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► **To cite this version:**

Thierry Montagu, André Ferrari. Variance Stabilizing Transformations for Intensity Estimators of Shot Noise. IEEE Signal Processing Letters, 2023, 30, pp.977-981. <10.1109/LSP.2023.3299191>. <hal-04267182v2>

HAL Id: hal-04267182

<https://hal.science/hal-04267182v2>

Submitted on 6 Apr 2024

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Variance Stabilizing Transformations for Intensity Estimators of Shot Noise

Thierry Montagu, André Ferrari

Abstract—This paper considers two estimators of the intensity λ of the Shot Noise process for high values of λ . They are based on the first and the second empirical cumulants of the Shot Noise. For both statistics, it is shown that their respective variance grows respectively linearly and quadratically with λ . We construct Variance Stabilizing Transforms (and their corresponding inverse transforms) for both estimators thanks to the Delta method so as to make their respective variance constant (regardless of λ). Asymptotic distributions of the variance stabilized estimators are established. Finally, variance stabilization property is validated by Monte-Carlo simulations.

Index Terms—Shot Noise, variance stabilization, Delta theorem, intensity estimator, pile-up effect, Campbell theorem, k -statistics, asymptotic normality.

I. INTRODUCTION

THE continuous time Shot Noise process is a well known stochastic process stemming from physics [1], [2]. This stochastic process can also be found in other application domains such as telecommunications [3], [4], biology (see [5] and references therein), and finance [6] as well. It is defined by:

$$X(t) = \sum_k u_k h(t - \tau_k)$$

where $h(t)$ is a known nonzero causal integrable function with finite support, representing a causal impulse response function ($h(t) = 0$ for all $t < 0$). The u_k are independent identically distributed (i.i.d.) random variables called *marks* of the Shot Noise and τ_k are arrival times of an (unobserved) Poisson process with intensity λ (see [7] for thorough details). John Gubner's works [8], [9] focused on Shot Noise probability distributions and intensity estimation [10], [11]. The estimation of the density of marks is sought in [12], [13].

We focus on the estimation of the intensity parameter λ which reflects the number of events (“shots”, like particles impinging a detector) that can occur in time: the higher λ , the more “shots” can be observed. As a direct consequence, the higher λ , the more likely events can occur simultaneously or at least very close to one another. As each pulse occurring at τ_k are each observed through $h(t - \tau_k)$, the larger the support of h , the more likely the two signals can superimpose and

sum: this phenomenon is known as “pile-up” effect. For recent works focusing on the estimation of the intensity of the Shot Noise with piled-up events using sparse signal representation of nuclear measurements see [14], [15]. In this work, we consider (very) high intensity regime, where all event pile-up so that one can not distinguish any single one.

We consider two estimators of the intensity λ for a stationary discretely sampled high intensity Shot Noise. The respective variances of these estimators are shown to increase with λ , a property called heteroskedasticity in statistics. One way to deal with heteroskedasticity is to seek to make the variance “stable” (constant) by constructing a Variance Stabilizing Transform (VST) [16], [17], [18]. Such a transformation makes the variance of the quantity of interest constant and asymptotically gaussian. One well known VST is the Anscombe transform [19] for the mean of Poisson random variables. Using a VST \mathcal{T} may require the knowledge of the inverse transform \mathcal{T}^{-1} (defined as $\mathcal{T}^{-1}(\mathcal{T}(x)) = x$). This is typically the case in signal/image processing where the denoising process is usually $[\mathcal{T} \rightarrow \text{Denoising} \rightarrow \mathcal{T}^{-1}]$ (see [20] for instance). Many applications make use of VST for Poisson denoising in signal processing [21, sec. 4], and in image processing such as astronomical imaging [22], radar imaging [23], biological imaging [24], [25], [26], [27], [28] or medical imaging [29], [30], [31].

The main contributions in this work are the variance stabilizing transforms established for two intensity estimators of the Shot Noise process. More precisely it is first noted that the respective variance of the first two cumulants estimators of the Shot Noise are linear and quadratic functions of the intensity λ in section II. Consequently, the two intensity estimators considered here inherit this property. Based on the Delta method, we derive two variance stabilizing transforms \mathcal{T}_1 and \mathcal{T}_2 (and their inverses) in section III which lead to two intensity estimators with constant variances. Asymptotic distributions of these variance-stabilized estimators are established in section IV and validated by Monte-Carlo simulations in section V.

II. INTENSITY ESTIMATORS

We consider a (very) high intensity discretely sampled Poisson Shot Noise process $X(kT_s) = X(k)$ assuming without loss of generality a sampling period $T_s = 1$. By high intensity, we mean that individual pulses $u_k h(k - \tau_k)$ can not be individually distinguished [32], [33]. Let define $c_j = E[u_k^j] \sum_n h^j(n)$ and assume c_1 and c_2 are known. Let

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κ_j denote the order j cumulant of $X(k)$ given by the Campbell theorem [7], [34, p. 223]

$$\kappa_j = \lambda c_j.$$

We seek to estimate λ based on the ‘‘plugin’’ intensity estimators:

$$\hat{\lambda}_j = \frac{\hat{\kappa}_j}{c_j} \quad (1)$$

for $j = 1$ and $j = 2$, and $c_j \neq 0$. These estimators, named first and second order intensity estimators in the following, are typically used in neutronics [35], [36], [37], [38].

A. First order estimator

We set $j = 1$ in (1) and use the empirical mean estimator for n observations

$$\hat{\kappa}_1 = \frac{1}{n} \sum_{k=1}^n X(k).$$

We seek an expression of the variance of $\hat{\kappa}_1$ taking into account that the samples $X(k)$ may be correlated. Correlation may stem from large support of h , since the covariance function of X is [34, p. 223]

$$\gamma(m) = \lambda E[u_k^2] \sum_{k \geq m} h(k)h(k-m). \quad (2)$$

Property 1. *Assuming that the impulse response $h(k)$ verifies $\exists 0 < \rho < 1, C > 0$ such that $\forall k \geq 0, |h(k)| \leq C\rho^k$, we have as $\rho \rightarrow 0$:*

$$\lim_{n \rightarrow +\infty} n\text{var}[\hat{\kappa}_1] = \kappa_2 + o(1).$$

Proof. Recall that (see [39, sec 4.2])

$$\lim_{n \rightarrow +\infty} n\text{var}[\hat{\kappa}_1] = \sum_{m=-\infty}^{\infty} \gamma(m)$$

with $\gamma(m)$ given by (2). Using the upper bound on $|h(k)|$ we obtain:

$$|\gamma(m)| \leq \lambda C^2 E[u_k^2] \frac{\rho^m}{1 - \rho^2} \quad (3)$$

and

$$\left| \sum_{m \neq 0} \gamma(m) \right| \leq \lambda C^2 E[u_k^2] \frac{2\rho}{(1 - \rho^2)(1 - \rho)}$$

and consequently as $\rho \rightarrow 0$

$$\lim_{n \rightarrow +\infty} n\text{var}[\hat{\kappa}_1] = \gamma(0) + o(1)$$

where $\gamma(0) = \kappa_2 = \lambda c_2$. \square

Property 1 suggests to approximate $\text{var}[\hat{\kappa}_1]$ for n sufficiently large and a sufficiently fast decay of the impulse response $h(k)$ by:

$$\text{var}[\hat{\kappa}_1] = \frac{\kappa_2}{n} = \lambda \frac{c_2}{n} \quad (4)$$

which is a linear function of λ . From (1), we clearly have $E[\hat{\lambda}_1] = \lambda$, and as a first order approximation

$$\text{var}[\hat{\lambda}_1] = \frac{\lambda c_2}{n c_1^2} \quad (5)$$

showing that the variance of $\hat{\lambda}_1$ is proportional to λ . We use this expression in section III-B.

B. Second order estimator

We set $j = 2$ in (1) and consider the empirical variance estimator:

$$\hat{\kappa}_2 = \frac{1}{n-1} \sum_{k=1}^n (X(k) - \hat{\kappa}_1)^2.$$

It is asymptotically unbiased (and, since it is a k -statistic, no other cumulant estimator with a smaller variance exist [40]). We seek an expression of the variance of $\hat{\kappa}_2$ as in the previous section.

Property 2. *Assuming that the impulse response $h(k)$ verifies $\exists 0 < \rho < 1, C > 0$ such that $\forall k \geq 0, |h(k)| \leq C\rho^k$ we have*

$$\lim_{n \rightarrow \infty} n\text{var}[\hat{\kappa}_2] = 2\kappa_2^2 + 2\kappa_4 + o(\rho)$$

Proof. According to [39, Th. 4.1], the asymptotic variance of the empirical variance (Bartlett formula) is:

$$\lim_{n \rightarrow \infty} n\text{var}[\hat{\kappa}_2] = \sum_{m=-\infty}^{+\infty} 2\gamma^2(m) + \text{Cum}_4(m, m, 0). \quad (6)$$

where $\text{Cum}_4(m, m, 0)$ is an order 4 cumulant of X .

In an analogous way to the proof of Property 1, (3) implies:

$$\left| \sum_{m \neq 0} \gamma^2(m) \right| \leq \lambda^2 C^4 E[u_k^2]^2 \frac{2\rho^2}{(1 - \rho^2)^3}. \quad (7)$$

Following the proof of (8) in [34, p. 223] we can show:

$$\text{Cum}_4(m, m, 0) = \lambda E[u_k^4] \sum_{k \geq m} h^2(k)h^2(k-m) \quad (8)$$

which implies

$$\begin{aligned} |\text{Cum}_4(m, m, 0)| &\leq \lambda C^4 E[u_k^4] \frac{\rho^{2m}}{1 - \rho^4} \\ \left| \sum_{m \neq 0} \text{Cum}_4(m, m, 0) \right| &\leq \lambda C^4 E[u_k^4] \frac{2\rho^2}{(1 - \rho^2)(1 - \rho^4)} \end{aligned} \quad (9)$$

Replacing (7) and (9) in (6) leads to

$$\lim_{n \rightarrow \infty} n\text{var}[\hat{\kappa}_2] = 2\gamma^2(0) + \text{Cum}_4(0, 0, 0) + o(\rho).$$

with $\text{Cum}_4(0, 0, 0) = \kappa_4 = \lambda c_4$. \square

In an analogous way to the first order estimator, Property 2 shows that for n sufficiently large and a sufficiently fast decay of the impulse response $h(k)$

$$\text{var}[\hat{\kappa}_2] = \frac{2\lambda^2 c_2^2}{n} + \frac{\lambda c_4}{n} \quad (10)$$

and the variance of $\hat{\lambda}_2$ is a second order polynomial function of λ

$$\text{var}[\hat{\lambda}_2] = \frac{2\lambda^2}{n} + \frac{\lambda c_4}{n c_2^2}, \quad (11)$$

which we will use in section III-C.

III. VARIANCE STABILIZING TRANSFORMS

A. Background

A VST is a data transformation whose main aim is to render the variance independent of the mean and provide constant (stable) variance. VST were introduced in the pioneering works [41], [42], [19]. For a statistic Z_n with mean $E[Z_n] = \mu$ and variance $\text{var}[Z_n] = g(\mu)$ where g is a continuous function, let define the VST \mathcal{T} as $Y_n = \mathcal{T}(Z_n)$. Using the Delta method (see for example [17]) for \mathcal{T} around the mean μ , we have $\mathcal{T}(Z_n) \approx \mathcal{T}(\mu) + \mathcal{T}'(\mu)(Z_n - \mu)$, whose variance writes $\text{var}[Y_n] \approx (\mathcal{T}'(\mu))^2 g(\mu)$. Equaling this last expression to an arbitrary fixed constant C^2 , set to 1 in the following, a VST $\mathcal{T}(Z_n)$ can be found by finding the primitive of $\mathcal{T}'(Z_n) \approx Cg^{-1/2}(Z_n)$

$$\mathcal{T}(Z_n) = \int^{Z_n} \frac{1}{\sqrt{g(x)}} dx. \quad (12)$$

B. VST for the first order estimator

Here $Z_n = \hat{\lambda}_1$ and $g(\lambda)$ is given by (5). Applying (12) gives

$$\lambda_1^* = \mathcal{T}_1(\hat{\lambda}_1) = \frac{2c_1}{\sqrt{c_2}} \sqrt{n\hat{\lambda}_1}. \quad (13)$$

The inverse algebraic transformation (obtained by squaring (13) then isolating $\hat{\lambda}_1$) is

$$\hat{\lambda}_1 = \mathcal{T}_1^{-1}(\lambda_1^*) = \frac{c_2}{4nc_1^2} \lambda_1^{*2}.$$

C. VST for the second order estimator

Here $Z_n = \hat{\lambda}_2$, the variance function g is given by (11). Applying (12) gives [43, formula 2.261]

$$\sqrt{\frac{n}{2}} \ln \left(2\sqrt{\frac{2}{n} \left(\frac{2x^2}{n} + \frac{c_4 x}{nc_2^2} \right)} + \frac{4x}{n} + \frac{c_4}{nc_2^2} \right) \quad (14)$$

$$\lambda_2^* = \mathcal{T}_2(\hat{\lambda}_2) = \frac{\sqrt{n}}{\sqrt{2}} \times \ln \left(\frac{2\sqrt{2}}{n} \sqrt{2\hat{\lambda}_2^2 + \frac{c_4}{c_2^2} \hat{\lambda}_2} + \frac{4}{n} \hat{\lambda}_2 + \frac{c_4}{nc_2^2} \right). \quad (15)$$

The inverse algebraic transform (obtained after taking exponential of (15) and isolating $\hat{\lambda}_2$) writes

$$\hat{\lambda}_2 = \mathcal{T}_2^{-1}(\lambda_2^*) = \frac{\sqrt{n} \left(\exp \left(\frac{2\sqrt{2}\lambda_2^*}{\sqrt{n}} \right)^2 - \frac{2c_4}{nc_2^2} \exp \left(\frac{\sqrt{2}\lambda_2^*}{\sqrt{n}} \right) + \frac{c_4^2}{n^2 c_2^4} \right)}{\sqrt{32} \exp \left(\frac{\sqrt{2}\lambda_2^*}{\sqrt{n}} \right)}.$$

IV. ASYMPTOTIC PROPERTIES

The Delta theorem [16, chap. 2] allows to establish asymptotic distributions of VST for a sequence of statistics Z_n such that ($\xrightarrow{\mathcal{D}}$ denoting convergence in distribution)

$$\sqrt{n}(Z_n - \theta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, g(\theta)), g'(\theta) \neq 0.$$

Moreover, if the process is linear (see [44, chap. 7]), asymptotic distributions of $\hat{\kappa}_1$ and $\hat{\kappa}_2$ are gaussian with covariances given by (4) and (10). We will assume this hypothesis holds in the following.

A. First order estimator

Assuming the empirical first cumulant of the stationary Shot Noise is asymptotically gaussian distributed and Property 1 holds:

$$\sqrt{n}(\hat{\kappa}_1 - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \lambda c_2)$$

as $n \rightarrow \infty$. In the same way the first order intensity estimator (see [45])

$$\sqrt{n}(\hat{\lambda}_1 - \lambda) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \lambda \frac{c_2}{c_1^2} \right).$$

According to the Delta theorem with $Z_n = \hat{\lambda}_1$, $\theta = \lambda$, and \mathcal{T}_1 given by (13), one obtains

Theorem 1. *Under the previous hypothesis,*

$$\sqrt{n}(\lambda_1^* - \mathcal{T}_1(\lambda_1)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

B. Second order estimator

Assuming the empirical second cumulant of the stationary Shot Noise is asymptotically gaussian distributed (see [46], [18, Th. 7.5.1]) and Property 2 holds:

$$\sqrt{n}(\hat{\kappa}_2 - \kappa_2) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2\kappa_2^2 + \kappa_4)$$

with $2\kappa_2^2 = 2\lambda^2 c_2^2$, as $n \rightarrow \infty$. The second order intensity estimator is also asymptotically gaussian according to

$$\sqrt{n}(\hat{\lambda}_2 - \lambda) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, 2\lambda^2 + \frac{c_4}{c_2^2} \lambda \right).$$

With $Z_n = \hat{\lambda}_2$, $\theta = \lambda$, \mathcal{T}_2 given by (15), one obtains according to the Delta theorem

Theorem 2. *Under the previous hypothesis,*

$$\sqrt{n}(\lambda_2^* - \mathcal{T}_2(\lambda_2)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

V. NUMERICAL EXPERIMENTS

In this section, we illustrate thanks to computer simulations the heteroskedasticity property of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ and the variance stabilization achieved by \mathcal{T}_1 and \mathcal{T}_2 wrt λ and n .

A. Setup

Shot Noise sample paths were generated with different intensities λ ranging from $5 \cdot 10^5$ to $2 \cdot 10^6$, with sampling period $T_s = 10^{-5}$ s. For each intensity, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are estimated as well as their respective variance. $\mathcal{T}_1(\hat{\lambda}_1)$ and $\mathcal{T}_2(\hat{\lambda}_2)$ are applied in order to stabilize variance for each λ .

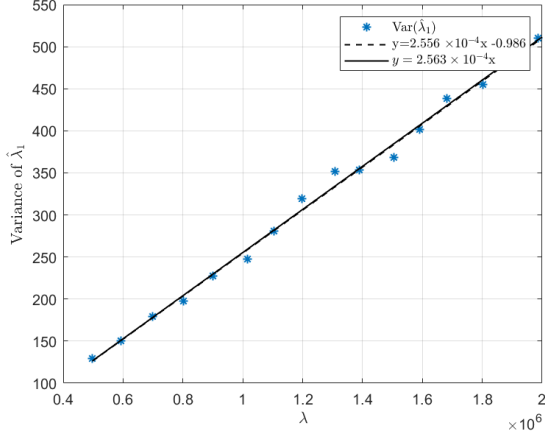


Figure 1. Linear variation of the variance of $\hat{\lambda}_1$ as a function of λ with $n = 2000$.

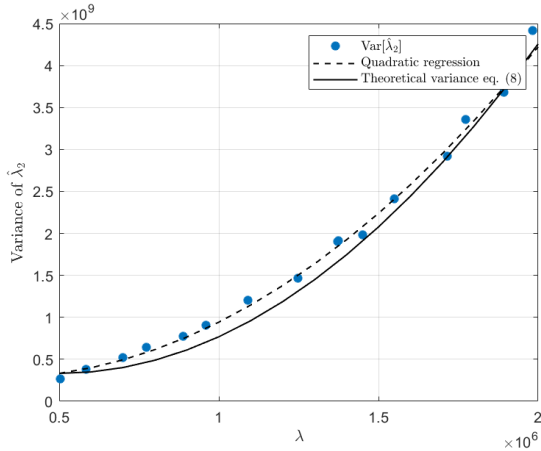


Figure 2. Quadratic variation of the variance of $\hat{\lambda}_2$ as a function of λ with $n = 2000$.

B. Parameter settings

The impulse response is $h(t) = \exp(-t)$, $t > 0$. Marks u_k were generated following a uniform distribution $\mathcal{U}(0, 300)$ with mean 150, so $c_1 = 150$, $c_2 = 150^2/2$.

C. Results

Figure 1 shows the linear relation between $\hat{\lambda}_1$ and its variance as expected by (4) and a linear regression of the estimated variances with $n = 2000$ Shot Noise samples. A good agreement between the theoretical/estimated values of the slope can be noticed. Figure 2 shows the quadratic relation between $\hat{\lambda}_2$ and its variance as expected by (10) and a quadratic regression (again with $n = 2000$ Shot Noise samples). The coefficient of order 2 monomial ($\frac{2}{n-1} = 10^{-3}$ is very close to the estimated one ($= 9.995 \times 10^{-4}$).

Variance stabilization wrt λ can be observed on the bottom of figure 3 and figure 4 for $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively: variance of both transformed quantities $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$ attain the asymptotic theoretical value 1. Boxplots were obtained based on 100000 Monte Carlo simulations with $n = 2000$ Shot Noise samples.

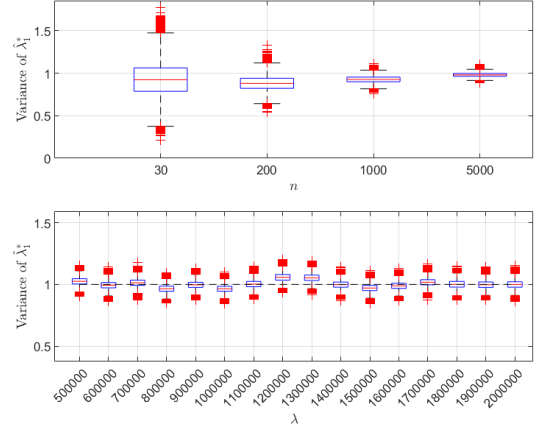


Figure 3. Top: Variance convergence for \mathcal{T}_1 as a function of n for (fixed) $\lambda = 2.10^6$. Bottom: Variance stabilization of $\hat{\lambda}_1^* = \mathcal{T}_1(\hat{\lambda}_1)$ for $n = 2000$. The dashed line corresponds to the theoretical asymptotic value of the variance after applying VST .

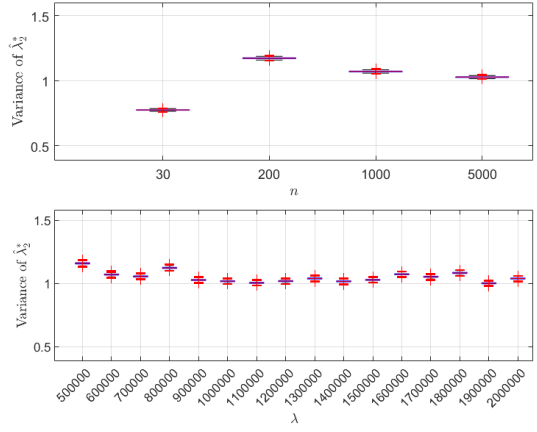


Figure 4. Top: Variance convergence for \mathcal{T}_2 as a function of n for (fixed) $\lambda = 2.10^6$. Bottom: Variance stabilization of $\hat{\lambda}_2^* = \mathcal{T}_2(\hat{\lambda}_2)$ for $n = 2000$. The dashed line corresponds to the theoretical asymptotic value of the variance after applying VST .

Variance stabilization wrt n is shown on top of figure 3 and 4 for $\hat{\lambda}_1$ and $\hat{\lambda}_2$ respectively: the more data the more stable the variance. For both quantities, good variance stabilization is obtained from $n = 1000$ and beyond.

VI. CONCLUSION

We have proved that the two intensity estimators $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of a Shot Noise process are heteroskedastic quantities whose variances vary linearly and quadratically, respectively. Variance stabilizing transforms \mathcal{T}_1 and \mathcal{T}_2 (and their inverse) were built, and asymptotic normality theorems were established. Numerical experiments were conducted in order to characterize their statistical properties. The VST-based denoising process of [20] adapted with \mathcal{T}_1 and \mathcal{T}_2 (and their inverses) targets typical real heteroskedastic neutron measurement signals based on Shot Noise empirical mean and variance (see [36], [38]). Application of this process is planned in a forthcoming work.

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