Localization and tracking control of autonomous vehicles in time-varying bearing formation

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***Abstract*— This letter proposes an observer-based formation tracking control approach for multi-agent velocity-controlled vehicles under the assumption that either relative or global position measurements are unavailable for all the vehicles. It is assumed that only some vehicles (at least one) have access to their own global position, and all vehicles are equipped with sensors capable of sensing the bearings relative to neighboring vehicles. Each vehicle estimates its global position using relative bearing measurements and estimates of neighboring vehicles received over a communications network. Then, a distributed output-feedback observer-based controller is designed relying on bearing measurements and the estimated global positions. In contrast with the literature on bearing-based localization and control, we relax the common assumption of so-called bearing rigidity, and, in addition, we do not assume that the interconnections are constant. To the best of our knowledge, the bearing-based localization-and-tracking control problem under such assumptions remains open. In support of our theoretical findings, some simulation results are presented to illustrate the performance of the proposed observer-based tracking controllers.**

I. INTRODUCTION

Multi-vehicle systems are in demand to accomplish missions in different challenging scenarios, such as infrastructure inspection, surveillance, precision agriculture, exploration of deep waters, land, and space, etc. [1]. During these coordinated tasks, it is always essential for a multi-agent system to have the ability to localize the position and track desired trajectories in a decentralized fashion. The different kinds of sensors and measurements gives, as a matter of fact, one of many ways to classify controllers for cooperative vehicles [2]. Two of the most common methods to acquire the localization information rely either on global positioning systems or on on-board proximity sensors. The former are particularly useful in outdoor environments, specifically when the vehicles in the swarm are too far apart for on-board sensors to work. The latter, however, are preferred in indoor and congested environments. There are many different sensors that deliver measurements of different kinds. For instance, vision sensors provide simple visual cues such as relative bearing (direction) measurements, which are robust to noise. Other sensors, such as ultra-wide band [3] deliver accurate relative distance measurements through radio communication. Thanks to the passive property of cameras, follower vehicles that see a leader may be able to estimate their location using their inter-agent bearing with respect to the said leader.

The main body of work on bearing-based localization [4], [5] relies on so-called bearing rigidity theory [6] (also termed parallel rigidity [7]), which establishes the conditions on the graph's topology and the agents' physical configuration that ensure the unique shape of the latter up to a translational and a scaling factors using constant inter-agent bearing measurements. Provided that the formation is bearing rigid, and at least two agents have access to their positions, distributed localization algorithms using bearing measurements are developed in [8] and [4]. These works mainly focus on formations with constant bearing measurements and under fixed interaction graph topologies. Multi-agent coordination, however, typically evolves in time with dynamic configuration and involves time-varying interaction topology among agents. This interaction may change dynamically because the visible neighbors may not always be the same. Some other works only consider specific dynamic formations or time-varying graph topologies. For instance, the bearing-based localization algorithm in [9] is limited to a three-agent formation, for which each agent has to perform a circular motion. In [10] and [11] the bearing-based observability under time-varying graph topologies is analyzed for the particular case of a group of agents defined in a two-dimensional space.

In this letter we address the bearing-based localization and the formation-tracking control problems for multi-agent systems defined in two and three dimensions. In particular, we consider that a set of inter-agent bearings can be time-varying, so the fundamental conditions of bearing-rigidity theory are not satisfied. Moreover, we relax the classical bearing-rigidity assumption and we lift the scale ambiguity of bearing formation. At any given instant, a given pair of bearings with respect to two neighbors may yield different estimates of the follower's position, as in the instance of a square configuration without a diagonal connection [4]. That is, the fundamental condition of localizability does not hold. This condition, however, is not necessary; it may be replaced with the relaxed property that a multi-agent system's formation be *Bearing Persistently Exciting* (BPE) [12]. Persistency of excitation (PE) is a concept that originated in the 1960s in the literature on adaptive control, but has also proved useful in control of autonomous vehicles. For instance, in smooth stabilization of nonholonomic vehicles [13], in the simultaneous localization and mapping problem [14], or in

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target localization and enclosed control of a group of agents in two-dimensional space [15].

We investigate the use of persistency of excitation in relation to the time-varying Bearing Laplacian matrix of multi-agent systems under any jointly connected undirected switching graphs. Based on the BPE property, we design a distributed localization algorithm using inter-agent bearings measures and the position measure of a single agent (the leader). Then, we use the estimated position to design the velocity control input to track desired time-varying trajectories for a multi-agent system under single-integrator dynamics. We show that both the estimation and position tracking errors converge asymptotically to zero, provided the desired formation is BPE.

The key advantages of the proposed approach are that the distributed observer-based formation tracking controller i) can be applied to multi-agent systems under any jointly connected undirected switching graph topologies and hence loosens the constraints on the graph topology often required in the classical bearing-based localization algorithms based on bearing rigidity [8] and [4], ii) needs only one leader with known position (instead of two as described in the existing literature [10], [8], [4]) to localize the formation's configuration using inter-agent bearings, and iii) achieves any BPE d-dimensional trajectory whereas only translational and scaling maneuvering can be realized in the existing literature on bearing-based formation maneuver control [16]. Relative to [12], we generalize the BPE property to encompass time-varying graph topologies, used to design desired trajectories of the formation. Furthermore, we stress that in [12] only the formation control problem is addressed, but not the localization problem nor tracking maneuvers.

The remainder of this letter is organized in six sections. In Section II we provide some mathematical preliminaries on graph theory. In Section III we formulate the bearing based estimation and tracking control problem that we address. In Section IV we provide the concepts of BPE formation under switching graph topologies. In Section V we present the integrated distributed bearing-based localization algorithm and the formation tracking controller. Simulation results are presented in Section VI and the letter is wrapped up with some final comments in Section VII.

II. Mathematical preliminaries

Notation: We denote by $\mathbb{S}^{d-1} := \{y \in \mathbb{R}^d : \|y\| = 1\}$ the $d-1$-Sphere ($d \geq 2$); $\|\cdot\|$ denotes the Euclidean norm. The null space and rank of a matrix are denoted by $\mathrm{null}(\cdot)$ and $\mathrm{rank}(\cdot)$, respectively. The operator $\mathrm{diag}(A_i) = \mathrm{blkdiag}\{A_1, \ldots, A_n\} \in \mathbb{R}^{nd\times nd}$ indicates the block diagonal matrix with elements given by $A_i \in \mathbb{R}^{d\times d}$, with $i \in \{1, 2, \ldots, n\}$.

On graph theory: The interaction topology of a n-agent system can be modeled as an undirected graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ ($n \geq 2$) is the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of undirected edges. Two vertices i and j are called adjacent (or neighbors) when $(i, j) \in \mathcal{E}$. The set of neighbors of agent i is denoted by $\mathcal{N}_i := \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. If $j \in \mathcal{N}_i$, it follows that $i \in \mathcal{N}_j$, since the edge set in an undirected graph consists of unordered vertex pairs. An undirected graph $\mathcal{G}$ is connected if there exists a path between every pair of vertices in $\mathcal{G}$ and in that case $m \geq n-1$. An undirected spanning tree is a connected subgraph of $\mathcal{G}$ without circuits and involving all the vertices of $\mathcal{G}$. An oriented graph is an undirected graph together with an orientation which is the assignment of a direction to each edge. The incidence matrix $H \in \mathbb{R}^{m\times n}$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows indexed by edges and columns by vertices: $[H]_{ki} = 1$ if vertex i is the head of the edge k, $[H]_{ki} = -1$ if it is the tail, and $[H]_{ki} = 0$ otherwise. Note that one always has $H\mathbf{1}_n = 0$, where $\mathbf{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^n$. For a connected graph, or equivalently a graph which has a spanning tree, $\mathrm{rank}(H) = n-1$. The graph Laplacian matrix is defined as

$$L := \bar{H}^\top \bar{H}, \text{ with } \bar{H} = H \otimes I_d, \tag{1}$$

where $\otimes$ denotes the Kronecker product, I_d the identity matrix of dimension d, and $\mathrm{null}(L) = \mathrm{span}(U)$ with $U = \mathbf{1}_n \otimes I_d$. If the graph is connected, one has $\mathrm{rank}(L) = \mathrm{rank}(\bar{H}) = dn - d$, $\mathrm{null}(L) = \mathrm{span}(U)$ and hence, by adopting λ_i as the ith eigenvalue of L under a non-increasing order, one ensures that λ_{dn-d} is the smallest positive eigenvalue of L.

For graphs with time-varying interaction topology we use $\mathcal{G}_\sigma$ to denote the corresponding time-dependent graph with $\sigma : [0, +\infty) \to \mathcal{P}$ a piece-wise constant function. The set $\mathcal{P}$ represents the set of indexes for all possible graphs, including non-connected ones, defined on vertices $\{1, \ldots, n\}$, with $|\mathcal{P}| = 2^{\frac{n(n-1)}{2}}$, where $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. We assume hereafter that there exists a switching sequence $\{t_\iota\}$, $\iota \in \mathbb{N}$ at which the index of the interaction topology $\sigma(t_\iota)$ changes. The time intervals $[t_\iota, t_{\iota+1})$ are assumed to be uniformly bounded and non-overlapping. We assume that $t_{\iota+1} - t_\iota \geq \tau_0 > 0$, where τ_0 is the dwell time during which $\sigma(t)$ is constant and the interaction topology keeps fixed. Define $\mathcal{E}_\sigma$ and $\mathcal{N}_\sigma^i$ as the corresponding edge set and neighbors set of agent i, respectively, at t.

III. Problem formulation

Consider a group of n vehicles modeled by the equation

$$\dot{p}_i = v_i, \quad i \in \{1, 2, \ldots, n\}. \tag{2}$$

The position of the ith vehicle with respect to a common inertial frame is given by the vector $p_i \in \mathbb{R}^d$ and the velocity $v_i \in \mathbb{R}^d$ is assumed to be the control input. It is required for these vehicles, to move in formation in a d-dimensional space ($d \in \{2, 3\}$), under the following standing hypothesis.

Assumption 1: At least, and possibly only, one agent in the formation can measure its own position p_i, without error. Such agent is referred to as the leader.

Without global position measurements, the position p_i is unknown to the follower vehicles. However, relative bearing measurements are obtained through onboard sensors. More

precisely, for the ith vehicle that "sees" the jth neighbor agent, with $j \in \mathcal{N}_i$, we denote by

$$p_{ij} := p_j - p_i, \tag{3}$$

the position vector of the jth vehicle relative to the ith and, if $\|p_{ij}\| \neq 0$, we denote by

$$g_{ij} := p_{ij}/\|p_{ij}\| \in \mathbb{S}^{d-1} \tag{4}$$

the bearing of agent j relative to agent i. Then, it is assumed that a bidirectional interconnection between adjacent nodes exists as described below.

Assumption 2: Each agent $i \in \mathcal{V}$ can measure the relative bearing vectors g_{ij}, with respect to its neighbors $j \in \mathcal{N}_i$, in a common inertial frame. In addition, each agent can communicate its position estimation to its neighbor agents. Assumption 2 implies that each agent knows its orientation matrix relative to a common inertial frame, so the relative bearing measured in its local frame can be represented in the inertial frame.

Under the latter assumption, defining the configuration $\boldsymbol{p} := [p_1^\top, ..., p_n^\top]^\top \in \mathbb{R}^{dn}$, the vehicles define a system of n connected agents, represented by a formation denoted $\mathcal{G}(\boldsymbol{p})$. Then, the bearing-based formation tracking control problem is posed as follows. By assigning one leader in the formation, 1) design a distributed localization algorithm using the velocity control input of each agent and the inter-agent bearings to estimate the agents' positions and 2) design an observer-based distributed controller relying on the estimated positions to asymptotically track any feasible desired formation.

Stated in such generic form, the bearing-based localization and formation tracking control problems have been addressed in a number of works—see the Introduction—but not necessarily simultaneously, as we do here. For instance, in [4] only the localization problem is addressed. Furthermore, a recurring assumption in the literature on bearing-based control is that the interconnections are constant—see *e.g*, [8], [16], [12]. In this letter we consider that the graph's topology changes with time, as a consequence of interconnections persistently switching on and off—see Figure 1 and Assumption 3 next.

Assumption 3: The group's topology is time-varying, represented by a switching graph $\mathcal{G}_\sigma$, as defined in Section II.

In our preliminary work [17], graphs with switching topologies are considered, but the localization problem is disregarded.

IV. Bearing Persistently Exciting Formation under Switching Graph Topologies

Given a group of interconnected agents in a configuration $\boldsymbol{p}$, the underlying formation $\mathcal{G}(\boldsymbol{p})$ is composed of nodes labeled $i \in \mathcal{V}$ and edges $(i,j) \in \mathcal{E}$, to this formation corresponds a so-called bearing Laplacian matrix L_B, defined as follows—cf. [4]. Consider an arbitrary orientation of the graph and denote

$$\bar{p}_k := p_{ij},\ k \in \{1, 2, \ldots, m\},\ (i,j) \in \mathcal{E} \tag{5}$$

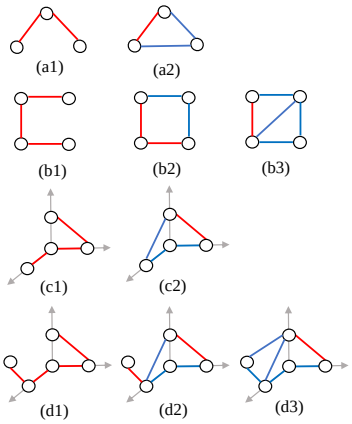

Fig. 1. Union of the graphs $\bigcup_{(t,t+T)} \mathcal{G}_{\sigma(t)}$ of BPE formations in two (a1-b3) and three-dimensional space (c1-d3). Red lines represent edges for which the corresponding bearing vector are PE and blue lines represent edges for which the corresponding bearing vectors are not necessarily PE. These are instances of time-varying formations covered by our main results.

as the edge vector with assigned direction such that i and j are, respectively, the initial and the terminal nodes of $\bar{p}_k$. Denote the corresponding bearing vector by

$$\bar{g}_k := \frac{\bar{p}_k}{\|\bar{p}_k\|} \in \mathbb{S}^{d-1},\ k \in \{1, 2, \ldots, m\}.$$

Then, the bearing Laplacian matrix is denoted as

$$L_B(\boldsymbol{p}) := \bar{H}^\top \Pi(\boldsymbol{p}) \bar{H}, \quad \Pi(\boldsymbol{p}) = \mathrm{diag}(\pi_{\bar{g}_k}), \tag{6}$$

where, for any $y \in \mathbb{S}^{d-1}$,

$$\pi_y := I - yy^\top \geq 0$$

denotes the operator that projects y onto its orthogonal plane.

Since $\mathrm{span}\{U, \boldsymbol{p}\} \subseteq \mathrm{null}(L_B(\boldsymbol{p}))$, it follows that $\mathrm{rank}(L_B) \leq dn - d - 1$. According to [4], if the formation is *Infinitesimal Bearing Rigid* (IBR) then $\mathrm{rank}(L_B) = dn - d - 1$ and $\mathrm{null}(L_B(\boldsymbol{p})) = \mathrm{span}\{U, \boldsymbol{p}\}$ for each fixed configuration $\boldsymbol{p}$. Then, it is possible to reconstruct the positions of follower agents, p_i, provided that two leaders know their global positions and all the bearing measurements and interconnections remain constant.

In this letter, we allow for only one leader know its global position and assume that some of the inter-agent bearings are time-varying such that the formation is *Bearing Persistently Exciting* (BPE) [12]. Besides, we consider the scenario in which the topology undergoes persistent switches. A BPE formation is a type of time-varying bearing formation, introduced first in [12], whose configuration can be uniquely determined up to a translation using only inter-agent bearings and velocity of each agents. The property defined in [12], for fixed topologies, is recalled below.

Definition 1 (BPE): A formation $\mathcal{G}(\boldsymbol{p})$ is called *Bearing Persistently Exciting* (BPE) if $\mathcal{G}$ is connected and the bearing Laplacian matrix is persistently exciting, *i.e.,* there exists $T > 0$ and $\mu > 0$ such that

$$\frac{1}{T}\int_t^{t+T} L_B(\boldsymbol{p}(\tau))d\tau \geq \mu L \quad \forall\, t \geq 0. \tag{7}$$

Remark 1: There is an obvious abuse of terminology in the previous definition since (7) does not imply that the bearing Laplacian is persistently exciting [18] along the trajectories, *i.e.*, that there exist $T > 0$ and $\mu > 0$ such that

$$\frac{1}{T}\int_{t}^{t+T} L_B(\boldsymbol{p}(\tau))d\tau \geq \mu I \quad \forall\, t \geq 0 \tag{8}$$

which cannot hold because L is not full rank.

Note that $\Pi(\boldsymbol{p})$ being persistently exciting along the trajectories implies (7), but not viceversa. For detailed properties of BPE formation under fixed graph topologies (including necessary conditions and sufficient and necessary conditions to guarantee a BPE formation), please refer to [12] and [19].

To provide a generalization of Definition 1 that encompasses switching-graph topologies $\mathcal{G}_\sigma$, we assume that there exists $T > \tau_0$ such that, for all $\tau \in [t,\ t+T]$ and any $t \geq 0$, the corresponding incidence matrix is $\bar{H}_\sigma(t) = W(t)\bar{H}$. In the latter $\bar{H}$ is the constant incidence matrix of the joint graph $\mathring{\mathcal{G}} = \cup_{\tau\in(t,t+T)}\mathcal{G}_{\sigma(\tau)}$ and $W(t) = \text{diag}(\bar{w}_k(t))$. In turn, $\bar{w}_k(t)$ is a binary valued-function that equals to 1 if the corresponding edge is connected and to 0 if otherwise. Then, we define the Laplacian $L_\sigma(t) := \bar{H}_\sigma(t)^\top \bar{H}_\sigma(t)$ and the bearing Laplacian $L_{B_\sigma}(\boldsymbol{p},t) := \bar{H}^\top \Pi_\sigma(\boldsymbol{p},t)\bar{H}$, where $\Pi_\sigma(\boldsymbol{p},t) = W(t)\Pi(\boldsymbol{p})W(t) = W(t)\Pi(\boldsymbol{p}) = \Pi(\boldsymbol{p})W(t)$.

Under a fixed graph topology, one can verify that $W = I_{dn}$, hence $L_\sigma = L$ and $L_{B_\sigma} = L_B$. For time-varying topologies, we have the following useful statement.

Lemma 1: Consider a formation $\mathcal{G}_\sigma(\boldsymbol{p})$, with $\sigma \in \mathcal{P}$. If the formation $\mathring{\mathcal{G}}(\boldsymbol{p})$ under the joint graph is BPE with parameters T and $\bar{\mu} > 0$ and

$$\frac{1}{T}\int_{t}^{t+T} \Pi_\sigma(\boldsymbol{p}(\tau),\tau)d\tau \geq \bar{\mu}\int_{t}^{t+T} \Pi(\boldsymbol{p}(\tau))d\tau,$$

then the formation $\mathcal{G}_\sigma(\boldsymbol{p})$ is also BPE. That is, $\forall t \geq 0$

$$\frac{1}{T}\int_{t}^{t+T} L_{B_\sigma}(\boldsymbol{p}(\tau),\tau)d\tau \geq \frac{\bar{\mu}}{T}\int_{t}^{t+T} L_B(\boldsymbol{p}(\tau))d\tau. \qquad \square$$

The statement follows from the observation that

$$L_{B_\sigma}(\boldsymbol{p},t) = \bar{H}^\top W(t)\Pi(\boldsymbol{p})W(t)\bar{H}. \tag{9}$$

The formation being BPE implies that some of the bearings between pairs of agents belonging to the formation are PE [19], *i.e.*, the shape of the multi-agent system may be time-varying or fixed. An illustration of a BPE formation is provided in Figure 2 below.

It shows an example of a 4-agent BPE formation with time-varying shape in which p_1, p_2 and p_3 are static while p_4 is oscillating along the black dashed line. Inter-agent bearings g_{21} and g_{23} are constant and g_{41} and g_{43} are PE during time interval $[t_1,t_3]$. For each time instant, the graph topology is not necessarily connected, and the time-varying sensing graph topology indicates that each bearing is measured by corresponding agents during a time interval greater than the dwell time τ_0 but may not be measured all the time from t_1 to t_3.

When the shape of a BPE formation is fixed, a similarity transformation involving a time-varying rotation has to be

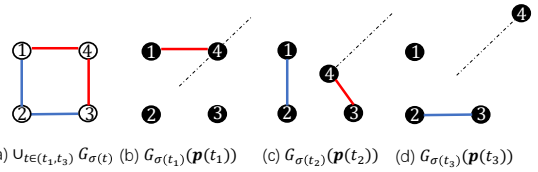

Fig. 2. An example of a BPE formation $\mathcal{G}_\sigma(\boldsymbol{p}(t))$ under a switching graph topology. Sub-figure (a) shows the union of the graphs during time t_1 to t_3. Sub-figures (b), (c), and (d) show three selected frames of the physical configuration $\boldsymbol{p}(t)$ and connections $\mathcal{G}_{\sigma(t)}$ of the formation under a timed sequence such that $t_1 < t_2 < t_3$, $t_2 - t_1 > \tau_0$ and $2\tau_0 < t_3 - t_1 < T$. The color blue/red on the connections indicates that the corresponding bearings are constant/PE, respectively. The dashed line indicates the trajectory of agent 4.

imposed on the whole system so that (7) hold. This particular case of BPE formation is defined as follows.

Definition 2: A formation $\mathcal{G}_\sigma(\boldsymbol{p})$ $(\sigma \in \mathcal{P})$ is called *Relaxed Bearing Rigid* if it is BPE and subject to a similarity transformation, *i.e.*, a rigid motion together with a rescaling: for each $t \geq 0$ and $i \in \mathcal{V}$, $p_i(t) = s(t)R(t)^\top p_i(0) + c(t)$ where $s(t) \in \mathbb{R}^+$ is a scaling factor, $c(t) \in \mathbb{R}^d$ is a translational factor, and $R(t) \in SO(d)$ is a time-varying rotation matrix[1].

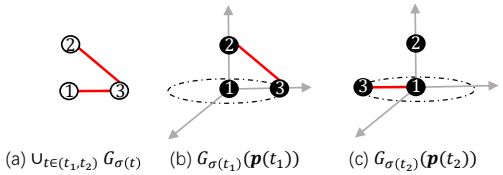

Fig. 3. An example of a RBR formation under a switching graph topology. Sub-figure (a) shows the union of the graphs during time t_1 to t_2. Sub-figures (b) and (c) show two selected frames of the physical configuration $\boldsymbol{p}(t)$ and connections $\mathcal{G}_{\sigma(t)}$ of the multi-agent system under a timed sequence such that $t_1 < t_2$ and $\tau_0 < t_2 - t_1 < T$. Connections in red represent PE bearings. The dashed line indicates the trajectory of agent 3.

Figure 3 shows an example of 3-agent RBR formation in three-dimensional space. The system is subject to a time-varying rotation motion around the z-axis. Agents 1 and 2 are static on the z-axis and agent 3 rotates around z-axis, as indicated by the black dashed line. Note that the corresponding inter-agent bearings g_{31} and g_{32} are both PE and $\underset{t\in(t_1,t_2)}{\cup}\mathcal{G}_{\sigma(t)}$ is as presented as in Figure 3a.

V. Bearing-based localization and formation tracking control under switching graph topologies

Without loss of generality, we consider a formation in which agent 1 is the leader, the unique agent in the formation that measures its own position p_1—cf. Assumption 1. The following hypothesis defines feasible desired BPE formations.

Assumption 4: The desired velocities $v_i^*(t) := \dot{p}_i^*(t)$ and desired positions $p_i^*(t)$ $(i \in \mathcal{V})$ are chosen such that, for all

[1] $SO(d)$ stands for Special Orthogonal group of dimension d.

t, $v_i^*(t)$ are bounded, the resulting desired bearings $g_{ij}^*(t)$ are well-defined and the desired formation $\mathcal{G}_\sigma(\boldsymbol{p}^*(t))$ under switching graph topologies is BPE.

Now, let $\hat{p}_i \in \mathbb{R}^d$ denote the estimate of p_i. Then, the observers of the leader's and each follower's positions are defined by the equations

$$\dot{\hat{p}}_1 = v_1 - k_o \sum_{j=1}^{n} w_{1j}(t)\pi_{g_{1j}}(\hat{p}_1 - \hat{p}_j) - k_o(\hat{p}_1 - p_1) \quad (10a)$$

$$\dot{\hat{p}}_i = v_i - k_o \sum_{j=1}^{n} w_{ij}(t)\pi_{g_{ij}}(\hat{p}_i - \hat{p}_j), \quad i \in \mathcal{V}\backslash\{1\}, \quad (10b)$$

where $k_o > 0$, $w_{ij}(t) := \bar{w}_k(t) = 1$, $j \in \mathcal{N}_{\sigma(t)}^i$ and $w_{ij}(t) = 0$ otherwise. Then, using the estimated position $\hat{p}_i$, let the velocity control input of the system (2) be defined as

$$v_i = -k_c(\hat{p}_i - p_i^*) + v_i^*, \quad i \in \mathcal{V}. \quad (11)$$

Let $\boldsymbol{v} = [v_1^\top, \ldots, v_n^\top]^\top$ and define the error variables $\boldsymbol{\delta} := \hat{\boldsymbol{p}} - \boldsymbol{p}$ and $\tilde{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{p}^*$ with $\hat{\boldsymbol{p}} = [\hat{p}_1^\top, \ldots, \hat{p}_n^\top]$ and $\boldsymbol{p}^* = [p_1^{*\top}, \ldots, p_n^{*\top}]^\top$. From (10), one has $\dot{\hat{\boldsymbol{p}}} = -k_o[\, L_{B_\sigma}(\boldsymbol{p}, t)\hat{\boldsymbol{p}} + A\boldsymbol{\delta}\,] + \boldsymbol{v}$ with $A = \mathrm{diag}(I_d, 0, \ldots, 0) \in \mathbb{R}^{dn\times dn}$. Since $\mathrm{span}(\boldsymbol{p}) \subseteq \mathrm{null}(L_{B_\sigma})$, we have $\boldsymbol{p}^\top L_{B_\sigma}(\boldsymbol{p}, t)\boldsymbol{p} \equiv 0$, so it is straightforward to verify that:

$$\dot{\boldsymbol{\delta}} = -k_o(L_{B_\sigma}(\boldsymbol{p}, t) + A)\boldsymbol{\delta} \quad (12)$$

$$\dot{\tilde{\boldsymbol{p}}} = -k_c(\tilde{\boldsymbol{p}} + \boldsymbol{\delta}). \quad (13)$$

Proposition 1: Consider a n-agent system $\mathcal{G}_\sigma(\boldsymbol{p}(t))$ defined in $\mathbb{R}^d$ under Assumptions 1–4. Then, for any initial condition satisfying

$$\|\tilde{\boldsymbol{p}}(0)\| < \min_{(i,j)\in\mathcal{E}} \|p_{ij}^*(t)\|/2, \quad (14a)$$

$$\|\boldsymbol{\delta}(0)\| < \min_{(i,j)\in\mathcal{E}} \|p_{ij}^*(t)\|/2, \quad (14b)$$

the observer (10) is well defined and the localization and formation-tracking errors $\boldsymbol{\delta}$ and $\tilde{\boldsymbol{p}}$ converge to zero.

Proof: First, we show that $\boldsymbol{\delta}$ and $\tilde{\boldsymbol{p}}$ are bounded and the bearing information $g_{ij}(t), \forall (i,j) \in \mathcal{E}$ is well defined $\forall t \geq 0$. Define the candidate Lyapunov function for $\boldsymbol{\delta}$-system (12), $\mathcal{W}_1 = \frac{1}{2}\|\boldsymbol{\delta}\|^2$, which satisfies

$$\dot{\mathcal{W}}_1 = -k_o\boldsymbol{\delta}^\top(L_{B_\sigma}(\boldsymbol{p}, t) + A)\boldsymbol{\delta} \leq 0, \quad (15)$$

since $L_{B_\sigma}(\boldsymbol{p}, t) + A \geq 0$ for all $t \geq 0$. Hence, $\boldsymbol{\delta}(t) \leq \boldsymbol{\delta}(0)$ for all $t \geq 0$.

Next, to analyze the trajectories of the $\tilde{\boldsymbol{p}}$-system (13), we use $\mathcal{W}_2 = \frac{1}{2}\|\tilde{\boldsymbol{p}}\|^2$, whose total derivative yields

$$\dot{\mathcal{W}}_2 = -k_c\tilde{\boldsymbol{p}}^\top(\tilde{\boldsymbol{p}} + \boldsymbol{\delta}) \leq -k_c\|\tilde{\boldsymbol{p}}\|(\|\tilde{\boldsymbol{p}}\| - \|\boldsymbol{\delta}\|). \quad (16)$$

We see that $\dot{\mathcal{W}}_2$ is negative definite for all $\|\tilde{\boldsymbol{p}}\| \geq \|\boldsymbol{\delta}\|$. Since $\|\boldsymbol{\delta}(t)\| \leq \|\boldsymbol{\delta}(0)\|$, one concludes that $\|\tilde{\boldsymbol{p}}(t)\| \leq \max\{\|\tilde{\boldsymbol{p}}(0)\|, \|\boldsymbol{\delta}(0)\|\}, \forall t \geq 0$.

To show that $g_{ij}(t), \forall (i,j) \in \mathcal{E}$ is well defined $\forall t \geq 0$, which in turn implies that (10) is well defined under the proposed initial condition, we have to prove that $p_{ij}, \forall (i,j) \in \mathcal{E}$ never crosses zero. Using the fact that $p_{ij} = \tilde{p}_j - \tilde{p}_i + p_{ij}^*$, one gets

$$\|p_{ij}(t)\| \geq \|p_{ij}^*(t)\| - \|\tilde{p}_i(t)\| - \|\tilde{p}_j(t)\| \geq \|p_{ij}^*(t)\| - 2\|\tilde{\boldsymbol{p}}(t)\|.$$

Combining this with (14) and the fact that $\|\tilde{\boldsymbol{p}}(t)\| \leq \max\{\|\tilde{\boldsymbol{p}}(0)\|, \|\boldsymbol{\delta}(0)\|\}, \forall t \geq 0$, one concludes that $\|\tilde{\boldsymbol{p}}(t)\| < \|p_{ij}^*(t)\|/2$ and hence $p_{ij}(t) \neq 0$ for all $t \geq 0$.

To show convergence to the equilibrium $(\boldsymbol{\delta}, \tilde{\boldsymbol{p}})$= $(0, 0)$, we first show that $\dot{\mathcal{W}}_1$ is uniformly continuous and converges to 0. To that end, we use Barbălat's Lemma [20, Lemma 9] , so we compute

$$\ddot{\mathcal{W}}_1 = -2\dot{\boldsymbol{\delta}}^\top(k_oL_{B_\sigma}(\boldsymbol{p}, t) + A)\boldsymbol{\delta} - \boldsymbol{\delta}^\top k_o\bar{H}^\top \mathrm{diag}(\dot{\pi}_{\bar{g}_k})\bar{H}\boldsymbol{\delta}.$$

Now, the fact that $\dot{\pi}_{\bar{g}_k} = -\frac{1}{\|\bar{p}_k\|}(\pi_{\bar{g}_k}\bar{v}_k\bar{g}_k^\top + \bar{g}_k\bar{v}_k^\top\pi_{\bar{g}_k})$ is bounded—since $\bar{v}_k = \dot{\bar{p}}_k = v_j - v_i, (i,j) \in \mathcal{E}$ is bounded and $\bar{p}_k$ (5) never crosses zero—implies that $\ddot{\mathcal{W}}_1$ is also bounded, so $\dot{\mathcal{W}}_1$ is uniformly continuous. It follows that $\dot{\mathcal{W}}_1$ converges to zero which implies $\|\boldsymbol{\delta}\|$ converges to a constant. From there, one ensures that $\boldsymbol{\delta}$ converges to a constant vector. Hence, from (13), one deduces that $\tilde{\boldsymbol{p}} + \boldsymbol{\delta}$ converges to 0, which indicates that $\hat{\boldsymbol{p}}$ converges to $\boldsymbol{p}^*$. Now, integrating by parts on both sides of

$$\dot{\mathcal{W}}_1 = -k_o\boldsymbol{\delta}^\top(L_{B_\sigma}(\boldsymbol{p}, t) + A)\boldsymbol{\delta} \to 0, \quad (17)$$

from t to $t + T$, we obtain

$$\left[\boldsymbol{\delta}(s)^\top \left(\int_t^s (L_{B_\sigma}(\boldsymbol{p}(\tau), \tau) + A)d\tau\right)\boldsymbol{\delta}(s)\right]\Bigg|_t^{t+T} - \mathrm{tr}\left[\int_t^{t+T}\left[\int_t^s (L_{B_\sigma}(\boldsymbol{p}(\tau), \tau) + A)d\tau\right]\left[\dot{\boldsymbol{\delta}}\boldsymbol{\delta}^\top + \boldsymbol{\delta}\dot{\boldsymbol{\delta}}^\top\right]ds\right].$$

Note that the inner integral on the last line of this expression is bounded since L_{B_σ} is bounded and the length of the integration interval T is bounded. Using the fact that $\dot{\mathcal{W}}_1$ and $\dot{\boldsymbol{\delta}} \to 0$, one ensures that:

$$\boldsymbol{\delta}^\top\left[\int_t^{t+T}(L_{B_\sigma}(\boldsymbol{p}(\tau), \tau) + A)d\tau\right]\boldsymbol{\delta} \to 0$$

Combining (17) and (9) with the fact that $\hat{\boldsymbol{p}}$ converges to $\boldsymbol{p}^*$, one verifies that $\hat{\boldsymbol{p}}(t)^\top\bar{H}^\top W(t)\Pi(\boldsymbol{p}(t))\bar{H}\hat{\boldsymbol{p}}(t) \to 0$ as $t \to \infty$ and hence $\boldsymbol{p}^*(t)^\top\bar{H}^\top W(t)\Pi(\boldsymbol{p}(t))\bar{H}\boldsymbol{p}^*(t) \to 0$. This latter ensures that $g_{ij} \to g_{ij}^*$. From there, and since $L_{B_\sigma}(\boldsymbol{p}^*)$ is PE one concludes that $L_{B_\sigma}(\boldsymbol{p})$ is also PE. This, in turn, implies that $\int_t^{t+T}(L_{B_\sigma}(\boldsymbol{p}(\tau), \tau) + A)d\tau > 0$ and hence one concludes that $\boldsymbol{\delta} \to 0$ and, in turn, $\tilde{\boldsymbol{p}} \to 0$. ■

Remark 2: In the special case when all the bearings are constant and the formation is *Persistently Bearing Rigid* [17] (i.e., not necessarily IBR for each instant but the formation under the joint graph is IBR), Proposition 1 still hold provided two leaders knowing their own positions.

VI. Simulation Results

In this section, simulation results are provided to illustrate the effectiveness of the proposed distributed observer-based formation tracking controller (10) (11). We consider a 4-agent system tracking the smooth time-varying trajectories with agent 1 as a leader. As we can see from Fig. 4, the formation first form a fixed pyramid shape rotating around the z-axis while translating along y-axis. Then, it tracks a time-varying shape which transforms from a pyramid shape to a square in the xy-plane. In the end, it continues

maintaining as the square shape while following circular trajectories.

Figure 5 shows the evolution of the estimation and position errors, which converge to zero asymptotically under the switching graph topologies. Note that the graph is not necessarily instantaneously connected. The gains are chosen as $k_o = 5$ and $k_c = 10$.

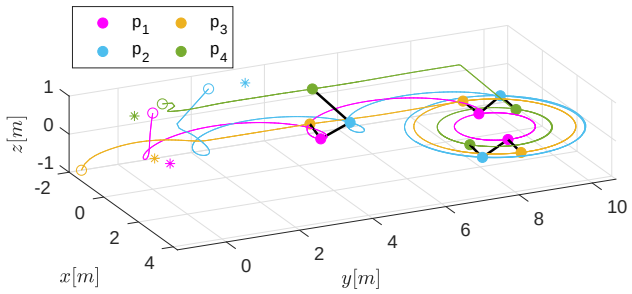

Fig. 4. 3D evolution of the formation. The stars and void circles represent initial estimates and initial positions, respectively. The colorful lines represent the trajectories of the formation. The filled circles indicate the configuration of the formation at $t = 6.2$, $t = 15$, and $t = 18.1$. The black lines are connections of the joint graph $\cup_{\tau\in(t,t+T)}\mathcal{G}_{\sigma(\tau)}$.

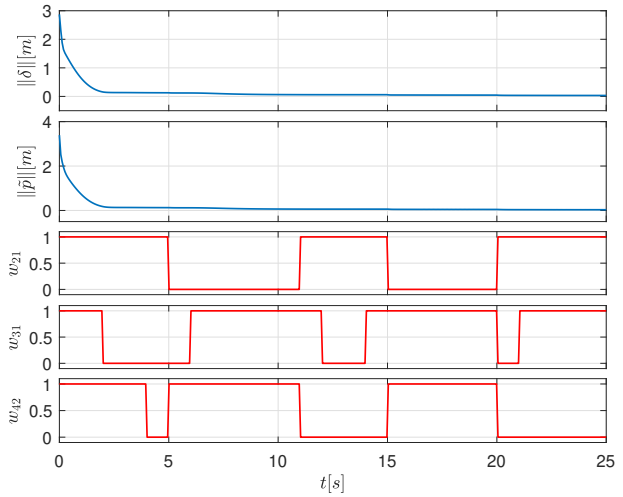

Fig. 5. The first subplot shows the time evolution of the norm of the estimation error $\|\boldsymbol{\delta}\|$ and the second subplot indicates the time evolution of the norm of the position error $\|\tilde{\boldsymbol{p}}\|$. The remaining subplots show the intermittent interconnections; from these plots the topology at any instant can be inferred.

VII. Conclusion

This paper solves an observer-based formation tracking problem for multi-agent systems defined in two and three-dimensional spaces. A distributed output-feedback observer-based tracking controller is designed for the system using inter-agent bearing measurements, the estimated global position of each agent, and the global position of at least one leader agent. The key distinction of the proposed method is that the estimation and tracking errors achieve asymptotically to zero under time-varying graph topologies without common assumptions of bearing rigidity. It also enables the multi-agent system to track a larger set of feasible desired formations, including rotational maneuvers and formations with time-varying shapes. Future work includes extending the observer design for multi-agent systems with each agent under a second-order system to estimate both velocity and position vectors for each agent. Attitude synchronization will also be considered to deal with the practical issue that the relative bearings are only measured in each agent's local frame without knowing its orientation matrix with respect to a common inertial frame.

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