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► **To cite this version:**

Clément Henry, Adrien Merlini, Francesco P. Andriulli. A Fast Direct Solver for Volume Integral Equations Based on Quasi-Helmholtz Laplacian Filters. 2023 IEEE Conference on Antenna Measurements and Applications (CAMA), Nov 2023, Genoa, Italy. 10.1109/CAMA57522.2023.10352886 . hal-04266848

**HAL Id: hal-04266848**

**<https://hal.science/hal-04266848>**

Submitted on 31 Oct 2023

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# A Fast Direct Solver for Volume Integral Equations Based on Quasi-Helmholtz Laplacian Filters

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**Abstract**—This contribution presents a fast direct solver for solving the electric flux volume integral equation (D-VIE) leveraging a new set of volume quasi-Helmholtz Laplacian filters. Numerical results will support the theory, showing the practical significance of the newly proposed technique for inverse scattering problems.

**Index Terms**—volume integral equations (VIEs), filtering theory, quasi-Helmholtz Projectors

## I. INTRODUCTION

Most of the imaging algorithms in bioelectromagnetics necessitate solving multiple forward problems to enforce Maxwell's equations in the imaged tissues and hence improve the convergence towards physical permittivity values. In such scenarios, direct solutions (as opposed to iterative ones) are preferred due to their superior performance when dealing with multiple right-hand-side (RHS) problems (e.g. in a multi-illumination imaging setting).

One effective approach among the forward solution strategies for complex biological tissues (e.g. a head) is the volume integral equation (VIE), which enables the modeling of time-harmonic electromagnetic scattering from tissues exhibiting a high degree of inhomogeneity and anisotropy. The VIE offers several advantages, including the need for discretization only in the scatterer and the automatic enforcement of the radiation conditions [1]. Although VIE methods require less unknowns than standard volume methods (e.g. finite element method), they give rise to linear systems of equations in which the matrix to be inverted is dense, which is computationally expensive to solve.

To address this challenge, various methods are commonly employed. Acceleration techniques like the fast multipole method [2] and the adaptive integral method [3] are utilized to achieve linear or quasi-linear memory and time complexities. However, these methods recast the dense matrix into a fast matrix-vector multiplication, which requires solving numerous iterative problems (one for each RHS).

This work proposes a novel method to obtain a fast direct solver for the electric flux volume integral equation (D-VIE). The approach is based on spectral filtering and enables the creation of a low-rank representation of the discretized D-VIE while simultaneously controlling the error on the solution. However, achieving this effect is not straightforward due to

the spectral pollution in discretized integral operators. We introduce a new family of volume quasi-Helmholtz Laplacian filters extended from their surface counterpart [4], [5] to efficiently address this spectral behavior and yield fast direct inversion. Numerical results will be presented to validate the theory and demonstrate the practical relevance of the newly proposed technique for inverse scattering problems in a biomedical context.

## II. MAIN FORMULATION

Consider the discretized D-VIE [6],

$$(\mathbf{G}_\epsilon + \mathbf{Z}_A + \mathbf{Z}_\phi) \boldsymbol{\alpha} = \mathbf{v}, \quad (1)$$

in which  $\mathbf{G}_\epsilon$ ,  $\mathbf{Z}_A$ , and  $\mathbf{Z}_\phi$  are the discretized identity scaled with the inverse permittivity, the vector potential, and the scalar potential operators of the D-VIE, respectively. To enable a linear-in-complexity direct inversion of the discretized D-VIE, we need to reformulate it as  $\mathbf{G}_\epsilon + \mathbf{C}$  in which  $\mathbf{C}$  is a compressible matrix. The compression of  $\mathbf{Z}_A + \mathbf{Z}_\phi$  is made possible here by leveraging the extension of the surface quasi-Helmholtz Laplacian filters [4], [5] in this volume setting. The combination  $\mathbf{F}_n$  of these filters acts as a spectral filter and allows to filter  $\mathbf{Z}_A$ ,  $\mathbf{Z}_\phi$ , and the RHS  $\mathbf{v}$  at the required spectral index. We obtain the resulting equation

$$(\mathbf{G}_\epsilon + \mathbf{F}_n (\mathbf{Z}_A + \mathbf{Z}_\phi)) \boldsymbol{\alpha} = \mathbf{F}_n \mathbf{v}, \quad (2)$$

in which the spectral index  $n$  is chosen to guarantee the desired error  $\|\mathbf{F}_n \mathbf{v} - \mathbf{v}\|/\|\mathbf{v}\|$  in the filtered RHS, which is directly related to the error on the solution. Since most sources have a band-limited spectrum with respect to the filter eigenvectors,  $n$  is generally small and independent from the number of unknowns, which enables a low-rank compression of  $\mathbf{F}_n (\mathbf{Z}_A + \mathbf{Z}_\phi)$ . This strategy will have the advantage of achieving both a well-posed system and a compressible formulation for which a fast direct solver will be provided.

## ACKNOWLEDGMENT

This work has been funded in part by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (ERC project 321, grant No.724846), in part by the Horizon Europe Research and innovation programme under the EIC Pathfinder grant

agreement N° 101046748 (project CEREBRO), and in part by the ANR Labex CominLabs under the project “CYCLE”.

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