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IRREDUCIBILITY CRITERIA FOR THE PREIMAGES OF A TRANSVERSE VARIETY UNDER ENDOMORPHISMS OF PRODUCTS OF ELLIPTIC CURVES

RICCARDO PENGO AND EVELINA VIADA

ABSTRACT. We provide two different proofs of an irreducibility criterion for the preimages of a transverse subvariety of a product of elliptic curves under a diagonal endomorphism of sufficiently large degree. For curves, we present an arithmetic proof of the aforementioned irreducibility result, which enlightens connections to methods used in the context of the Torsion Anomalous Conjecture. On the other hand, we generalize the result for higher dimensional varieties using a more geometric approach. Finally, we give some applications of these results. More precisely, we establish the irreducibility of some explicit families of polynomials, we provide new estimates for the normalized heights of certain intersections and images, and we give new lower bounds for the essential minima of preimages.

1. INTRODUCTION

The aim of the present note is to prove by two different methods a criterion for the irreducibility of the preimage of a transverse subvariety of a product of elliptic curves under a diagonal endomorphism. It is well known that to establish the irreducibility of a variety is a difficult matter. Here, we show that the assumption of transversality of the variety is a necessary and sufficient condition to ensure the irreducibility of its preimage by diagonal endomorphisms of large degree. In particular, we make this largeness explicit with a precise bound depending only on the partial degrees of the variety. Such a question, which has a geometric flavor, can in fact be approached by arithmetic means, at least in the case of curves. A key role is played by the precise formulas for the degrees of the preimages of subvarieties, proven by Hindry in [Hin88, Lemme 6] (see also [Lemma 13](#)).

Notation. We fix an algebraic closure $\overline{\mathbb{Q}}$ of the rational numbers \mathbb{Q} . In this article, a *variety* is an algebraic variety defined over $\overline{\mathbb{Q}}$.

Elliptic curves. We fix an integer $N \geq 1$ and N elliptic curves E_1, \dots, E_N defined over $\overline{\mathbb{Q}}$. We write

$$A_N := E_1 \times \cdots \times E_N$$

for their product. Moreover, for each $i \in \{1, \dots, N\}$ we fix a Weierstrass model

$$(1) \quad E_i: y^2 = x^3 + A_i x + B_i,$$

where A_i and B_i are algebraic integers. These Weierstrass models induce the embedding

$$(2) \quad A_N \hookrightarrow (\mathbb{P}^2)^N \hookrightarrow \mathbb{P}^m$$

where $m = 3^N - 1$, which is given by the composition of the product of the natural inclusions $E_i \hookrightarrow \mathbb{P}^2$ determined by the Weierstrass equations (1), together with the Segre embedding $(\mathbb{P}^2)^N \hookrightarrow \mathbb{P}^m$.

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Endomorphisms. We write $\text{End}(A_N)$ for the endomorphisms of A_N considered as an abelian variety. We recall as well that $\phi \in \text{End}(A_N)$ is an *isogeny* if it is surjective and has finite kernel. In this case, the *degree* of ϕ is defined as $\deg(\phi) := |\ker(\phi)|$. Every endomorphism $\phi \in \text{End}(A_N)$ can be represented by a matrix $\phi = (\phi_{i,j})_{i,j=1}^N$, where each $\phi_{i,j}: E_i \rightarrow E_j$ is either an isogeny or the trivial map. In particular, every tuple $(\alpha_1, \dots, \alpha_N) \in \prod_i \text{End}(E_i)$ induces a diagonal endomorphism

$$(3) \quad [\alpha_1, \dots, \alpha_N] := \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix}.$$

Transversality. A subvariety $V \subset A_N$ is a *translate* (respectively a *torsion variety*) if it is a finite union of translates of proper algebraic subgroups of A_N by points (respectively by torsion points). Moreover, an irreducible subvariety $V \subset A_N$ is *transverse*, (respectively *weak-transverse*), if it is not contained in any translate, (respectively in any torsion variety).

Main results. We are now ready to state the main results of this note. The following theorem provides a first irreducibility criterion for the preimages of transverse curves $C \subseteq A_N$.

Theorem A. *Let $p_1, \dots, p_N \in \mathbb{Z}$ be prime numbers defining the endomorphism $[p_1, \dots, p_N]$ of A_N . Then, for every transverse curve $C \subseteq A_N$ such that*

$$|p_j| \geq \deg(C)N3^{N-1}$$

for each $j \in \{1, \dots, N\}$, the preimage

$$[p_1, \dots, p_N]^{-1}(C)$$

is transverse.

Remark 4. Transversality is a necessary assumption, as the following counterexample easily shows. Suppose that $N = N_1 + N_2$ and let $A_{N_1} = E_1 \times \dots \times E_{N_1}$ and $A_{N_2} = E_{N_1+1} \times \dots \times E_N$. Moreover, fix a curve $C \subseteq A_{N_1}$, and let $O \in A_{N_2}(\mathbb{Q})$ be the origin. Then, for every $n \in \mathbb{Z}$, the preimage of $C \times \{O\}$ by the map $[n, \dots, n]$ is given by $\mathcal{D} \times A_{N_2}[n]$, where \mathcal{D} is the preimage of C under the multiplication by n on A_{N_1} . Even if \mathcal{D} will be irreducible when $|n|$ is large enough, the product $\mathcal{D} \times A_{N_2}[n]$ is clearly reducible, because its irreducible components are given by $\mathcal{D} \times \{T\}$, where $T \in A_{N_2}[n]$ runs over the n -th torsion points of A_{N_2} .

The previous theorem will be proven in [Section 3](#). Our proof is arithmetic in nature, and relies on the arithmetic Bézout theorem of Philippon [[Phi95](#)] and the comparison between the height and the essential minimum of a variety, proven by Zhang [[Zha95a](#)]. Moreover, this proof was inspired by some of the techniques used in the proof of the Torsion Anomalous Conjectures for curves inside a product of elliptic curves (see [[Via08](#)] and [[CVV19](#)]).

The aforementioned conjecture, proposed by Bombieri, Masser and Zannier [[BMZ07](#)], is still open for general varieties, despite several partial results. We refer to [[Zan12](#)] or [[Via17](#)] for an introduction to the question. (Weak)-transversality is the central geometric assumption in this conjecture. Several works have pointed out the connection between (weak)-transversality and the arithmetic of a variety. Other results, like [[Via21](#)], shed some light on how difficult and important it is to determine whether a curve is or not transverse. This inspired our [Theorem A](#).

On the other hand, our arithmetic proof of [Theorem A](#) does not generalize easily to higher dimensional varieties, as we point out in [Remark 37](#). This prompted us to find a different, geometric proof of [Theorem A](#), which generalizes to higher dimensional varieties and yields slightly different bounds. The best version of the bounds that we obtain is portrayed in [Theorem 38](#), and involves the notion of multiprojective degree, which we recall in [Section 2.4](#). To easy notations, we prefer to state here a weaker version of our main result, which is a corollary of [Theorem 38](#), as we show in [Lemma 39](#).

Theorem B. *Let $V \subseteq A_N$ be a transverse variety with finite stabilizer. Moreover, let*

$$(\alpha_1, \dots, \alpha_N) \in \prod_{j=1}^N \text{End}(E_j)$$

be a tuple of isogenies such that for every prime $p \mid \deg(\alpha_1) \cdots \deg(\alpha_N)$ we have that

$$p > \dim(V)! \deg(V).$$

Then, the preimage

$$[\alpha_1, \dots, \alpha_N]^{-1}(V)$$

is transverse.

Remark 5. The general case of any transverse variety V , can be easily reduced to the case of finite stabilizer via an isogeny. More precisely, let $M \in \mathbb{Z}_{\geq 1}$ and E_1, \dots, E_M be elliptic curves. Then, for every variety $V \subseteq A_M := E_1 \times \cdots \times E_M$ there exist an integer $N \leq M$, an isogeny $\phi: A_M \rightarrow A_M$ and a variety $V_0 \subseteq A_N$ with finite stabilizer, such that $\phi(V) = V_0 \times A_{M-N}$. Therefore, up to a fixed isogeny, the study of the transversality of $f^{-1}(V)$ can be reduced to the study of the transversality of $[\alpha_1, \dots, \alpha_N]^{-1}(V_0)$ and the study of the subgroups $[\alpha_{N+1}, \dots, \alpha_M]^{-1}(A_{M-N}) = A_{M-N}$. Since this last part is invariant, we see that [Theorem B](#) (and [Theorem 38](#)) suffice to understand the transversality of the preimage of a general transverse variety.

This theorem can be stated in a more direct way when $[\alpha_1, \dots, \alpha_N]$ is an integral multiple of the identity, as we show in [Corollary 41](#). In [Section 5](#) we give some applications of our main results. More precisely we show that the equations defining the preimages considered in [Theorems A](#) and [B](#) can quickly become very complicated. In particular, the irreducibility guaranteed by our results is difficult to check by other means. Thus, using our theorems, we provide new families of irreducible polynomials. Finally, we show how our theorems can be used to improve the upper bound on the height of certain intersections of A_N obtained using the arithmetic Bézout theorem ([17](#)), and the lower bound on the essential minimum of a subvariety provided by a theorem of Galateau [[Gal10](#)], portrayed in ([54](#)).

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2. SOME GEOMETRIC PRELIMINARIES

The aim of this section is to prove some basic lemmas which will be used in the proofs of both of our main results.

2.1. Subvarieties and transversality. First of all, we observe that any translation of a transverse subvariety of A_N by a point $P \in A_N(\mathbb{Q})$ is again transverse. Moreover, the following lemma links transversality to irreducibility.

Lemma 6. *Fix $N, M \in \mathbb{Z}_{\geq 1}$. Then, for any surjective morphism of abelian varieties $\phi: A_N \rightarrow A_M$, the preimage $\phi^{-1}(V)$ of a transverse variety $V \subseteq A_M$ is transverse if and only if it is irreducible.*

Proof. Transversality implies irreducibility by definition. On the other hand, suppose that $\phi^{-1}(V)$ is irreducible, and assume by contradiction that there exists a proper translate $B + P \supsetneq \phi^{-1}(V)$, where $B \subsetneq A_N$ is a proper algebraic subgroup, and $P \in A_N(\overline{\mathbb{Q}})$. Then $B \supseteq \ker(\phi)$, and so $\phi(B) \neq A_M$ thanks to the subgroup correspondence theorem. Therefore, we have that

$$V = \phi(\phi^{-1}(V)) \subseteq \phi(B + P) = \phi(B) + \phi(P),$$

because ϕ is surjective by assumption. Hence, V is not transverse, because $\phi(B) + \phi(P)$ is a proper translate. This contradicts our assumptions, and implies therefore that $\phi^{-1}(V)$ must be transverse, as we wanted to show. \square

This shows that, to study the transversality of preimages of transverse varieties it is sufficient to study their irreducibility.

2.2. Action of the Kernel on the fibers. The following lemma describes some invariants of the irreducible components of the preimage of a transverse subvariety under an isogeny.

Lemma 7. *Let $V \subseteq A_N$ be a transverse subvariety, and let ϕ be an isogeny of A_N . Then, the action of the subgroup $\ker(\phi)$ on the irreducible components of $\phi^{-1}(V)$ is transitive. In particular, each irreducible component of $\phi^{-1}(V)$ surjects on V and all have the same dimension, degree and stabilizer.*

Proof. First of all, observe that there exists an irreducible component $W \subseteq \phi^{-1}(V)$ of maximal dimension such that $\phi(W) = V$. Indeed, if this was not the case, $\phi(\phi^{-1}(V)) = V$ would be a closed subvariety of V which has strictly smaller dimension, which is clearly absurd.

Now, let $W' \subseteq \phi^{-1}(V)$ be any other irreducible component, and pick any point $P' \in W'(\overline{\mathbb{Q}})$. Then, there exists a point $P \in W(\overline{\mathbb{Q}})$ such that $\phi(P) = \phi(P')$, which implies that $P' = P + T$ for some point $T \in \ker(\phi)$. Therefore, the irreducible variety $W + T$ passes through P' , which implies that $W + T \subseteq W'$ because W' is an irreducible component. Since $\dim(W + T) = \dim(W) \geq \dim(W')$ by construction, this implies that $W' = W + T$, that $\phi(W') = \phi(W) = V$ and that the action of $\ker(\phi)$ on the irreducible components of $\phi^{-1}(V)$ is transitive. This shows in particular that all the components have the same dimension and degree, because these two quantities are invariant under translations.

To conclude, show that any variety $W \subseteq A_N$ and any translate $W + T$ by a point $T \in A_N(\overline{\mathbb{Q}})$ have the same stabilizer. Indeed, if $Q \in \text{Stab}(W)$ then $Q + (W + T) = (Q + W) + T = W + T = W'$, which implies that $\text{Stab}(W) \subseteq \text{Stab}(W')$. Similarly, any $Q' \in \text{Stab}(W + T)$ stabilizes $W = (W + T) - T$, which allows us to conclude that $\text{Stab}(W) = \text{Stab}(W')$. \square

2.3. Composing multiplication maps. The aim of this subsection is to prove a result which relates the irreducibility of the preimages of a transverse variety $V \subseteq A_N$ by different multiplication maps. To do so, we need to single out a specific class of factorizations of the diagonal endomorphisms $[\alpha_1, \dots, \alpha_N]$ introduced in (3). First of all, let us observe that each of these endomorphisms can be factored as

$$(8) \quad [\alpha_1, \dots, \alpha_n] = \begin{pmatrix} \alpha_1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix}.$$

In other words, we have that $[\alpha_1, \dots, \alpha_N] = [\alpha_1]_1 \circ \cdots \circ [\alpha_N]_N$, where we define

$$[\alpha]_j := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \alpha & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}$$

for every $j \in \{1, \dots, N\}$ and $\alpha \in \text{End}(E_j)$.

More generally, we say that a factor of $[\alpha_1, \dots, \alpha_N]$ is *admissible* if it is the composition of some of the factors $[\alpha_1]_1, \dots, [\alpha_N]_N$. For example, writing

$$[\alpha_1, \dots, \alpha_5] = \begin{pmatrix} 1 & & & & \\ & \alpha_2 & & & \\ & & 1 & & \\ & & & \alpha_4 & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \alpha_5 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \alpha_3 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

yields a factorization of $[\alpha_1, \dots, \alpha_5]$ into admissible factors.

With this definition at hand, we show in the following lemma that the transversality of the preimage of a transverse variety by a diagonal endomorphism is equivalent to the transversality of each of its preimages under the factors of any admissible factorization.

Lemma 9. *Let $\phi = [\alpha_1, \dots, \alpha_N]$ be a diagonal endomorphism of A_N . Fix moreover a transverse variety $V \subseteq A_N$, and a factorization $\phi = f_1 \circ \dots \circ f_r$ into admissible factors. Then,*

- $\phi^{-1}(V)$ is transverse if and only if $f_j^{-1}(V)$ is transverse for every $j \in \{1, \dots, r\}$.

Equivalently,

- $\phi^{-1}(V)$ is irreducible if and only if $f_j^{-1}(V)$ is irreducible for every $j \in \{1, \dots, r\}$.

Proof. The equivalence of the two statements is due to [Lemma 6](#). We now prove the transversality statement.

First of all, let us assume that $\phi^{-1}(V)$ is transverse, which implies by definition that $\phi^{-1}(V)$ is irreducible. We also observe that all the endomorphisms f_1, \dots, f_r commute, because they are represented by diagonal matrices. Therefore, for every $j \in \{1, \dots, r\}$ the variety

$$f_j^{-1}(V) = (f_1 \circ \dots \circ f_{j-1} \circ f_{j+1} \circ \dots \circ f_r)(\phi^{-1}(V))$$

will also be irreducible, since irreducibility is preserved under images. This allows us to conclude that $f_j^{-1}(V)$ is transverse, thanks to [Lemma 6](#).

Conversely, suppose that $f_j^{-1}(V)$ is transverse for every $j \in \{1, \dots, r\}$. In order to prove that $\phi^{-1}(V)$ is transverse, we will prove its irreducibility, which suffices thanks to [Lemma 6](#). To this aim, we proceed by induction on r .

To deal with the base case, let us suppose that $r = 2$. Then, for any given irreducible component $W \subseteq \phi^{-1}(V)$ we have that $f_1^{-1}(V) = f_2(W)$, thanks to [Lemma 7](#). Analogously, $f_2^{-1}(V) = f_1(W)$. Therefore, we see that

$$\text{Stab}(W) \supseteq \text{Stab}(f_1^{-1}(V)) + \text{Stab}(f_2^{-1}(V)),$$

which also implies that $\text{Stab}(W) \supseteq \ker(f_1) + \ker(f_2) = \ker(\phi)$, because the preimage of any variety by an endomorphism is stabilized by its kernel. This implies the irreducibility of $\phi^{-1}(V)$, as follows. We observe that $\phi(W) = V$ because V is irreducible, which implies that $W = \phi^{-1}(V)$, because W is stabilized by $\ker(\phi)$.

To conclude this proof, let us proceed with the inductive step of our proof. If $r \geq 3$, we know by the basis of the induction that $g^{-1}(V)$ is irreducible, where $g := f_{N-1} \circ f_N$. Therefore, we see by the inductive hypothesis that $\phi^{-1}(V) = (f_1 \circ \dots \circ f_{N-2} \circ g)^{-1}(V)$ is irreducible, as we wanted to show. \square

Remark 10. In particular, the previous lemma shows that if $\phi = [\alpha_1, \dots, \alpha_n]$ and $V \subseteq A_N$ is transverse, the irreducibility of $[\alpha_j]_j^{-1}(V)$ for every $j \in \{1, \dots, N\}$ suffices to guarantee that V is itself irreducible and transverse.

2.4. Degrees of preimages. In this subsection, we first recall a central result of Hindry (see [\[Hin88, Lemma 6\]](#)) concerning the geometric degree of preimages of algebraic varieties under group homomorphisms. Meanwhile we introduce some relevant notation and we prove some preliminary lemmas on degrees.

In order to do so, let us recall that for every zero cycle $\xi := \sum_{j=1}^k a_j P_j$ on an algebraic variety X , where $a_1, \dots, a_k \in \mathbb{Z}$ and $P_1, \dots, P_k \in X(\overline{\mathbb{Q}})$, we define $\deg_0(\xi) := \sum_{j=1}^k a_j$. Now, fix an embedding $\iota: X \hookrightarrow \mathbb{P}^n$, given by a very ample divisor $H \in \text{Div}(X)$. Then, we have an associated notion of degree of a closed subvariety $Y \subseteq X$, which is given by $\deg(Y) := \deg_0(Y \cdot H^{\dim(V)})$. Moreover, suppose that $X = X_1 \times \dots \times X_r$, and that the embedding ι factors as

$$\iota: X = X_1 \times \dots \times X_r \hookrightarrow \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \hookrightarrow \mathbb{P}^n$$

where the first embedding is the product of some embeddings $\iota_j: X_j \hookrightarrow \mathbb{P}^{n_j}$, for $j \in \{1, \dots, r\}$, which correspond to some very ample divisors $H'_j \in \text{Div}(X_j)$, and the second embedding

$$\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \hookrightarrow \mathbb{P}^n$$

is the Segre embedding, so that $n = (n_1 + 1) \cdots (n_r + 1) - 1$. Then, the first embedding allows one to define a new notion of multiprojective degree. More precisely, for every tuple $I = (i_1, \dots, i_r) \in \mathbb{N}^r$ and every subvariety $Y \subseteq X$ such that $i_1 + \dots + i_r = \dim(Y)$, one defines the multiprojective degree

$$(11) \quad \deg_I(Y) := \deg_0(Y \cdot H_1^{i_1} \cdots H_n^{i_n})$$

where $H_j := H'_j \times \prod_{i \neq j} X_i$ for every $j \in \{1, \dots, r\}$. Moreover, we have that $H = H_1 + \dots + H_n$, as follows from the basic properties of the Segre embedding. Therefore, we see from the multinomial theorem that the degree of any closed subvariety $Y \subseteq X$ can be expressed as

$$(12) \quad \deg(Y) = \dim(Y)! \sum_I \frac{\deg_I(Y)}{I!},$$

where the sum runs over all the r -tuples $I = (i_1, \dots, i_r) \in \mathbb{N}^r$ such that $i_1 + \dots + i_r = \dim(Y)$, and $I! := i_1! \cdots i_r!$. This and further properties of multiprojective degrees, can be found in [Phi81, § 3].

Now, let us point out that these multiprojective degrees are useful if one wants to express the degree of the preimage of a subvariety $V \subseteq A_N$ in terms of the degree of the subvariety itself, as expressed by the following lemma.

Lemma 13. *Let $V \subseteq A_N$ be a subvariety, and $(\alpha_1, \dots, \alpha_N) \in \prod_{j=1}^N \text{End}(E_j)$. Then, we have that*

$$(14) \quad \deg([\alpha_1, \dots, \alpha_N]^{-1}(V)) = \dim(V)! \sum_J \left(\prod_{k \notin J} \deg(\alpha_k) \right) \deg_{I_J}(V)$$

where the sum runs over the subsets $J \subseteq \{1, \dots, N\}$ with $|J| = \dim(V)$, and

$$I_J = (i_{J,1}, \dots, i_{J,N}) \in \mathbb{N}^N$$

is the tuple defined by setting $i_{J,j} = 1$ if $j \in J$, and $i_{J,j} = 0$ otherwise.

Proof. First of all, let us observe that for every $j \in \{1, \dots, N\}$ the divisor $H'_j \in \text{Div}(E_j)$ corresponding to the embedding $E_j \hookrightarrow \mathbb{P}^2$ is simply given by $H'_j = 3(0: 1: 0)$. Therefore, the divisor $H \in \text{Div}(A_N)$ corresponding to the embedding (2) is given by $H = H_1 + \dots + H_N$, where

$$(15) \quad H_j := 3 \cdot E_1 \times \dots \times E_{j-1} \times \{(0: 1: 0)\} \times E_{j+1} \times \dots \times E_N$$

for every $j \in \{1, \dots, N\}$. In particular, we see that $H_j^2 = 0$ for every $j \in \{1, \dots, n\}$, because in the j -th factor H_j^2 reduces to the intersection of two generic points, which is empty. Therefore, we see that $\deg_I(V) = 0$ for every $I = (i_1, \dots, i_N) \in \mathbb{N}^N$ such that there exists $j \in \{1, \dots, N\}$ with $i_j \geq 2$. Since all the remaining tuples are of the form I_J for some subset $J \subseteq \{1, \dots, N\}$, and $I_J! = 1$ for each of these tuples, we see from (12) that

$$\deg([\alpha_1, \dots, \alpha_N]^{-1}(V)) = \dim(V)! \sum_J \deg_{I_J}([\alpha_1, \dots, \alpha_N]^{-1}(V))$$

where the sum runs over all the subsets $J \subseteq \{1, \dots, N\}$ such that $|J| = \dim(V)$. To conclude our proof, it suffices to observe that

$$\deg_{I_J}([\alpha_1, \dots, \alpha_N]^{-1}(V)) = \deg_{I_J}(V) \cdot \prod_{k \notin J} \deg(\alpha_k)$$

for every $J \subseteq \{1, \dots, N\}$ such that $|J| = \dim(V)$, as follows from [Hin88, Lemma 6]. \square

To conclude this subsection, we point out that, for varieties $V \subseteq A_N$ with finite stabilizers, the multiprojective degrees $\deg_{I_J}(V)$ are actually degrees of zero divisors, as we show in the following lemma.

Lemma 16. *Let V be a transverse variety in A_N with finite stabilizer. Then, there exists a closed point $P \in A_N(\overline{\mathbb{Q}})$ such that for every tuple $I = (i_1, \dots, i_N) \in \mathbb{N}^N$ with $i_1 + \dots + i_N = \dim(V)$, the support of the intersection*

$$(V + P) \cdot H_1^{i_1} \cdots H_N^{i_N}$$

is a finite set of points.

Proof. Suppose that there exists an intersection $V \cdot H_1^{i_1} \cdots H_N^{i_N}$ that is empty or of positive dimension. Up to a reordering of the coordinates, we can assume that this is given by the index

$$I = (1, \dots, 1, 0, \dots, 0)$$

where the firsts $\dim(V)$ indices are 1. Then, let $H := H_1 \cap \dots \cap H_{\dim V}$, and observe that the intersection $V \cap H$ is also empty or of positive dimension. Now, the image of V under the canonical projection

$$\pi_I: A_N \rightarrow A_N/H = E_1 \times \dots \times E_{\dim V}$$

is a closed subvariety of A_N/H , because π_I is closed. Moreover, $\dim(\pi_I(V)) = \dim(V)$. Indeed, if we had by contradiction that $\dim \pi_I(V) < \dim V$, the fiber $\pi_I^{-1}(Q)$ of a generic point $Q \in \pi_I(V)$ would be stable under the action of a positive dimensional subgroup $S \subseteq H$. Therefore, we would have that $S \subset \text{Stab}(V)$, which contradicts the assumption that $\text{Stab}(V)$ is finite. Therefore

$$\dim(\pi_I(V)) = \dim(V) = \dim(A_N/H),$$

which implies that $\pi|_V$ is surjective, and that its generic fiber is zero dimensional. Since the I are finitely many, there exists a point $P \in A_N(\overline{\mathbb{Q}})$ such that for all I the fiber $\pi_I^{-1}(0) \cap (V + P)$ is generic and therefore zero dimensional. \square

This lemma, combined with the fact that transversality is preserved by translations, shows that we can assume without loss of generality that all the multiprojective degrees of a subvariety $V \subseteq A_N$ with finite stabilizers correspond to the degree of a finite set of points.

3. PREIMAGES OF TRANSVERSE CURVES: AN ARITHMETIC APPROACH

In this section we prove [Theorem A](#) using an arithmetic method. Among others, we use the arithmetic Bézout theorem (17) together with Zhang's inequality (19). Thus, we use arithmetic information in order to understand the transversality of a variety, which is a geometric notion.

3.1. Some Diophantine inequalities. The aim of this subsection is to recall the three fundamental inequalities (17), (19) and (21), and to recall some notation needed in the sequel.

Heights. To start, we let $h_2: \mathbb{P}^m(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ denote the *Faltings (or Fubini-Study) height* of points, which is defined as

$$h_2(P) := \sum_{v \in M_K^0} \frac{[K_v: \mathbb{Q}_v]}{[K: \mathbb{Q}]} \log \max_{j=1, \dots, m} \{|P_j|_v\} + \sum_{v \in M_K^\infty} \frac{[K_v: \mathbb{Q}_v]}{2[K: \mathbb{Q}]} \log \left(\sum_{j=1}^m |P_j|^2 \right)$$

where K is any number field over which P is defined, and M_K^0 (respectively M_K^∞) denotes the set of finite (resp. infinite) places of K . This height can be extended to subvarieties $V \subseteq \mathbb{P}^m$ in several ways, and in this paper we follow the convention introduced by Philippon in [Phi91, Section 2.B]. Moreover, for every subvariety $V \subseteq A_N$, we let $\hat{h}(V)$ denote the *Néron-Tate height* of V associated to our fixed embedding (2), which is defined as in [Phi91, Page 281]. We also let $h: \mathbb{P}^m(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$ denote the *logarithmic Weil height* of points, defined as

$$h(P) := \sum_{v \in M_K} \frac{[K_v: \mathbb{Q}_v]}{[K: \mathbb{Q}]} \log \max_{j=1, \dots, m} \{|P_j|_v\},$$

where K is any number field over which P is defined, and M_K denotes its set of places. Finally, if K is a number field we let $h_\infty: K \rightarrow \mathbb{R}$ denote the *Archimedean contribution to the Weil height*, which is defined to be

$$h_\infty(x) := \sum_{v \in M_K^\infty} \frac{[K_v: \mathbb{Q}_v]}{[K: \mathbb{Q}]} \log \max(|x|_v, 1).$$

We note in particular that this function *depends* on the number field K .

The arithmetic Bézout theorem. Suppose now that $X, Y \subseteq \mathbb{P}^m$ are irreducible subvarieties, and let Z_1, \dots, Z_g be the irreducible components of $X \cap Y$. Then, the arithmetic Bézout theorem, which was proven by Philippon in [Phi95, Theorem 3], implies that

$$(17) \quad \sum_{i=1}^g h_2(Z_i) \leq \deg(X)h_2(Y) + \deg(Y)h_2(X) + c_0(\dim X, \dim Y, m) \deg(X) \deg(Y),$$

where the function $c_0: \mathbb{N}^3 \rightarrow \mathbb{Q}$ admits the explicit expression

$$(18) \quad \begin{aligned} c_0(d_1, d_2, m) &= \left(\sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \frac{1}{2(i+j+1)} \right) + \left(m - \frac{d_1 + d_2}{2} \right) \log 2 \\ &= \frac{1}{2} ((d_1 + d_2 + 1)\mathcal{H}_{d_1+d_2+2} - (d_1 + 1)\mathcal{H}_{d_1+1} - (d_2 + 1)\mathcal{H}_{d_2+1}) \\ &\quad + \left(m - \frac{d_1 + d_2}{2} \right) \log 2, \end{aligned}$$

which features the harmonic numbers $\mathcal{H}_k := \sum_{j=1}^k \frac{1}{j}$.

Zhang's inequality. We will now recall another seminal inequality in Diophantine geometry, which was proven by Zhang in [Zha95a, Theorem 1.10], and relates the height of a subvariety $V \subseteq A_N$ to the heights of its points. More precisely, Zhang's theorem implies that

$$(19) \quad \mu_2(V) \leq \frac{h_2(V)}{\deg(V)} \leq (1 + \dim V)\mu_2(V),$$

where $\mu_2(V)$ denotes the *essential minimum* of V with respect to the Faltings height h_2 , which is defined to be the infimum of all the real numbers $\theta \in \mathbb{R}_{\geq 0}$ such that the subset $\{P \in V(\overline{\mathbb{Q}}) : h_2(P) \leq \theta\}$ is Zariski dense in V .

Differences of heights. To conclude this subsection, we will recall an explicit inequality between the Faltings height $h_2(P)$ and the Néron-Tate height $\hat{h}(P)$ of a point $P \in A_N(\overline{\mathbb{Q}})$, which was proven in [CVV19, Proposition 3.2]. More precisely, if E is an elliptic curve defined over $\overline{\mathbb{Q}}$ by the Weierstrass equation $E: y^2 = x^3 + Ax + B$, we set

$$(20) \quad \begin{aligned} c_1(E) &:= \frac{h(A) + h(B)}{2} + \frac{h(\Delta(E)) + h_\infty(j(E))}{4} + \frac{h(j(E))}{8} + 3.724 \\ c_2(E) &:= \frac{h(A) + h(B)}{2} + \frac{h(\Delta(E)) + h_\infty(j(E))}{4} + 4.015 \end{aligned}$$

where $j(E)$ denotes the j -invariant of E and $\Delta(E)$ denotes the discriminant of the Weierstrass equation we fixed above. Moreover, in these formulas, the function h_∞ is taken with respect to the number field $K = \mathbb{Q}(j(E))$. Finally, we set $c_1(A_N) := c_1(E_1) + \cdots + c_1(E_N)$ and we analogously define $c_2(A_N) := c_2(E_1) + \cdots + c_2(E_N)$. Then, we have that

$$(21) \quad -c_1(A_N) \leq h_2(P) - \hat{h}(P) \leq c_2(A_N)$$

for every point $P \in A_N(\overline{\mathbb{Q}})$. We also recall that, if E is defined over \mathbb{Q} and $P \in E(\mathbb{Q})$, one can take the better constants

$$\begin{aligned} c_1(E) &:= \min \left(\frac{\log(|A| + |B| + 3)}{2} + \frac{\log|\Delta(E)| + \log \max(|j(E)|, 1)}{4} + \frac{h(j(E))}{8} + 2.919, \right. \\ &\quad \left. 3h(1: A^{1/2}: B^{1/3}) + 4.709 \right) \\ c_2(E) &:= \min \left(\frac{\log(|A| + |B| + 3)}{2} + \frac{\log|\Delta(E)| + \log \max(|j(E)|, 1)}{4} + 3.21, \right. \\ &\quad \left. \frac{3}{2}h(1: A^{1/2}: B^{1/3}) + 2.427 \right) \end{aligned}$$

instead of those defined in (20).

3.2. An irreducibility criterion for curves. We are almost ready to prove [Theorem A](#). Before doing that, we will need to specialize some of the results recalled in the previous paragraphs to the case of curves. First of all, the following lemma shows how to combine the inequalities (19) and (21) in order to give an upper bound for the Weil height of some particularly simple varieties.

Lemma 22. *For every $j \in \{1, \dots, N\}$ and every $Q_j \in E_j(\overline{\mathbb{Q}})$, let*

$$X := E_1 \times \cdots \times E_{j-1} \times \{Q_j\} \times E_{j+1} \times \cdots \times E_N,$$

then

$$(23) \quad h_2(X) \leq N3^{N-1}(h_2(Q_j) + c_3(E_j)),$$

where $c_3(E_j) := c_1(E_j) + c_2(E_j)$.

Proof. First of all, observe that $\deg(X) = 3^{N-1}$ and $\mu_2(X) \leq \hat{h}(Q_j) + c_2(E_j)$, because (21) implies that the set $\{P \in X(\overline{\mathbb{Q}}) : h_2(P) \leq \hat{h}(Q_j) + c_2(E_j)\}$ contains the set

$$\{P \in X(\overline{\mathbb{Q}}) : \hat{h}(P) \leq \hat{h}(Q_j)\},$$

and the latter contains the set $E_1(\overline{\mathbb{Q}})_{\text{tors}} \times \cdots \times E_{i-1}(\overline{\mathbb{Q}})_{\text{tors}} \times \{Q_j\} \times E_{i+1}(\overline{\mathbb{Q}})_{\text{tors}} \times \cdots \times E_N(\overline{\mathbb{Q}})_{\text{tors}}$, which is Zariski dense in X . Therefore, (19) implies that

$$h_2(X) \leq (1 + \dim(X)) \deg(X) \mu_2(X) = N3^{N-1} \mu_2(X) \leq N3^{N-1} (\hat{h}(Q_j) + c_2(E_j)),$$

which can be combined with (21) to see that

$$h(X) \leq N3^{N-1} (h(Q_j) + c_3(E_j))$$

as wished. \square

Now, let us observe that the formula (14), which expresses the degree of a preimage of a subvariety $V \subseteq A_N$ in terms of the degree of V itself, can be slightly simplified when V is a curve and $[\alpha_1, \dots, \alpha_N] = [\alpha]_j$ for some $\alpha \in \mathbb{Z}$ and $j \in \{1, \dots, N\}$, as the following lemma shows.

Lemma 24. *Let $C \subseteq A_N$ be an irreducible curve and fix some $j \in \{1, \dots, N\}$ and $\alpha \in \text{End}(E_j)$. Then, we have that*

$$(25) \quad \deg([\alpha]_j^{-1}(C)) = d_j + \deg(\alpha) \sum_{i \neq j} d_i$$

where $d_i := 3 \deg_0(C \cdot (E_1 \times \dots \times E_{i-1} \times \{(0:1:0)\} \times E_{i+1} \times \dots \times E_N))$ for every $i \in \{1, \dots, N\}$.

Proof. It suffices to observe that when C is a curve the only subsets $J \subseteq \{1, \dots, N\}$ which provide a nontrivial contribution to (14) are the singletons $J = \{i\}$. Indeed, setting $\alpha_j = \alpha$ and $\alpha_k = 1$ if $k \neq j$, we see that $\prod_{k \notin \{i\}} \deg(\alpha_k) = \prod_{k \notin \{i\}} \deg(\alpha_k) = \deg(\alpha)$ for every $i \in \{1, \dots, N\} \setminus \{j\}$, and that $\prod_{k \notin \{j\}} \deg(\alpha_k) = 1$. Moreover, we have that $\deg_{L_{\{i\}}}(V) = d_i$ for every $i \in \{1, \dots, N\}$, as follows directly from the definition of multiprojective degree (11), combined with the explicit formula (15) for the embedding divisors associated to the multiprojective embedding given by (2). \square

We are finally ready to prove [Theorem A](#).

Proof (of Theorem A). Combining [Lemma 6](#) with [Lemma 9](#), we see that it suffices to prove that for every $j \in \{1, \dots, N\}$ the curve $[p_j]_j^{-1}(C)$ is irreducible.

Therefore, let us fix $j \in \{1, \dots, N\}$, and let us suppose by contradiction that $[p_j]_j^{-1}(C)$ is reducible. Then, the number of components of $[p_j]_j^{-1}(C)$ is either p_j or p_j^2 . Hence, if C' denotes any irreducible component of $[p_j]_j^{-1}(C)$ which has minimal degree, we see from (25) that

$$(26) \quad \deg(C') \leq \frac{\deg([p_j]_j^{-1}(C))}{p_j} \leq d'_j \leq p_j \deg(C),$$

where $d'_j := d_j + p_j \sum_{k \neq j} d_k$.

Now, the height of C' can be bounded using Zhang's inequality (19). More precisely, we have that

$$(27) \quad \mu_2(C') \leq \mu_2(C) + c_3(E_j)$$

where $c_3(E_j) := c_1(E_j) + c_2(E_j)$ is the constant introduced in [Lemma 22](#). Indeed, this follows by combining (21) with the fact that $\widehat{h}(P) \leq \widehat{h}([p_j]_j(P))$ for every point $P \in C(\overline{\mathbb{Q}})$. Therefore,

$$(28) \quad h_2(C') \leq 2 \deg(C') \mu_2(C') \leq 2p_j \deg(C) (\mu_2(C) + c_3(E_j)) \leq 2p_j (h_2(C) + c_3(E_j) \deg(C)),$$

where the first and third inequality follow from Zhang's inequality (19), while the second one follows from (26) and (27).

Let us now fix a set of points $\mathcal{Q} \subseteq C(\overline{\mathbb{Q}})$ such that the set of Faltings heights $h_2(\mathcal{Q}) \subseteq \mathbb{R}_{\geq 0}$ is unbounded. Such a set surely exists, because any projection of C onto one of the factors of A_N is surjective, since C is transverse. Moreover, thanks to the pigeonhole principle, we can assume, up to shrinking \mathcal{Q} , that there exists $i \in \{1, \dots, N\}$ such that for every $k \in \{1, \dots, N\}$ and every point $Q = (Q_1, \dots, Q_N) \in \mathcal{Q}$ we have that $h_2(Q_k) \leq h_2(Q_i)$.

Suppose now that $i \neq j$. Then, the arithmetic Bézout theorem yields an upper bound for the Faltings height of the i -th coordinate $Q_i \in E_i(\overline{\mathbb{Q}})$ of a point $Q = (Q_1, \dots, Q_N) \in \mathcal{Q}$. More precisely, fix such a point $Q \in \mathcal{Q}$. Then, there exists a point $Q'_j \in E_j(\overline{\mathbb{Q}})$ such that $[p_j](Q'_j) = Q_j$ and $Q' \in C'(\overline{\mathbb{Q}})$, where $Q' := (Q_1, \dots, Q_{j-1}, Q'_j, Q_{j+1}, \dots, Q_N)$. Note that [Lemma 6](#) implies that C' is transverse, because

it is an irreducible component of the preimage of a transverse curve. Therefore, we see that $\{Q'\}$ is a component of the intersection $X'_j \cap C'$, where $X'_j := E_1 \times \cdots \times E_{j-1} \times \{Q'_j\} \times E_{j+1} \times \cdots \times E_N$. We apply the arithmetic Bézout theorem (17) to the intersection $X'_j \cap C'$, obtaining

$$h_2(Q'_j) + \sum_{k \neq j} h_2(Q_k) = h_2(\{Q'\}) \leq h_2(X'_j) \deg(C') + h_2(C') \deg(X'_j) + c_4 \deg(C') \deg(X'_j),$$

where $c_4 := c_0(1, N-1, 3^{N-1})$. Combining this with (23), we see that

$$(29) \quad h_2(Q'_j) + \sum_{k \neq j} h_2(Q_k) \leq 3^{N-1}(Nh_2(Q'_j) \deg(C') + h_2(C') + (c_3(E_j) + c_4) \deg(C')),$$

because $\deg(X'_j) = 3^{N-1}$. Moreover, (21) implies that

$$(30) \quad h_2(Q'_j) \leq \hat{h}(Q'_j) + c_2(E_j) = \frac{\hat{h}(Q_j)}{p_j^2} + c_2(E_j) \leq \frac{h_2(Q_j)}{p_j^2} + c_3(E_j).$$

Finally, we have that

$$(31) \quad h_2(Q_i) \leq h_2(Q'_j) + \sum_{k \neq j} h_2(Q_k),$$

because we are assuming that $i \neq j$, and we also know that

$$(32) \quad h_2(Q_j) \leq h_2(Q_i)$$

because $Q \in \mathcal{Q}$. Combining the upper bound given by (29) together with the inequalities (26), (28), (30), (31) and (32), we obtain

$$(33) \quad 3^{1-N} h_2(Q_i) \leq N \frac{d'_j}{p_j^2} h_2(Q_i) + (2(\mu_2(C) + c_3(E_j)) + c_4) d'_j.$$

Therefore, the height of Q_i is uniformly bounded above, because we have by assumption that

$$p_j^2 \geq (\deg(C)N3^{N-1})^2 > d'_j N 3^{N-1}.$$

This contradicts the fact that $h_2(Q) \leq Nh_2(Q_i)$ is unbounded, and allows us to conclude that the preimage $[p_j]_j^{-1}(C)$ cannot be reducible whenever $i \neq j$.

Suppose, on the other hand, that $i = j$, and fix a point $Q = (Q_1, \dots, Q_N) \in \mathcal{Q}$. Then, the inequality (29) still holds true, but it is not sufficient anymore to get to a contradiction. Instead, we will need to combine (29) with another application of the arithmetic Bézout theorem. More precisely, since C is transverse, the point $\{Q\}$ is a component of each of the intersections $X_l \cap C$, where $l \in \{1, \dots, N\} \setminus \{i\}$ and X_l is defined as $X_l := E_1 \times \cdots \times E_{l-1} \times \{Q_l\} \times E_{l+1} \times \cdots \times E_N$. Therefore, we see that

$$(34) \quad h_2(Q_l) + h_2(Q_i) \leq h_2(\{Q\}) \leq 3^{N-1}(Nh_2(Q_l) \deg(C) + h_2(C) + (c_3(E_l) + c_4) \deg(C)),$$

where the second inequality follows from a combination of (17) and (23). This implies that there exists a constant $c_5 \in \mathbb{R}_{>0}$, depending on C , such that

$$(35) \quad h_2(Q_j) = h_2(Q_i) \leq (3^{N-1}N \deg(C) - 1)h_2(Q_l) + c_5$$

for every $l \in \{1, \dots, N\} \setminus \{i\}$. In particular, we see that for every $l \in \{1, \dots, N\}$ the set

$$\{h_2(Q_l) : Q = (Q_1, \dots, Q_N) \in \mathcal{Q}\} \subseteq \mathbb{R}_{\geq 0}$$

is unbounded. Now, choose any $l \in \{1, \dots, N\} \setminus \{i\}$, and we see that an inequality similar to (33) holds true. More precisely, we have that $h_2(Q_l) \leq h_2(Q'_j) + \sum_{k \neq j} h_2(Q_k)$, which allows us to see that

$$(36) \quad 3^{1-N} h_2(Q_l) \leq N \frac{d'_j}{p_j^2} h_2(Q_j) + (2(\mu(C) + c_3(E_j)) + c_4) d'_j$$

by combining once again (29) together with the inequalities (26), (28) and (30). Combining this with (35) guarantees that

$$3^{1-N}h_2(Q_l) \leq N \frac{d'_j}{p_j^2} (3^{N-1}N \deg(C) - 1)h_2(Q_l) + (2(\mu(C) + c_3(E_j)) + c_4)d'_j + N \frac{d'_j}{p_j^2} c_5.$$

As before, this allows us to conclude that $h_2(Q_l)$ is bounded, because

$$p_j^2 \geq (\deg(C)N3^{N-1})^2 > d'_j N 3^{N-1} (3^{N-1}N \deg(C) - 1),$$

by assumption. Since this contradicts what we have shown before, $[p_j]_j^{-1}(C)$ must be irreducible even when $i = j$, as we wanted to prove. \square

Remark 37. We remark that a generalisation of this proof to a transverse variety $V \subseteq A_N$ is not directly possible. Indeed, to prove something analogous to [Theorem A](#) it would be sufficient to prove that, given a prime p , the preimage $[p, \dots, p, 1, \dots, 1]^{-1}(V)$ is irreducible when p is sufficiently big, where the diagonal endomorphism $[p, \dots, p, 1, \dots, 1]$ has $\dim(V)$ components equal to p . To do so, one would be tempted to consider a point $Q' \in V'$, where V' is some irreducible component of $[p, \dots, p, 1, \dots, 1]^{-1}(V)$. Then, Q' would be an irreducible component of the intersection

$$V' \cap (\{Q'\} \times A_{N-\dim(V)}),$$

to which one could apply the arithmetic Bézout theorem (17). However, the gain one obtains by considering the height of Q' is just $\frac{1}{p^2}$, which is not sufficient to overtake the degree of V' , which can only be bounded by $p^{2(\dim(V)-1)} \deg(V)$.

The previous remark prompted us to use a more geometric approach to study the transversality of preimages of higher dimensional varieties. This is also a hint for the difficulties that one encounters when trying to extend to higher dimensional varieties the methods used in the proof of the Torsion Anomalous Conjecture for curves.

4. PREIMAGES OF TRANSVERSE SUBVARIETIES: A GEOMETRIC APPROACH

The aim of this section is to give a geometric proof of the following result, which guarantees that the preimage by suitable group homomorphisms of a transverse subvariety $V \subseteq A_N$ with finite stabilizer remains transverse.

Theorem 38. *Let $V \subseteq A_N$ be a transverse subvariety with finite stabilizer, and let*

$$(\alpha_1, \dots, \alpha_N) \in \prod_{j=1}^N \text{End}(E_j)$$

be a tuple of isogenies. Moreover, suppose that for every $j \in \{1, \dots, N\}$ there exists a tuple of indices $I_j = (i_{1,j}, \dots, i_{N,j})$ consisting of $\dim(V)$ ones and $N - \dim(V)$ zeros, such that $i_{j,j} = 1$ and

$$\gcd(\deg(\alpha_j), \dim(V)! \deg_{I_j}(V)) = 1.$$

Then, the preimage

$$[\alpha_1, \dots, \alpha_N]^{-1}(V)$$

is transverse.

This theorem is stronger than our main theorem in the introduction, as specified in the following lemma.

Lemma 39. *Theorem 38 implies Theorem B.*

Proof. We observe that if we have a tuple $(\alpha_1, \dots, \alpha_j) \in \prod_{j=1}^N \text{End}(E_j)$ such that for every index $j \in \{1, \dots, N\}$ and every prime $p \mid \deg(\alpha_j)$ we have that $p > \dim(V)! \deg(V)$, then in particular $p > \dim(V)! \deg_I(V)$ for every $I \subseteq \{1, \dots, N\}$ such that $|I| = \dim(V)$, thanks to (12). Therefore, we see that $p \nmid \dim(V)! \deg_I(V)$ for any $I \subseteq \{1, \dots, N\}$ such that $|I| = \dim(V)$, which clearly implies that the hypotheses of [Theorem 38](#) are satisfied. \square

We are finally ready to prove [Theorem 38](#), using most of the results that we proved in [Section 2](#).

Proof (of [Theorem 38](#)). First of all, let us observe that each of the tuples I_j will necessarily be of the form I_{J_j} for some subset $J_j \subseteq \{1, \dots, N\}$, because our assumptions imply that $\deg_{I_j}(V) \neq 0$. Moreover, we consider the endomorphisms $f_j = [\beta_{1,j}, \dots, \beta_{N,j}]$, where $\beta_{k,j} := \alpha_j$ if $k \in J_j$, and $\beta_{k,j} = 1$ otherwise. Then, thanks to [Lemma 9](#), it suffices to show that each of the varieties $f_1^{-1}(V), \dots, f_N^{-1}(V)$ is irreducible.

Therefore, let us fix any $j \in \{1, \dots, N\}$, and let us show that $f_j^{-1}(V)$ is irreducible. To do so, let us suppose by contradiction that the variety $f_j^{-1}(V)$ is irreducible, and let $d > 1$ be the number of its irreducible components. Then, d divides $\deg(f_j) = \deg(\alpha_j)^{\dim(V)}$. Moreover, [Lemma 7](#) implies that $\deg(f_j^{-1}(V)) = d \deg(W)$, where W is any irreducible component of $f_j^{-1}(V)$. This shows that $\deg(\alpha_j)$ and $\deg(f_j^{-1}(V))$ are not coprime. On the other hand, [Lemma 13](#) implies that

$$\deg(f_j^{-1}(V)) - \dim(V)! \deg_{I_j}(V) = \dim(V)! \sum_{J \neq J_j} \left(\prod_{k \in J} \deg(\beta_{k,j}) \right) \deg_{I_J}(V),$$

where the sum on the right hand side runs over all the subsets $J \subseteq \{1, \dots, N\}$ such that $|J| = \dim(V)$ and $J \neq J_j$. These two conditions show that for every such J there exists $k \notin J$ such that $\beta_{k,j} = \alpha_j$, as one sees from the definition of f_j . Therefore, we see that

$$(40) \quad \deg(\alpha_j) \mid \deg(f_j^{-1}(V)) - \dim(V)! \deg_{I_j}(V),$$

which implies that $\deg(\alpha_j)$ and $\dim(V)! \deg_{I_j}(V)$ are not coprime. However, this contradicts our assumptions, and allows us to conclude that $f_j^{-1}(V)$ must be irreducible, as we wanted to show. \square

In particular, we see that [Theorem 38](#) implies the following result for endomorphisms which are multiples of the identity.

Corollary 41. *Let $V \subseteq A_N$ be a transverse variety with finite stabilizer, and let $n \in \mathbb{Z}$ be an integer such that for every $j \in \{1, \dots, N\}$ there exists a tuple $I = (i_1, \dots, i_N) \in \mathbb{N}^N$ with $i_j = 1$ such that*

$$\gcd(n, \dim(V)! \deg_I(V)) = 1.$$

Then, the preimage $[n, \dots, n]^{-1}(V)$ is transverse.

In particular, if $p \in \mathbb{Z}$ is a prime such that $p \nmid \dim(V)! \deg_I(V)$ and

$$(42) \quad p \nmid \gcd\{\deg_I(V) : I = (i_1, \dots, i_N) \in \{0, 1\}^N, i_1 + \dots + i_N = \dim(V), i_j = 1\}$$

for every $j \in \{1, \dots, N\}$, then $[p, \dots, p]^{-1}(V)$ is transverse.

Proof. The first part of this corollary is precisely obtained by setting $\alpha_1 = \dots = \alpha_N = n$ in [Theorem 38](#), so there is nothing to prove. For the second part, suppose by contradiction that setting $\alpha_1 = \dots = \alpha_N = p$ yields a tuple of isogenies which does not satisfy the assumptions of [Theorem 38](#). This implies necessarily that $p \mid \deg_I(V)$ for every tuple $I = (i_1, \dots, i_N) \in \{0, 1\}^N$ such that $i_1 + \dots + i_N = \dim(V)$, because for each of these tuples there exists $j \in \{1, \dots, N\}$ such that $i_j = 1$, since $\dim(V) \geq 1$. However, this divisibility property contradicts our assumption (42), and this allows us to conclude that the tuple of isogenies (p, \dots, p) satisfies the assumptions of [Theorem 38](#), which implies that $[p, \dots, p]^{-1}(V)$ is transverse. \square

Finally, we provide a corollary to [Theorem 38](#) for curves.

Corollary 43. *Let $C \subseteq A_N$ be a transverse curve, and let $(\alpha_1, \dots, \alpha_N) \in \prod_{j=1}^N \text{End}(E_j)$ be a tuple of isogenies. Moreover, suppose that for every $j \in \{1, \dots, N\}$ we have that*

$$\gcd(\deg(\alpha_j), \deg_j(C)) = 1$$

where $\deg_1(C) := \deg_{(1,0,\dots,0)}(C), \dots, \deg_N(C) := \deg_{(0,\dots,0,1)}(C)$. Then, $[\alpha_1, \dots, \alpha_N]^{-1}(C)$ is transverse.

5. EXAMPLES AND APPLICATIONS

The aim of this section is to provide some applications of our main results.

5.1. Transversality of specific subvarieties. Let us see how our main results allow one to prove that some specific subvarieties of a product of elliptic curves are irreducible.

First of all, we give an example for $N = 2$. More precisely, for every $n \in \mathbb{Z}_{\geq 1}$ we consider the curve $C_n \subseteq E_1 \times E_2$ which is the projective closure of the affine curve

$$(44) \quad C_n^\circ: \begin{cases} y_2 &= x_1^n \\ y_1^2 &= x_1^3 + A_1 x_1 + B_1 \\ y_2^2 &= x_2^3 + A_2 x_2 + B_2 \end{cases}$$

inside $\mathbb{A}^2 \times \mathbb{A}^2$. Then, [CVV19, Theorem 6.2] shows that C_n is transverse, and computes that

$$\begin{aligned} \deg_{(1,0)}(C_n) &= 9 \\ \deg_{(0,1)}(C_n) &= 6n, \end{aligned}$$

which implies that $\deg(C_n) = 6n + 9$.

Let us see how one can make explicit the equations of the preimages of C_n . To this end, we recall that for every elliptic curve E embedded in \mathbb{P}^2 via a short Weierstrass equation $y^2z = x^3 + Axz^2 + Bz^3$, and every $\alpha \in \mathbb{Z}$, there exist three polynomials $r_\alpha, s_\alpha, t_\alpha \in \mathbb{Q}[x, z, A, B]$, which are homogeneous in x and z , such that

$$(45) \quad [\alpha]_E(P) = (r_\alpha(x, z, A, B)t_\alpha(x, z, A, B) : s_\alpha(x, z, A, B)y : t_\alpha(x, z, A, B)^3 z)$$

for every point $P = (x : y : z) \in E \setminus \ker([\alpha]_E)$, as shown in [Was08, § 2.9]. Moreover, the polynomial t_α is the homogenized version of the α -th division polynomial of E , and we always have that

$$\gcd(r_\alpha, s_\alpha) = \gcd(s_\alpha, t_\alpha) = 1,$$

whereas $\gcd(r_\alpha, t_\alpha) = 1$ if and only if $2 \nmid \alpha$. On the other hand, when $2 \mid \alpha$ we have that

$$\begin{aligned} r_\alpha &= (x^3 + Axz^2 + Bz^3)\tilde{r}_\alpha \\ t_\alpha &= (x^3 + Axz^2 + Bz^3)\tilde{t}_\alpha \end{aligned}$$

where $\tilde{r}_\alpha, \tilde{t}_\alpha \in \mathbb{Q}[x, z, A, B]$ are homogeneous in x and z , and we have that $\gcd(\tilde{r}_\alpha, \tilde{t}_\alpha) = 1$. Using these facts, we see that (45) still holds for every point $(x : y : z) \in \ker([\alpha]_E)$, unless $2 \mid \alpha$ and $P \in \ker([2]_E)$, in which case the morphism appearing in (45) is not well defined. To avoid this issue, one can multiply the three polynomials appearing on the right hand side of (45) by yz , and then divide everything by $x^3 + Axz^2 + Bz^3$, to obtain the following formula

$$(46) \quad [\alpha]_E(P) = (r_\alpha(x, z, A, B)\tilde{t}_\alpha(x, z, A, B)yz : s_\alpha(x, z, A, B) : \tilde{t}_\alpha(x, y, A, B)t_\alpha(x, y, A, B)^2 yz^2),$$

which is valid for every point $P := (x : y : z) \in E$.

Substituting a dehomogenized version of the equations (45) and (46) inside (44), we see that for every pair of odd integers $\alpha_1, \alpha_2 \in \mathbb{Z}$, the preimage $[\alpha_1, \alpha_2]^{-1}(C_n)$ is given by the projective closure of the affine curve $([\alpha_1, \alpha_2]^{-1}(C_n))^\circ = V \cap (E_1^\circ \times E_2^\circ) \subseteq \mathbb{A}^2 \times \mathbb{A}^2$, where $E_i^\circ : y_i^2 = x_i^3 + A_i x_i + B_i$ for every $i \in \{1, 2\}$, while

$$V : y_2 s_{\alpha_2}(x_2, 1, A_2, B_2) t_{\alpha_1}(x_1, 1, A_1, B_1)^{2n} = r_{\alpha_1}(x_1, 1, A_1, B_1)^n t_{\alpha_2}(x_2, 1, A_2, B_2)^3.$$

On the other, hand, if for example α_1 is even and α_2 is odd, an affine model for the curve $[\alpha_1, \alpha_2]^{-1}(C_n)$ is given by $([\alpha_1, \alpha_2]^{-1}(C_n))^\circ = W \cap (E_1^\circ \times E_2^\circ) \subseteq \mathbb{A}^2 \times \mathbb{A}^2$, where

$$\begin{aligned} W &: y_2 s_{\alpha_2}(x_2, 1, A_2, B_2) t_{\alpha_1}(x_1, 1, A_1, B_1)^n \tilde{t}_{\alpha_1}(x_1, 1, A_1, B_1)^n \\ &= \tilde{r}_{\alpha_1}(x_1, 1, A_1, B_1)^n t_{\alpha_2}(x_2, 1, A_2, B_2)^3. \end{aligned}$$

In particular, the equations describing these preimages can be quite complicated, because the size of the coefficients of the polynomials r_α , s_α and t_α grows with respect to $|\alpha|$. For example, we have that

$$\begin{aligned} r_2(x, 1, A, B) &= x^7 - Ax^5 - 7Bx^4 - A2x^3 - 10ABx^2 + (A^3 - 8B^2)x + A^2B \\ s_2(x, 1, A, B) &= x^9 + 6Ax^7 + 21Bx^6 + 21ABx^4 + (12B^2 - 6A^3)x^3 - 9A^2Bx^2 \\ &\quad - (A^4 + 12AB^2)x - A^3B - 8B^3 \\ t_2(x, 1, A, B) &= 2(x^3 + Ax + B) \end{aligned}$$

whereas

$$\begin{aligned} r_3(x, 1, A, B) &= x^9 - 12Ax^7 - 96Bx^6 + 30A^2x^5 - 24ABx^4 + (36A^3 + 48B^2)x^3 + 48A^2Bx^2 \\ &\quad + (9A^4 + 96AB^2)x + 8A^3B + 64B^3 \\ s_3(x, 1, A, B) &= x^{12} + 22Ax^{10} + 220Bx^9 - 165A^2x^8 - 528ABx^7 - (92A^3 + 1776B^2)x^6 \\ &\quad + 264A^2Bx^5 - (185A^4 + 960AB^2)x^4 - (80A^3B - 320B^3)x^3 \\ &\quad - (90A^5 + 624A^2B^2)x^2 - (132A^4B + 896AB^3)x - 3A^6 - 96A^3B^2 - 512B^4 \\ t_3(x, 1, A, B) &= 3x^4 + 6Ax^2 + 12Bx - A^2. \end{aligned}$$

Therefore, we see that checking whether the curves $[\alpha, 1]^{-1}(C_n)$ and $[1, \alpha]^{-1}(C_n)$ are irreducible can be difficult, even with the help of a computer.

On the other hand, [Theorem A](#) shows that if $p \in \mathbb{Z}$ is a prime such that $|p| \geq 6(6n + 9)$ then the two curves $[p, 1]^{-1}(C_n)$ and $[1, p]^{-1}(C_n)$ are transverse (and in particular irreducible), because we have that $\deg(C_n) = 6n + 9$, as we recalled above. These results can in fact be improved using [Corollary 43](#). More precisely, since $\deg_{(1,0)}(C_n) = 9$ and $\deg_{(0,1)}(C_n) = 6n$ we see that $[\alpha, 1]^{-1}(C_n)$ is transverse whenever $3 \nmid \alpha$, whereas $[1, \alpha]^{-1}(C_n)$ is transverse if $\gcd(\alpha, 6n) = 1$. The following example gives an explicit illustration of this transversality criterion.

Example 47. Let $A_1 = A_2 = 0$ and $B_1 = B_2 = 1$, so that $E_1 = E_2$ is an elliptic curve of conductor 36. Then, the previous considerations imply that the projective closure of the affine curve

$$(48) \quad ([2, 1]^{-1}(C_3))^\circ : \begin{cases} y_2(x_1^3 + 1)^3 = (x_1^4 - 8x_1)^3 \\ y_1^2 = x_1^3 + 1 \\ y_2^2 = x_2^3 + 1 \end{cases}$$

is irreducible inside $(\mathbb{P}^2)^2$. Analogously, we see that the curve $[1, 5]^{-1}(C_3)$, which is the projective closure of the affine curve $V^\circ \cap (E_1^\circ \times E_2^\circ)$, where $E_i^\circ : y_i^2 = x_i^3 + 1$ for $i \in \{1, 2\}$, and

$$\begin{aligned} V^\circ &: y_2(x_2^{36} + 4692x_2^{33} - 884544x_2^{30} + 1880320x_2^{27} - 94222080x_2^{24} \\ &\quad - 1437769728x_2^{21} - 3534606336x_2^{18} - 8883929088x_2^{15} - 6868500480x_2^{12} \\ &\quad - 1853358080x_2^9 - 497025024x_2^6 - 742391808x_2^3 + 16777216)^3 = \\ &= x_1^3(x_2^{12} + 76x_2^9 - 48x_2^6 - 320x_2^3 - 256/5)^3 \end{aligned}$$

is irreducible in $(\mathbb{P}^2)^2$.

Input: A subvariety $V \subseteq \mathbb{A}^4$, two elliptic curves E_1, E_2 and a diagonal endomorphism $f: E_1 \times E_2 \rightarrow E_1 \times E_2$.

```

A.<x1,y1,x2,y2> = AffineSpace(QQ,4)
R.<x,y,z> = PolynomialRing(QQ)
A.inject_variables ()
def Preimage(V,E1,E2,f):
E1 = E1.short_weierstrass_model ()
E2 = E2.short_weierstrass_model ()
e1 = (R(E1.defining_polynomial ())).subs(x=x1,y=y1,z=1)
e2 = (R(E2.defining_polynomial ())).subs(x=x2,y=y2,z=1)
f1 = [h.subs(x=x1,y=y1) for h in E1.scalar_multiplication (f[0]).rational_maps ()]
f2 = [h.subs(x=x2,y=y2) for h in E2.scalar_multiplication (f[1]).rational_maps ()]
L = V.defining_polynomials ()
M = [e1,e2]
for l in L:
M.append((l.subs({x1:f1[0],y1:f1[1],x2:f2[0],y2:f2[1]}).numerator ()))
return A.subscheme(M)
S = Preimage(V,E1,E2,f).irreducible_components ()

```

Output: The set S of affine models of the irreducible components of $f^{-1}(C)$, where $C := \overline{V} \cap (E_1 \times E_2)$ is the intersection of the closure of V inside $(\mathbb{P}^2)^2$, denoted \overline{V} , with $E_1 \times E_2$.

ALGORITHM 5.1. SAGEMATH code to compute the irreducible components of the preimage of a curve $C \subseteq A_2$.

Computational aspects. Since these curves are defined by explicit equations, one could also try to check their irreducibility using a software such as SAGEMATH. This can be easily done using the code portrayed in [Algorithm 5.1](#). However, even an irreducibility check in this simple case turns out to be very expensive from a computational point of view.

Remark 49. Let us note that the SAGEMATH command `rational_maps()` applied to the scalar multiplications f_1 and f_2 in [Algorithm 5.1](#) allows one to obtain the canonical, simplified form of the isogenies f_1 and f_2 , which corresponds to our choice of the polynomials r_α, s_α and t_α . On the other hand, if one uses the other natural command `as_morphism().defining_polynomials()`, the projective equations that one obtains are not reduced to the lowest terms. In particular, writing [Algorithm 5.1](#) with such a command would yield preimages which contain always components of the form $\ker(f_1) \times E_2$ and $E_1 \times \ker(f_2)$, which clearly cannot happen.

Higher dimensions. To conclude, let us note that one can consider more generally curves $C \subseteq A_N$ when $N \geq 3$. For instance, [\[Via21, Theorem 2\]](#) shows that for every family of non-constant polynomials $p_1, \dots, p_{N-1} \in \overline{\mathbb{Q}}[t]$, the curve defined in A_N by the affine equations

$$\begin{cases} y_1 = p_1(x_2) \\ \dots \\ y_{N-1} = p_{N-1}(x_N) \end{cases}$$

is transverse. Therefore, carrying out a computation analogous to the one that we performed above, we can apply our [Theorems A](#) and [B](#) to this family of curves, in order to show that some of their preimages under diagonal endomorphisms are irreducible.

5.2. Bounding the height of intersections. Our main results can be used to bound explicitly the Faltings height of an irreducible component of intersections of the form $\phi^{-1}(C) \cap B$, where $B \subseteq A_N$ is

a subgroup, $C \subseteq A_N$ is a transverse curve and $\phi: A_N \rightarrow A_N$ is a diagonal endomorphism such that $\phi(B) = B$.

More precisely, assume to know the degree of $\phi^{-1}(C)$. For example, one might know its equations. Let C_0 be an irreducible component of $\phi^{-1}(C)$. Then, the arithmetic Bézout theorem (17) implies that (50)

$$\begin{aligned} h_2(C_0 \cap B) &\leq \deg(C_0)h_2(B) + \deg(B)h_2(C_0) + c_0(1, d, 3^N - 1) \deg(C_0) \deg(B), \\ &\leq \deg(\phi^{-1}(C))h_2(B) + \deg(B)h_2(\phi^{-1}(C)) + c_0(1, d, 3^N - 1) \deg(\phi^{-1}(C)) \deg(B), \end{aligned}$$

where $d = \dim(B)$. Unfortunately without further knowledge on the number of components of $\phi^{-1}(C)$, for the degree of C_0 , one has to use the trivial bound $\deg(C_0) \leq \deg(\phi^{-1}(C))$. On the other hand, only the irreducibility of $\phi^{-1}(C)$ ensures that C_0 is stabilized by $\ker(\phi)$. So without our theorem it is not easy to improve (50).

However, Theorem 38 shows that in a large number of cases, $\phi^{-1}(C)$ is irreducible. Moreover it is stable under the action of $\ker(\phi)$, because we assumed that $\phi(B) = B$. Thus $\phi^{-1}(C) \cap B$ consists of $\deg(\phi)$ irreducible components of the same height. Therefore, by the arithmetic Bézout theorem we obtain

$$(51) \quad \deg(\phi)h_2(P) \leq \deg(C_0)h_2(B) + \deg(B)h_2(C_0) + c_0(1, d, 3^N - 1) \deg(C_0) \deg(B),$$

for every $P \in (\phi^{-1}(C) \cap B)(\overline{\mathbb{Q}})$ which can considerably improve (50) when ϕ has a big degree. Note that here it is central the fact that we explicitly know the degree and irreducibility of $\phi^{-1}(C)$, and not only (or, in fact, not necessarily) the degree of C .

5.3. Lower bounds for the essential minima of preimages. Our main theorem can be applied also to get new lower bounds for the essential minima, with respect to the Néron-Tate height, of the images of certain curves $C \subseteq A_N$ by some endomorphisms $\Phi: A_N \rightarrow A_N$. In particular, we obtain these bounds by applying a result of Galateau [Gal10] in an indirect way, which requires to combine it with our Theorem A and Theorem 38. As we will show, this allows us to improve the lower bounds obtained from a direct application of Galateau's result.

More precisely, fix an integer $r \geq 2$ and an endomorphism $\Phi: A_N \rightarrow A_N$ which admits the matrix representation

$$(52) \quad \Phi = \left(\begin{array}{c|c} \alpha \cdot \text{Id}_r & L \\ \hline \mathbf{0} & \text{Id}_{N-r} \end{array} \right),$$

where $\alpha \in \mathbb{Z}$ and $L = (\ell_{i,j}): E_{r+1} \times \cdots \times E_N \rightarrow E_1 \times \cdots \times E_r$ is a morphism of abelian varieties with components $\ell_{i,j}: E_j \rightarrow E_i$, for $j \in \{r+1, \dots, N\}$ and $i \in \{1, \dots, r\}$. Moreover, let us suppose that $\alpha^2 \geq d_L := \max_{i,j}(\deg(\ell_{i,j}))$. Then, if we define the automorphism $\Psi: A_N \rightarrow A_N$ as

$$\Psi := \left(\begin{array}{c|c} \text{Id}_r & L \\ \hline \mathbf{0} & \text{Id}_{N-r} \end{array} \right),$$

we have the obvious relation $\Phi \circ f = [\alpha, \dots, \alpha] \circ \Psi$, where $f := [1, \dots, 1, \alpha, \dots, \alpha]$ is the diagonal endomorphism with the first r entries equal to one and the last $N - r$ entries equal to α . In particular, we have that

$$\Phi(C) = [\alpha, \dots, \alpha](\Psi(f^{-1}(C))).$$

Since the Néron-Tate height is a quadratic form, this implies that

$$(53) \quad \hat{\mu}(\Phi(C)) = \alpha^2 \hat{\mu}(\Psi(f^{-1}(C))),$$

where $\hat{\mu}$ denotes the essential minimum with respect to the Néron-Tate height $\hat{h}: A_N(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}$, which is defined as

$$\hat{\mu}(V) := \inf\{\theta \in \mathbb{R}_{>0}: \{P \in V(\overline{\mathbb{Q}}): \hat{h}(P) \leq \theta\} \text{ is Zariski dense in } V\}$$

for any irreducible subvariety $V \subseteq A_N$.

Now, thanks to our [Theorem A](#) and [Theorem 38](#) we know that for every α which is big enough with respect to the multiprojective degrees of C , the curve $f^{-1}(C)$ is transverse. This implies that the curve

$$D := \Psi(f^{-1}(C))$$

is also transverse, because Ψ is an automorphism. Hence, we can apply to D a theorem of Galateau [\[Ga10\]](#), which provides a lower bound for the essential minimum of any transverse subvariety $V \subseteq A_N$.

More precisely, [\[Ga10, Corollaire 1.2\]](#) shows that

$$(54) \quad \hat{\mu}(V) \geq \frac{c_6(A_N)}{\deg(V)^{1/\text{codim}(V)} \log(3 \deg(V))^{\lambda(N, \text{codim}(V))}},$$

where $c_6(A_N) \in \mathbb{R}_{>0}$ is an effectively computable constant depending only on A_N , and where one defines $\lambda(N, k) := (5N(k+1))^{k+1}$. To make this lower bound more explicit when $V = D$, note that

$$(55) \quad \deg(D) \leq 3N^2 d_L \deg(f^{-1}(C)) \leq 3N^3 d_L \alpha^{2(N-r)} \deg(C) \leq 3N^3 \alpha^{2(N+1-r)} \deg(C)$$

where the first upper bound can be proven using Bézout's classical theorem, as explained in the proof of [\[Via08, Lemma 13.2\]](#), while the second upper bound follows from [Lemma 13](#), because we set two entries of the diagonal endomorphism f to be equal to one. Combining [\(53\)](#) with [\(54\)](#), with $V = D$, and [\(55\)](#), we see that

$$(56) \quad \hat{\mu}(\Phi(C)) = \alpha^2 \hat{\mu}(D) \geq \frac{c_7(A_N, \deg(C))}{\log(d_L |\alpha|)^{\lambda(N, N-1)}} (\alpha^{2(r-1)} / d_L)^{\frac{1}{N-1}} \geq \frac{c_7(A_N, \deg(C))}{\log(d_L |\alpha|)^{\lambda(N, N-1)}} \alpha^{\frac{2(r-2)}{N-1}}$$

for some constant $c_7(A_N, \deg(C)) \in \mathbb{R}_{>0}$ which depends on A_N and on the degree of C . In particular, if we let Φ vary by letting $|\alpha| \rightarrow +\infty$, while leaving C fixed, we see that the essential minimum of the images $\Phi(C)$ will tend to infinity as a power of $|\alpha|$, unless $r = 2$. In this case, if $d_L \gg \alpha^2 / \log(|\alpha|)$ then the lower bound for $\hat{\mu}(\Phi(C))$ portrayed in [\(56\)](#) will decrease as a power of $1/\log(d_L)$.

The lower bound [\(56\)](#) that we obtained is much better than what would come out of a direct application of Galateau's inequality [\(54\)](#) to the curve $\Phi(C)$. More precisely, without further knowledge on Φ , we know only that

$$\deg(\Phi(C)) \leq 3N^2 \alpha^2 \deg(C),$$

as follows again from an application of Bézout's classical theorem, which is explained in the proof of [\[Via08, Lemma 13.2\]](#). Combining this upper bound with [\(54\)](#), where we set $V = \Phi(C)$, we get the lower bound

$$\hat{\mu}(\Phi(C)) \geq c_8(A_N, \deg(C)) \frac{1}{\alpha^{2/(N-1)} (\log|\alpha|)^{\lambda(N, N-1)}}$$

for some constant c_8 depending on A_N and $\deg(C)$. In particular, this lower bound tends to zero as $|\alpha| \rightarrow +\infty$, and is evidently seen to be worse than [\(56\)](#).

Lower bounds for the essential minimum such as the ones obtained in this subsection are known to be essential in the study of the torsion anomalous conjecture, as shown in [\[Via08, CVV16\]](#). In particular, let us observe that, up to torsion and to a reordering of the variables, every subgroup $B \subseteq A_N$ of codimension $r \geq 2$ is of the form $B = \ker(\eta)$, where $\eta = (\text{Id}_r \times L): A_N \rightarrow E_1 \times \cdots \times E_r$ is a morphism of abelian varieties defined as the first r rows of the endomorphism $\Phi: A_N \rightarrow A_N$ introduced in [\(52\)](#). Hence, the results obtained in this subsection provide an explicit link between the irreducibility statements proven in this paper and the torsion anomalous conjecture, which will be investigated further in future work.

REFERENCES

- [BMZ07] E. Bombieri, D. Masser, and U. Zannier, *Anomalous subvarieties—structure theorems and applications*, Int. Math. Res. Not. IMRN (2007), no. 19, Art. ID rnm057, 33. MR 2359537 (2008k:11060)
- [CVV16] S. Checcoli, F. Veneziano and E. Viada, *On the explicit torsion anomalous conjecture*. Trans. Amer. Math. Soc. 369 (2017), 6465–6491.

- [CVV19] S. Checcoli, F. Veneziano, and E. Viada, *The Explicit Mordell Conjecture for families of curves*, Forum of Mathematics, Sigma, vol. 7, p. e31, Jan. 2019.
- [Ga10] A. Galateau, *Une minoration du minimum essentiel sur les variétés abéliennes*, Commentarii Mathematici Helvetici, vol. 85, no. 4, pp. 775–812, 2010.
- [Hin88] M. Hindry, *Autour d'une conjecture de Serge Lang*, Invent. Math. 94 (1988), 575–603.
- [Phi81] P. Philippon, *Lemmes de zéros dans les groupes algébriques commutatifs*, Bulletin de la Société Mathématique de France, vol. 114, pp. 355–383, 1986, doi:10.24033/bsmf.2060.
- [Phi91] P. Philippon, *Sur des hauteurs alternatives. I*, Math. Ann. 289 (1991), no. 2, 255–284.
- [Phi95] P. Philippon, *Sur des hauteurs alternatives. III*, J. Math. Pures Appl. (9) **74** (1995), no. 4, 345–365. MR 1341770 (97a:11098)
- [Via08] E. Viada, *The intersection of a curve with a union of translated codimension-two subgroups in a power of an elliptic curve*, Algebra Number Theory **2** (2008), no. 3, 249–298.
- [Via17] E. Viada, *Explicit height bounds and the effective Mordell-Lang conjecture*, Riv. Mat. Univ. Parma (N.S.), vol. 7, no. 1, pp. 101–131, 2016.
- [Via21] E. Viada, *A criterion for transversality of curves and an application to the rational points*, Ann. Math. Québec, vol. 45, no. 2, pp. 453–464, Oct. 2021.
- [Was08] L. C. Washington, *Elliptic curves*, Second Edition. In: Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2008.
- [Zan12] U. Zannier, *Some Problems of Unlikely Intersections in Arithmetic and Geometry* (AM-181). With six appendices by David Masser. Princeton University Press, 2012.
- [Zha95a] S. Zhang, *Small points and adelic metrics*, J. Algebraic Geom., vol. 4, no. 2, pp. 281–300, 1995.

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