_{i+1}) - \hat{z}(t - \frac{h_1+h_2}{r}, \theta_i) &= \\ &= \frac{h_1 + h_2}{r} \left( -\frac{\partial}{\partial \theta} \hat{z}(t, \theta) + \frac{\partial}{\partial t} \hat{z}(t, \theta) \right) + \mathcal{O}\left(\frac{1}{r}\right). \end{aligned}$$

Thus, when  $r \rightarrow +\infty$ , the set of ordinary differential equations (9) become a partial differential equation of the form

$$\begin{aligned} \frac{\partial}{\partial t} \hat{z}(t, \theta) &= A\hat{z}(t, \theta) + \phi(\hat{z}(t, \theta), u(t + \theta - h_2)) \\ &\quad - \lambda_r^* \cdot \left( \frac{\partial}{\partial \theta} \hat{z}(t, \theta) - \frac{\partial}{\partial t} \hat{z}(t, \theta) \right), \end{aligned}$$

where  $t \geq 0$ ,  $\theta \in (-h_1, h_2)$ , and  $\lambda_r^* = \lim_{r \rightarrow \infty} \lambda_r \frac{h_1 + h_2}{r}$ . Solving for the partial derivative with respect to time gives

$$\begin{aligned} \frac{\partial}{\partial t} \hat{z}(t, \theta) = & -k \frac{\partial}{\partial \theta} \hat{z}(t, \theta) + (1+k)A\hat{z}(t, \theta) \\ & + (1+k)\phi(\hat{z}(t, \theta), u(t+\theta-h_2)), \end{aligned} \quad (10)$$

where  $k = \frac{\lambda_r^*}{1-\lambda_r^*}$  is the *distributed gain*.

**Remark 2** The well-posedness of system (10) with the boundary condition (6) is guaranteed by setting  $k > 0$  [11]. Conversely, for any  $k > 0$  we can set  $\lambda_r = \frac{r}{h_1+h_2} \frac{k}{1+k}$ .  $\lrcorner$

Define the error variable

$$\tilde{z}(t, \theta) = x(t+\theta) - \hat{z}(t, \theta), \quad (11)$$

Using (1) and (10), the dynamics of  $\tilde{z}$  are expressed as

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{z}(t, \theta) = & -k \frac{\partial}{\partial \theta} \tilde{z}(t, \theta) + (1+k)A\tilde{z}(t, \theta) \\ & + (1+k)\tilde{\Phi}_1(\tilde{z}(t+\theta), x(t+\theta), u(t+\theta-h_2)). \end{aligned} \quad (12)$$

The rationale behind the previous equation is that, for any  $k > 0$ , information is transported from  $\theta = -h_1$  to  $\theta = h_2$ . Later, we will formally show the convergence of the error to zero in terms of the  $\mathcal{L}^2(-h_1, h_2)$  norm.

#### 2.4 Filter

Lastly, to implement a control strategy, we need to propose an estimation  $\hat{w}(t)$  of the instantaneous state  $x(t+h_2)$ . The idea is to have an estimate, which avoids implementation issues and can be adjusted independently. To do so, referring to early advances in control and observation of time-delay systems [26], we add an output filter. Consider then the system

$$\dot{\hat{w}}(t) = -\mu\hat{w}(t) + \phi(\hat{z}(t, h_2), u(t)) + (A + \mu I)\hat{z}(t, h_2), \quad (13)$$

where  $\mu > 0$  is the filter gain. Define the error variable

$$\tilde{w}(t) = x(t+h_2) - \hat{w}(t) \quad (14)$$

and compute its dynamics

$$\begin{aligned} \dot{\tilde{w}}(t) = & -\mu\tilde{w}(t) + (A + \mu I)\tilde{z}(t, h_2) \\ & + \tilde{\Phi}_1(\tilde{z}(t, h_2), x(t+h_2), u(t)). \end{aligned} \quad (15)$$

We will select  $\mu$  according to the desired performance.

**Remark 3** The stability of system (12) in the sense of the  $\mathcal{L}^2(-h_1, h_2)$  norm does not guarantee the convergence of  $\tilde{z}(t, h_2)$  to zero. The filter state  $\tilde{w}(t)$  enables us

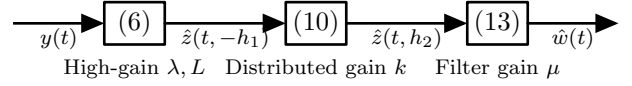


Fig. 1. Whole predictor block diagram.

to work in the space  $\mathcal{L}^2(-h_1, h_2) \times \mathbb{R}^n$  and certify convergence at boundary.  $\lrcorner$

#### 2.5 Complete predictor

The complete predictor is illustrated in Fig. 1. It includes the delay-free observer (6), the distributed predictor (10), and the filter (13). Summarising,

$$\begin{aligned} \dot{\hat{w}}(t) = & -\mu\hat{w}(t) + (A + \mu I)\hat{z}(t, h_2) \\ & + \phi(\hat{z}(t, h_2), u(t)), \end{aligned} \quad (16a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \hat{z}(t, \theta) = & -k \frac{\partial}{\partial \theta} \hat{z}(t, \theta) + (1+k)A\hat{z}(t, \theta) \\ & + (1+k)\phi(\hat{z}(t, \theta), u(t+\theta-h_2)), \end{aligned} \quad (16b)$$

$$\begin{aligned} \dot{\hat{z}}(t, -h_1) = & A\hat{z}(t, -h_1) + \Lambda(\lambda)L(C\hat{z}(t, -h_1) - y(t)) \\ & + \phi(\hat{z}(t, -h_1), u(t-h_1-h_2)). \end{aligned} \quad (16c)$$

**Remark 4** As the function  $\phi$  is triangular, there exists a (possibly nonlinear) map

$$T : \left( y(t), \dots, \frac{d^{n-1}}{dt^{n-1}} y(t) \right) \mapsto x(t-h_1).$$

Thus, it is possible to remove the high-gain observer and use the boundary condition

$$\hat{z}(t, -h_1) = T \left( y(t), \dots, \frac{d^{n-1}}{dt^{n-1}} y(t) \right).$$

It is also possible to avoid the filter (13) by setting  $\hat{w}(t) = \hat{z}(t, h_2)$ . However, even if stability can be assessed, the high-gain observer and the filter play an important role when put into practice. It allows the whole predictor system to be robust with respect to measurement noise and to face several implementation issues.  $\lrcorner$

Recall the error (7) and define the scaled errors

$$e_z(t, \theta) = \Lambda(\lambda^{-1})\tilde{z}(\lambda^{-1}t, \theta)$$

and

$$e_w(t) = \Lambda(\lambda^{-1})\tilde{w}(\lambda^{-1}t).$$

As before, this transformation is useful for showing how, via high-gain, the linear terms dominate  $\tilde{\Phi}_\lambda$ . From (12) and (15) we obtain the filter error dynamics

$$\begin{aligned} \lambda \dot{e}_w(t) = & -\mu e_w(t) + (\lambda A + \mu I)e_z(t, h_2) \\ & + \tilde{\Phi}_\lambda(e_z(t, h_2), \xi(t, h_2), u(\lambda^{-1}t)). \end{aligned} \quad (17a)$$

For  $t \geq 0$  and  $\theta \in (-h_1, h_2)$ , we obtain the distributed predictor error dynamics,

$$\begin{aligned} \lambda \frac{\partial}{\partial t} e_z(t, \theta) &= -k \frac{\partial}{\partial \theta} e_z(t, \theta) + (1+k)\lambda A e_z(t, \theta) \\ &+ (1+k)\tilde{\Phi}_\lambda(e_z(t, \theta), \xi(t, \theta), u(\lambda^{-1}t + \theta - h_2)), \end{aligned} \quad (17b)$$

with boundary condition  $e_z(t, -h_1) = e(t)$ , and the observer error dynamics

$$\begin{aligned} \lambda \dot{e}(t) &= \lambda(A + LC)e(t) \\ &+ \tilde{\Phi}_\lambda(e(t), \xi(t, -h_1), u(\lambda^{-1}t - h_1 - h_2)). \end{aligned} \quad (17c)$$

**Remark 5** A predictor based on partial differential equations was proposed in [19, Chapter 3] in a backstepping framework. The current approach is different since we are not trying to target a transport equation with a zero boundary condition (which would correspond to  $e_z(t, h_2) = 0$ ,  $k = -1$ ). In fact, the backstepping approach in the nonlinear case requires the computation of a kernel, a solution of a nonlinear differential equation. Here, the parameter  $k$  will play a role in the convergence rate, and equation (16b) is not a transport equation for  $k \neq -1$ . We can study the stability of the error system in the original coordinates without using a backstepping transformation.  $\dashv$

In the sequel, the stability of this error system is analysed. The main objective is to find the observer parameters  $(\lambda, L, k, \mu)$  that stabilise the origin of the error system exponentially fast.

### 3 Stability analysis

In this section, we use a Lyapunov functional to show that the prediction error converges to zero exponentially fast.

#### 3.1 Exponential stability

To alleviate the notation, allow us to define the aggregate error

$$\mathcal{E}(t, \cdot) = \begin{pmatrix} e(t) & e_w(t) & e_z(t, \cdot) \end{pmatrix}$$

and the aggregate norm

$$\|\mathcal{E}(t, \cdot)\|_A = \sqrt{|e(t)|^2 + |e_w(t)|^2 + \|e_z(t, \cdot)\|_A^2}.$$

**Lemma 2 ([24])** *If there exist  $\alpha_1, \alpha_2, \alpha_3 > 0$  and a functional  $\mathcal{V} : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L}^2(-h_1, h_2) \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} \alpha_1 \|\mathcal{E}(t, \cdot)\|_A^2 &\leq \mathcal{V}(\mathcal{E}(t, \cdot)) \leq \alpha_2 \|\mathcal{E}(t, \cdot)\|_A^2, \\ \dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) &\leq -\alpha_3 \mathcal{V}(\mathcal{E}(t, \cdot)), \end{aligned}$$

where  $\dot{\mathcal{V}}$  is the Dini derivative of  $\mathcal{V}$  along the trajectories of (17). Then,  $\mathcal{V}$  is a Lyapunov functional for system (17) and, for all  $t \geq 0$ ,

$$\|\mathcal{E}(t, \cdot)\|_A \leq \frac{\alpha_2}{\alpha_1} e^{-\alpha_3 t} \|\mathcal{E}(0, \cdot)\|_A.$$

Our first main result, stated below, guarantees the existence of a predictor for any desired decay rate.

**Theorem 1** *Under Assumption 1, for any  $\alpha > 0$  and  $k > 0$ , there exist parameters  $(\lambda, L, \mu)$  such that the solution of system (17) satisfies*

$$\|\mathcal{E}(t, \cdot)\|_A^2 \leq \beta e^{-\alpha t} \|\mathcal{E}(0, \cdot)\|_A^2, \quad (18)$$

where

$$\beta = \frac{\bar{P} + p \exp(\varrho(h_1 + h_2)) + q}{\min\{\underline{P}, p, q\}}, \quad (19)$$

and  $P, p$ , and  $q$  are given in (21).

*Proof :* Consider the functional [8]

$$\mathcal{V}(\mathcal{E}(t, \cdot)) = e(t)^\top P e(t) + q |e_w(t)|^2 + p \|\rho(\cdot) e_z(t, \cdot)\|_A^2, \quad (20)$$

where  $P \succ 0$ ,  $p, q > 0$  and  $\rho(\theta) = \exp(-\frac{\varrho(\theta - h_2)}{2})$  with  $r > 0$  to be fixed later. This functional satisfies

$$\begin{aligned} \mathcal{V}(\mathcal{E}(t, \cdot)) &\geq \min\{\underline{P}, p, q\} \|\mathcal{E}(t, \cdot)\|_A^2, \\ \mathcal{V}(\mathcal{E}(t, \cdot)) &\leq (\bar{P} + p \exp(\varrho(h_1 + h_2)) + q) \|\mathcal{E}(t, \cdot)\|_A^2. \end{aligned}$$

Its time derivative along the trajectories of system (17),  $\mathcal{W}(\mathcal{E}(t, \cdot)) := \lambda \dot{\mathcal{V}}(\mathcal{E}(t, \cdot))$ , is provided in (†). It follows from (8) that

$$|\tilde{\Phi}_\lambda(e_z(t, \theta), \xi(t, \theta), u(\lambda^{-1}t + \theta - h_2))| \leq \gamma |e_z(t, \theta)|$$

for all  $\theta \in [-h_1, h_2]$ . Hence,

$$\begin{aligned} \mathcal{W}(\mathcal{E}(t, \cdot)) &\leq 2e^\top(t) (\lambda P(A + LC) + \gamma \bar{P} I) e(t) \\ &- 2q\mu |e_w(t)|^2 + 2q(\gamma + \lambda \bar{A} + \mu) |e_w(t)| |e_z(t, h_2)| \\ &- \left[ pk\rho^2(\theta) |e_z(t, \theta)|^2 \right]_{-h_1}^{h_2} - pkr \int_{-h_1}^{h_2} |e_z(t, \theta)|^2 d\theta \\ &+ 2p(1+k)(\lambda \bar{A} + \gamma) \int_{-h_1}^{h_2} \rho^2(\theta) |e_z(t, \theta)|^2 d\theta. \end{aligned}$$

Noting that  $\bar{A} = 1$ ,  $e_z(t, -h_1) = e(t)$  and applying Young inequality on the crossed term, we obtain

$$\begin{aligned} \mathcal{W}(\mathcal{E}(t, \cdot)) &\leq 2\lambda e^\top(t) P(A + LC) e(t) - q\mu |e_w(t)|^2 \\ &+ 2(\gamma \bar{P} + pk \exp(\varrho(h_1 + h_2))) |e(t)|^2 \\ &+ \left( q \frac{(\gamma + \lambda + \mu)^2}{\mu} - pk \right) |e_z(t, h_2)|^2 \\ &+ p(2(1+k)(\lambda + \gamma) - k\varrho) \|\rho(\cdot) e_z(t, \cdot)\|_A^2. \end{aligned}$$

---


$$\begin{aligned} \mathcal{W}(\mathcal{E}(t, \cdot)) &= 2e^\top(t)P \left( \lambda(A + LC)e(t) + \tilde{\Phi}_\lambda(e(t), \xi(t, -h_1), u(\lambda^{-1}t - h_1 - h_2)) \right) - 2q\mu|e_w(t)|^2 \\ &+ 2qe_w^\top(t) \left( \tilde{\Phi}_\lambda(e_z(t, h_2), \xi(t, h_2), u(\lambda^{-1}t)) + (\lambda A + \mu I)e_z(t, h_2) \right) - pk \int_{-h_1}^{h_2} \rho^2(\theta) \frac{\partial}{\partial \theta} |e_z(t, \theta)|^2 d\theta \\ &+ 2p(1+k) \int_{-h_1}^{h_2} \rho^2(\theta) e_z^\top(t, \theta) \left( \lambda A e_z(t, \theta) + \tilde{\Phi}_\lambda(e_z(t, \theta), \xi(t, \theta), u(\lambda^{-1}t + \theta - h_2)) \right) d\theta. \quad (\dagger) \end{aligned}$$


---

We can now select

$$\begin{aligned} \mu &= \lambda\alpha, \quad q = \frac{\alpha}{\lambda}, \quad pk = (\gamma + 1 + \alpha)^2, \\ \varrho &= \lambda \left( \frac{\alpha + 2(1+k)(1+\gamma)}{k} \right), \quad (21) \end{aligned}$$

and  $\lambda$  is to be determined later. Moreover, we choose matrices  $L$  and  $P \succ 0$  such that (5) holds with  $Q = Q_\alpha$ ,

$$Q_\alpha = (1 + 2(\gamma + 1 + \alpha) \exp(\varrho(h_1 + h_2))) I.$$

Thus, we have

$$\dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) \leq -\alpha \mathcal{V}(\mathcal{E}(t, \cdot)) - (\lambda - 2\gamma\bar{P}) |e(t)|^2.$$

The proof is concluded by choosing a sufficiently large high-gain parameter  $\lambda \geq 2\gamma\bar{P}$  and invoking Lemma 2.  $\square$

Theorem 1 shows that, for system (1), it is possible to construct a predictor whose error goes to zero at any prescribed decay rate.

In the following subsection, our stability statement is enriched and quantified in the presence of disturbances and noises.

### 3.2 Input-to-state stability

Allow us to augment the model (1) as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \phi(x(t), u(t - h_2)) + \eta_x(t), \\ y(t) &= Cx(t - h_1) + \eta_y(t), \end{aligned} \quad (22)$$

where  $\eta_x$  and  $\eta_y$  are the perturbation and measurement noise, assumed to be bounded for any  $t \geq 0$ .

With the same observer (16), the error model for the filter is

$$\begin{aligned} \lambda \dot{e}_w(t) &= -\mu e_w(t) + (\lambda A + \mu I)e_z(t, h_2) + \delta_x(t, h_2) \\ &+ \tilde{\Phi}_\lambda(e_z(t, h_2), \xi(t, h_2), u(\lambda^{-1}t)), \quad (23a) \end{aligned}$$

where  $\delta_x(t, \theta) = \Lambda(\lambda^{-1})\eta_x(\lambda^{-1}t + \theta)$ . The distributed predictor error is now

$$\begin{aligned} \lambda \frac{\partial}{\partial t} e_z(t, \theta) &= -k \frac{\partial}{\partial \theta} e_z(t, \theta) + (1+k)\lambda A e_z(t, \theta) + \delta_x(t, \theta) \\ &+ (1+k)\tilde{\Phi}_\lambda(e_z(t, \theta), \xi(t, \theta), u(\lambda^{-1}t + \theta - h_2)), \end{aligned} \quad (23b)$$

and the observer error is

$$\begin{aligned} \lambda \dot{e}(t) &= \lambda(A + LC)e(t) + \delta_x(t, -h_1) + L\delta_y(t) \\ &+ \tilde{\Phi}_\lambda(e(t), \xi(t, -h_1), u(\lambda^{-1}t - h_1 - h_2)), \end{aligned} \quad (23c)$$

where  $\delta_y(t) = \eta_y(\lambda^{-1}t)$ .

The robustness with respect to perturbations and measurement noises is addressed via Lyapunov analysis and the following dissipation lemma.

**Lemma 3 ([24])** *If there exist  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$  and  $\mathcal{V} : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L}^2(-h_1, h_2) \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} \alpha_1 \|\mathcal{E}(t, \cdot)\|_A^2 &\leq \mathcal{V}(\mathcal{E}(t, \cdot)) \leq \alpha_2 \|\mathcal{E}(t, \cdot)\|_A^2, \\ \dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) &\leq -\alpha_3 \mathcal{V}(\mathcal{E}(t, \cdot)) + \alpha_4 \eta_\infty, \end{aligned}$$

where  $\dot{\mathcal{V}}(\mathcal{E}(t, \cdot))$  is the Dini derivative of  $\mathcal{V}$  along the trajectories of (23). Then,  $\mathcal{V}$  is an input-to-state Lyapunov functional for system (23) and, for all  $t \geq 0$ ,

$$\|\mathcal{E}(t, \cdot)\|_A^2 \leq \max \left\{ \frac{\alpha_2}{\alpha_1} e^{-\frac{\alpha_3}{2} t} \|\mathcal{E}(t, \cdot)\|_A^2, \frac{2\alpha_4}{\alpha_1 \alpha_3} \eta_\infty \right\}.$$

Our second main result guarantees the input-to-state stability with respect to uncertainties and provides estimates of the corresponding input-to-state gain.

**Theorem 2** *Let  $\alpha > 0$ . Under Assumption 1, for any  $k \geq 0$ , there exists observer gains  $(\lambda, L, \mu)$  such that the solution of system (23) satisfies*

$$\|\mathcal{E}(t, \cdot)\|_A \leq \beta e^{-\alpha t} \|\mathcal{E}(t, \cdot)\|_A + \frac{\beta(1 + \bar{L})}{(\alpha\lambda)^2} \eta_\infty, \quad (24)$$

$$\eta_\infty = \sup_{t \geq 0} \max \{ |\delta_x(t, 0)|^2, |\delta_y(t)|^2 \},$$

where  $\beta$  in (19) and  $P, p$ , and  $q$  set as in (25).

*Proof*: The time derivative of the functional (20) along the trajectories of system (23) is

$$\begin{aligned} \lambda \dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) &= \mathcal{W}(\mathcal{E}(t, \cdot)) + 2qe_w^\top(t)\delta_x(t, h_2) \\ &+ 2p \int_{-h_1}^{h_2} \rho^2(\theta)e_z^\top(t, \theta)\delta_x(t, \theta)d\theta \\ &+ 2e^\top(t)P(\delta_x(t, -h_1) + L\delta_y(t)), \end{aligned}$$

where  $\mathcal{W}$  is again given in (†). Applying Young's inequality yields

$$\begin{aligned} \lambda \dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) &\leq \mathcal{W}(\mathcal{E}(t, \cdot)) + \alpha\lambda\mathcal{V}(\mathcal{E}(t, \cdot)) \\ &+ \frac{\bar{P} + p \exp(\varrho(h_1 + h_2)) + q + \bar{P}\bar{L}}{\alpha\lambda} \eta_\infty. \end{aligned}$$

Similarly to the proof of Theorem 1, replacing  $\alpha$  by  $3\alpha$ , we select first

$$\begin{aligned} \mu &= 3\lambda\alpha, \quad q = \frac{3\alpha}{\lambda}, \quad pk = (\gamma + 1 + 3\alpha)^2, \\ \varrho &= \lambda \left( \frac{3\alpha + 2(1+k)(1+\gamma)}{k} \right). \end{aligned} \quad (25)$$

Then, select matrices  $L$  and  $P \succ 0$  such that (5) holds with  $Q = Q_{3\alpha}$  and, finally,  $\lambda = 2\gamma\bar{P}$ . We end up with the following inequality:

$$\dot{\mathcal{V}}(\mathcal{E}(t, \cdot)) \leq -2\alpha\mathcal{V}(\mathcal{E}(t, \cdot)) + \frac{\beta \min\{\bar{P}, p, q\}(1 + \bar{L})}{\alpha\lambda^2} \eta_\infty.$$

According to Lemma 3,  $\mathcal{V}$  is an input-to-state Lyapunov functional, and the inequality (24) is satisfied.  $\square$

Theorem 2 shows that the proposed observer is robust with respect to perturbations and measurement noises. For any decay rate  $\alpha > 0$ , the input-to-state gain is proportional to  $\frac{1}{\alpha^2}$ , i.e., high convergence rates lead to high input-to-state gains. Such a trade-off between performance and robustness is discussed in the numerical section.

## 4 Numerical example

Consider a two-dimensional bioreactor system [9] in the canonical form with input and output delays

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_1(t)u(t - h_2), \\ \dot{x}_2(t) &= -x_2(t)u(t - h_2) + x_1(t)x_2(t), \\ y(t) &= x_1(t - h_1) + \eta_y(t), \end{aligned} \quad (26)$$

where  $u \in \mathcal{U} = [u_m, u_M]$ ,  $u_m > 0$ . Assumption 1 is met with  $\gamma_\varphi = u_m$ . In the sequel, let  $u(t) = 0.2 + 0.1 \sin(\pi t)$  and  $h_1 = 0.5$ ,  $h_2 = 0.1$ .

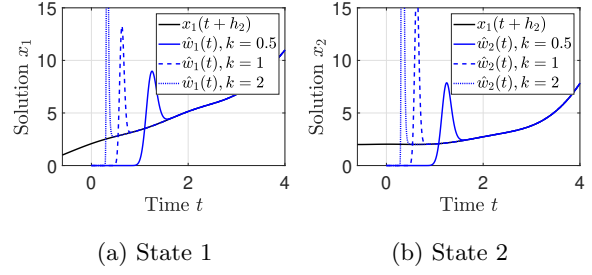


Fig. 2. Estimation of the state  $x(t + h_2)$  for  $k \in \{0.5, 1, 2\}$ .

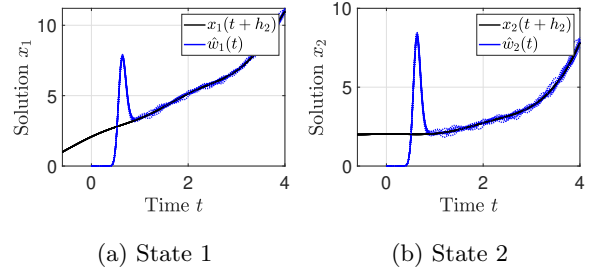


Fig. 3. Estimation of the state  $x(t + h_2)$  with output noise.

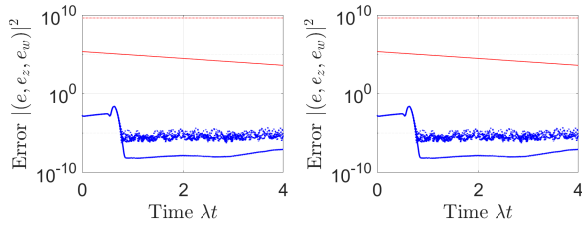
### 4.1 Without noise ( $\eta = 0$ )

Consider the observer (16) with the gains  $(\lambda, L, \mu)$  set in (21) and with  $\alpha = 1$ . Consider three possible gains  $k \in \{0.5, 1, 2\}$ . For initial conditions  $x(0) = (1, 2)$ , the state  $x(t)$  is depicted in black in Fig. 2. The estimated states  $(\hat{z}(t, -h_1), \hat{z}(t, \theta), \hat{w}(t))$  of the observer are computed for  $t \in [0, 4]$  using an Euler explicit scheme and null initial conditions. We can see that the solution  $\hat{w}(t)$ , plotted in blue in Fig. 2, converges towards the expected solution. Indeed, Theorem 1 guarantees the exponential convergence of the error for any  $k \geq 0$ . However, note that the damping ratio increases when  $k$  increases. The initial error seems to decay more quickly but it suffers a large overshoot. Lastly, the error norm represented in Fig 4a and the inequality (18) is confirmed. Note that the error norm reaches an ultimate bound due to numerical implementation.

### 4.2 With measurement noise ( $\eta_y \in [-0.1, 0.1]$ )

Consider the observer (16) with the gains  $(\lambda, L, \mu)$  set in (25), with  $k = 1$  and  $\alpha = 1$ . The simulation is run ten times with several random noises  $\eta_y \in [-0.1, 0.1]$ . As illustrated in Fig. 3, the estimated state  $\hat{w}(t)$  is still close to the expected solution  $\hat{x}(t + h_2)$ . As proven in Theorem 2, the input-to-state stability is obtained and our observer is robust to measurement noise. In steady state, an upper bound of the error norm is given by  $0.1 \frac{\beta(1+\bar{L})}{\lambda^2} \sim 10^{10}$ . In Fig 4b, the bound expressed in (18) is satisfied. It is very conservative and would benefit from improvement.





(a) No noise,  $k \in \{0.5, 1, 2\}$       (b) Noise,  $k = 1$

Fig. 4. Upper bound for the error norm  $\|\mathcal{E}(t, \cdot)\|_A$ .

## 5 Conclusions

In this paper, we have designed an observer for nonlinear time-delay systems. The nonlinear part is handled by a large gain observer and the input and output delays are managed by an infinite number of sub-predictors. Therefore, we have proposed a new high-gain distributed observer. Using a Lyapunov analysis, we have succeeded in expressing explicitly the different observation gains required for the error system to be exponentially stable and input-to-state stable with respect to state or measurement noises. An example taken from bioprocesses corroborates our statements.

Future research works will be dedicated to more general robustness results, involving the robustness to state parameters, delay uncertainties, or time-varying delays [20]. The influence of time sampling or spacial discretisation should also be studied in detail to obtain an accurate numerical method to solve the hyperbolic partial differential equation involved in the observer.

## References

- [1] T. Ahmed-Ali, E. Cherrier, and F. Lamnabhi-Lagarrigue. Cascade high gain predictors for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1), 01 2012.
- [2] V. Andrieu, L. Praly, and A. Astolfi. High-gain observers with updated gain and homogeneous correction terms. *Automatica*, 45(2):422–428, 2009.
- [3] G. Bastin. *On-line estimation and adaptive control of bioreactors*, volume 1. Elsevier, 2013.
- [4] G. Besançon. *High-gain observation with disturbance attenuation and application to robust fault detection*, volume 39. 2003.
- [5] G. Besançon. *Nonlinear Observers and Applications*. Springer, 2007.
- [6] D.E. Contois. Kinetics of bacterial growth: relationship between population density and specific growth rate of continuous cultures. *Microbiology*, 21(1):40–50, 1959.
- [7] I. Estrada-Sánchez, M. Velasco-Villa, and H. Rodríguez-Cortés. Prediction-based control for nonlinear systems with input delay. *Mathematical Problems in Engineering*, 2017(5):1–11, 10 2017.
- [8] E. Fridman and U. Shaked. An improved stabilization method for linear time-delay systems. *IEEE transactions on automatic control*, 47(11):1931–1937, 2002.
- [9] J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control*, 37:875–880, 1992.
- [10] A. Germani, C. Manes, and P. Pepe. A new approach to state observation of nonlinear systems with delayed output. *IEEE Transactions on Automatic Control*, 47(1):96–101, 2002.
- [11] J.K. Hale and S.M.V. Lunel. Stability and control of feedback systems with time delays. *International Journal of Systems Science*, 34(8-9):497–504, 2003.
- [12] H. Hammouri, B. Targui, and F. Armanet. High gain observer based on a triangular structure. *International Journal of Robust and Nonlinear Control*, 12(6):497–518, 2002.
- [13] I. Karafyllis and M. Krstic. Stabilization of nonlinear delay systems using approximate predictors and high-gain observers. *Automatica*, 49(12):3623–3631, 2013.
- [14] I. Karafyllis and M. Krstic. *Predictor Feedback for Delay Systems: Implementations and Approximations*. Systems and Control: Foundations and Applications. Birkhäuser, 2017.
- [15] H.K. Khalil. *Nonlinear Systems*. Pearson Education. Prentice Hall, 2002.
- [16] V. Kharitonov. An extension of the prediction scheme to the case of systems with both input and state delay. *Automatica*, 50(1):211–217, 2014.
- [17] V. Kharitonov. Predictor-based controls: the implementation problem. *Differential Equations*, 51(13):1675–1682, 2015.
- [18] S.J. Kong and D. Bresch-Pietri. Prediction-based controller for linear systems with stochastic input delay. *Automatica*, 138:110149, 2022.
- [19] M. Krstic. *Delay compensation for nonlinear, adaptive, and PDE systems*. Systems and Control: Foundations and Applications. Birkhäuser, 2009.
- [20] V. Léchappé, E. Moulay, and F. Plestan. Prediction-based control of LTI systems with input and output time-varying delays. *Systems and Control Letters*, 112:24–30, 2018.
- [21] A.Z. Manitius and A.W. Olbrot. Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4):541–553, 08 1979.
- [22] F. Mazenc and M. Malisoff. Stabilization of nonlinear time-varying systems through a new prediction based approach. *IEEE Transactions on Automatic Control*, 62(6):2908–2915, 2017.
- [23] F. Mazenc and M. Malisoff. Continuous discrete sequential observers for time-varying systems under sampling and input delays. *IEEE Transactions on Automatic Control*, 65(4):1704–1709, 2020.
- [24] A. Mironchenko. *Input-to-State Stability*. Communications and Control Engineering. Springer, 2023.
- [25] A. Mironchenko and C. Priour. Input-to-state stability of infinite-dimensional systems: Recent results and open questions. *SIAM Review*, 62(3):529–614, 2020.
- [26] S. Mondié and W. Michiels. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 48(12):2207–2212, 2003.
- [27] M. Najafi, S. Hosseinnia, F. Sheikholeslam, and M. Karimadini. Closed-loop control of dead time systems via sequential sub-predictors. *International Journal of Control*, 86(4):599–609, 2013.
- [28] J.E. Normey-Rico and E.F. Camacho. Dead-time compensators: A survey. *Control Engineering Practice*, 16:407–428, 2008.

- [29] B. Rojas-Ricca, F. Castaños, and S. Mondié. Dominant-pole placement for predictor synthesis, 2023.
- [30] B. Zhou, Q. Liu, and F. Mazenc. Stabilization of linear systems with both input and state delays by observer-predictors. *Automatica*, 83:368–377, 2017.