



HAL
open science

Non-linear non-zero-sum Dynkin games with Bermudan strategies

Miryana Grigorova, Marie-Claire Quenez, Yuan Peng

► **To cite this version:**

Miryana Grigorova, Marie-Claire Quenez, Yuan Peng. Non-linear non-zero-sum Dynkin games with Bermudan strategies. 2023. hal-04267335

HAL Id: hal-04267335

<https://hal.science/hal-04267335>

Preprint submitted on 1 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Non-linear non-zero-sum Dynkin games with Bermudan strategies

Miryana Grigorova^{1*} Marie-Claire Quenez^{2 †}

Peng Yuan^{3 ‡}

^{1,3}University of Warwick

²Université Paris Cité

November 1, 2023

Abstract: In this paper, we study a non-zero-sum game with two players, where each of the players plays what we call Bermudan strategies and optimizes a general non-linear assessment functional of the pay-off. By using a recursive construction, we show that the game has a Nash equilibrium point.

1 Introduction

Game problems with linear evaluations between a finite number of players are by now classical problems in stochastic control and optimal stopping (cf., e.g., [1], [3], [5], [9], [13], [14], [15], [16], [17], [18], [25], [26] and [28]) with various applications, in particular in economics and finance (cf., e.g., [13], [14], [22] and [25]). In the recent years game problems with *non-linear* evaluation functionals have attracted considerable interest: cf. [2] for the case of non-linear functionals of the form of worst case expectations over a set of possibly singular measures; [6], [7], [8] and [10] for the case of non-linear functionals induced by backward stochastic differential equations (BSDEs). Most of the works dealing with *non-linear* games have focused on the *zero-sum* case (cf.,

*Corresponding Author. Department of Statistics, University of Warwick, E-mail:miryana.grigorova@warwick.ac.uk

[†]LPSM, University Paris-Cité

[‡]Department of Statistics, University of Warwick

e.g., [2], [6], [7], [8] and [10]). *Non-zero-sum* games are notoriously more intricate than their zero-sum counterparts even in the case of linear evaluations (cf., e.g., [14], [16], [17], [27], [29] and [31]). Non-zero-sum games with non-linear functionals have been considered in [12] in the discrete-time framework and with non-linear functionals induced by Backward SDEs with Lipschitz driver, in [21] in the continuous time framework and with non-linear functionals of the form of expected exponential utilities.

In the current paper, we address the question of existence of a Nash equilibrium point in a framework with general non-linear evaluations and with a set of stopping strategies which is in between the discrete time and the continuous time stopping strategies. The results of [12] can be seen as a particular case of the current paper.

The paper is organised as follows: In Section 2, we introduce the framework, including the set of optimal stopping strategies of the agents (namely the Bermudan strategies), the pay-off as well as the properties on the risk functionals ρ^1 and ρ^2 of agent 1 and agent 2. In Section 3, we present our main results and show that the non-linear non-zero-sum game with Bermudan strategies has a Nash equilibrium point.

2 The framework

Let $T > 0$ be a **fixed finite** terminal horizon.

Let (Ω, \mathcal{F}, P) be a (complete) probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$.

In the sequel, equalities and inequalities between random variables are to be understood in the P -almost sure sense. Equalities between measurable sets are to be understood in the P -almost sure sense.

Let \mathbb{N} be the set of natural numbers, including 0. Let \mathbb{N}^* be the set of natural numbers, excluding 0.

We first define the so-called Bermudan stopping strategies (introduced in [11]).

Let $(\theta_k)_{k \in \mathbb{N}}$ be a sequence of stopping times satisfying the following properties:

- (a) The sequence $(\theta_k)_{k \in \mathbb{N}}$ is non-decreasing, i.e. for all $k \in \mathbb{N}$, $\theta_k \leq \theta_{k+1}$, a.s.
- (b) $\lim_{k \rightarrow \infty} \uparrow \theta_k = T$ a.s.

Moreover, we set $\theta_0 = 0$.

We note that the family of σ -algebras $(\mathcal{F}_{\theta_k})_{k \in \mathbb{N}}$ is non-decreasing (as the sequence (θ_k) is non-decreasing). We denote by Θ the set of stopping times τ of the form

$$\tau = \sum_{k=0}^{+\infty} \theta_k \mathbf{1}_{A_k} + T \mathbf{1}_{\bar{A}}, \quad (1)$$

where $\{(A_k)_{k=0}^{+\infty}, \bar{A}\}$ form a partition of Ω such that, for each $k \in \mathbb{N}$, $A_k \in \mathcal{F}_{\theta_k}$, and $\bar{A} \in \mathcal{F}_T$.

The set Θ can also be described as the set of stopping times τ such that for almost all $\omega \in \Omega$, either $\tau(\omega) = T$ or $\tau(\omega) = \theta_k(\omega)$, for some $k = k(\omega) \in \mathbb{N}$.

Note that the set Θ is closed under concatenation, that is, for each $\tau \in \Theta$ and each $A \in \mathcal{F}_\tau$, the stopping time $\tau \mathbf{1}_A + T \mathbf{1}_{A^c} \in \Theta$. More generally, for each $\tau \in \Theta$, $\tau' \in \Theta$ and each $A \in \mathcal{F}_{\tau \wedge \tau'}$, the stopping time $\tau \mathbf{1}_A + \tau' \mathbf{1}_{A^c}$ is in Θ . The set Θ is also closed under pairwise minimization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \wedge \tau' \in \Theta$) and under pairwise maximization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \vee \tau' \in \Theta$). Moreover, the set Θ is closed under monotone limit, that is, for each non-decreasing (resp. non-increasing) sequence of stopping times $(\tau_n)_{n \in \mathbb{N}} \in \Theta^{\mathbb{N}}$, we have $\lim_{n \rightarrow +\infty} \tau_n \in \Theta$.

We note also that all stopping times in Θ are bounded from above by T .

Remark 1. *We have the following canonical writing of the sets in (1):*

$$\begin{aligned} A_0 &= \{\tau = \theta_0\}; \\ A_{n+1} &= \{\tau = \theta_{n+1}, \theta_{n+1} < T\} \setminus (A_n \cup \dots \cup A_0); \text{ for all } n \in \mathbb{N}^* \\ \bar{A} &= (\cup_{k=0}^{+\infty} A_k)^c \end{aligned}$$

From this writing, we have: if $\omega \in A_{k+1} \cap \{\theta_k < T\}$, then $\omega \notin \{\tau = \theta_k\}$.

For each $\tau \in \Theta$, we denote by Θ_τ the set of stopping times $\nu \in \Theta$ such that $\nu \geq \tau$ a.s. The set Θ_τ satisfies the same properties as the set Θ . We will refer to the set Θ as the set of **Bermudan stopping strategies**, and to the set Θ_τ as the set of Bermudan stopping strategies, greater than or equal to τ (or the set of Bermudan stopping strategies from time τ perspective). For simplicity, the set Θ_{θ_k} will be denoted by Θ_k .

Definition 1. *We say that a family $\phi = (\phi(\tau), \tau \in \Theta)$ is admissible if it satisfies the following conditions*

1. *for all $\tau \in \Theta$, $\phi(\tau)$ is a real valued random variable, which is \mathcal{F}_τ -measurable.*
2. *for all $\tau, \tau' \in \Theta$, $\phi(\tau) = \phi(\tau')$ a.s. on $\{\tau = \tau'\}$.*

Moreover, for $p \in [1, +\infty]$ fixed, we say that an admissible family ϕ is p -integrable, if for all $\tau \in \Theta$, $\phi(\tau)$ is in L^p .

Let $\phi = (\phi(\tau), \tau \in \Theta)$ be an admissible family. For a stopping time τ of the form (1), we have

$$\phi(\tau) = \sum_{k=0}^{+\infty} \phi(\theta_k) \mathbf{1}_{A_k} + \phi(T) \mathbf{1}_{\bar{A}} \quad \text{a.s.} \quad (2)$$

Given two admissible families $\phi = (\phi(\tau), \tau \in \Theta)$ and $\phi' = (\phi'(\tau), \tau \in \Theta)$, we say that ϕ is *equal to* ϕ' and write $\phi = \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) = \phi'(\tau)$ a.s. We say that ϕ *dominates* ϕ' and write $\phi \geq \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) \geq \phi'(\tau)$ a.s.

Let $p \in [1, +\infty]$. We introduce the following properties on the non-linear operators $\rho_{S,\tau}[\cdot]$, which will appear in the sequel.

For $S \in \Theta$, $S' \in \Theta$, $\tau \in \Theta$, for η, η_1 and η_2 in $L^p(\mathcal{F}_\tau)$, for $\xi = (\xi(\tau))$ an admissible p -integrable family:

- (i) $\rho_{S,\tau} : L^p(\mathcal{F}_\tau) \longrightarrow L^p(\mathcal{F}_S)$
- (ii) (*admissibility*) $\rho_{S,\tau}[\eta] = \rho_{S',\tau}[\eta]$ a.s. on $\{S = S'\}$.
- (iii) (*knowledge preservation*) $\rho_{\tau,S}[\eta] = \eta$, for all $\eta \in L^p(\mathcal{F}_S)$, all $\tau \in \Theta_S$.
- (iv) (*monotonicity*) $\rho_{S,\tau}[\eta_1] \leq \rho_{S,\tau}[\eta_2]$ a.s., if $\eta_1 \leq \eta_2$ a.s.
- (v) (*consistency*) $\rho_{S,\theta}[\rho_{\theta,\tau}[\eta]] = \rho_{S,\tau}[\eta]$, for all S, θ, τ in Θ such that $S \leq \theta \leq \tau$ a.s.
- (vi) (*"generalized zero-one law"*) $I_A \rho_{S,\tau}[\xi(\tau)] = I_A \rho_{S,\tau'}[\xi(\tau')]$, for all $A \in \mathcal{F}_S$, $\tau \in \Theta_S$, $\tau' \in \Theta_S$ such that $\tau = \tau'$ on A .
- (vii) (*monotone Fatou property with respect to terminal condition*)
 $\rho_{S,\tau}[\eta] \leq \liminf_{n \rightarrow +\infty} \rho_{S,\tau}[\eta_n]$, for $(\eta_n), \eta$ such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n \rightarrow +\infty} \uparrow \eta_n = \eta$ a.s.
- (viii) (*left-upper-semicontinuity (LUSC) along Bermudan stopping times with respect to the terminal condition and the terminal time*), that is,

$$\limsup_{n \rightarrow +\infty} \rho_{S,\tau_n}[\phi(\tau_n)] \leq \rho_{S,\nu}[\limsup_{n \rightarrow +\infty} \phi(\tau_n)],$$

for each non-decreasing sequence $(\tau_n) \in \Theta_S^{\mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} \uparrow \tau_n = \nu$ a.s., and for each p -integrable admissible family ϕ such that $\sup_{n \in \mathbb{N}} |\phi(\tau_n)| \in L^p$.

- (ix) $\limsup_{n \rightarrow +\infty} \rho_{\theta_n, T}[\eta] \leq \rho_{T, T}[\eta]$, for all $\eta \in L^p(\mathcal{F}_T)$.

These assumptions on ρ ensure that the one-agent's non-linear optimal stopping problem admits a solution and that the first hitting time (when the value family "hits" the pay-off family) is optimal (cf. [11] for more details).

3 The game problem

We consider two agents, agent 1 and agent 2, whose pay-offs are defined via four admissible families $X^1 = (X^1(\tau))_{\tau \in \Theta}$, $X^2 = (X^2(\tau))_{\tau \in \Theta}$, $Y^1 = (Y^1(\tau))_{\tau \in \Theta}$ and $Y^2 = (Y^2(\tau))_{\tau \in \Theta}$. We assume that X^1, X^2, Y^1 and Y^2 are p -integrable families such that

$$(A1) \quad X^1 \leq Y^1, \quad X^2 \leq Y^2 \quad (\text{that is, for each } \tau \in \Theta, \quad X^1(\tau) \leq Y^1(\tau), \text{ and } X^2(\tau) \leq Y^2(\tau)).$$

$$(A2) \quad X^1(T) = Y^1(T), \quad X^2(T) = Y^2(T).$$

$$(A3) \quad \text{ess sup}_{\tau \in \Theta} X^1(\tau) \in L^p, \quad \text{ess sup}_{\tau \in \Theta} X^2(\tau) \in L^p, \\ \text{ess sup}_{\tau \in \Theta} Y^1(\tau) \in L^p \text{ and } \text{ess sup}_{\tau \in \Theta} Y^2(\tau) \in L^p.$$

$$(A4) \quad \limsup_{k \rightarrow +\infty} X^1(\theta_k) \leq X^1(T), \quad \limsup_{k \rightarrow +\infty} X^2(\theta_k) \leq X^2(T).$$

The set of stopping strategies of each agent at time 0 is the set Θ of Bermudan stopping times. If the first agent plays $\tau_1 \in \Theta$ and the second agent plays $\tau_2 \in \Theta$, the pay-off of agent 1 (resp. agent 2) at time $\tau_1 \wedge \tau_2$ is given by:

$$I^1(\tau_1, \tau_2) := X^1(\tau_1)1_{\{\tau_1 \leq \tau_2\}} + Y^1(\tau_2)1_{\{\tau_2 < \tau_1\}} \\ (\text{resp. } I^2(\tau_1, \tau_2) := X^2(\tau_2)1_{\{\tau_2 < \tau_1\}} + Y^2(\tau_1)1_{\{\tau_1 \leq \tau_2\}}),$$

where we have adopted the convention: when $\tau_1 = \tau_2$, it is the first agent who is responsible for stopping the game. The agents evaluate their respective pay-offs via possibly different evaluation functionals. Let $\rho^1 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agent 1, and let $\rho^2 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agents 2. If agent 1 plays $\tau_1 \in \Theta$, and agent 2 plays $\tau_2 \in \Theta$, then the assessment (or evaluation) of agent 1 (resp. agent 2) at time 0 of his/her pay-off is given by:

$$J_1(\tau_1, \tau_2) := \rho_{0, \tau_1 \wedge \tau_2}^1 [X^1(\tau_1)1_{\{\tau_1 \leq \tau_2\}} + Y^1(\tau_2)1_{\{\tau_2 < \tau_1\}}]. \\ (\text{resp. } J_2(\tau_1, \tau_2) := \rho_{0, \tau_1 \wedge \tau_2}^2 [X^2(\tau_2)1_{\{\tau_2 < \tau_1\}} + Y^2(\tau_1)1_{\{\tau_1 \leq \tau_2\}}]).$$

We assume that both ρ^1 and ρ^2 satisfy the properties (i) - (ix). We will investigate the problem of existence of a Nash equilibrium strategy (τ_1^*, τ_2^*) .

Definition 2. *A pair of Bermudan stopping times $(\tau_1^*, \tau_2^*) \in \Theta \times \Theta$ is called a Nash equilibrium strategy (or a Nash equilibrium point) for the above non-zero-sum non-linear Bermudan Dynkin game if: $J_1(\tau_1^*, \tau_2^*) \geq J_1(\tau_1, \tau_2^*)$, for any $\tau_1 \in \Theta$, and $J_2(\tau_1^*, \tau_2^*) \geq J_2(\tau_1^*, \tau_2)$, for any $\tau_2 \in \Theta$.*

In other words, any unilateral deviation from the strategy (τ_1^*, τ_2^*) by one of the agent (the strategy of the other remaining fixed) does not render the deviating agent better off.

Theorem 1. *Under assumptions (i) - (ix) on ρ^1 and ρ^2 , there exists a Nash equilibrium point (τ_1^*, τ_2^*) for the game described above.*

We will construct a sequence $(\tau_{2n+1}, \tau_{2n})_{n \in \mathbb{N}}$ (by induction), for which we will show that it converges to a Nash equilibrium point.

We set $\tau_1 := T$ and $\tau_2 := T$. We suppose that $\tau_{2n-1} \in \Theta$ and $\tau_{2n} \in \Theta$ have been defined. We set, for each $k \in \mathbb{N}$,

$$\xi^{2n+1}(\theta_k) := X^1(\theta_k)1_{\{\theta_k < \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} \leq \theta_k\}}. \quad (3)$$

Moreover, $\xi^{2n+1}(T) := Y^1(\tau_{2n})$. This definition is “consistent” with the above, as by (3), $1_{\{\theta_k = T\}}\xi^{2n+1}(\theta_k) = 1_{\{\theta_k = T\}}Y^1(\tau_{2n})$.

For $\tau \in \Theta$ of the form $\tau = \sum_{k \in \mathbb{N}} \theta_k 1_{A_k} + T 1_{\bar{A}}$, where $((A_k), \bar{A})$ is a partition, A_k is \mathcal{F}_{θ_k} -measurable for each $k \in \mathbb{N}$, and \bar{A} is \mathcal{F}_T -measurable,

$$\xi^{2n+1}(\tau) := \sum_{k \in \mathbb{N}} \xi^{2n+1}(\theta_k) 1_{A_k} + \xi^{2n+1}(T) 1_{\bar{A}}. \quad (4)$$

We note that $\xi^{2n+1}(\theta_k)$ is the pay-off at $\theta_k \wedge \tau_{2n}$ of agent 1 (up to the equality $\{\theta_k = \tau_{2n}\}$) if agent 1 plays θ_k and agent 2 plays τ_{2n} .

We also note that:

$$\xi^{2n+1}(\tau) = X^1(\tau)1_{\{\tau < \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} \leq \tau\}}.$$

Thus, $\xi^{2n+1}(\tau)$ is the pay-off at $\tau \wedge \tau_{2n}$ of agent 1 (up to the equality $\{\tau = \tau_{2n}\}$) if agent 1 plays τ and agent 2 plays τ_{2n} .

For each $S \in \Theta$, we define

$$\begin{aligned} V^{2n+1}(S) &:= \text{ess sup}_{\tau \in \Theta_S} \rho_{S, \tau \wedge \tau_{2n}}^1 [\xi^{2n+1}(\tau)] \\ \tilde{\tau}_{2n+1} &:= \text{ess inf } \tilde{\mathcal{A}}^1, \text{ where } \tilde{\mathcal{A}}^1 := \{\tau \in \Theta : V^{2n+1}(\tau) = \xi^{2n+1}(\tau)\} \\ \tau_{2n+1} &:= (\tilde{\tau}_{2n+1} \wedge \tau_{2n-1}) 1_{\{\tilde{\tau}_{2n+1} \wedge \tau_{2n-1} < \tau_{2n}\}} + \tau_{2n-1} 1_{\{\tilde{\tau}_{2n+1} \wedge \tau_{2n-1} \geq \tau_{2n}\}}. \end{aligned} \quad (5)$$

Assuming that $\limsup_{k \rightarrow +\infty} X^1(\theta_k) \leq X^1(T)$ (from (A4)) ensures that $\limsup_{k \rightarrow +\infty} \xi^{2n+1}(\theta_k) \leq \xi^{2n+1}(T)$. This is a technical condition on the pay-off which we use to apply Theorem 2.3 in [11].

We recall that under the assumptions of Theorem 2.3 in [11], the Bermudan

stopping time $\tilde{\tau}_{2n+1}$ is optimal for the optimal stopping problem with value $V^{2n+1}(0)$, that is

$$V^{2n+1}(0) = \rho_{0, \tilde{\tau}_{2n+1} \wedge \tau_{2n}}^1[\xi(\tilde{\tau}_{2n+1})] = \sup_{\tau \in \Theta} \rho_{0, \tau \wedge \tau_{2n}}^1[\xi^{2n+1}(\tau)]. \quad (6)$$

We also recall that $V^{2n+1}(T) = \xi^{2n+1}(T)$, under the assumption of knowledge preservation on ρ .

Remark 2. *i) It is not difficult to show, by induction, that for each $n \in \mathbb{N}$, $(\xi^{2n+1}(\tau))_{\tau \in \Theta}$ is an admissible L^p -integrable family, and τ_{2n+1} is a Bermudan stopping time (for the latter property, we use that Θ has the property of stability by concatenation of two Bermudan stopping times).*

ii) For each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau \wedge \tau_{2n})$.

Indeed, we have:

$$\begin{aligned} \xi^{2n+1}(\theta_k) &= X^1(\theta_k)1_{\{\theta_k < \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} \leq \theta_k\}} \\ &= X^1(\theta_k \wedge \tau_{2n})1_{\{\theta_k \wedge \tau_{2n} < \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} \leq \theta_k \wedge \tau_{2n}\}} = \xi^{2n+1}(\theta_k \wedge \tau_{2n}). \end{aligned} \quad (7)$$

Now, let $\tau \in \Theta$ be of the form $\tau = \sum_{k \in \mathbb{N}} \theta_k 1_{A_k} + T 1_{\bar{A}}$. By definition of $\xi^{2n+1}(\tau)$, of $\xi^{2n+1}(T)$ and by Eq. (7), we have:

$$\begin{aligned} \xi^{2n+1}(\tau) &= \sum_{k \in \mathbb{N}} \xi^{2n+1}(\theta_k) 1_{A_k} + \xi^{2n+1}(T) 1_{\bar{A}} \\ &= \sum_{k \in \mathbb{N}} \xi^{2n+1}(\theta_k \wedge \tau_{2n}) 1_{A_k} + Y^1(\tau_{2n}) 1_{\bar{A}} = \xi^{2n+1}(\tau \wedge \tau_{2n}). \end{aligned}$$

Proposition 1. *i) $\xi^{2n+1}(\tau) 1_{\{\tau_{2n} \leq \theta_k\}} = Y^1(\tau_{2n}) 1_{\{\tau_{2n} \leq \theta_k\}}$.*

ii) $V^{2n+1}(\theta_k) 1_{\{\tau_{2n} \leq \theta_k\}} = Y^1(\tau_{2n}) 1_{\{\tau_{2n} \leq \theta_k\}}$.

iii) $V^{2n+1}(\tau) 1_{\{\tau_{2n} \leq \tau\}} = Y^1(\tau_{2n}) 1_{\{\tau_{2n} \leq \tau\}}$.

iv) For each $n \in \mathbb{N}$, $\tilde{\tau}_{2n+1} = \text{ess inf}\{\tau \in \Theta : V^{2n+1}(\tau) = X^1(\tau)\} \wedge \tau_{2n}$. In particular, $\tilde{\tau}_{2n+1} \leq \tau_{2n}$.

Proof. *i) On the set $\{\tau = T\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(T) = Y^1(\tau_{2n})$. On the set $\{\tau = \theta_k < T\}$, by the second statement in Remark 2, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\theta_k) = \xi^{2n+1}(\theta_k \wedge \tau_{2n})$. Hence, on the set $\{\tau = \theta_k < T\} \cap \{\tau_{2n} \leq \theta_k\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau_{2n}) = Y^1(\tau_{2n})$, which proves the desired property.*

ii) We have:

$$\begin{aligned} 1_{\{\tau_{2n} \leq \theta_k\}} V^{2n+1}(\theta_k) &= 1_{\{\tau_{2n} \leq \theta_k\}} \text{ess sup}_{\tau \in \Theta_k} \rho_{\theta_k, \tau \wedge \tau_{2n}}[\xi^{2n+1}(\tau \wedge \tau_{2n})] \\ &= \text{ess sup}_{\tau \in \Theta_k} 1_{\{\tau_{2n} \leq \theta_k\}} \rho_{\theta_k, \tau \wedge \tau_{2n}}[\xi^{2n+1}(\tau \wedge \tau_{2n})] \\ &= \text{ess sup}_{\tau \in \Theta_k} 1_{\{\tau_{2n} \leq \theta_k\}} \rho_{\theta_k, \tau \wedge \tau_{2n} \wedge \theta_k}[\xi^{2n+1}(\tau \wedge \tau_{2n} \wedge \theta_k)], \end{aligned}$$

where we have used the “genrealized zero-one law” to obtain the last equality.

For any $\tau \in \Theta_k$, $\tau \wedge \tau_{2n} \wedge \theta_k = \tau_{2n} \wedge \theta_k \leq \theta_k$. Hence,

$$\begin{aligned} 1_{\{\tau_{2n} \leq \theta_k\}} \rho_{\theta_k, \tau \wedge \tau_{2n} \wedge \theta_k} [\xi^{2n+1}(\tau \wedge \tau_{2n} \wedge \theta_k)] &= 1_{\{\tau_{2n} \leq \theta_k\}} \rho_{\theta_k, \tau_{2n} \wedge \theta_k} [\xi^{2n+1}(\tau_{2n} \wedge \theta_k)] \\ &= 1_{\{\tau_{2n} \leq \theta_k\}} \xi^{2n+1}(\tau_{2n} \wedge \theta_k), \end{aligned}$$

where we have used the knowledge-preserving property of ρ to obtain the last equality.

Finally, we get

$$1_{\{\tau_{2n} \leq \theta_k\}} V^{2n+1}(\theta_k) = 1_{\{\tau_{2n} \leq \theta_k\}} \xi^{2n+1}(\tau_{2n}) = 1_{\{\tau_{2n} \leq \theta_k\}} Y^1(\tau_{2n}).$$

iii) Let $\tau \in \Theta$ be the form $\tau = \sum_{k \in \mathbb{N}} \theta_k 1_{A_k} + T 1_{\bar{A}}$. Then, by admissibility, we have

$$V^{2n+1}(\tau) = \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) 1_{A_k} + V^{2n+1}(T) 1_{\bar{A}}.$$

Hence,

$$\begin{aligned} V^{2n+1}(\tau) 1_{\{\tau_{2n} \leq \tau\}} &= \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) 1_{A_k \cap \{\tau_{2n} \leq \tau\}} + V^{2n+1}(T) 1_{\bar{A} \cap \{\tau_{2n} \leq \tau\}} \\ &= \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) 1_{A_k \cap \{\tau_{2n} \leq \theta_k\}} + V^{2n+1}(T) 1_{\bar{A} \cap \{\tau_{2n} \leq T\}} \\ &= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) 1_{A_k \cap \{\tau_{2n} \leq \theta_k\}} + \xi^{2n+1}(T) 1_{\bar{A} \cap \{\tau_{2n} \leq T\}}, \end{aligned}$$

where we have used the previous property (ii) to obtain the last equality.

Hence, we get

$$\begin{aligned} V^{2n+1}(\tau) 1_{\{\tau_{2n} \leq \tau\}} &= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) 1_{A_k \cap \{\tau_{2n} \leq \theta_k\}} + \xi^{2n+1}(T) 1_{\bar{A} \cap \{\tau_{2n} \leq T\}} \\ &= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) 1_{A_k \cap \{\tau_{2n} \leq \theta_k\}} + Y^1(\tau_{2n}) 1_{\bar{A} \cap \{\tau_{2n} \leq T\}} = Y^1(\tau_{2n}) 1_{\{\tau_{2n} \leq \tau\}}. \end{aligned}$$

iv) By the previous property (iii), we have, $V^{2n+1}(\tau) = \xi^{2n+1}(\tau)$ if and only if $V^{2n+1}(\tau) 1_{\{\tau < \tau_{2n}\}} = \xi^{2n+1}(\tau) 1_{\{\tau < \tau_{2n}\}}$. Hence,

$$\tilde{\tau}_{2n+1} = \text{ess inf} \{ \tau \in \Theta : V^{2n+1}(\tau) = X^1(\tau) \} \wedge \tau_{2n}.$$

□

Similarly to (3), (4) and (5), we define:

$$\xi^{2n+2}(\theta_k) := X^2(\theta_k) 1_{\{\theta_k < \tau_{2n+1}\}} + Y^2(\tau_{2n+1}) 1_{\{\tau_{2n+1} \leq \theta_k\}}, \text{ and}$$

$$\xi^{2n+2}(T) := Y^2(\tau_{2n+1}).$$

For $\tau \in \Theta$ of the form $\tau = \sum_{k \in \mathbb{N}} \theta_k 1_{A_k} + T 1_{\bar{A}}$, we define

$$\begin{aligned} \xi^{2n+2}(\tau) &:= \sum_{k \in \mathbb{N}} \xi^{2n+2}(\theta_k) 1_{A_k} + \xi^{2n+2}(T) 1_{\bar{A}} \\ V^{2n+2}(S) &:= \text{ess sup}_{\tau \in \Theta_S} \rho_{S, \tau \wedge \tau_{2n+1}}^2 [\xi^{2n+2}(\tau)] \\ \tilde{\tau}_{2n+2} &:= \text{ess inf } \tilde{\mathcal{A}}^2, \text{ where } \tilde{\mathcal{A}}^2 := \{\tau \in \Theta : V^{2n+2}(\tau) = \xi^{2n+2}(\tau)\} \\ \tau_{2n+2} &:= (\tilde{\tau}_{2n+2} \wedge \tau_{2n}) 1_{\{\tilde{\tau}_{2n+2} \wedge \tau_{2n} < \tau_{2n+1}\}} + \tau_{2n} 1_{\{\tilde{\tau}_{2n+2} \wedge \tau_{2n} \geq \tau_{2n+1}\}}. \end{aligned} \quad (8)$$

The random variable $\xi^{2n+2}(\tau)$ is exactly the pay-off at $\tau \wedge \tau_{2n+1}$ of agent 2, if agent 1 plays τ_{2n+1} and agent 2 plays τ . Hence, $V^{2n+2}(S)$ is the optimal value at time S for agent 2, when agent 1's strategy is fixed to τ_{2n+1} . Assuming that $\limsup_{k \rightarrow +\infty} X^2(\theta_k) \leq X^2(T)$ leads to $\limsup_{k \rightarrow +\infty} \xi^{2n+2}(\theta_k) \leq \xi^{2n+2}(T)$, which we use in applying Theorem 2.3 in [11]. By Theorem 2.3 in [11], the Bermudan stopping time $\tilde{\tau}_{2n+2}$ is optimal for the problem with value $V^{2n+2}(0)$, that is,

$$V^{2n+2}(0) = \rho_{0, \tilde{\tau}_{2n+2} \wedge \tau_{2n+1}}^2 [\xi^{2n+2}(\tilde{\tau}_{2n+2})] = \sup_{\tau \in \Theta} \rho_{0, \tau \wedge \tau_{2n+1}}^2 [\xi^{2n+2}(\tau)].$$

Remark 3. Let us recall that (cf. [11]) $\tilde{\tau}_n = \text{ess inf} \{\tau \in \Theta : V^n(\tau) = \xi^n(\tau)\}$ satisfies the property: $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n)$. (This is due to the property of stability of Θ by monotone limit and to the right-continuity-along Bermudan stopping strategies of the families $(V^n(\tau))$ and $(\xi^n(\tau))$).

Remark 4. By analogy with Remark 2, we have:

- i) $(\xi^{2n+2}(\tau))$ is a admissible L^p -integrable family;
- ii) for each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+2}(\tau) = \xi^{2n+2}(\tau \wedge \tau_{2n+1})$.

Proposition 2. We assume that ρ satisfies the usual “zero-one law”. Then, for all $m \geq 1$, $\tilde{\tau}_{m+2} \leq \tau_m$.

Proof. We suppose, by way of contradiction, that there exists $m \geq 1$ such that $P(\tilde{\tau}_{m+2} > \tau_m) > 0$, and we set $n := \min\{m \geq 1 : P(\tilde{\tau}_{m+2} > \tau_m) > 0\}$. We have $\tilde{\tau}_{n+1} \leq \tau_{n-1}$, by definition of n . This observation, together with the definition of τ_{n+1} and with the inequality of part (iv) of Proposition 1 gives:

$$\begin{aligned} \tau_{n+1} &= (\tilde{\tau}_{n+1} \wedge \tau_{n-1}) 1_{\{\tilde{\tau}_{n+1} \wedge \tau_{n-1} < \tau_n\}} + \tau_{n-1} 1_{\{\tilde{\tau}_{n+1} \wedge \tau_{n-1} \geq \tau_n\}} \\ &= \tilde{\tau}_{n+1} 1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tau_{n-1} 1_{\{\tilde{\tau}_{n+1} \geq \tau_n\}} \\ &= \tilde{\tau}_{n+1} 1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tau_{n-1} 1_{\{\tilde{\tau}_{n+1} = \tau_n\}} \end{aligned} \quad (9)$$

For similar reasons, we have

$$\tau_n = \tilde{\tau}_n 1_{\{\tilde{\tau}_n < \tau_{n-1}\}} + \tau_{n-2} 1_{\{\tilde{\tau}_n = \tau_{n-1}\}}. \quad (10)$$

For the easing of the presentation, we set $\Gamma := \{\tau_n < \tilde{\tau}_{n+2}\}$.

On the set Γ , we have:

- 1) $\tau_n < \tilde{\tau}_{n+2} \leq \tau_{n+1}$, the last inequality being due to property (iv) of Proposition 1.
- 2) $\tau_{n+1} = \tau_{n-1}$. This is due to (1), together with Eq. (9).
- 3) $\xi^{n+2} = \xi^n$. This is a consequence of (2) and the definitions of ξ^{n+2} and ξ^n .
- 4) $\tau_n = \tilde{\tau}_n$.

We prove that $\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \emptyset$, which together with Eq. (10), gives the desired statement.

Due to Eq. (10), we have $\{\tilde{\tau}_n = \tau_{n-1}\} = \{\tilde{\tau}_n = \tau_{n-1}\} \cap \{\tau_n = \tau_{n-2}\}$. Thus, we have

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1}, \tau_n = \tau_{n-2} < \tilde{\tau}_{n+2}\}.$$

Now, we have $\tilde{\tau}_n \leq \tau_{n-2}$ (due to the definition of n). Hence,

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1} \leq \tau_{n-2} = \tau_n < \tilde{\tau}_{n+2}\} = \emptyset,$$

where the equality with \emptyset is due to $\tilde{\tau}_{n+2} \leq \tau_{n-1}$.

We note that combining properties (1) and (4) gives $\tilde{\tau}_n < \tilde{\tau}_{n+2}$ on Γ . We will obtain a contradiction with this property. To this end, we will show that:

$$1_\Gamma V^{n+2}(\tilde{\tau}_n) = 1_\Gamma \xi^{n+2}(\tilde{\tau}_n). \quad (11)$$

By definition of $\tilde{\tau}_n$ and by Remark 3, we have:

$$V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n).$$

This property, together with property (3) on Γ , gives $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n) = \xi^{n+2}(\tilde{\tau}_n)$ on Γ . In order to show Eq. (11), it suffices to show

$$1_\Gamma V^{n+2}(\tilde{\tau}_n) = 1_\Gamma V^n(\tilde{\tau}_n).$$

By property (4) on Γ and Proposition 4 (applied with $A = \Gamma \in \mathcal{F}_{\tau_n}$ and $\tau = \tau_n$), we have

$$1_\Gamma V^{n+2}(\tilde{\tau}_n) = 1_\Gamma V^{n+2}(\tau_n) = 1_\Gamma V_\Gamma^{n+2}(\tau_n).$$

Due to property (3) on Γ , V_Γ^{n+2} and V_Γ^n have the same pay-off, and by applying again Proposition 4 and property (4) on Γ , we have

$$1_\Gamma V_\Gamma^{n+2}(\tau_n) = 1_\Gamma V_\Gamma^n(\tau_n) = 1_\Gamma V^n(\tau_n) = 1_\Gamma V^n(\tilde{\tau}_n).$$

We have shown that $1_\Gamma V^{n+2}(\tilde{\tau}_n) = 1_\Gamma V^n(\tilde{\tau}_n)$, which is the desired equality. Hence, we get $\tilde{\tau}_{n+2} \leq \tilde{\tau}_n$ on Γ (as by definition $\tilde{\tau}_{n+2} = \text{ess inf}\{\tau \in \Theta : V^{n+2}(\tau) = \xi^{n+2}(\tau)\}$). However, this is in contradiction with the property $\tilde{\tau}_{n+2} > \tilde{\tau}_n$ on Γ . The proof is complete. \square

Lemma 1. *i) For all $n \geq 2$, $\tau_{n+1} = \tilde{\tau}_{n+1}1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tau_{n-1}1_{\{\tilde{\tau}_{n+1} = \tau_n\}}$.*

ii) For all $n \geq 2$, $\tilde{\tau}_{n+1} = \tau_{n+1} \wedge \tau_n$.

iii) On $\{\tau_n = \tau_{n-1}\}$, $\tau_m = T$, for all $m \in \{1, \dots, n\}$.

Proof. i) This property follows from the definition of τ_{n+1} , together with Proposition 2, and with property (iv) of Proposition 1.

ii) By using (i), we get

$$\begin{aligned} \tau_{n+1} \wedge \tau_n &= (\tilde{\tau}_{n+1} \wedge \tau_n)1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tau_n)1_{\{\tilde{\tau}_{n+1} = \tau_n\}} \\ &= \tilde{\tau}_{n+1}1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tilde{\tau}_{n+1})1_{\{\tilde{\tau}_{n+1} = \tau_n\}} \\ &= \tilde{\tau}_{n+1}1_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tilde{\tau}_{n+1}1_{\{\tilde{\tau}_{n+1} = \tau_n\}}, \end{aligned}$$

where we have used Proposition 2 for the last equality.

Finally, by using property (iv) of Proposition 1, we get $\tau_{n+1} \wedge \tau_n = \tilde{\tau}_{n+1}$.

iii) To prove this property, we proceed by induction. The property is true for $n = 2$. We suppose that the property is true at rank $n - 1$ (where $n \geq 3$), that is on $\{\tau_{n-1} = \tau_{n-2}\}$, $\tau_m = T$, for all $m \in \{1, \dots, n - 1\}$.

From the expression for τ_n from statement (i), we get

$$\tau_n = \tilde{\tau}_n1_{\{\tilde{\tau}_n < \tau_{n-1}\}} + \tau_{n-2}1_{\{\tilde{\tau}_n = \tau_{n-1}\}}.$$

Hence, $\tau_n = \tau_{n-2}$ on the set $\{\tau_n = \tau_{n-1}\}$. We conclude by the induction hypothesis. \square

Lemma 2. *The following inequalities hold true:*

i) $J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$, for all $\tau \in \Theta$.

ii) $J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2})$, for all $\tau \in \Theta$.

Proof. Let us first prove statement i):

By (A1), we have $X^1 \leq Y^1$; it follows

$$X^1(\tau)1_{\{\tau \leq \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} < \tau\}} \leq \xi^{2n+1}(\tau).$$

Hence, by monotonicity, and by definition of $V^{2n+1}(0)$, we have

$$J_1(\tau, \tau_{2n}) \leq V^{2n+1}(0) \tag{12}$$

We will now show that $V^{2n+1}(0) = J_1(\tau_{2n+1}, \tau_{2n})$, which will complete the proof of statement i).

We have

$$\begin{aligned} J_1(\tau_{2n+1}, \tau_{2n}) &= \rho_{0, \tau_{2n+1} \wedge \tau_{2n}}^1 (X^1(\tau_{2n+1})1_{\{\tau_{2n+1} \leq \tau_{2n}\}} + Y^1(\tau_{2n})1_{\{\tau_{2n} < \tau_{2n+1}\}}) \\ &= \rho_{0, \tau_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tau_{2n+1})), \end{aligned}$$

where we have used iii) from Lemma 1, and $X^1(T) = Y^1(T)$ from (A2) to show the last equality.

On the other hand, by ii) from Remark 2 and ii) from Lemma 1,

$$\rho_{0, \tau_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tau_{2n+1})) = \rho_{0, \tau_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tau_{2n+1} \wedge \tau_{2n})) = \rho_{0, \tilde{\tau}_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tilde{\tau}_{2n+1})).$$

By optimality of $\tilde{\tau}_{2n+1}$ for $V^{2n+1}(0)$ (cf. Eq. (6)), we get

$$\rho_{0, \tilde{\tau}_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tilde{\tau}_{2n+1})) = V^{2n+1}(0).$$

Hence, we conclude

$$J_1(\tau_{2n+1}, \tau_{2n}) = \rho_{0, \tau_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tau_{2n+1})) = \rho_{0, \tilde{\tau}_{2n+1} \wedge \tau_{2n}}^1 (\xi^{2n+1}(\tilde{\tau}_{2n+1})) = V^{2n+1}(0). \tag{13}$$

From Eq. (13) and Eq. (12), we get

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n}).$$

Let us now prove statement ii):

We have

$$J_2(\tau_{2n+1}, \tau) \leq V^{2n+2}(0), \tag{14}$$

by definition of $V^{2n+2}(0)$ (cf. Eq. (8)).

We will now show that $J_2(\tau_{2n+1}, \tau_{2n+2}) = V^{2n+2}(0)$, which will complete the proof.

By definition of $\xi^{2n+2}(\tau_{2n})$, by ii) from Remark 4, and by ii) from Lemma 1, we have

$$\begin{aligned} J_2(\tau_{2n+1}, \tau_{2n+2}) &= \rho_{0, \tau_{2n+1} \wedge \tau_{2n+2}}^2 (X^2(\tau_{2n+2})1_{\{\tau_{2n+2} < \tau_{2n+1}\}} + Y^2(\tau_{2n+1})1_{\{\tau_{2n+1} \leq \tau_{2n+2}\}}) \\ &= \rho_{0, \tau_{2n+1} \wedge \tau_{2n+2}}^2 (\xi^{2n+2}(\tau_{2n+2})) \\ &= \rho_{0, \tau_{2n+1} \wedge \tau_{2n+2}}^2 (\xi^{2n+2}(\tau_{2n+2} \wedge \tau_{2n+1})) \\ &= \rho_{0, \tilde{\tau}_{2n+2} \wedge \tau_{2n+1}}^2 (\xi^{2n+2}(\tilde{\tau}_{2n+2})). \end{aligned}$$

By Eq. (8), we have

$$\rho_{0, \tilde{\tau}_{2n+2} \wedge \tau_{2n+1}}^2 (\xi^{2n+2}(\tilde{\tau}_{2n+2})) = V^{2n+2}(0).$$

Hence, we conclude

$$J_2(\tau_{2n+1}, \tau_{2n+2}) = \rho_{0, \tau_{2n+1} \wedge \tau_{2n+2}}^2 (\xi^{2n+2}(\tau_{2n+2})) = \rho_{0, \tilde{\tau}_{2n+2} \wedge \tau_{2n+1}}^2 (\xi^{2n+2}(\tilde{\tau}_{2n+2})) = V^{2n+2}(0). \quad (15)$$

From Eq. (15) and Eq. (14), we get

$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}).$$

□

Remark 5. As a by-product of the previous proof, we find that τ_{2n+1} is optimal for the problem with value $V^{2n+1}(0)$, and τ_{2n+2} is optimal for the problem with value $V^{2n+2}(0)$.

Definition 3. We define $\tau_1^* = \lim_{n \rightarrow +\infty} \tau_{2n+1}$, and $\tau_2^* = \lim_{n \rightarrow +\infty} \tau_{2n}$.

Proposition 3. We assume that ρ^1 and ρ^2 satisfy properties (i) – (vii), and the following additional property: for $i \in \{1, 2\}$,

$$\limsup_{n \rightarrow +\infty} \rho_{0, \nu_n}^i [\xi(\nu_n)] = \rho_{0, \nu}^i [\xi(\nu)], \quad (16)$$

for any sequence $(\nu_n) \subset \Theta^{\mathbb{N}}$, $\nu \in \Theta$, such that $\nu_n \downarrow \nu$. We have:

- i) For all $\tau \in \Theta$, $\lim_{n \rightarrow +\infty} J_1(\tau, \tau_{2n}) = J_1(\tau, \tau_2^*)$.
- ii) For all $\tau \in \Theta$, $\lim_{n \rightarrow +\infty} J_2(\tau_{2n+1}, \tau) = J_2(\tau_1^*, \tau)$.
- iii) For all $\tau \in \Theta$, $\lim_{n \rightarrow +\infty} J_1(\tau_{2n+1}, \tau_{2n+2}) = J_1(\tau_1^*, \tau_2^*)$.
- iv) For all $\tau \in \Theta$, $\lim_{n \rightarrow +\infty} J_2(\tau_{2n+1}, \tau_{2n+2}) = J_2(\tau_1^*, \tau_2^*)$.

Proof. Let us first show statement i):

Let us recall the following notation:

for a fixed $\tau \in \Theta$,

$$I^1(\tau, \nu) := X^1(\tau)1_{\{\tau \leq \nu\}} + Y^1(\nu)1_{\{\nu < \tau\}},$$

$$I^1(\tau, \tau_2^*) := X^1(\tau)1_{\{\tau \leq \tau_2^*\}} + Y^1(\tau_2^*)1_{\{\tau_2^* < \tau\}}.$$

With this notation, we have

$$J_1(\tau, \tau_{2n}) = \rho_{0, \tau \wedge \tau_{2n}}^1 [I(\tau, \tau_{2n})], \quad \text{and} \quad J_1(\tau, \tau_2^*) = \rho_{0, \tau \wedge \tau_2^*}^1 [I(\tau, \tau_2^*)].$$

We note that the sequence (τ_{2n}) and $(\tau \wedge \tau_{2n})$ converges from above to τ_2^* and $\tau \wedge \tau_2^*$, respectively. Moreover, for each $\tau \in \Theta$, the family $(I^1(\tau, \nu))_{\nu \in \Theta}$ is admissible. Indeed, for each $\nu \in \Theta$, $I^1(\tau, \nu)$ is \mathcal{F}_ν -measurable. Moreover, if $\{\nu = \nu'\}$, $I(\tau, \nu) = I(\tau, \nu')$ a.s. Hence, as any admissible family in our framework is right-continuous along Bermudan stopping strategies (cf. Remark 2.10 in [11]), we get

$$\lim_{n \rightarrow +\infty} I(\tau, \tau_{2n}) = I(\tau, \tau_2^*).$$

Hence, by property (16) on ρ^1 , we get

$$\limsup_{n \rightarrow +\infty} \rho_{0, \tau \wedge \tau_{2n}}^1 [I(\tau, \tau_{2n})] = \rho_{0, \tau \wedge \tau_2^*}^1 [I(\tau, \tau_2^*)].$$

Now, let us prove statement ii):

For $\tau \in \Theta$, we recall the following notation:

$$I^2(\nu, \tau) := X^2(\tau)1_{\{\tau < \nu\}} + Y^2(\nu)1_{\{\nu \leq \tau\}},$$

The family $(I^2(\nu, \tau))_{\nu \in \Theta}$ is admissible. Indeed, for each $\nu \in \Theta$, $I^2(\nu, \tau)$ is \mathcal{F}_ν -measurable. Moreover, on $\{\nu_1 = \nu_2\}$, $I^2(\nu_1, \tau) = I^2(\nu_2, \tau)$ a.s.

As (τ_{2n+1}) converges from above to τ_1^* , and as $(I^2(\nu, \tau))_{\nu \in \Theta}$ is right-continuous along Bermudan stopping strategies (cf. Remark 2.10 in [11]), we get

$$\lim_{n \rightarrow +\infty} I^2(\tau_{2n+1}, \tau) = I^2(\tau_1^*, \tau).$$

By property (16) on ρ^2 , we get

$$\limsup_{n \rightarrow +\infty} \rho_{0, \tau_{2n+1} \wedge \tau}^2 [I^2(\tau_{2n+1}, \tau)] = \rho_{0, \tau_1^* \wedge \tau}^2 [I^2(\tau_1^*, \tau)].$$

We now prove statement iii).

The proof relies again on the Bermudan structure on Θ . For any sequence $(\tau_n) \in \Theta^{\mathbb{N}}$ converging from above to $\tau \in \Theta$, we have: for almost each $\omega \in \Omega$, there exists $n_0 = n_0(\omega)$ such that for all $n \geq n_0$, $\tau_n(\omega) = \tau(\omega)$ (cf. Remark

10 in [11]).

Hence, for almost each $\omega \in \Omega$, there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$, $\tau_{2n+1}(\omega) = \tau_1^*(\omega)$, $\tau_{2n+2}(\omega) = \tau_2^*(\omega)$ and

$$I(\tau_{2n+1}, \tau_{2n+2})(\omega) = X^1(\tau_1^*)(\omega)1_{\{\tau_1^* \leq \tau_2^*\}}(\omega) + Y^1(\tau_2^*)(\omega)1_{\{\tau_2^* < \tau_1^*\}}(\omega).$$

By property (16) on ρ^1 , we get

$$\lim_{n \rightarrow +\infty} J_1(\tau_{2n+1}, \tau_{2n+2}) = J_1(\tau_1^*, \tau_2^*).$$

The proof of iv) is based on the same arguments. □

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. By combining Lemma 2 and Proposition 3, we get:

$$J_1(\tau, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$

$$J_2(\tau_1^*, \tau) \leq J_2(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$

Hence, (τ_1^*, τ_2^*) is a Nash equilibrium point. □

We have thus shown that the non-linear non-zero-sum Dynkin game with Bermudan strategies admits a Nash equilibrium point.

4 Appendix

Proposition 4. (*Localisation property*) Let $(\xi(\tau))_{\tau \in \Theta}$ be a given admissible p -integrable family. Let $(V(\tau))_{\tau \in \Theta}$ be the value family of the optimal stopping problem: for $S \in \Theta$,

$$V(S) = \text{ess sup}_{\tau \in \Theta_S} \rho_{S,\tau}[\xi(\tau)].$$

Let $S \in \Theta$, and let A be in \mathcal{F}_S . We consider the pay-off family $(\xi(\tau)1_A)_{\tau \in \Theta_S}$, and we denote by $(V_A(\tau))_{\tau \in \Theta_S}$ the corresponding value family, defined by:

$$V_A(\tau) = \text{ess sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[\xi(\nu)1_A].$$

If ρ satisfies the usual “zero-one law” (that is $1_A \rho_{S,\tau}[\eta] = 1_A \rho_{S,\tau}[1_A \eta]$ for all $A \in \mathcal{F}_S$, for all $\eta \in L^p(\mathcal{F}_\tau)$), then for each $\tau \in \Theta_S$,

$$1_A V_A(\tau) = 1_A V(\tau).$$

Proof. By the definition of $V(\tau)$ and the usual “zero-one law”, we have

$$\begin{aligned} 1_A V(\tau) &= 1_A \text{ess sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[\xi(\nu)] = \text{ess sup}_{\nu \in \Theta_\tau} 1_A \rho_{\tau,\nu}[\xi(\nu)] \\ &= \text{ess sup}_{\nu \in \Theta_\tau} 1_A \rho_{\tau,\nu}[1_A \xi(\nu)] = 1_A \text{ess sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[1_A \xi(\nu)] = 1_A V_A(\tau). \end{aligned}$$

□

References

- [1] M. Alario-Nazaret, J. P. Lepeltier, and B. Marchal, *Dynkin games*, Stochastic Differential Systems, Lecture Notes in Control and Information Sciences, Vol. 43, 1982, pp. 23-32.
- [2] E. Bayraktar and S. Yao, *On the robust Dynkin game*, Ann. Appl. Probab., Vol. 27(3), 2017, pp. 1702-1755.
- [3] J.-M. Bismut, *Sur un probleme de Dynkin*, Z.Wahrsch. Verw. Gebiete 39, 1977, pp. 31-53.
- [4] S. Crepey and A. Matoussi, *Reflected and Doubly Reflected BSDEs with jumps*, Annals of App. Prob. 18(5), 2008, pp. 2041-2069.
- [5] J. Cvitanic and I. Karatzas, *Backward stochastic differential equations with reflection and Dynkin games*, Annals of Prob. 24 (4), 1996, pp. 2024-2056.
- [6] R. Dumitrescu, M.-C. Quenez, and A. Sulem, *Game options in an imperfect market with default*, SIAM Journal on Financial Mathematics 8, 2017, pp. 532-559.
- [7] R. Dumitrescu, M.-C. Quenez, and A. Sulem, *Mixed Generalized Dynkin Games and Stochastic control in a Markovian framework*, Stochastics 89, 2017, pp. 400-429.
- [8] R. Dumitrescu, M.-C. Quenez, and A. Sulem, *Generalized Dynkin games and doubly reflected BSDEs with jumps*, Electron. J. Probab. 21, no. 64, 2016, pp. 1-32.
- [9] E. B. Dynkin, *Game variant of a problem of optimal stopping*, Soviet Math. Dokl. 10, 1969, pp. 16-19.
- [10] M. Grigorova, P. Imkeller, E. Offen, Y. Ouknine, and M.-C. Quenez, *Doubly Reflected BSDEs and \mathcal{E}^f -Dynkin games: beyond the right-continuous case*, Electron. J. Probab. 23 pp. 1 - 38, 2018. (<https://doi.org/10.1214/18-EJP225>)
- [11] M. Grigorova, M.-C. Quenez and P. Yuan, *Optimal stopping: Bermudan strategies meet non-linear evaluations*, 2301.11102, (<https://arxiv.org/pdf/2301.11102.pdf>)

- [12] M. Grigorova and M.-C. Quenez, *Optimal stopping and a non-zero-sum Dynkin game in discrete time with risk measures induced by BSDEs*, Stochastics: An International Journal of Probability and Stochastic Processes, Vol. 89, 2017, pp. 259-279.
- [13] S. Hamadene, *Mixed zero-sum stochastic differential game and American game options*, SIAM J. Control Optim. 45(2), 2006, pp. 496-518.
- [14] S. Hamadene and J. Zhang, *The continuous time nonzero-sum Dynkin game problem and application in game options*, SIAM Journal on Control and Optimization 48, 2015, pp. 3659-3669.
- [15] S. Hamadene and M. Hassani, *BSDEs with two reflecting barriers driven by a Brownian motion and a Poisson noise and related Dynkin game*, Electron. J. Probab. 11(5), 2006, pp. 121-145.
- [16] S. Hamadene and M. Hassani, *The multiplayer nonzero-sum Dynkin game in continuous time*, SIAM Journal on Control and Optimization 52, 2014, 821-835.
- [17] S. Hamadene and M. Hassani, *The multi-player nonzero-sum Dynkin game in discrete time*, Math. Methods Oper. Res. 79, 2014, pp. 179-194.
- [18] S. Hamadene and J.-P. Lepeltier, *Reflected BSDEs and mixed game problem*, Stochastic Process. Appl. 85, 2000, pp. 177-188.
- [19] J. Kallsen and C. Kühn, *Pricing derivatives of American and game type in incomplete markets*, Finance Stoch., 8(2), 2004, pp. 261-284.
- [20] J. Kallsen and C. Kühn, *Convertible bonds: financial derivatives of game type*, Exotic Option Pricing and Advanced Levy Models, 2005, pp. 277-291.
- [21] K. Kentia and C. Kühn, *Nash Equilibria for Game Contingent Claims with Utility-Based Hedging*, SIAM Journal on Control and Optimization. vol. 56, 2018, pp. 3948-3972.
- [22] Y. Kifer, *Game options*, Finance and Stochastics 4, 2000, pp. 443-463.
- [23] Y. Kifer, *Dynkin games and Israeli options*, ISRN Probability and Statistics, Article ID: 856458, vol. 2013.
- [24] M. Kobylanski and M.-C. Quenez, *Optimal stopping time problem in a general framework*, Electron. J. Probab. 17, 2012.

- [25] M. Kobylanski, M.-C. Quenez, and E. Rouy-Mironescu, *Optimal multiple stopping time problem*, Ann. Appl. Probab. 21, 2011, pp. 1365-1399.
- [26] M. Kobylanski, M.-C. Quenez and M. Roger de Campagnolle, *Dynkin games in a general framework*, Stochastics 86(2), 2013, pp. 304-329.
- [27] R. Laraki and E. Solan, *Equilibrium in two-player non-zero-sum Dynkin games in continuous time*, Stochastics 85, 2013, pp. 997-1014.
- [28] J.P. Lepeltier and M.A. Maingueneau, *Le jeu de Dynkin en theorie generale sans l'hypothese de Mokobodski*, Stochastics 13, 1984, pp. 25-44.
- [29] H. Morimoto, *Non-zero-sum discrete parameter stochastic games with stopping times*, Probability Theory and Related Fields 72, 1986, pp. 155-160.
- [30] T. Nie, E. Kim and M. Rutkowski, *Arbitrage-Free Pricing of Game Options in Nonlinear Markets*, 1807.05448. Available at (<https://arxiv.org/pdf/1807.05448.pdf>)
- [31] Y. Ohtsubo, *A non-zero-sum extension of Dynkin's stopping problem*, Mathematics of Operations Research 12, 1987, pp. 277-296.
- [32] M.-C. Quenez and A. Sulem, *BSDEs with jumps, optimization and applications to dynamic risk measures*, Stochastic Processes and their Applications 123, 2013, pp. 3328-3357.
- [33] M.-C. Quenez and A. Sulem, *Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps*, Stochastic Processes and their Applications 124(9), 2014, pp. 3031-3054.