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Hyperbolic isometries of the fine curve graph of higher genus surfaces

Pierre-Antoine Guihéneuf, Emmanuel Militon

November 6, 2023

Abstract

We prove that for a homeomorphism f that is isotopic to the identity on a closed hyperbolic surface, the following are equivalent:

- f acts hyperbolically on the fine curve graph;
- f is isotopic to a pseudo-Anosov map relative to a finite f-invariant set;
- the ergodic homological rotation set of f has nonempty interior.

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1 Introduction

Let S be a closed orientable surface with genus ≥ 2. We denote $\mathrm{Homeo}(S)$ the space of homeomorphisms of S, and $\mathrm{Homeo}_0(S)$ the connected component of Id_S in $\mathrm{Homeo}(S)$.

The *fine curve graph* $C^\dagger(S)$ of S was introduced by Bowden, Hensel, and Webb [BHW22] to give a counterpart of the classical *curve graph* adapted to the study of the group of all homeomorphisms of S.

Definition 1. The *fine curve graph* on the surface S is the graph $C^\dagger(S)$ whose vertices are essential[1] simple loops. There is an edge between two vertices α and β if and only if the loops α and β are disjoint.

As a consequence of the Gromov hyperbolicity of the classical curve graphs for punctured surfaces, it was proved in [BHW22] that the fine curve graph $C^\dagger(S)$ is Gromov

[1] *I.e.* non contractible.

hyperbolic. The authors also prove that this graph is connected and has infinite diameter. This enables them to use large scale geometry techniques to study $\mathrm{Homeo}(S)$ via its action on $C^\dagger(S)$. As an application, they prove that, for any closed surface S of genus ≥ 1, the commutator length and the fragmentation norm on $\mathrm{Homeo}_0(S)$ are unbounded, answering a question posed by Burago, Ivanov, and Polterovich [BIP08].

In the same way as the mapping class group $\mathrm{Map}(S)$ acts on $C(S)$ by isometries, the whole homeomorphism group $\mathrm{Homeo}(S)$ acts on $C^\dagger(S)$ by isometries. Gromov has classified isometries of Gromov hyperbolic spaces (see [Gro87, paragraph 8] or [BH99]) according to the asymptotic translation length, defined for an isometry g of a Gromov hyperbolic space X as

$$|g|_X = \lim_{n\to+\infty} \frac{1}{n} d_X\big(x, g^n(x)\big).$$

It is a standard exercise to see that this limit exists and is independent of x. This independence immediately implies that the asymptotic translation length is a conjugacy invariant of isometries of X. Gromov classification is then as follows: for g an isometry of a Gromov hyperbolic space, g is

- **Hyperbolic** (or loxodromic) if the asymptotic translation length is positive;
- **Parabolic** if the asymptotic translation length is zero but g has no finite diameter orbits, and
- **Elliptic** if g has finite diameter orbits.

There is an equivalent reformulation of this trichotomy in terms of fixed points on the Gromov boundary of X, but we do not require this point of view in the present work.

The present work aims at starting the classification of the type of isometry actions of $f \in \mathrm{Homeo}_0(S)$ on $C^\dagger(S)$ in terms of rotational properties of f.[2] Such a classification is now completed in the case of the torus due to works of Bowden, Hensel, Mann, Militon and Webb [BHM+22] and Guihéneuf and Militon [GM23]. In particular, we have the following.

Theorem 2 ([BHM+22], Theorem 1.3)**.** *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. The following are equivalent:*

- *f acts hyperbolically on $C^\dagger(\mathbb{T}^2)$;*
- *the* rotation set *of f has non-empty interior;*
- *there is a finite, f-invariant set $F \subset \mathbb{T}^2$ such that the restriction of f to $\mathbb{T}^2 \setminus F$ represents a pseudo-Anosov mapping class.*

In the present work we get a counterpart of this statement for closed surfaces of genus $g \geq 2$, in terms of ergodic rotation set $\rho_{erg}(f) \subset H_1(S, \mathbb{R})$ of f, which is defined in the next section.

Theorem A. *Let $f \in \mathrm{Homeo}_0(S)$. The following are equivalent:*

[2] Note that by a theorem of Long, Margalit, Pham, Verberne and Yao [LMP+21], any isometry of $C^\dagger(S)$ is induced by the action of some homeomorphism.

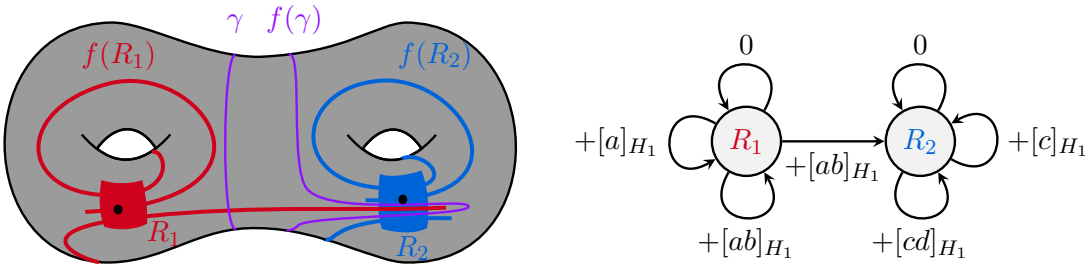


Figure 1: Example of a homeomorphism of the surface of genus 2 having homological rotation set with nonempty interior but homological ergodic rotation set included in the union of two planes. The black dots are contractible fixed points, the thick lines represent the images by f of the rectangles R_1 and R_2; the intersections of all rectangles are supposed to be markovian.

1. *The following does not hold. Either $\rho_{erg}(f)$ is contained in a hyperplane of $H_1(S,\mathbb{R})$, or there exists two nonempty rational subspaces E and F of $H_1(S,\mathbb{R})$, which are orthogonal for the intersection form $\wedge$ (defined after the theorem), and such that $\rho_{erg}(f) \subset E \cup F$.*

2. *The ergodic rotation set $\rho_{erg}(f)$ has nonempty interior in $H_1(S,\mathbb{R})$.*

3. *There exists a finite f-invariant subset $F \subset S$ such that f is isotopic to a pseudo-Anosov homeomorphism relative to F (this notion is recalled in Section 4).*

4. *The homeomorphism f acts hyperbolically on $C^\dagger(S)$.*

The intersection form $\wedge$ on $H_1(S,\mathbb{R})$ can be obtained from the cup product on $H^1(S,\mathbb{R}) \times H^1(S,\mathbb{R}) \to H^2(S,\mathbb{R})$ that induces by Poincaré duality a symplectic form

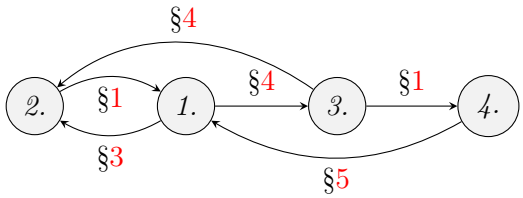
$$\wedge : H_1(S,\mathbb{R}) \times H_1(S,\mathbb{R}) \to \mathbb{R}.$$

This intersection form has a geometrical interpretation in the case of elements of $H_1(S,\mathbb{Z})$: if α and β are closed curves, then $[\alpha]_{H_1} \wedge [\beta]_{H_1}$ is the algebraic intersection number between α and β. For more details about these facts, see [Lel19, Section 1.1].

Example 3. No similar characterization hold with the more classical homological rotation set (in the sense of Definition 4), as seen in the example of Figure 1 which is a homeomohphism of the suface of genus 2 having homological rotation set with nonempty interior but homological ergodic rotation set included in the union of two planes.
Indeed, the fact that a separating curve γ is mapped to a curve disjoint from $f(\gamma)$ implies that the rotation vectors of recurrent points are included in the union of two planes that are the homologies of the surfaces respectively on the left and on the right of γ.
The rotation set of f in the sense of Definition 4 contains the rotation set of the labeled Markov shift represented on the right (associated to the different markovian intersections between R_1, R_2 and their images by f). Here, a and b are generators of the fundamental group of the surface on the left of γ, and c and d are generators of the fundamental group of the surface on the right of γ.
Hence, this homeomorphism has rotation set with nonempty interior but by Theorem A does not act hyperbolically on $C^\dagger(S)$.

Plan of the proof

The diagram of the proof of implications of theorem A is the following:



The implication *2.* $\implies$ *1.* is trivial.

The implication *1.* $\implies$ *2.* is a consequence of the shape of the ergodic rotation set that we study in Section 3 (Proposition 10). These results are based on a theorem by Lellouch [Lel19] that we state as Theorem 8. The fact that not only $\overline{\rho_{erg}(f)}$ but also $\rho_{erg}(f)$ has nonempty interior is done in Lemma 14 of Section 4.

The implication *1.* $\implies$ *3.* is a consequence of the Nielsen-Thurston classification. The proof is standard and similar to the ones due to Llibre-MacKay [LM91] and Pollicott [Pol92]. We prove it in Section 4, using results of Section 3.

The implication *3.* $\implies$ *4.* is standard and is a consequence of [BHW22, Lemma 4.2] by Bowden, Hensel and Webb. Indeed, by this lemma, the asymptotic translation length of the action of f on the fine curve graph of S is at least the asymptotic translation length of the action of its mapping class on the curve graph of $S \setminus F$. But the latter does not vanish when it is a pseudo-Anosov element by [MM99, Proposition 4.6] by Masur and Minsky.

The most difficult implication is *4.* $\implies$ *1.*. We will actually prove that if *1.* does not hold, then f cannot act hyperbolically on $C^\dagger(M)$. The last section of this article (Section 5) is devoted to the proof of this implication; it uses crucially the results of using results of Section 3.

The next section (Section 2) is devoted to some standard facts about the homological rotation set.

Acknowledgements

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2 Some useful results about the homological rotation set

Let us start by defining the homological rotation set that we use in this article. The definition we use here is different from the one which is used by Lellouch in [Lel19]. More precisely, the latter rotation set is the convex hull of the rotation set that we define here, which is not necessarily convex. We also prove standard results about this homological rotation set which will be useful later.

We fix an isotopy $(f_t)_{t\in[0,1]}$ with $f_0 = \mathrm{Id}_S$ and $f_1 = f$. We extend this isotopy to $\mathbb{R}$ by setting, for any $t \in \mathbb{R}$, $f_t = f_{t-\lfloor t\rfloor} \circ f^{\lfloor t\rfloor}$, where $\lfloor t\rfloor$ is the lower integer part of t.

We endow the surface S with a hyperbolic metric. For any points x and y of S, we fix a minimizing geodesic $g_{x,y}$ joining x to y.

For any point x in S and any integer $n \geq 0$, we define a cycle by concatenating the path $(f_t(x))_{t\in[0,n]}$ with the path $g_{f^n(x),x}$. We denote by $c_{x,n} \in H_1(S,\mathbb{R})$ the homology class of this cycle.

Definition 4. We call *homological rotation set* of f the subset of $H_1(S,\mathbb{R})$ which consists of limit values of sequences of the form $(\frac{1}{n}c_{x_n,n})$, where (x_n) is a sequence of points of S.

This homological rotation set is, by construction, a compact subset of $H_1(S,\mathbb{R})$.

Let $x \in S$. If the sequence $(\frac{1}{n}c_{x,n})_n$ converges, we say that the point x admits a homological rotation vector. In this case, the limit $r(x)$ of the sequence $(\frac{1}{n}c_{x,n})_n$ is called the homological rotation vector of x. Observe that any periodic point of f admits a homological rotation vector.

We can also associate rotation vectors to ergodic f-invariant measures, by the following lemma.

Lemma 5. *Let μ be an ergodic f-invariant Borel probability measure on S. Then there exists $r(\mu) \in \rho_{H_1}(f)$ such that μ-almost any point x of S admits a rotation vector which is equal to $r(\mu)$. We call $r(\mu)$ the homological rotation vector associated to μ.*

We denote by $\rho_{erg}(f)$ the subset of $\rho_{H_1}(f)$ consisting of rotation vectors associated to ergodic measures.

Proof. For any point $x \in S$, we denote by $I(x)$ the path $(f_t(x))_{t\in[0,1]}$. We see the cohomology group $H^1(S,\mathbb{R})$ as the quotient of the space of smooth closed 1-forms on S by the space of exact 1-forms on S. The map d from $H_1(S,\mathbb{R})$ to the algebraic dual $\left(H^1(S,\mathbb{R})\right)^*$, which, to an element $\gamma \in H_1(S,\mathbb{R})$, associates the linear form $\omega \mapsto \int_\gamma \omega$, is an isomorphism.

Hence it suffices to prove that for μ-almost every point x of S and for any $\omega \in H^1(S,\mathbb{R})$,

$$\lim_{n\to+\infty} \frac{1}{n}\int_{c(x,n)} \omega = \int_S \int_{I(y)} \omega \, \mathrm{d}\mu(y).$$

Observe that the right-hand side of the above equality vanishes if ω is exact as the measure μ is f-invariant.

But this is a straightforward consequence of Birkhoff ergodic theorem applied to the bounded functions $x \mapsto \int_{I(x)} \omega$ and of the fact that

$$\lim_{n\to+\infty} \frac{1}{n}\int_{g_{f^n(x),x}} \omega = 0.$$

This last assertion holds as the length of $g_{f^n(x),x}$ is bounded from above by the diameter of S. □

Lemma 6. *Any extremal point in* $\mathrm{conv}(\rho_{H_1}(f))$ *is a homological rotation vector associated to some ergodic f-invariant Borel probability measure on S.*

Proof. We will use once again the isomorphism

$$\begin{array}{rccc} d: & H_1(S,\mathbb{R}) & \longrightarrow & \left(H^1(S,\mathbb{R})\right)^* \\ & \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right). \end{array}$$

Moreover, we saw during the proof of Lemma 5 that, for any ergodic invariant probability measure μ,

$$r(\mu) = d^{-1}\left(\omega \mapsto \int_S \int_{I(x)} \omega \, \mathrm{d}\mu(x)\right). \tag{2.1}$$

Let us denote by $\mathcal{M}(f)$ the set of f-invariant Borel probability measures on S.

Let

$$\rho_{mes}(f) = d^{-1}\left(\left\{\omega \mapsto \int_S \int_{I(x)} \omega \, \mathrm{d}\mu(x),\ \mu \in \mathcal{M}(f)\right\}\right).$$

This subset of $H_1(S, \mathbb{R})$ is the rotation set which is used in Lellouch's thesis [Lel19].

To prove the lemma, we need the following claim.

Claim 7.

$$\mathrm{conv}(\rho_{H_1}(f)) = \mathrm{conv}(\rho_{erg}(f)) = \rho_{mes}(f).$$

Proof of Claim 7. The right-hand side equality is a consequence of the fact that the set $\mathcal{M}(f)$ is the convex hull of the set of ergodic f-invariant probability measures on S, and the linearity of the map d used to define $\rho_{mes}(f)$ (see (2.1)).

As, for any ergodic measure μ, the vector $d^{-1}\left(\omega \mapsto \int_S \int_{I(x)} \omega \, \mathrm{d}\mu(x)\right)$ belongs to $\rho_{erg}(f) \subset \rho_{H_1}(f)$ and as the set $\mathcal{M}(f)$ is the convex hull of ergodic measures, we obtain that

$$\rho_{mes}(f) \subset \mathrm{conv}(\rho_{H_1}(f)).$$

Conversely, fix $r \in \rho_{H_1}(f)$ and take a sequence of integers $n_k \to +\infty$ and a sequence (x_k) of points of S such that

$$\lim_{k \to +\infty} \frac{1}{n_k} c_{x_k, n_k} = r.$$

Extracting a subsequence if necessary, we can suppose that the sequence

$$(\mu_k)_k = \left(\frac{1}{n_k} \sum_{l=0}^{n_k - 1} \delta_{f^l(x_k)}\right)_k$$

converges to a f-invariant probability measure μ for the weak-$*$ topology. Then, for any closed 1-form ω,

$$\begin{aligned} d(r)(\omega) &= \lim_{k \to +\infty} \frac{1}{n_k} \int_{c_{x_k, n_k}} \omega \\ &= \lim_{k \to +\infty} \int_M \int_{I(x)} \omega \, \mathrm{d}\mu_k \\ &= \int_S \int_{I(x)} \omega \, \mathrm{d}\mu(x). \end{aligned}$$

This proves that $r \in \rho_{mes}(f)$. □

We now finish the proof of Lemma 6.

Take any extremal point $r \in \mathrm{conv}(\rho_{H_1}(f)) = \rho_{mes}(f)$. Observe that the subset

$$\left\{\mu \in \mathcal{M}(f) \mid \omega \mapsto \int_M \int_{I(x)} \omega \, \mathrm{d}\mu(x) = d(r)\right\}$$

of $\mathcal{M}(f)$ is convex. Hence any extremal point of this subset is an extremal point of $\mathcal{M}(f)$. Any such extremal point is an ergodic probability measure μ with $r(\mu) = r$. $\square$

The following theorem is a straightforward consequence of Theorem C p. 16 of [Lel19]. We denote by $\wedge$ the intersection form on $H_1(S, \mathbb{R})$.

Theorem 8 (Lellouch)**.** *Let r_1 and r_2 be vectors in $\rho_{erg}(f)$. Suppose that $r_1 \wedge r_2 \neq 0$. Then, for any $r \in \operatorname{conv}\{0, r_1, r_2\}$ and any $\epsilon > 0$, there exists $r' \in \rho_{H_1}(f) \cap B(r, \epsilon)$ that is associated to a periodic point.*

3 A decomposition of the rotation set

Definition 9. We define on $\rho_{erg}(f) \setminus \{0\}$ the equivalence relation $\sim$ by: $v \sim w$ if and only if there exists a sequence $(v_j)_{1 \leq j \leq m} \in \rho_{erg}(f) \setminus \{0\}$ such that $v_1 = v$, $v_n = w$ and, for any $j < m$, either v_j and v_{j+1} are collinear or $v_j \wedge v_{j+1} \neq 0$.

The fact that this is an equivalence relation is straightforward.

Observe that two distinct classes for this equivalence relation are orthogonal for the intersection form $\wedge$.

Taking the vector spaces generated by each of those equivalence classes, we obtain the following decomposition.

Proposition 10. *Let $f \in \mathrm{Homeo}_0(S)$. There exist pairwise orthogonal vector subspaces $V_1, V_2, \ldots, V_n$ of $H_1(S, \mathbb{R})$, with $n \leq g$, as well as a subset $\mathcal{L}$ of $H_1(S, \mathbb{R})$ such that:*

1. $\rho_{erg}(f) \subset \mathcal{L} \cup \bigcup_{i=1}^{n} V_i$.
2. *For any $1 \leq i \leq n$, V_i is a rational subspace of $H_1(S, \mathbb{R})$ whose dimension is at least 2.*
3. *The set $\mathcal{L}$ is a union of pairwise orthogonal lines containing 0 which are all orthogonal to any V_i.*
4. *For any i, the set $\overline{V_i \cap \rho_{erg}(f)}$ is a convex set containing 0, with nonempty interior in V_i, and with a dense subset of elements realized by periodic orbits.*
5. *For any $1 \leq i \leq n$ and any two vectors v, w in $\rho_{erg}(f) \cap V_i \setminus \{0\}$, $v \sim w$.*

Figure 2 displays an example where the ergodic rotation set is reduced to the set $\mathcal{L}$ but is uncountable and dense in a 2-dimensional convex subset of $H_1(S, \mathbb{R})$.

Note that a more precise decomposition of the ergodic homological rotation set, for transforming nonempty interior pieces in the set $\mathcal{L}$ into classes that are alike to the V_i appearing in proposition 10, is the subject of a work in progress by García, Guihéneuf and Lessa [GGL23].

Proof. Denote $(C_i)_{i \in I}$ the classes for the relation $\sim$ which generate a vector space with dimension at least 2, and define $V_i = \mathrm{Span}(C_i)$. Define $\mathcal{L}$ as the union of the lines generated by the other classes.

Let us prove that that the family $(V_i)_{i \in I}$ is in direct sum. For $i \in I$, consider $v_i \in V_i$ and $\lambda_i \in \mathbb{R}$, and suppose that $\sum_{i \in J} \lambda_i v_i = 0$, with J a finite subset of I. For some $j \in J$,

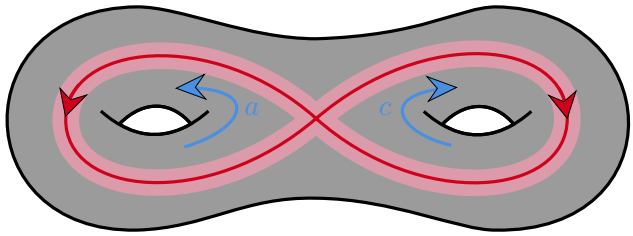

Figure 2: In this example, the homeomorphism f is the time one of a time dependent vector field which is identically zero outside the light red neighbourhood of the red closed curve. The red closed curve represents the trajectory of a periodic orbit of f under the isotopy I. One easily sees that the homological rotation set of f must be included in $\langle [a]_{H_1}, [c]_{H_1} \rangle$, which is a totally isotropic subspace of $H_1(S, \mathbb{R})$ for $\wedge$. However, by [GM22, Theorem E], the closure $\overline{\rho_{erg}(f)}$ of the ergodic rotation set has nonempty interior in this subspace. This gives an example where $\rho_{erg}(f) = \mathcal{L}$ but is uncontable.

by the definition of the V_j there exists $w_j \in V_j$ such that $v_j \wedge w_j \neq 0$. As the spaces V_i are pairwise orthogonal, we get that

$$0 = \left(\sum_{i \in J} \lambda_i v_i \right) \wedge w_j = \sum_{i \in J} \lambda_i (v_i \wedge w_j) = \lambda_j (v_j \wedge w_j),$$

and as $v_j \wedge w_j \neq 0$, this shows that $\lambda_j = 0$. Hence, the family $(V_i)_{i \in I}$ is in direct sum, and the cardinality of I can be at most g. We denote this family of vector spaces by $(V_i)_{1 \leq i \leq n}$, with $n \leq g$.

Point 1. of the proposition is immediate.

Let us prove point 2. Take $1 \leq i \leq n$. Let $\mu_1, \dots, \mu_k \in \rho_{erg}(f) \setminus \{0\}$ such that $(r(\mu_\ell))_{1 \leq \ell \leq k}$ form a basis of the vector subspace V_i associated to the equivalence class C_i. Then by definition of the equivalence relation $\sim$, for any $1 \leq \ell < k$, there exists $(\mu_\ell^j)_{1 \leq j \leq m_\ell} \in \mathcal{M}_{erg}(f)$ such that $\mu_\ell^1 = \mu_\ell$, $\mu_\ell^{m_\ell} = \mu_{\ell+1}$ and $r(\mu_\ell^j) \wedge r(\mu_\ell^{j+1}) \neq 0$ for any $1 \leq j < m_\ell$.

By Theorem 8, for any $\varepsilon > 0$, for any $1 \leq \ell \leq k$, there exists a periodic measure ν_ℓ such that $r(\nu_\ell) \in \rho_{H_1}(f) \cap B(r(\mu_\ell), \varepsilon)$. By taking ε sufficiently small, one can moreover suppose that for $1 \leq \ell < k$ one has $r(\nu_\ell) \wedge r(\mu_\ell^2) \neq 0$, and $r(\nu_k) \wedge r(\mu_{k-1}^{m_{k-1}-1}) \neq 0$. This implies that for any $1 \leq \ell \leq k$, one has $r(\nu_\ell) \in C_i$. Moreover, by taking ε small enough, one can ensure that the family $(r(\nu_\ell))_{1 \leq \ell \leq k}$ is free. Hence this family is free and has the same cardinality as a basis of V_i, so it is a basis of V_i, made of rational vectors.

Point 3. follows from the construction of $\mathcal{L}$.

To prove point 4. it suffices to prove that, given $r(\mu_1), r(\mu_2) \in C_i$, then the whole triangle spanned by $r(\mu_1), r(\mu_2)$ and 0 is accumulated by points of C_i which are realized by periodic orbits. By definition, there exists $(\mu^j)_{1 \leq j \leq m} \in \mathcal{M}_{erg}(f)$ such that $\mu^1 = \mu_1$, $\mu^m = \mu_2$ and $r(\mu^j) \wedge r(\mu^{j+1}) \neq 0$ for any $1 \leq j < m$. Up to shortening the family if necessary, one can suppose that $r(\mu^j) \wedge r(\mu^{j+2}) = 0$ for any $1 \leq j < m-1$. Let $\varepsilon > 0$, and let us build a sequence of periodic measures $(\nu^j)_{2 \leq j \leq m}$ by recurrence, satisfying:

- For any $2 \leq j \leq m$, $r(\nu^j) \in B(r(\mu_1), \varepsilon)$;

- For any $2 \leq j \leq m$, $r(\nu^j) \wedge r(\mu^j) \neq 0$.

Let us explain the first step of the construction, the other ones being identical. By Theorem 8, as $r(\mu^2) \wedge r(\mu^1) \neq 0$, for any $0 < \varepsilon' < \varepsilon/2$, the ball $B(\varepsilon/2r(\mu^2) + (1 - \varepsilon/2)r(\mu^1), \varepsilon')$ contains a periodic measure ν^1. If ε' is chosen small enough, then $r(\nu^1) \wedge r(\mu^3)$ is close enough to $(\varepsilon/2r(\mu^2) + (1-\varepsilon/2)r(\mu^1)) \wedge r(\mu^3) = \varepsilon/2r(\mu^2) \wedge r(\mu^3) \neq 0$, and in particular $r(\nu^1) \wedge r(\mu^3) \neq 0$.

Finally, the measure ν^m satisfies $r(\nu^m) \in B(r(\mu_1), \varepsilon)$ and $r(\nu^m) \wedge r(\mu_2) \neq 0$. It then suffices to apply Theorem 8 to ν^m and μ_2 to approximate any element of the triangle spanned by $r(\mu_1), r(\mu_2)$ and 0 by the rotation vector of a periodic measure, with distance proportional to ε (with the proportion coefficient depending on the norm of $r(\mu_1)$ and $r(\mu_2)$).

Point 5. is a direct consequence of the definition of the vector subspaces V_i. □

Recall that the intersection form on $H_1(S, \mathbb{R})$ defines a symplectic form on $H_1(S, \mathbb{R})$. Recall also that a symplectic vector subspace of $H_1(S, \mathbb{R})$ is a subspace on which the restricted intersection form is a symplectic form. We will need the following lemma.

Lemma 11. *Suppose that $\rho_{erg}(f)$ is not contained in a hyperplane. Then $\mathcal{L} = \emptyset$ and each of the subspaces V_i which appear in the decomposition given by Proposition 10 is a symplectic subspace of $H_1(S, \mathbb{R})$.*

As a consequence, we deduce that each subspace V_i is even-dimensional.

Proof. Observe that the set $\mathcal{L}$ has to be empty. Otherwise, the set $\rho_{erg}(f)$ would be contained in a hyperplane which consists of vectors which are orthogonal for the intersection form to one of the lines of $\mathcal{L}$.

Fix a subspace V_{i_0} given by Proposition 10. We prove the following statement by induction on $n \geq 1$. If $\dim(V_{i_0}) \geq 2n - 1$, then there exists a family $(e_i)_{1 \leq i \leq 2n}$ of independent vectors of V_{i_0} such that

1. For any $1 \leq i \leq n$, $e_{2i-1} \wedge e_{2i} = 1$ and $e_{2i} \wedge e_{2i+1} = 0$.
2. For any $1 \leq i, j \leq 2n$, if $|i - j| \geq 2$, then $e_i \wedge e_j = 0$.

Suppose $n = 1$. As the dimension of V_{i_0} is at least 2, we can find a nonzero vector $e_1 \in C_{i_0}$. By definition of C_{i_0} and as V_{i_0} is at least 2 dimensional by definition, there exists $e_2' \in V_{i_0}$ such that $e_1 \wedge e_2' \neq 0$. Changing e_2' into an appropriate collinear vector, we can find e_2 such that $e_1 \wedge e_2 = 1$. This last relation implies that the vectors e_1 and e_2 are independent.

Suppose that $\dim(V_{i_0}) \geq 2n+1$ and that there exists a family $(e_i)_{1 \leq i \leq 2n}$ of independent vectors of V_{i_0} which satisfy the above conditions. By hypothesis on the dimension of V_{i_0}, we can find a vector v of V_{i_0} which does not belong to the subspace generated by $(e_i)_{i \leq 2n}$. Now let

$$e_{2n+1} = v + \sum_{i=1}^{n} (v \wedge e_{2i-1}) e_{2i} - \sum_{i=1}^{n} (v \wedge e_{2i}) e_{2i-1}.$$

By construction of e_{2n+1}, we have $e_{2n+1} \neq 0$ and e_{2n+1} is orthogonal to the space generated by the e_i's, for $i \leq 2n$. Now, we claim that there exists a vector $w \in V_{i_0}$ such that $e_{2n+1} \wedge w \neq 0$. Indeed, otherwise, V_{i_0} would be contained in the orthogonal

of e_{2n+1} and hence, by Proposition 10, $\rho_{erg}(f)$ would be contained in a hyperplane, a contradiction.

By the same trick used to construct e_{2n+1}, we can suppose that w is orthogonal to all the vectors e_i, with $i \leq 2n$. Finally, we can find a vector e_{2n+2} which is collinear to w such that $e_{2n+1} \wedge e_{2n+2} = 1$. The vectors e_i, for $1 \leq i \leq 2n+2$, have to be independent. □

4 Big rotation set homeomorphisms are pseudo-Anosov

Suppose that case 1. of Theorem A holds. By Proposition 10, the set $\rho_{erg}(f)$ has nonempty interior in $H_1(S, \mathbb{R})$, and a dense subset of $\mathrm{Conv}\,(\rho_{erg}(f))$ belongs to $\rho_{erg}(f)$ and is realized by a periodic orbit. We shall prove that in this case, the homeomorphism f is isotopic to a pseudo-Anosov homeomorphism relative to a finite set. We will obtain it by classical arguments of [LM91] and [Pol92].

Let us first recall some definitions of Nielsen-Thurston theory.

Assume that $h \in \mathrm{Homeo}(\Sigma)$ is a homeomorphism of a compact surface Σ, possibly with boundary or punctures. We call h *periodic* if there exists $n > 0$ such that $h^n = \mathrm{Id}_\Sigma$. We call h *pseudo-Anosov* if there exist h-invariant measurable foliations with associated uniformly expanding transverse measures (see [Thu88] for details). These two types of homeomorphism are distinct, and in particular periodic homeomorphisms have zero topological entropy while pseudo-Anosov homeomorphisms have nonzero topological entropy. Given a finite h-invariant set F (in other words, F is a finite union of h-periodic orbits), we call h *pseudo-Anosov relative to* F if $h|_{\Sigma \setminus F}$ is pseudo-Anosov.

The key theorem of Nielsen-Thurston theory is the following:

Theorem 12 (Nielsen-Thurston classification)**.** *Every homeomorphism $f \in \mathrm{Homeo}(\Sigma)$ is isotopic to a homeomorphism $h \in \mathrm{Homeo}(\Sigma)$ such that:*

(i) h leaves invariant a finite family (possibly empty) of disjoint simple closed curves $C_1, \ldots, C_n$ on Σ;

(ii) No curve C_i is homotopic to a boundary curve of S;

(iii) We can decompose $\Sigma = \bigcup_{j=1}^{d} \Sigma_j$, where $\Sigma_1, \ldots, \Sigma_d$ are closed surfaces with disjoint interiors obtained by cutting the surface S along the curves $\{C_1, \ldots, C_n\}$;

(iv) For each $1 \leq j \leq d$, the homeomorphism $h|_{\Sigma_j}$ is either periodic or pseudo-Anosov.

In [LM91], Llibre and MacKay prove that if $S = \mathbb{T}^2$ and if $f \in \mathrm{Homeo}_0(S)$ has a rotation set with nonempty interior, then f is isotopic to a pseudo-Anosov relative to a finite set.

In [Pol92], Pollicott shows the following:

Theorem 13 ([Pol92], Theorem 2)**.** *Let S be a compact closed surface of genus $g \geq 2$, and $f \in \mathrm{Homeo}_0(S)$. Assume that there exist $2g+1$ periodic points $x_1, \ldots, x_{2g+1}$ whose rotation vectors $\rho_1, \ldots, \rho_{2g+1} \in H_1(S, \mathbb{R})$ do not lie on a hyperplane, then f is isotopic to some $h \in \mathrm{Homeo}_0(S)$ having an invariant set $S_1 \subset S$ that is a closed surface with boundary, such that $h|_{S_1}$ is pseudo-Anosov relative to a finite set.*

Hence, the only thing we have to prove is that in our case (*i.e.* under the additional assumption that $\rho_{erg}(f)$ has nonempty interior in $H_1(S, \mathbb{R})$) we have $S_1 = S$.

Proof. Let $f \in \mathrm{Homeo}_0(S)$ be such that case 1. of Theorem A holds, *i.e.* that the following does not hold: either $\rho_{erg}(f)$ is contained in an hyperplane of $H_1(S,\mathbb{R})$ or there exists two nonempty rational subspaces E and F of $H_1(S,\mathbb{R})$, which are orthogonal for the intersection form, and such that $\rho_{erg}(f) \subset E \cup F$.

By Proposition 10, the set $\rho_{erg}(f)$ has nonempty interior in $H_1(S,\mathbb{R})$, and a dense subset of $\mathrm{Conv}\,(\rho_{erg}(f))$ belongs to $\rho_{erg}(f)$ and is realized by a periodic orbit.

Let us build $x_1,\dots,x_k$ a finite set of f-periodic orbits, whose associated rotation vectors are $\rho_1,\dots,\rho_k \in H_1(S,\mathbb{R})$, and suppose that these vectors span $H_1(S,\mathbb{R})$, and suppose that these vectors are not contained in any union $E \cup F$ of two linear subspaces $E,F \subset H_1(S,\mathbb{R})$ of positive codimension. To see that such a family does exist, consider a family $\tilde{\rho}_0 = 0, \tilde{\rho}_1,\dots,\tilde{\rho}_{2g}$ a family of $\rho_{erg}(f)$ that is a basis of $H_1(S,\mathbb{R})$ (it exists by hypothesis made on f). Now, for any partition $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$ of $\{0,\dots,2g\}$, consider $\rho_{\mathcal{P}} \in \rho_{erg}(f) \setminus \big(\mathrm{span}\{\tilde{\rho}_i \mid i \in \mathcal{P}_1\} \cup \mathrm{span}\{\tilde{\rho}_j \mid j \in \mathcal{P}_2\}\big)$, that exists by the hypothesis made on f. Finally, consider a family $x_1,\dots,x_k$ of f-periodic orbits, such that the associated rotation vectors $\rho_1,\dots,\rho_k \in H_1(S,\mathbb{R})$ are close to the vectors $\tilde{\rho}_0,\tilde{\rho}_1,\dots,\tilde{\rho}_{2g},(\tilde{\rho}_{\mathcal{P}})_{\mathcal{P}\text{ partition}}$. If these two family are sufficiently close then $\rho_1,\dots,\rho_k$ span $H_1(S,\mathbb{R})$, and are not contained in any union $E \cup F$ of two linear subspaces $E,F \subset H_1(S,\mathbb{R})$ of positive codimension.[3]

Now, set $F := \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{k} f^n(x_j)$ and $\Sigma := S \setminus F$. As the points x_j are periodic, the set F is finite. Apply Nielsen-Thurston classification (Theorem 12) to $h := f|_\Sigma$, and suppose there is at least one simple closed curve C appearing in the decomposition *(i)*. In the proof of [Pol92, Theorem 2], Pollicott shows that this implies that C must be a separating curve; we can write $S = S_1 \cup S_2$, with S_1 and S_2 being two surfaces with disjoint interiors and boundaries equal to C. But in this case, this implies that $\rho_{erg}(h) \subset H_1(S_1,\mathbb{R}) \cup H_1(S_2,\mathbb{R})$ (in this equality, h is seen as a homeomorphism of S, easily obtained by extending h to F by $f|_F$), which is impossible because there is some vector $\rho_i \in \rho_{erg}(h)$ with $\rho_i \notin H_1(S_1,\mathbb{R}) \cup H_1(S_2,\mathbb{R})$. This shows that the set of curves appearing in Theorem 12 is empty, hence that f is isotopic to a pseudo-Anosov relative to F. □

Applying the arguments of [MZ91] (that are based on Nielsen-Thurston theory) in the higher genus case, we get the following.

Lemma 14. *If $f \in \mathrm{Homeo}_0(S)$ is pseudo-Anosov relative to a finite subset of S, then any element of $\mathrm{int}(\mathrm{conv}(\rho_{erg}(f)))$ is realized as the rotation vector of an ergodic measure. In particular, $\rho_{erg}(f)$ has nonempty interior.*

Remark 15. By a repeated use of [GM22, Theorem F], it is possible to replace the invariant finite set F in the above proof by a single periodic orbit. Hence, any homeomorphism in $\mathrm{Homeo}_0(S)$ whose ergodic rotation set has nonempty interior is isotopic to a pseudo-Anosov homeomorphism relative to one periodic orbit.

5 Small rotation set homeomorphisms do not act hyperbolically on the fine curve graph

Suppose that 1. does not hold in the statement of the theorem.

[3] Of course there should be some much more economical way (in terms of the number k of orbits) to build the family $x_1,\dots,x_k$, but we do not need any estimation on this number k for our purpose.

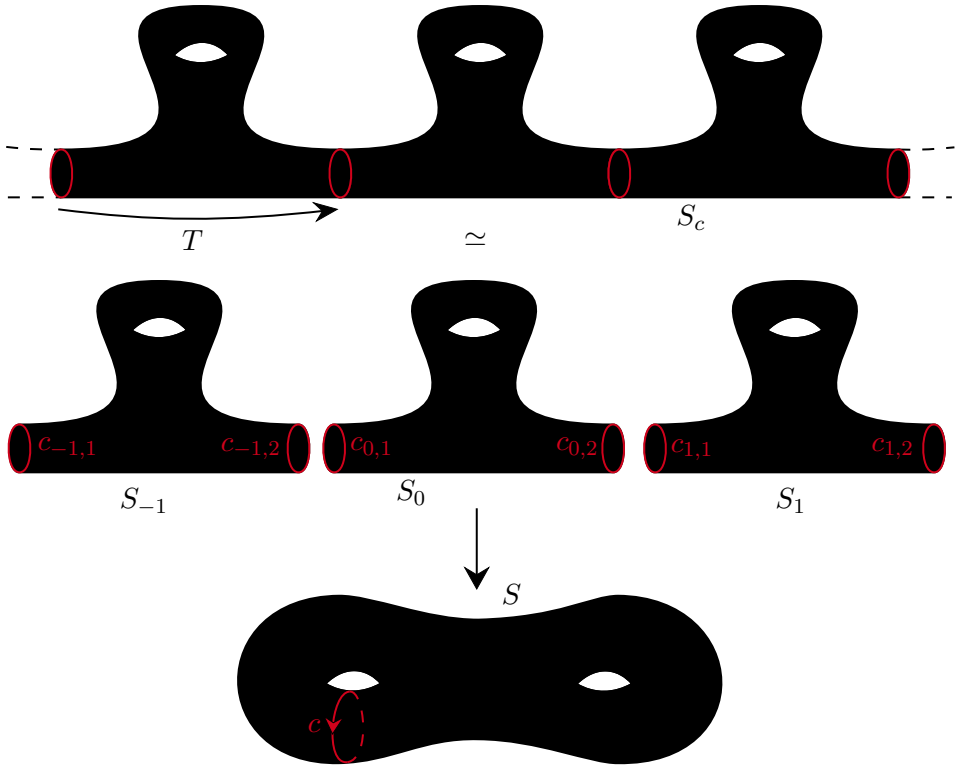

Figure 3: The covering map $S_c \to S$ of the first case.

First case: the rotation set $\rho_{erg}(f)$ is contained in a rational hyperplane H.

By [Mey76] and [Sch76], there exists a nonzero primitive integral vector $v_1 \in H^\perp$ and a simple nonseparating closed curve c that represents v_1. The morphism

$$\begin{array}{rcl} H_1(S,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ a & \longmapsto & a \wedge v_1 \end{array}$$

defines a covering map $S_c \to S$. Another way to see this covering map is the following (see Figure 3). Cut the surface S along c to obtain a surface S' with two boundary components c_1, c_2. Take a family $(S_i)_{i\in\mathbb{Z}}$ of surfaces, where each of the surfaces S_i is a copy of the surface S' and denote by $c_{i,1}$, and $c_{i,2}$ the boundary components of S_i which correspond respectively to c_1 and c_2. To form the surface S_c, glue, for any i, the curve $c_{i,2}$ with the curve $c_{i+1,1}$ in such a way that corresponding points of c are glued together. Using the map $S' \to S$, we obtain a covering map $S_c \to S$. With this second vision, we can define a map $\ell : S_c \to \mathbb{Z}$, which, to any point of $S_i \setminus S_{i+1}$, associates the value i.

The group of automorphisms of this covering map is infinite cyclic and is generated by an automorphism which we will denote by T. We choose T in such a way that, for any i, T sends S_i to S_{i+1}

Fix also a lift $f_c : S_c \to S_c$ of f. This lift commutes with T. Using this lift, we can define as follows new rotation sets for f, which are adapted to this lift.

We define $\rho^t_{v_1}(f)$ as the subset of $\mathbb{R}$ consisting of limit values of sequences of the form

$$\left(\frac{\ell(f_c^n(x_n)) - \ell(x_n)}{n}\right),$$

where $(x_n)_n$ is any sequence of points of S_c. For any point x of S, choose a lift x_c of x in such a way that $x \mapsto x_c$ is a measurable map. Let us denote by $\mathcal{M}(f)$ the set of f-invariant Borel probability measures on S and by $\mathcal{M}_{erg}(f)$ the subset of $\mathcal{M}(f)$ consisting of ergodic probability measures.

Lemma 16.

$$\begin{aligned}\rho^t_{v_1}(f) &= \left\{ \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x) \mid \mu \in \mathcal{M}(f) \right\} \\ &= \mathrm{Conv}\left(\left\{ \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x) \mid \mu \in \mathcal{M}_{erg}(f) \right\} \right).\end{aligned}$$

Proof. The equality

$$\begin{aligned}&\left\{ \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x) \mid \mu \in \mathcal{M}(f) \right\} \\ &\qquad = \mathrm{Conv}\left(\left\{ \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x) \mid \mu \in \mathcal{M}_{erg}(f) \right\} \right)\end{aligned}$$

holds as $\mathcal{M}(f)$ is the convex hull of $\mathcal{M}_{erg}(f)$, and as the map $\mu \mapsto \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x)$ is affine.

Take a real number $r \in \rho^t_{v_1}(f)$. Then there exists a sequence of integers $(n_k)_k$ and a sequence $(x'_k)_k$ of points of S_c such that

$$\lim_{k \to +\infty} \frac{\ell(f_c^{n_k}(x'_k)) - \ell(x'_k)}{n_k} = r.$$

For any k, let x_k be the projection of x'_k on the surface S and denote

$$\nu_k = \frac{1}{n_k} \sum_{k=0}^{n_k - 1} \delta_{f^k(x_k)}.$$

Then

$$\frac{\ell(f_c^{n_k}(x'_k)) - \ell(x'_k)}{n_k} = \int_S \ell(f_c(x_c)) - \ell(x_c) d\nu_k(x).$$

Let μ be a weak-* limit of the probability measures ν_k. Observe that $\mu \in \mathcal{M}(f)$ and that

$$r = \int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x).$$

This shows the inclusion $\subset$ for the first equality of the lemma.

Finally, if $\mu \in \mathcal{M}_{erg}(f)$, apply Birkhoff ergodic theorem to the function $x \mapsto \ell(f_c(x_c)) - \ell(x_c)$ to obtain that

$$\int_S \ell(f_c(x_c)) - \ell(x_c)\,\mathrm{d}\mu(x) \in \rho^t_{v_1}(f).$$

This shows the reverse inclusion for the first equality of the lemma. □

As a consequence of the Lemma 16 and Lemma 6, we obtain the following result.

Lemma 17.

$$\begin{aligned}\rho^t_{v_1}(f) &= \{a \wedge v_1 \mid a \in \rho_{H_1}(f)\} \\ &= \mathrm{Conv}\big(\{a \wedge v_1 \mid a \in \rho_{erg}(f)\}\big).\end{aligned}$$

Proof. Take any differentiable map $\lambda : S_c \to \mathbb{R}$ such that, for any point z on S_c, $\lambda(T(z)) = \lambda(z) + 1$. The differential of this map is invariant under T and defines a closed 1-form ω on the surface S. Observe that, for any closed curve $\gamma : [0,1] \to S$,

$$\int_\gamma \omega = \ell(\gamma_c(1)) - \ell(\gamma_c(0)) = [\gamma] \wedge v_1,$$

where γ_c is a lift of γ to the surface S_c.

Then, by Lemma 6,

$$d(\rho_{H_1}(f))(\omega) = \{a \wedge v_1 \mid a \in \rho_{H_1}(f)\} = \mathrm{Conv}\left(\{a \wedge v_1 \mid a \in \rho_{erg}(f)\}\right)$$

as $d(\rho_{H_1}(f))(\omega)$ is a connected and hence a convex subset of $\mathbb{R}$.

It remains to prove that $d(\rho_{H_1}(f))(\omega) = \rho^t_{v_1}(f)$. This last equality is a consequence of the fact that any converging sequence of the form $(\frac{1}{n_k} d(c_{x_k,n_k})(\omega))_k$, where $n_k \to +\infty$ and $(x_k)_k$ is a sequence of points on the surface S, has the same limit as the sequence $(\frac{1}{n_k}(\ell(f_c^{n_k}(x_{k,c})) - \ell(x_{k,c})))_k$. □

By Lemma 17, if $\rho_{erg}(f)$ is contained in $v_1^\perp$, then $\rho^t_{v_1}(f) = \{0\}$.

We now prove that, in this case, f does not act hyperbolically on the fine curve graph of S.

For any simple closed curve α that is isotopic to c, we define $\mathcal{C}_c(\alpha)$ as the number of lifts of c to S_c that a given lift of α to S_c meets. This quantity does not depend on the chosen lift of α.

The following lemma is analogous to Lemma 4.5 in [BHM+22].

Lemma 18. *For any simple closed curve α on S which is isotopic to c,*

$$\mathcal{C}_c(\alpha) + 1 \geq d_{C^\dagger}(\alpha, c).$$

Before proving the lemma, let us see why it allows us to complete the proof in this case. As $\rho^t_{v_1}(f) = \{0\}$,

$$\lim_{n \to +\infty} \frac{1}{n} \mathcal{C}_c(f^n(c)) = 0.$$

Indeed, otherwise, we would find a sequence of integers $n_k \to +\infty$ and a sequence of points (x_k) of c such that

$$\lim_{k \to +\infty} \frac{1}{n_k} \big(\ell(f_c^{n_k}(x_{k,c})) - \ell(x_{k,c})\big) \neq 0,$$

a contradiction with $\rho^t_{v_1}(f) = \{0\}$. Lemma 18 implies then that

$$\lim_{n \to +\infty} \frac{1}{n} d_{C^\dagger}(f^n(c), c) = 0$$

so that f does not act hyperbolically on $C^\dagger(S)$.

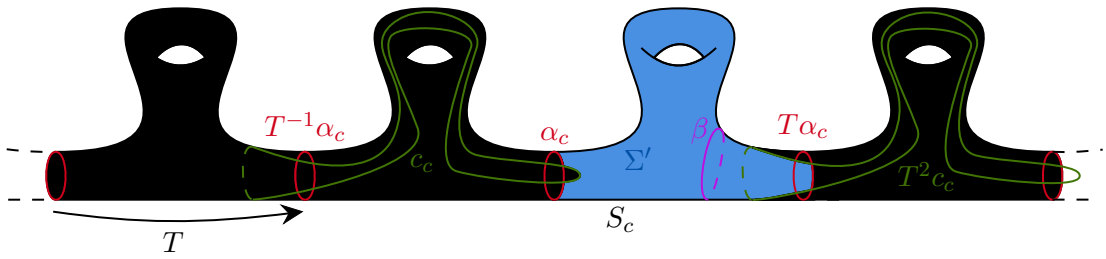

Figure 4: Construction of the curve β in S_c for the proof of Lemma 18.

Proof of Lemma 18. The proof is similar to the proof of [BHM+22, Lemma 4.5]. We prove it by induction on $n = \mathcal{C}_c(\alpha)$.

For $n = 0$, the curves α and c have to be disjoint so that $d_{C^\dagger}(\alpha, c) = 1$.

Suppose that the inequality holds for any curve α with $\mathcal{C}_c(\alpha) = n$. Fix now a curve α such that $\mathcal{C}_c(\alpha) = n + 1$. We will find a curve β such that $d_{C^\dagger}(\alpha, \beta) = 1$ and $\mathcal{C}_c(\beta) = n$, which will complete the induction.

The curve β is constructed as follows (see Figure 4). Fix a lift α_c of α to S_c and denote by c_c the lift of c which meets α_c but such that no lift of the form $T^k(c_c)$, with $k < 0$, meets α. Hence $T^{n+1}(c_c)$ does not meet α_c either. Now take the connected component Σ of $S_c \setminus (c_c \cup \alpha_c)$ which contains $T^k(\alpha_c)$ for any $k > 0$. Consider now the surface Σ' which is the bounded connected component of $\Sigma \setminus (T(\alpha_c) \cup T^{n+1}(c_c))$. This surface has one boundary component which is made of pieces of α_c and c_c and another boundary component which is made of pieces of $T(\alpha_c)$ and $T^{n+1}(c_c)$. The interior of this surface does not meet any lift of α and meets only the lifts $T^k(c_c)$ of c, with $1 \leq k \leq n$. Let β_c be any closed curve isotopic to $T(\alpha)$ which belongs to the interior of Σ'. The projection β of β_c on S will satisfy the wanted requirements. □

Second case: the rotation set $\rho_{erg}(f)$ is contained in a hyperplane which is not rational.

Take a (non-rational) vector v of $H_1(S, \mathbb{R})$ such that $\rho_{erg}(f) \subset v^\perp$. Then, by Claim 7, $\rho_{H_1}(f) \subset v^\perp$. Also, fix a euclidean norm $\|.\|$ on $H_1(S)$.

We will approximate this hyperplane by rational hyperplanes and use transverse rotation sets (as in the first case) to prove that f does not act hyperbolically on the fine curve graph of S.

Suppose for a contradiction that f acts hyperbolically on the fine curve graph. By definition, for any element $c \in C^\dagger(S)$

$$\lim_{n \to +\infty} d_{C^\dagger}(f^n(c), c) = \lambda > 0$$

and this constant λ does not depend on the chosen c.

As the set $\rho_{H_1}(f)$ is bounded, there exists a constant $C_1 > 0$ such that, for any $\rho \in \rho_{H_1}(f)$ and any vector x in $H_1(S, \mathbb{R})$,

$$|\rho \wedge x| \leq C_1 \|x\| .$$

By Dirichlet's theorem, there exists infinitely many integers q such that there exists

an integral vector $v_q \in H_1(S,\mathbb{Z})$ satisfying:

$$\|qv - v_q\| \leq \frac{2g}{q^{\frac{1}{2g}}}.$$

For any such integer q, we obtain that, for any $\rho \in \rho_{H_1}(f)$

$$|\rho \wedge v_q| \leq C_1 \frac{2g}{q^{\frac{1}{2g}}}.$$

Fix q large enough so that the right-hand side of this inequality is smaller than $\frac{\lambda}{4}$. Also, taking a smaller vector which is collinear to v_q instead of v_q if necessary, we can suppose that v_q is irreducible. Let c be a simple closed curve of S that represents v_q.

Then, taking the same notation as in the first case,

$$\rho^t_{v_q}(f) \subset \left[-\frac{\lambda}{4}, \frac{\lambda}{4}\right].$$

This implies that all the limit values of the sequence

$$\left(\frac{1}{n}\mathcal{C}_c\big(f^n(c)\big)\right)_n$$

belong to the interval $[0, \frac{\lambda}{2}]$.

Lemma 18 implies then that

$$\lim_{n\to+\infty} \frac{1}{n} d_{C^\dagger}(f^n(c), c) \leq \frac{\lambda}{2},$$

a contradiction.

Third case: no hyperplane contains $\rho_{erg}(f)$ and there exist two orthogonal and complementary rational subspaces E and F of $H_1(S,\mathbb{R})$ such that $\rho_{erg}(f) \subset E \cup F$.

Recall that we have fixed an isotopy $(f_t)_{t\in[0,1]}$ between Id and f. Let x be a periodic point of f of period q. We call *geodesic associated to* x, and denote γ_x, the closed geodesic that is freely homotopic to the closed curve $(f_t(x))_{t\in[0,q]}$ (if this path is non contractible). We also denote $r(x) = r\left(\frac{1}{q}\sum_{i=0}^{q-1}\delta_{f^i(x)}\right)$ the homological rotation vector associated to the periodic point x.

Claim 19. *There exist four periodic points x_E, y_E, x_F, y_F of respective periods $q_{x_E}, q_{y_E}, q_{x_F}, q_{y_F}$ such that:*

- *$r(x_E), r(y_E) \in E \setminus \{0\}$ and $r(x_F), r(y_F) \in F \setminus \{0\}$;*
- *γ_{x_E} and γ_{y_E} are simple and have a single intersection point, and γ_{x_F} and γ_{y_F} are simple and have a single intersection point;*
- *$(\gamma_{x_E} \cup \gamma_{y_E}) \cap (\gamma_{x_F} \cup \gamma_{y_F}) = \emptyset$.*

Proof. By Lemma 11, the set $\mathcal{L}$ appearing in Proposition 10 is empty, and each of the subspaces V_i which appear in the decomposition given by Proposition 10 is a symplectic subspace of $H_1(S,\mathbb{R})$. This implies that E and F each contain a finite (and nonempty) unions of the V_i's. Moreover, by point 4. of Proposition 10, the subspaces E and F each contain the rotation vector of a periodic orbit: there exist periodic points x_E and x_F of respective periods q_E and q_F such that $r(x_E) \in E \setminus \{0\}$ and $r(x_F) \in F \setminus \{0\}$.

Let us first show that the closed paths $(f_t(x_E))_{t\in[0,q_E]}$ and $(f_t(x_F))_{t\in[0,q_F]}$ can be supposed to be homotopic to simple closed geodesics.

Suppose that this is not the case and that for example $(f_t(x_E))_{t\in[0,q_E]}$ is homotopic to a closed geodesic $(\gamma(s))_{s\in\mathbb{S}^1}$ with a transverse self-intersection. Let $s_1, s_2 \in \mathbb{S}^1$ be two successive intersection times: $\gamma(s_1) = \gamma(s_2)$ and $\gamma|_{[s_1,s_2]}$ is simple. We denote $\tilde{S}$ the universal cover of S, and $\tilde{f}$ the canonical lift of f to $\tilde{S}$ (*i.e.* the one commuting with deck transformations). Let $\tilde{\gamma}$ be a lift of γ to $\tilde{S}$, and T_1 and T_2 be the deck transformations of $\tilde{S}$ associated to respectively $\tilde{\gamma}|_{[s_1,s_2]}$ and $\tilde{\gamma}|_{[s_2,s_1]}$. By [GM22, Theorem E], there exists $m > 0$ such that a dense subset (containing the extremal points) of the segment $[1/m[T_1]_{H_1}, 1/m[T_2]_{H_1}]$ is realised by rotation vectors of periodic orbits of f. Note that the line $\langle r(x_E)\rangle = \langle [T_1]_{H_1} + [T_2]_{H_1}\rangle$ meets $[1/m[T_1]_{H_1}, 1/m[T_2]_{H_1}]$. This implies (because $\rho_{erg}(f) \subset E \cup F$) that $[1/m[T_1]_{H_1}, 1/m[T_2]_{H_1}] \subset E$. If $[T_1]_{H_1} \neq 0$ (in other words, if T_1 is not a commutator), then there exists a periodic point x'_E realising the rotation vector $1/m[T_1]_{H_1} \in E\setminus\{0\}$. Otherwise, there exists a periodic point y_E realising the rotation vector $1/m[T_2]_{H_1} \in E \setminus \{0\}$, and we can apply the same reasoning as before to the point y_E. Iterating this process, we finally find a periodic point x'_E of period q'_E, realising a rotation vector in $E \setminus \{0\}$ and such that the closed path $(f_t(x'_E))_{t\in[0,q'_E]}$ is homotopic to simple closed geodesic (the process terminates because the initial closed geodesic associated to x_E has a finite number of self-intersections).

Denote by γ_{x_E} and γ_{x_F} the simple closed geodesics homotopic to respectively $(f_t(x_E))_{t\in[0,q_{x_E}]}$ and $(f_t(x_F))_{t\in[0,q_{x_F}]}$. By [GM22, Theorem F], these geodesics γ_{x_E} and γ_{x_F} do not intersect. Indeed, otherwise, denote $\tilde{\gamma}_{x_E}$ and $\tilde{\gamma}_{x_F}$ lifts of γ_{x_E} and γ_{x_F} to $\tilde{S}$ that intersect, and T_E and T_F deck transformations of $\tilde{S}$ associated to these lifts of closed geodesics. Then by [GM22, Theorem F] there would exist $m > 0$ such that $1/(2m)\big([T_E]_{H_1} + [T_F]_{H_1}\big) \in \rho_{erg}(f)$, which contradicts the fact that $\rho_{erg}(f) \subset E \cup F$.

Let us now explain how to get the periodic point y_E, the obtaining of y_F being identical.

As $r(x_E) \in E\setminus\{0\}$, and as E is symplectic (Lemma 11), there exists $r_0 \in \rho_{erg}(f)\cap E$ with $r_0 \wedge r(x_E) \neq 0$. By Proposition 10, this rotation vector r_0 is accumulated by rotation vectors in E of periodic orbits: there exists a periodic point y_E such that $r(y_E) \wedge r(x_E) \neq 0$ and $r(y_E) \in E$.

Let us first show that one can suppose that the cardinality of $\gamma_{x_E} \cap \gamma_{y_E}$ is 1. if this is not the case, we parametrize the closed geodesics γ_{x_E} and γ_{y_E} by $\mathbb{S}^1$. Then there exists $s_{x,1}, s_{x,2}, s_{y,1}, s_{y,2} \in \mathbb{S}^1$ such that $s_{y,1}$ and $s_{y,2}$ are two successive intersection times of γ_{y_E} with γ_{x_E} such that these intersections have the same orientation, in other words:

- $\gamma_{x_E}(s_{x,1}) = \gamma_{y_E}(s_{y,1})$ and $\gamma_{x_E}(s_{x,2}) = \gamma_{y_E}(s_{y,2})$;
- $\gamma_{y_E}|_{(s_{y,1},s_{y,2})} \cap \gamma_{x_E} = \emptyset$;
- γ_{y_E} crosses γ_{x_E} either from left to right at both $s_{y,1}$ and $s_{y,2}$, or from right to left at both $s_{y,1}$ and $s_{y,2}$.

The last property comes from the fact that $r(x_E) \wedge r(y_E) \neq 0$.

By [GM22, Theorem F], there is a periodic point y'_E of period $q_{y'_E}$ such that its closed path under the isotopy $(f_t(y'_E))_{t\in[0,q_{y'_E}]}$ is homotopic to the closed curve $\gamma_{y_E}|_{[s_{y,1},s_{y,1}+1]}\gamma_{x_E}|_{[s_{x,1},s_{x,1}+1]}$. This closed path has a transverse self-intersection, so by [GM22, Theorem E] there is a periodic orbit y''_E of period $q_{y''_E}$ such that its closed path under the isotopy $(f_t(y''_E))_{t\in[0,q_{y''_E}]}$ is homotopic to the closed curve $\gamma_{y_E}|_{[s_{y,1},s_{y,2}]}\gamma_{x_E}|_{[s_{x,2},s_{x,1}+1]}$. By the hypothesis made on the orientation of crossings of γ_{y_E} and γ_{x_E} at $s_{y,1}$ and $s_{y,2}$, this closed curve intersects γ_{x_E} exactly once.

Finally, if the closed geodesic $\gamma_{y''_E}$ is not simple, let us denote $\overline{s_x}, \overline{s_y} \in \mathbb{S}^1$ and $\overline{s_{y,-1}} \neq \overline{s_{y,1}} \in \mathbb{S}^1$ such that:

- $\gamma_{x_E}(\overline{s_x}) = \gamma_{y''_E}(\overline{s_y})$;
- $\gamma_{y''_E}(\overline{s_{y,-1}}) = \gamma_{y''_E}(\overline{s_{y,1}})$;
- $\overline{s_y} \in [\overline{s_{y,-1}}, \overline{s_{y,1}}]$.

To find these times, note that the first condition determines uniquely $\overline{s_y} \in \mathbb{S}^1$, and choose $\overline{s_{y,1}}$ the smallest $s \geq \overline{s_y}$ at which $\gamma_{y''_E}$ intersects itself transversally.

Applying [GM22, Theorem E] again, we get a periodic point y'''_E of f such that $\gamma_{y'''_E}$ intersects γ_{x_E} once and has a number of self-intersections strictly smaller than the one of $\gamma_{y''_E}$. Iterating this process if necessary, we get a periodic point y'''_E of f such that $\gamma_{y'''_E}$ intersects γ_{x_E} once and is simple. In particular, this implies that $r(x_E) \wedge r(y'''_E) \neq 0$, and so that $r(y'''_E) \in E \setminus \{0\}$.

The last point of the claim is obtained identically to the proof we have already made of the fact that γ_{x_E} and γ_{x_F} do not intersect (this is a direct consequence of [GM22, Theorem F]). □

This is a classical fact (*e.g.* [FM12, 1.3.3]) that all pairs of geodesics intersecting only once are homeomorphic[4]. This allows to get a closed geodesic γ separating the geodesics γ_{x_E} and γ_{y_E} from all the other geodesics of S that do not intersect $\gamma_{x_E} \cup \gamma_{y_E}$. In particular, it separates γ_{x_E} and γ_{y_E} from γ_{x_F} and γ_{y_F}.

Remark 20. By using the proofs of the claim and of Lemma 11, it is possible to show that, when $\rho_{erg}(f)$ is not contained in any hyperplane, each subspace V_i appearing in the decomposition given by Proposition 10 has a symplectic basis consisting of vectors with integral coefficients. By [MP78], every symplectic automorphism of $H_1(S,\mathbb{Z})$ is induced by a diffeomorphism of S. This implies that, for any i, there exists a subsurface $S_i \subset S$ with boundary such that $H_1(S_i,\mathbb{R}) = V_i$. Moreover, the surfaces S_i can be chosen to be pairwise disjoint.

We will use a covering map involving γ so that we can define an analogue of a transverse rotation set associated to γ, similarly to what we did in case 1.

Let us describe how to obtain such a covering map. An example of such covers is given in Figure 5. Denote by S_E and S_F the two connected components of $S \setminus \gamma$, where S_E contains γ_{x_E} and S_F contains γ_{x_F}.

To simplify notations, denote $\alpha = \gamma_{x_E}$ and $\beta = \gamma_{x_F}$; they are respectively contained in S_E and S_F and collinear to nontrivial rotation vectors in $\rho_{erg}(f)$. Cut the surface S along

[4]Note that on the contrary, there are pairs of disjoint geodesics that are not homeomorphic, for example if the first pair is bounding (*i.e.* the union of the geodesics separates the surface) and the other is not.

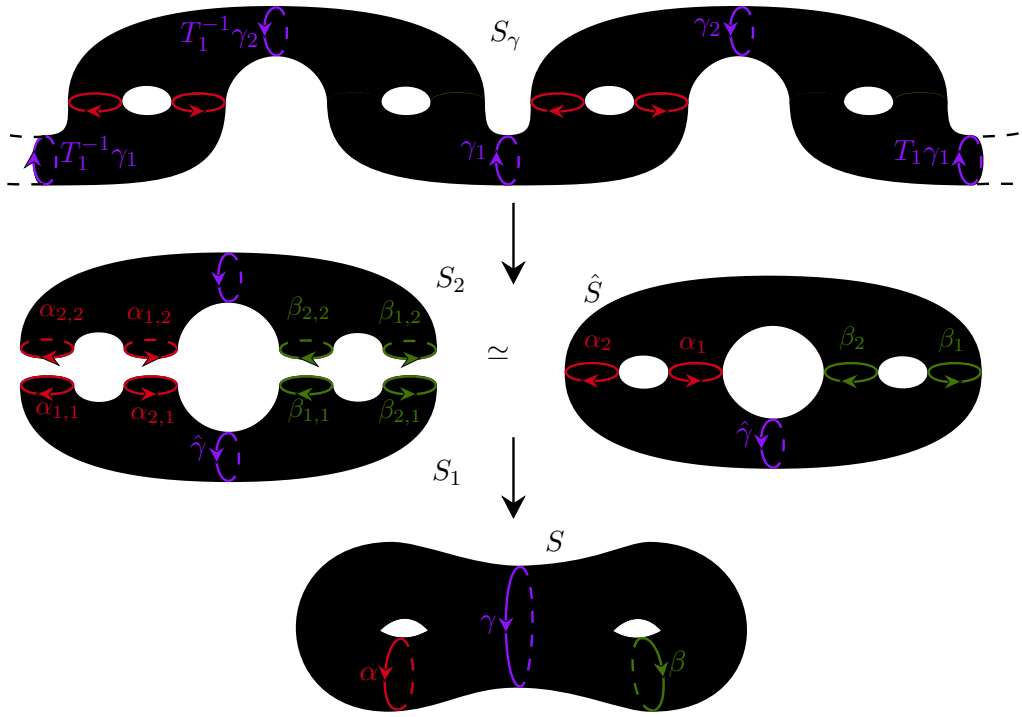

Figure 5: The covers used in the proof of the third case.

those two simple closed curves to obtain a surface $\mathcal{S}$ with four boundary components α_1, α_2 (those two first curves correspond to the closed curve α of S), β_1, β_2. Take two copies $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{S}$. For $i = 1, 2$ denote by $\alpha_{1,i}$,$\alpha_{2,i}$,$\beta_{1,i}$,$\beta_{2,i}$ for $\mathcal{S}_i$ the boundary components which correspond respectively to α_1, α_2, β_1, β_2. Now, glue $\alpha_{1,1}$ to $\alpha_{2,2}$, $\alpha_{1,2}$ to $\alpha_{2,1}$, $\beta_{1,1}$ to $\beta_{2,2}$, $\beta_{1,2}$ to $\beta_{2,1}$, in such a way that corresponding points of α or β are glued together, to obtain a closed surface $\hat{S}$. The map $\hat{S} \to S$ is a degree 2 covering map. This map can be also obtained as the covering map corresponding to the kernel of the map

$$\begin{array}{rcl} H_1(S,\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ h & \longmapsto & h \wedge (a+b) \mod 2, \end{array}$$

where a and b respectively represent the homology classes of α and β.

Denote by $\hat{T}$ the nontrivial deck transformation of this covering map. Fix a lift $\hat{\gamma}$ of the curve γ to $\hat{S}$ and observe that the closed curve $\hat{\gamma}$ is not separating in $\hat{S}$. Denote by $\hat{f}$ a lift of f to $\hat{S}$. As $\hat{\gamma}$ is nonseparating, we can use the transverse rotation set $\rho^t_{\hat{\gamma}}(\hat{f})$ of $\hat{f}$ relative to $\hat{\gamma}$ that we used in the first case.

Suppose first that $\rho^t_{\hat{\gamma}}(\hat{f}) = \{0\}$. In this case, we will prove that f does not act hyperbolically on $\mathcal{C}^\dagger(S)$. The proof in this case will be similar to the proof in the first case.

Denote by S_γ the cyclic covering over $\hat{S}$ corresponding to the map

$$\begin{array}{rcl} H_1(\hat{S},\mathbb{R}) & \to & \mathbb{Z} \\ a & \mapsto & a \wedge [\hat{\gamma}] \end{array}.$$

Denote by T_1 a generator of the group of deck transformations of the covering map $S_\gamma \to \hat{S}$. Fix a lift γ_1 of $\hat{\gamma}$ to S_γ and denote by γ_2 the lift of $T(\hat{\gamma})$ that lies in the bounded component of $S_\gamma \setminus (\gamma_1 \cup T_1\gamma_1)$.

Finally, for any simple closed curve σ which is isotopic to γ, we define

$$C_\gamma(\sigma) = \mathrm{Card}\left(\{n \in \mathbb{Z} \mid T_1^n\gamma_1 \cap \tilde{\sigma} \neq \emptyset\}\right) + \mathrm{Card}\left(\{n \in \mathbb{Z} \mid T_1^n\gamma_2 \cap \tilde{\sigma} \neq \emptyset\}\right),$$

where $\tilde{\sigma}$ is a lift of σ to S_γ. Observe that this quantity does not depend on the chosen lift σ.

Lemma 21. *For any simple closed curve α on S which is isotopic to γ,*

$$\mathcal{C}_\gamma(\alpha) + 1 \geq d_{C^\dagger}(\alpha, \gamma).$$

Proof. The proof of this lemma is almost identical to the proof of Lemma 18. □

Now, similarly to the first case, as $\rho^t_{\hat{\gamma}}(\hat{f}) = \{0\}$,

$$\lim_{n\to+\infty} \frac{1}{n}\mathcal{C}_\gamma(f^n(\gamma)) = 0.$$

Hence

$$\lim_{n\to+\infty} \frac{1}{n} d_{C^\dagger}(f^n(\gamma), \gamma) = 0$$

and f does not act hyperbolically on $C^\dagger(S)$.

Suppose now that $\rho^t_{\hat{\gamma}}(\hat{f}) \neq \{0\}$. Then we can find an ergodic $\hat{f}$-invariant probability measure with non-zero rotation vector $\hat{c}$. Observe that this rotation vector has a nonzero intersection with the homology class $\hat{a}$ of one of lifts $\hat{\alpha}$ of α to $\hat{S}$ and with the homology class $\hat{b}$ of one of the lifts $\hat{\beta}$ of β to $\hat{S}$. By Theorem 8, we can suppose, up to changing $\hat{c}$, $\hat{b}$ and $\hat{a}$ for nontrivial colinear vectors, that the homology classes $\hat{c}$, $\frac{1}{2}(\hat{c}+\hat{a})$, $\frac{1}{2}(\hat{c}+\hat{b})$ and $\frac{1}{3}(\hat{c}+\hat{b}+\hat{a})$ are limits of homology rotation vectors realized by periodic orbits. Hence the homology classes c (which is the projection of $\hat{c}$ on $H_1(S,\mathbb{R})$), $\frac{1}{2}(c+a)$, $\frac{1}{2}(c+b)$ and $\frac{1}{3}(c+b+a)$ are elements of $\overline{\rho_{erg}(f)}$. Take a homology class $a' \in \rho_{erg}(f) \cap E$ which is realized by a f-periodic orbit such that $a' \wedge a \neq 0$ and a homology class $b' \in \rho_{erg}(f) \cap F$ which is realized by a f-periodic orbit such that $b' \wedge b \neq 0$.

Lemma 22. *One of the following holds.*

1. $c \wedge a' \neq 0$ *and* $c \wedge b' \neq 0$.
2. $\frac{1}{2}(c+a) \wedge a' \neq 0$ *and* $\frac{1}{2}(c+a) \wedge b' \neq 0$.
3. $\frac{1}{2}(c+b) \wedge a' \neq 0$ *and* $\frac{1}{2}(c+b) \wedge b' \neq 0$.
4. $\frac{1}{3}(c+b+a) \wedge a' \neq 0$ *and* $\frac{1}{3}(c+b+a) \wedge b' \neq 0$.

Proof. Observe that as $a' \wedge b = a \wedge b' = 0$, the cases of the lemma are equivalent to the following ones:

2. $(c+a) \wedge a' \neq 0$ and $c \wedge b' \neq 0$.
3. $c \wedge a' \neq 0$ and $(c+b) \wedge b' \neq 0$.

4. $(c+a) \wedge a' \neq 0$ and $(c+b) \wedge b' \neq 0$.

We then argue by case disjunction.

a) If $c \wedge a' \neq 0$ and $c \wedge b' \neq 0$, then 1. holds.

b) If $c \wedge a' \neq 0$ and $c \wedge b' = 0$, then $(c+b) \wedge b' = b \wedge b' \neq 0$, so 3. holds.

c) If $c \wedge a' = 0$ and $c \wedge b' \neq 0$, then $(c+a) \wedge a' = a \wedge a' \neq 0$ so 2. holds.

d) If $c \wedge a' = 0$ and $c \wedge b' = 0$, then $(c+a) \wedge a' = a \wedge a' \neq 0$, and $(c+b) \wedge b' = b \wedge b' \neq 0$, so 4. holds.

□

In any case of Lemma 22, we have obtained a vector in $\rho_{erg}(f) \setminus \{0\}$ which is not orthogonal to a vector in E and which is not orthogonal to a vector in F. This contradicts the hypothesis $\rho_{erg}(f) \subset E \cup F$, with E and F orthogonal.

This finishes the proof of Theorem A. □

References

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss., vol. 319, Berlin: Springer, 1999.

[BHM+22] Jonathan Bowden, Sebastian Hensel, Kathryn Mann, Emmanuel Militon, and Richard Webb, *Rotation sets and actions on curves*, Adv. Math. **408 B** (2022), 33.

[BHW22] Jonathan Bowden, Sebastian Hensel, and Richard Webb, *Quasi-morphisms on surface diffeomorphism groups*, J. Am. Math. Soc. **35** (2022), no. 1, 211–231.

[BIP08] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Groups of diffeomorphisms in honor of Shigeyuki Morita on the occasion of his 60th birthday, Tokyo: Mathematical Society of Japan, 2008, pp. 221–250.

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups (pms-49)*, Princeton University Press, 2012.

[GGL23] Alejo García, Pierre-Antoine Guihéneuf, and Pablo Lessa, *Ergodic homological rotation sets for higher genus surfaces*, 2023, in preparation.

[GM22] Pierre-Antoine Guihéneuf and Emmanuel Militon, *Homotopic rotation sets for higher genus surfaces*, 2022, to appear in the memoirs of the AMS.

[GM23] Pierre-Antoine Guihéneuf and Emmanuel Militon, *Parabolic isometries of the fine curve graph of the torus*, 2023.

[Gro87] Mikhaïl Gromov, *Hyperbolic groups*, Essays in group theory, Publ., Math. Sci. Res. Inst. 8, 75-263 (1987), 1987.

[Lel19] Gabriel Lellouch, *Sur les ensembles de rotation des homéomorphismes de surface en genre ≥ 2*, Ph.D. thesis, Sorbonne Université, April 2019, to appear in *Mémoires de la SMF*.

[LM91] Jaume Llibre and Robert S. MacKay, *Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity*, Ergodic Theory Dynam. Systems **11** (1991), no. 1, 115–128.

[LMP+21] Adele Long, Dan Margalit, Anna Pham, Yvon Verberne, and Claudia Yao, *Automorphisms of the fine curve graph*, 2021, To appear in *Transactions of the AMS*.

[Mey76] Mark D. Meyerson, *Representing homology classes of closed orientable surfaces*, Proc. Amer. Math. Soc. **61** (1976), no. 1, 181–182 (1977).

[MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I: Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.

[MP78] William H. Meeks, III and Julie Patrusky, *Representing homology classes by embedded circles on a compact surface*, Illinois J. Math. **22** (1978), no. 2, 262–269. MR 474304

[MZ91] Michał Misiurewicz and Krystyna Ziemian, *Rotation sets and ergodic measures for torus homeomorphisms*, Fundam. Math. **137** (1991), no. 1, 45–52.

[Pol92] Mark Pollicott, *Rotation sets for homeomorphisms and homology*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 881–894.

[Sch76] James A. Schafer, *Representing homology classes on surfaces*, Canad. Math. Bull. **19** (1976), no. 3, 373–374.

[Thu88] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Am. Math. Soc., New Ser. **19** (1988), no. 2, 417–431.