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Explicit maximal totally real embeddings

BY

NEFTON PALI and BRUNO SALVY

Abstract

For real analytic compact manifolds equipped with a covariant derivative operator acting on the real analytic sections of its tangent bundle, a construction of canonical maximal totally real embeddings is known from previous works by Guillemin-Stenzel, Lempert, Lempert-Szőke, Szőke and Bielawski. The construction is based on the use of Jacobi fields which are far from being explicit. As a consequence, the form of the corresponding complex structure has been a mystery since the very beginning. A quite simple recursive expression for such complex structures has been provided in the first author's work "On maximal totally real embeddings" [Pali]. In our series of articles we focus on the case of torsion free connections. In the present article we provide a fiber-wise Taylor expansion of the canonical complex structure which is expressed in terms of symmetrization of curvature monomials and a rather simple and explicit expression of the coefficients of the expansion. Our main argument applies to far more general settings that can be useful for the study of open questions in the theory of the embeddings in consideration. In this article we also provide evidence for some remarkable canonical vanishing of some of the integrability equations in quite general settings.

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1 Introduction and statement of the main result

1.1 Previous works

We remind first some notations and definitions from the work [Pali]. We recall first that over a smooth manifold M , the vector bundle $T_{T_M}|_M \simeq T_M \oplus T_M$, is a complex one with the canonical complex structure $J^{\text{can}} : (u, v) \mapsto (-v, u)$ acting on the fibers. Any almost complex structure which is a continuous extension of J^{can} in a neighborhood of M inside T_M makes M an (almost complex) maximally totally real sub-manifold of T_M . We denote by

$$T \in C^\infty(T_M, \pi^*T_M^* \otimes_{\mathbb{R}} T_{T_M}),$$

the canonical section which at the point $\eta \in T_M$ takes the value $T_\eta : E_p \rightarrow T_{E_p, \eta}$, which is the canonical isomorphism map. We remind now the following definition from [Pali].

Definition 1.1. *Let M be a smooth manifold. An M -totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the image of the zero section 0_M is a couple (α, B) with $\alpha \in C^\infty(U, \pi^*T_M^* \otimes_{\mathbb{R}} T_{T_M})$ and with $B \in C^\infty(U, \pi^*\text{GL}(T_M))$ such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^*T_M}$ over U and such that $\alpha_{0_p} = d_p 0_M$, $B_{0_p} = \mathbb{I}_{T_{M,p}}$, for all $p \in M$. With $A := \alpha + iTB$, the almost complex structure J_A associated to (α, B) is the one which satisfies $T_{T_M, J_A, \eta}^{0,1} = A_\eta(\mathbb{C}T_{M, \pi(\eta)}) \subset \mathbb{C}T_{T_M, \eta}$ for all $\eta \in U \subseteq T_M$.*

Every almost complex smooth extension of the canonical complex structure J^{can} of $T_{T_M}|_M$ over a neighborhood of M inside T_M writes, over a sufficiently small neighborhood $U \subseteq T_M$ of M , as the almost complex structure associated to a unique M -totally real almost complex structure over U .

Definition 1.2. *We denote by Circ the circular operator*

$$(\text{Circ} \theta)(v_1, v_2, v_3, \bullet) = \theta(v_1, v_2, v_3, \bullet) + \theta(v_2, v_3, v_1, \bullet) + \theta(v_3, v_1, v_2, \bullet),$$

acting on the first three entries of any q -tensor θ , with $q \geq 3$. We also define the permutation operation $\theta_2(v_1, v_2, \bullet) := \theta(v_2, v_1, \bullet)$.

For any complex covariant derivative operator ∇ acting on the smooth sections of $\mathbb{C}T_M$ we denote by H^∇ the associated horizontal distribution. We define the operator

$$d_1^\nabla : C^\infty(M, T_M^* \otimes_{\mathbb{R}}^k \mathbb{C}T_M) \rightarrow C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^* \otimes_{\mathbb{R}}^{(k-1)} \mathbb{C}T_M),$$

with $k \geq 1$ as follows

$$d_1^\nabla A(\xi_1, \xi_2, \mu) := \nabla_{\xi_1} A(\xi_2, \mu) - \nabla_{\xi_2} A(\xi_1, \mu),$$

with $\xi_1, \xi_2 \in T_M$ and with $\mu \in T_M^{\oplus(k-1)}$. In the case ∇ acts on the smooth sections of T_M , we denote by $(\eta, t) \mapsto \Phi_t^\nabla(\eta) \in T_M$, the geodesic flow of ∇ starting at a point $\eta \in T_M$.

For any two sections $A \in C^\infty(M, T_M^* \otimes_{\mathbb{R}}^{(k+1)} \mathbb{C}T_M)$ and $B \in C^\infty(M, T_M^* \otimes_{\mathbb{R}}^{(l+1)} \mathbb{C}T_M)$ we define the exterior product

$$A \wedge_1 B \in C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^* \otimes_{\mathbb{R}}^{(k+l-1)} \mathbb{C}T_M)$$

as

$$(A \wedge_1 B)(\xi_1, \xi_2, \eta, \mu) := A(\xi_1, B(\xi_2, \eta), \mu) - A(\xi_2, B(\xi_1, \eta), \mu),$$

with $\xi_1, \xi_2 \in T_M$, $\eta \in T_M^{\oplus l}$ and $\mu \in T_M^{\oplus(k-1)}$. We denote by $\text{Sym}_{r_1, \dots, r_s}$ the symmetrizing operator (without normalizing coefficient!) acting on the entries r_1, \dots, r_s of a multi-linear form. We use the convention that a sum and a product running over an empty set are equal respectively to 0 and 1.

We observe that the argument given in the proof of corollary 2 in [Pali] applies without modifications in the case of a general torsion free covariant derivative operator acting on the smooth sections of the tangent bundle. For the convenience of the reader we provide in the appendix, the argument in this more general setting. We therefore have the following statement.

Corollary 1.3. *Let M be a smooth manifold equipped with a torsion free covariant derivative operator ∇ acting on the smooth sections of the tangent bundle T_M , let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers, let $J \equiv J_A$ be an M -totally real almost complex structure over U , real analytic along the fibers of U and consider the fiber-wise Taylor expansion at the origin*

$$T_\eta^{-1}(H^\nabla - \bar{A})_\eta \cdot \xi = i\xi + \sum_{k \geq 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a sufficiently small neighborhood of the image of the zero section, with $\xi \in T_{M, \pi(\eta)}$ arbitrary with $S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ and with $\eta^k := \eta^{\times k} \in T_{M, \pi(\eta)}^{\oplus k}$.

Then the almost complex structure J is integrable over U and for any $\eta \in U$, the complex curve $\psi_\eta: t + is \mapsto s\Phi_t^\nabla(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J -holomorphic if and only if $S_1 = 0$,

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2, \dots, k+1} \Theta_k^\nabla,$$

for all $k \geq 2$, with $\Theta_2^\nabla := 2R^\nabla$, with

$$\Theta_k^\nabla := -2i (id_1^\nabla)^{k-3} (\nabla R^\nabla)_2 + \sum_{r=3}^{k-1} (r+1)! \sum_{p=2}^{r-1} (id_1^\nabla)^{k-1-r} (pS_p \wedge_1 S_{r-p+1}),$$

for all $k \geq 3$ and the equations $\text{Circ Sym}_{3, \dots, k+1} \Theta_k^\nabla = 0$, are satisfied for all $k \geq 4$.

We remind that by the results due to Guillemin-Stenzel [Gu-St], Lempert [Lem], Lempert-Szöke [Le-Sz1, Le-Sz2], Szöke [Szo] and Bielawski [Bie] in particular, the complex structures in the statement of corollary 1.3 exist in the case M is a compact real analytic manifold and ∇ is a torsion free covariant derivative operator acting on the real analytic sections of T_M .

By corollary 1.3, the vanishing of the tensor $\text{Circ Sym}_{3, \dots, k+1} \Theta_k^\nabla$, for all $k \geq 4$, is certainly true in the above analytic case. We feel however that a proof independent of [Gu-St, Lem, Le-Sz1, Le-Sz2, Szo, Bie] provide a more general statement useful for the applications.

The first author has shown [Pali] that the equation $\text{Circ Sym}_{3, \dots, k+1} \Theta_k^\nabla = 0$ is satisfied for $k = 4$ even in the more general case of a torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle.

We wish to point out that in this more general case it is not possible to have a statement similar to corollary 1.3, simply because there are no geodesics in M associated to the complex covariant derivative operator.

However this complex case is rather important to the applications to micro-local analysis since quite often the operators there are expressed in terms of complex covariant derivative operators. In this article we build a general formalism which allows us to give a more compact proof of the vanishing of the tensor $\text{Circ Sym}_{3, \dots, k+1} \Theta_k^\nabla$ in the case $k = 4$, and which allows us to provide also a proof in the case $k = 5$.

We have checked by using computer algebra [Pal-Sal] the above vanishing also for the cases $k = 6, 7$. It is quite likely that the vanishing is always satisfied in the torsion free complex covariant derivative case.

1.2 Statement of the main result

We provide now more notations useful for the statement of our main theorem.

Notation 1.4. We denote by $A \vDash_0 p$ any element $A \equiv (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N$ with $N \in \mathbb{Z}_{>0}$ arbitrary such that $|A| := \sum_{j=1}^N a_j = p$. We set $l_A := N$. We denote also $A! := a_1! \cdots a_N!$,

$$|A|_j := \sum_{s=1}^j a_s,$$

$$\|A\|_j := |A|_j + j,$$

for any $1 \leq j \leq N$, and we set $\|A\| := \|A\|_N + N$. For any $l = 1, \dots, N$, we define $A_l^- := (a_1, \dots, a_l)$ $A_l^+ := (a_{l+1}, \dots, a_N)$ and

$$A'_{l,h} := (A_h^-)_l^+ = (a_{l+1}, \dots, a_h),$$

for any $l, h \in \mathbb{Z}_{\geq 0}$ with $l \leq h \leq N$. In all this article, for notational simplicity, we use the identification $(v_1, \dots, v_p) \equiv (1, \dots, p)$, for a given element $(v_1, \dots, v_p) \in T_M^{\oplus p}$.

Let $R^{(a)} := \nabla^a R^\nabla$. We denote by $R^{(a)}(1, \dots, a+1, \bullet, a+2)$ the endomorphism

$$v \longmapsto R^{(a)}(1, \dots, a+1, v, a+2).$$

Given a family of endomorphisms $(T_j)_{j=1}^p$ we denote by

$$\prod_{j=1}^p T_j := T_1 \circ \cdots \circ T_p.$$

We also define the curvature monomial of multi-degree A as

$$R^A(1, \dots, \|A\| + 1) := \left[\prod_{j=1}^{l_A} R^{(a_j)}(\|A\|_{j-1} + 2, \dots, \|A\|_j + 1, \bullet, \|A\| - j + 2) \right] \cdot 1,$$

i.e. the curvature monomial R^A writes as

$$\begin{aligned} & R^A(1, \dots, \|A\| + 1) \\ &= R^{(a_1)}(2, \dots, a_1 + 2, R^{(a_2)}(a_1 + 3, \dots, a_1 + a_2 + 3, R^{(a_3)}(a_1 + a_2 + 4, \dots, a_1 + a_2 + a_3 + 4, R^{(a_4)}(\dots \\ & \dots, R^{(a_{l_A})}(\|A\|_{l_A-1} + 2, \dots, \|A\|_{l_A} + 1, 1, \|A\|_{l_A} + 2), \|A\|_{l_A} + 3), \dots), \|A\| + 1). \end{aligned}$$

We denote by $\lambda \vDash l$ any $\lambda \in \mathbb{Z}_{\geq 1}^p$ such that $|\lambda| = l$ and we define for any $A \in \mathbb{Z}_{\geq 0}^l$ the coefficient

$$\begin{aligned} \mathcal{C}_A &:= \sum_{0 \leq H \leq A} \frac{(-1)^{|H|} C(H)}{H! (A-H)!}, \quad \text{with } l_H = l, \\ C(H) &:= \sum_{\lambda \vDash l_H} (-1)^{l_\lambda} \|H_{\lambda_1}^-\| \prod_{j=1}^{l_\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1} \frac{1}{\|H'_{s,|\lambda|_j}\| (\|H'_{s,|\lambda|_j}\| + 1)}. \end{aligned}$$

With these notation we can state our main theorem.

Theorem 1.5. (*Main theorem: Explicit maximal totally real embeddings*)

Let M be a smooth manifold equipped with a torsion free covariant derivative operator ∇ acting on the smooth sections of the tangent bundle T_M , let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers, let $J \equiv J_A$ be an M -totally real almost complex structure over U , real analytic along the fibers of U and consider the fiber-wise Taylor expansion at the origin

$$T_\eta^{-1}(H^\nabla - \bar{A})_\eta \cdot \xi = i\xi + \sum_{k \geq 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a sufficiently small neighborhood of the image of the zero section, with $\xi \in T_{M, \pi(\eta)}$ arbitrary with $S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ and with $\eta^k := \eta^{\times k} \in T_{M, \pi(\eta)}^{\oplus k}$.

Then the almost complex structure J is integrable over U and for any $\eta \in U$, the complex curve $\psi_\eta : t + is \mapsto s\Phi_t^\nabla(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J -holomorphic if and only if $S_1 = 0$,

$$S_k = \frac{1}{i^{k+1} k!} \sum_{\|D\|=k} \mathbf{C}_D \text{Sym}_{2, \dots, k+1} R^D, \quad \text{with } D \geq 0,$$

for all $k \geq 2$ and the equations $\text{CircSym}_{3, \dots, k+1} \Theta_k^\nabla = 0$, with

$$\Theta_k^\nabla := -2i (id_1^\nabla)^{k-3} (\nabla R^\nabla)_2 + \sum_{r=3}^{k-1} (r+1)! \sum_{p=2}^{r-1} (id_1^\nabla)^{k-1-r} (pS_p \wedge_1 S_{r-p+1}),$$

are satisfied for all $k \geq 6$.

The main theorem is a direct consequence of corollary 1.3, combined with the following general result (as well as the vanishing for $k = 4, 5$ that we provide below).

Theorem 1.6. *Let M be a smooth manifold and let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$. Then for all $k \geq 2$, a section $S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ satisfies*

$$S_k = \frac{i}{(k+1)! k!} \text{Sym}_{2, \dots, k+1} \Theta_k^\nabla,$$

with $\Theta_2^\nabla := 2R^\nabla$ and with

$$\Theta_k^\nabla := -2i (id_1^\nabla)^{k-3} (\nabla R^\nabla)_2 + \sum_{r=3}^{k-1} (r+1)! \sum_{p=2}^{r-1} (id_1^\nabla)^{k-1-r} (pS_p \wedge_1 S_{r-p+1}),$$

for all $k \geq 3$, if and only if

$$S_k = \frac{1}{i^{k+1} k!} \sum_{\|D\|=k} \mathbf{C}_D \text{Sym}_{2, \dots, k+1} R^D, \quad \text{with } D \geq 0,$$

for all $k \geq 2$.

In the auxiliary preprint [Pal-Sal], we provide a Maple program verifying the main equivalent statement in the proof of the theorem 1.6. We warmly thank François Guenard for providing an alternative verification of the same statement in "Mathematica", that we also include there. In [Pal-Sal], we provide as well verifications of the vanishing of the integrability equations for the cases $k = 4, \dots, 7$.

2 Proof of theorem 1.6

In all the article, with the exception of the appendix, we assume ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$.

2.1 Expliciting the powers of the 1-differential. Part I

Notation 2.1. For any $A \in T_M^{*\otimes p} \otimes \text{End}_{\mathbb{C}}(\mathbb{C}T_M)$ and for any $\theta \in T_M^{*\otimes q} \otimes \mathbb{C}T_M$, the product operations of tensors $A \cdot \theta, A \lrcorner \theta \in T_M^{*\otimes(p+q)} \otimes \mathbb{C}T_M$ are defined by

$$(A \cdot \theta)(u_1, \dots, u_p, v_1, \dots, v_q) := A(u_1, \dots, u_p) \cdot \theta(v_1, \dots, v_q),$$

$$(A \lrcorner \theta)(u_1, \dots, u_p, v_1, \dots, v_q) := \sum_{j=1}^q \theta(v_1, \dots, A(u_1, \dots, u_p) \cdot v_j, \dots, v_q).$$

We denote $R^\nabla \cdot \theta := R^\nabla \cdot \theta - R^\nabla \lrcorner \theta$.

We denote by Alt_2 , the alternating operator (without normalizing coefficient!) acting on the first two entries of any tensor. We recall the following well known fact (see [Pali]).

Lemma 2.2. For any complex covariant derivative operator ∇ acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$ and for any tensor $\theta \in C^\infty(M, T_M^{*\otimes k} \otimes_{\mathbb{R}} \mathbb{C}T_M)$, holds the commutation identity

$$\text{Alt}_2 \nabla^2 \theta = R^\nabla \cdot \theta. \quad (2.1)$$

We observe now two other elementary lemmas. Their proof is left to the reader.

Lemma 2.3. For any p -tensor θ ,

$$(d_1^\nabla)^k \theta(1, \dots, k+p) = \sum_{\sigma \in S_{k+1}^*} \varepsilon_\sigma \nabla^k \theta(\sigma_1, \dots, \sigma_{k+1}, k+2, \dots, k+p),$$

where S_{k+1}^* is the set of permutations σ of the set $\{1, \dots, k+1\}$ such that if $1 \leq j < k < l \leq k+1$ then $\sigma_j < \sigma_k$ and $\sigma_j < \sigma_l$.

Lemma 2.4. For any p -tensor θ ,

$$(d_1^\nabla)^k \theta(1, \dots, k+p)$$

$$= \sum_{\substack{j_{r-1} < l_r < j_r \leq k+1 \\ j_0=0, r \geq 1}} (-1)^{\sum_{r \geq 1} (j_r - l_r)} \nabla^k \theta(1, \dots, \hat{l}_1, \dots, j_1, l_1, j_1+1, \dots,$$

$$, \dots, \hat{l}_2, \dots, j_2, l_2, j_2+1, \dots, \hat{l}_r, \dots, j_r, l_r, j_r+1, \dots, k+2, \dots, k+p).$$

The notation a, \dots, b for integers $a < b$ denotes the increasing by one sequence from the left to the right hand side. We ignore this notation when $a > b$. We start by showing the following fundamental proposition.

Proposition 2.5. *Let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of $\mathbb{C}T_M$ with curvature operator $R^\nabla(\cdot, \cdot) \cdot \equiv R^\nabla(\cdot, \cdot, \cdot)$. Then for all integers $k \geq 1$,*

$$\text{Sym}_{2, \dots, k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \text{Sym}_{2, \dots, k+4} \Phi_k, \quad (2.2)$$

$$\text{Sym}_{3, \dots, k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \text{Sym}_{3, \dots, k+4} (T_k + Q_k + V_k), \quad (2.3)$$

with

$$\Phi_k(1, \dots, k+4) := -(-1)^k \nabla^{k+1} R^\nabla(2, \dots, k+3, 1, k+4),$$

$$T_k(1, \dots, k+4) := (-1)^k \nabla^{k+1} R^\nabla(3, \dots, k+3, 1, 2, k+4),$$

$$Q_k(1, \dots, k+4) := -(-1)^k \sum_{j=2}^{k+1} \nabla^{j-2} (R^\nabla \cdot \nabla^{k+1-j} R^\nabla)(3, \dots, j, 2, j+1, \dots, k+3, 1, k+4),$$

$$V_k(1, \dots, k+4) := (-1)^k \sum_{j=2}^{k+1} \nabla^{j-2} (R^\nabla \cdot \nabla^{k+1-j} R^\nabla)(3, \dots, j, 1, j+1, \dots, k+3, 2, k+4).$$

Proof. We notice first that the symmetrization of a tensor with two alternating entries vanishes. Applying this fact to the covariant derivatives of R^∇ we infer the identities

$$\text{Sym}_{2, \dots, k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \text{Sym}_{2, \dots, k+4} \varphi_k,$$

$$\text{Sym}_{3, \dots, k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \text{Sym}_{3, \dots, k+4} (\varphi_k + \psi_k),$$

with

$$\varphi_k(1, \dots, k+4) := (-1)^k \nabla^k (\nabla R^\nabla)_2(2, \dots, k+1, 1, k+2, k+3, k+4),$$

$$\psi_k(1, \dots, k+4) := -(-1)^k \nabla^k (\nabla R^\nabla)_2(1, 3, \dots, k+1, 2, k+2, k+3, k+4),$$

which rewrite as

$$\begin{aligned} \varphi_k(1, \dots, k+4) &= (-1)^k \nabla^{k+1} R^\nabla(2, \dots, k+2, 1, k+3, k+4) \\ &= -(-1)^k \nabla^{k+1} R^\nabla(2, \dots, k+3, 1, k+4), \end{aligned}$$

$$\begin{aligned} \psi_k(1, \dots, k+4) &= -(-1)^k \nabla^{k+1} R^\nabla(1, 3, \dots, k+2, 2, k+3, k+4) \\ &= (-1)^k \nabla^{k+1} R^\nabla(1, 3, \dots, k+3, 2, k+4), \end{aligned}$$

thanks to the alternating property of R^∇ . We deduce in particular the identity (2.2). Using the identity (2.1) we infer

$$\begin{aligned}\varphi_k(1, \dots, k+4) &= -(-1)^k \nabla^{k+1} R^\nabla(3, 2, 4, \dots, k+3, 1, k+4) \\ &\quad - (-1)^k (R^\nabla \cdot \nabla^{k-1} R^\nabla)(2, \dots, k+3, 1, k+4), \\ \psi_k(1, \dots, k+4) &= (-1)^k \nabla^{k+1} R^\nabla(3, 1, 4, \dots, k+3, 2, k+4) \\ &\quad + (-1)^k (R^\nabla \cdot \nabla^{k-1} R^\nabla)(1, 3, \dots, k+3, 2, k+4).\end{aligned}$$

We now show by finite induction on p , the identities

$$\begin{aligned}\varphi_k(1, \dots, k+4) &= -(-1)^k \nabla^{k+1} R^\nabla(3, \dots, p+1, 2, p+2, \dots, k+3, 1, k+4) \\ &\quad - (-1)^k \sum_{j=2}^p \nabla^{j-2} (R^\nabla \cdot \nabla^{k+1-j} R^\nabla)(3, \dots, j, 2, j+1, \dots, k+3, 1, k+4), \\ \psi_k(1, \dots, k+4) &= (-1)^k \nabla^{k+1} R^\nabla(3, \dots, p+1, 1, p+2, \dots, k+3, 2, k+4) \\ &\quad + (-1)^k \sum_{j=2}^p \nabla^{j-2} (R^\nabla \cdot \nabla^{k+1-j} R^\nabla)(3, \dots, j, 1, j+1, \dots, k+3, 2, k+4),\end{aligned}$$

for any integer p , with $2 \leq p \leq k+1$. We assume them true for $p < k+1$. Applying the covariant derivative $\nabla_{3, \dots, p+1}^{p-1}$ to the identity (2.1), with $\theta := \nabla^{k+2-p} R^\nabla$, we infer the conclusion of the induction. If we set $p := k+1$ in the previous identities we obtain

$$\begin{aligned}(\varphi_k + \psi_k)(1, \dots, k+4) &= -(-1)^k \nabla^{k+1} R^\nabla(3, \dots, k+2, 2, k+3, 1, k+4) \\ &\quad - (-1)^k \nabla^{k+1} R^\nabla(3, \dots, k+2, 1, 2, k+3, k+4) \\ &\quad + (Q_k + V_k)(1, \dots, k+4),\end{aligned}$$

thanks to the alternating property of R^∇ . Then the identity (2.3) follows from the differential Bianchi identity. \square

2.2 Equivalent definitions of the tensor S_k

We start by noticing a few elementary equivalent definitions of the tensors S_k introduced in the statement of corollary 1.3. We assume more in general here that ∇ acts on the smooth sections of $\mathbb{C}T_M$. For notational simplicity we use the identification $\nabla^k S \equiv S^{(k)}$ for any tensor S .

Definition 2.6. *We define for all $k \geq -1$*

$$\Phi_k(1, \dots, k+4) := -(-1)^k R^{(k+1)}(2, \dots, k+3, 1, k+4).$$

Using the identity (2.2) we infer the following equivalent definition for S_k .

Lemma 2.7. *The tensors S_k defined in the statement of corollary 1.3 satisfy for all $k \geq 2$,*

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2,\dots,k+1} \theta_k,$$

$$\theta_k := 2i^k \Phi_{k-3} + k! \sum_{p=2}^{k-2} (p S_p \wedge S_{k-p}) + \sum_{r=3}^{k-2} (r+1)! \sum_{p=2}^{r-1} \rho_{k,r,p},$$

$$\rho_{k,r,p}(1, \dots, k+1) := (-i)^{k-1-r} (p S_p \wedge S_{r-p+1})^{(k-1-r)}(2, \dots, k-r, 1, k-r+1, \dots, k+1).$$

Remark 2.8. Notice that

$$\theta_4 = 2\Phi_1 + 4!2S_2 \wedge S_2,$$

and

$$\text{Sym}_{2,3,4,5}(S_2 \wedge S_2) = \text{Sym}_{2,3,4,5} \mu_4,$$

$$\mu_4(1, \dots, 5) := -S_2(2, S_2(1, 3, 4), 5).$$

This implies

$$\text{Sym}_{2,3,4,5}(S_2 \wedge S_2) = \frac{1}{18} \text{Sym}_{2,3,4,5} \tilde{\mu}_4,$$

$$\tilde{\mu}_4(1, \dots, 5) = R(2, R(1, 3, 4), 5).$$

Remark 2.9. Notice that

$$\theta_5 = 2i\Phi_2 + 5!2S_2 \wedge S_3 + 5!3S_3 \wedge S_2 + 4!\rho_{5,3,2},$$

$$\rho_{5,3,2}(1, \dots, 6) = -2i(S_2 \wedge S_2)'(2, 1, 3, 4, 5, 6),$$

and

$$\text{Sym}_{2,3,4,5,6} \theta_5 = \text{Sym}_{2,3,4,5,6} (2i\Phi_2 + \mu_5),$$

$$\begin{aligned} \mu_5(1, \dots, 6) &:= -5!2S_2(2, S_3(1, 3, 4, 5), 6) - 5!3S_3(2, S_2(1, 3, 4), 5, 6) \\ &+ 2i4!S_2'(2, 3, S_2(1, 4, 5), 6) + 2i4!S_2(3, S_2'(2, 1, 4, 5), 6). \end{aligned}$$

$$\text{Sym}_{2,3,4,5,6} \mu_5 = \text{Sym}_{2,3,4,5,6} \tilde{\mu}_5$$

$$\tilde{\mu}_5(1, \dots, 6) := -6iR(2, R'(3, 1, 4, 5), 6) - 6iR'(2, 3, R(1, 4, 5), 6).$$

Remark 2.10. We observe the equality

$$\begin{aligned} &(S_p \wedge_1 S_{r-p+1})(1, k-r+1, \dots, k+1) \\ &:= S_p(1, S_{r-p+1}(k-r+1, \dots, k-p+2), k-p+3, \dots, k+1) \\ &- S_p(k-r+1, S_{r-p+1}(1, k-r+2, \dots, k-p+2), k-p+3, \dots, k+1). \end{aligned}$$

Now, the fact that the tensors θ_k have at least one couple of alternating entries combined with the fact that the total symmetrization of such tensors vanishes implies the following equivalent definition of S_k .

Lemma 2.11. *The tensors S_k defined in the statement of corollary 1.3 satisfy for all $k \geq 2$,*

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2,\dots,k+1} \tilde{\theta}_k,$$

$$\tilde{\theta}_k := 2i^k \Phi_{k-3} + \mu_k$$

$$\mu_k := - \sum_{r=3}^{k-1} (r+1)!(-i)^{k-1-r} \sum_{p=2}^{r-1} p \sum_{I \subseteq \{2,\dots,k-r\}} \rho_{k,r,p}^I,$$

with

$$\rho_{k,r,p}^I(1, \dots, k+1)$$

$$:= S_p^{(|I|)}(I, k-r+1, S_{r-p+1}^{(|\mathbb{C}I|)}(\mathbb{C}I, 1, k-r+2, \dots, k-p+2), k-p+3, \dots, k+1).$$

(Notice that in the case $r = k - 1$ the set I is empty). Using the elementary properties of the symmetrization operators we infer the following equivalent definition of S_k .

Lemma 2.12. *The tensors S_k defined in the statement of corollary 1.3 satisfy for all $k \geq 2$,*

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2,\dots,k+1} \theta_k,$$

$$\theta_k := 2i^k \Phi_{k-3} + \mu_k,$$

$$\mu_k := \sum_{r=3}^{k-1} \sum_{p=2}^{r-1} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{I \subseteq \{2,\dots,k-r\}} \sum_{j=k-p+2}^{k+1} \rho_{k,r,p,j}^I,$$

with

$$\rho_{k,r,p,j}^I(1, \dots, k+1)$$

$$:= \theta_p^{(|I|)}(I, k-r+1, k-p+3, \dots, j, \theta_{r-p+1}^{(|\mathbb{C}I|)}(\mathbb{C}I, 1, k-r+2, \dots, k-p+2), j+1, \dots, k+1).$$

For any $A \equiv (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N$, we define the multinomial

$$\binom{|A|}{A} := \frac{|A|!}{A!}.$$

Proposition 2.13. *For any integer $k \geq 2$,*

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2,\dots,k+1} \tilde{\theta}_k,$$

with

$$\tilde{\theta}_k = \sum_{D \equiv_0 k-2l_D} C_D R^D,$$

where the coefficients C_D are given by $C_D := 2(-i)^k$ for $l_D = 1$ and by the recursive formula

$$C_D = - \sum_{\substack{1 \leq h \leq l_D - 1 \\ 0 \leq A \leq D \\ l_A = l_D}} (-i)^{|D-A|} \binom{|D-A|}{D-A} \frac{\|A\|! C_{A_h^-} C_{A_h^+}}{(\|A_h^-\| + 1)! (\|A_h^+\| + 1)!}, \quad (2.4)$$

for $l_D \geq 2$.

Proof. We assume by induction on $k \geq 3$ that

$$S_p = \frac{i}{(p+1)! p!} \text{Sym}_{2, \dots, p+1} \tilde{\theta}_p,$$

for any $p = 2, \dots, k-1$, with $\tilde{\theta}_p$ under the form

$$\tilde{\theta}_p = \sum_{A \in \mathbb{F}_0^{p-2l_A}} C_A R^A,$$

for some coefficients C_A , with obviously $\tilde{\theta}_2 = 2R$ and $C_A = 2(-i)^p$ in the case $l_A = 1$, thanks to the equivalent definition of S_p given in lemma (2.12). We show that $\tilde{\theta}_k$ writes under the form claimed in the statement of the proposition 2.13 with the coefficients C_D given by the recursive formula (2.4). For this purpose we notice first that the straightforward argument showing the equivalent definition (2.12) of S_k implies for all $k \geq 3$ the identity

$$S_k = \frac{i}{(k+1)! k!} \text{Sym}_{2, \dots, k+1} \hat{\theta}_k,$$

with

$$\hat{\theta}_k := 2i^k \Phi_{k-3} + \hat{\mu}_k,$$

$$\hat{\mu}_k := \sum_{r=3}^{k-1} \sum_{p=2}^{r-1} \frac{(r+1)! (-i)^{k-1-r}}{(r-p+2)! (p+1)!} \sum_{I \subseteq \{2, \dots, k-r\}} \sum_{j=k-p+2}^{k+1} \hat{\rho}_{k,r,p,j}^I,$$

and with

$$\begin{aligned} & \hat{\rho}_{k,r,p,j}^I(1, \dots, k+1) \\ & := \tilde{\theta}_p^{(|I|)}(I, k-r+1, k-p+3, \dots, j, \tilde{\theta}_{r-p+1}^{(|\mathbb{C}I|)}(\mathbb{C}I, 1, k-r+2, \dots, k-p+2), j+1, \dots, k+1). \end{aligned}$$

For any integer $a \geq 1$, we denote by $[a] := \{1, \dots, a\}$ and we set $[0] := \emptyset$. For any integer $b \geq 1$, we denote by $\text{Map}(a, b)$ the set of maps $f: [a] \rightarrow [b]$.

We observe now that for any $A \in \mathbb{F}_0^{p-2l_A}$ and any integer $q \geq 0$, the Leibniz identity implies

$$(R^A)^{(a)}(1, \dots, q+p+1) = \sum_{f \in \text{Map}(q, l_A)} \left[\prod_{j=1}^{l_A} R_{f,j}^{A,q} \right] \cdot (q+1),$$

with

$$R_{f,j}^{A,q} := R^{(a_j + |f^{-1}(j)|)}(f^{-1}(j), \|A\|_{j-1} + q + 2, \dots, \|A\|_j + q + 1, \bullet, p - j + q + 2).$$

We write now

$$\tilde{\theta}_p^{(|I|)} = \sum_{A \models_0 p - 2l_A} C_A (R^A)^{(|I|)},$$

and we notice that the shape of $(R^A)^{(|I|)}$ shows that $\hat{\rho}_{k,r,p,k-h+1}^I$, with $h := l_A$, are the only terms with non vanishing symmetrization of the variables $2, \dots, k+1$. Indeed we consider the factor

$$R^{(a_h + |f^{-1}(h)|)}(f^{-1}(h), p-h+1-a_h, \dots, p-h+1, 1, p-h+2), \quad (2.5)$$

with $f \in \text{Map}(I, h)$ in the expression of $(R^A)^{(|I|)}(I, 1, \dots, p+1)$ and we perform the change of variables

$$\begin{array}{ccccccc} 1 & 2 & \dots & j-k+p-1 & & j-k+p & & j-k+p & \dots & p+1 \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k-r+1 & k-p+3 & \dots & j & & \tilde{\theta}_{r-p+1}^{(|\mathbb{C}I|)}(\mathbb{C}I, 1, k-r+2, \dots, k-p+2) & & j+1 & \dots & k+1 \end{array}$$

in the factor (2.5) with $j \in \{k-p+2, \dots, k+1\}$. (We recall that we ignore the standard increasing notation a, \dots, b when $a > b$.) The only case when the symmetrization of the variables $2, \dots, k+1$ does not annihilate the factor (2.5), is when the index $j \in \{k-p+2, \dots, k+1\}$ satisfies the equality

$$j-k+p = p-h+1,$$

i.e. only when $j = k-h+1$. This shows the required statement about $\hat{\rho}_{k,r,p,k-h+1}^I$.

We infer that for all $k \geq 3$ the equality hold

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2, \dots, k+1} \check{\theta}_k,$$

with

$$\check{\theta}_k := 2i^k \Phi_{k-3} + \check{\mu}_k,$$

with

$$\check{\mu}_k := \sum_{\substack{3 \leq r \leq k-1 \\ 2 \leq p \leq r-1}} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A \models_0 p-2l_A \\ B \models_0 r-p+1-2l_B \\ I \subseteq \{2, \dots, k-r\}}} C_A C_B \check{\rho}_{k,r,p}^{I,A,B},$$

with

$$\check{\rho}_{k,r,p}^{I,A,B}(1, \dots, k+1)$$

$$:= (R^A)^{(|I|)}(I, k_r+1, k_p+3, \dots, k_{l_A}+1, (R^B)^{(|\mathbb{C}I|)}(\mathbb{C}I, 1, k_r+2, \dots, k_p+2), k_{l_A}+2, \dots, k+1),$$

and with $k_s := k-s$. Using the expression

$$R^A(x, y, \dots, p+y-1) = \left[\prod_{j=1}^{l_A} R^{(a_j)}(\|A\|_{j-1}+y, \dots, \|A\|_j+y-1, \bullet, p-j+y) \right] \cdot x,$$

for any integers $x, y > 0$ and using the change of variables

$$\begin{array}{ccccccc} k_p + 3 & \dots & k_{l_A} + 1 & & k_{l_A} + 2 & & k_{l_A} + 3 \dots k + 2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k_p + 3 & \dots & k_{l_A} + 1 & R^B(1, k_r + 2, \dots, k_p + 2) & k_{l_A} + 2 & \dots & k + 1 \end{array}$$

we infer the equality

$$\begin{aligned} & R^A(k_r + 1, k_p + 3, \dots, k_{l_A} + 1, R^B(1, k_r + 2, \dots, k_p + 2), k_{l_A} + 2, \dots, k + 1) \\ &= \left[\prod_{j=1}^{l_A-1} R^{(a_j)}(\|A\|_{j-1} + k_p + 3, \dots, \|A\|_j + k_p + 2, \bullet, k_j + 2) \right] \cdot \\ & \cdot R^{(a_{l_A})}(k_{l_A} + 2 - a_{l_A}, \dots, k_{l_A} + 1, R^B(1, k_r + 2, \dots, k_p + 2), k_r + 1, k_{l_A} + 2). \end{aligned}$$

Indeed notice that $|A| + l_A = p - l_A = k_{l_A} + 2 - (k_p + 2)$ and $p - j + (k_p + 3) - 1 = k_j + 2$. Expanding $R^B(1, k_r + 2, \dots, k_p + 2)$ we obtain

$$\begin{aligned} & R^A(k_r + 1, k_p + 3, \dots, k_{l_A} + 1, R^B(1, k_r + 2, \dots, k_p + 2), k_{l_A} + 2, \dots, k + 1) \\ &= \left[\prod_{j=1}^{l_A-1} R^{(a_j)}(\|A\|_{j-1} + k_p + 3, \dots, \|A\|_j + k_p + 2, \bullet, k_j + 2) \right] \cdot \\ & \cdot R^{(a_{l_A})}(k_{l_A} + 2 - a_{l_A}, \dots, k_{l_A} + 1, \bullet, k_r + 1, k_{l_A} + 2) \cdot \\ & \cdot \left[\prod_{t=1}^{l_B} R^{(b_t)}(\|B\|_{t-1} + k_r + 2, \dots, \|B\|_t + k_r + 1, \bullet, k_p - t + 3) \right] \cdot 1, \end{aligned}$$

since $\|B\| - t + k_r + 2 = k_p - t + 3$.

For any integer $a \geq 1$, we denote by $[a]_1 := \{2, \dots, a + 1\}$ and we set $[0]_1 := \emptyset$. For any integer $b \geq 1$, we denote by $\text{Map}([a]_1, b)$ the set of maps $f: [a]_1 \rightarrow [b]$. Using the Leibniz formula and the alternating property of R we infer the identity

$$\check{\mu}_k = \sum_{\substack{3 \leq r \leq k-1 \\ 2 \leq p \leq r-1}} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A \in \mathbb{F}_0^{p-2l_A} \\ B \in \mathbb{F}_0^{r-p+1-2l_B} \\ f \in \text{Map}([k_r-1]_1, l_A+l_B)}} C_A C_B \check{\rho}_{k,r,p}^{f,A,B}, \quad (2.6)$$

with

$$\begin{aligned} & - \check{\rho}_{k,r,p}^{f,A,B}(1, \dots, k + 1) \\ &:= \left[\prod_{j=1}^{l_A-1} R^{(a_j+|f^{-1}(j)|)}(f^{-1}(j), \|A\|_{j-1} + k_p + 3, \dots, \|A\|_j + k_p + 2, \bullet, k_j + 2) \right] \cdot \\ & \cdot R^{(a_{l_A}+|f^{-1}(l_A)|)}(f^{-1}(l_A), k_{l_A} + 2 - a_{l_A}, \dots, k_{l_A} + 1, k_r + 1, \bullet, k_{l_A} + 2) \cdot \\ & \cdot \left[\prod_{t=1}^{l_B} R^{(b_t+|f^{-1}(l_A+t)|)}(f^{-1}(l_A+t), \|B\|_{t-1} + k_r + 2, \dots, \|B\|_t + k_r + 1, \bullet, k_p - t + 3) \right] \cdot 1. \end{aligned}$$

We observe the equality

$$-\text{Sym}_{2,\dots,k+1} \check{\rho}_{k,r,p}^{f,A,B} = \text{Sym}_{2,\dots,k+1} R^D, \quad (2.7)$$

with $l_D := l_A + l_B$ and with $D \vDash_0 k - 2l_D$ given by

$$d_j := \begin{cases} a_j + |f^{-1}(j)|, & j = 1, \dots, l_A, \\ b_{j-l_A} + |f^{-1}(j)|, & j = l_A + 1, \dots, l_D. \end{cases} \quad (2.8)$$

We deduce that $\tilde{\theta}_k$ writes under the form claimed in the statement of the proposition 2.13.

We denote by abuse of notations $D - A - B \geq 0$ when $d_j - a_j \geq 0$ for all $j = 0, \dots, l_A$ and when $d_j - b_{j-l_A} \geq 0$ for all $j = l_A + 1, \dots, l_D$.

Given A, B, D as before we denote by $\text{Map}_{A,B}^D([k-r-1]_1, l_D)$ the sub-set of $\text{Map}([k-r-1]_1, l_D)$ given by the elements f which satisfy condition (2.8). We denote also by

$$M_{A,B,k,r}^D := |\text{Map}_{A,B}^D([k-r-1]_1, l_D)|.$$

Using (2.6) and (2.7), we infer the recursive formula

$$C_D = \sum_{\substack{3 \leq r \leq k-1 \\ 2 \leq p \leq r-1}} \frac{(r+1)!(-i)^{k+1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A \vDash_0 p-2l_A \\ B \vDash_0 r-p+1-2l_B \\ l_A+l_B=l_D \\ D-A-B \geq 0}} C_A C_B M_{A,B,k,r}^D,$$

We remind that in the case $l_D = 1$ hold the formula $C_D = 2(-i)^k$ for all $k \geq 2$.

Remark 2.14. Let $A \vDash_0 a$ with $l_A = q$ and consider the sub-set $\text{Map}_A(a, q)$ of $\text{Map}(a, q)$ given by the elements f such that $|f^{-1}(j)| = a_j$, for all $j = 1, \dots, q$. Then

$$|\text{Map}_A(a, q)| = \binom{a}{A}.$$

The fact that in our set-up $|D| - |A| - |B| = k - r - 1$, allows to apply the previous remark to the set $\text{Map}_{A,B}^D([k-r-1]_1, l_D)$. We infer the formula

$$M_{A,B,k,r}^D = (k-r-1)! \prod_{j=1}^{l_A} \frac{1}{(d_j - a_j)!} \prod_{j=l_A+1}^{l_D} \frac{1}{(d_j - b_{j-l_A})!}.$$

We conclude the explicit recursive formula

$$C_D = \sum_{\substack{3 \leq r \leq k-1 \\ 2 \leq p \leq r-1 \\ A \vDash_0 p-2l_A \\ B \vDash_0 r-p+1-2l_B \\ l_A+l_B=l_D \\ D-A-B \geq 0}} \frac{(-i)^{k+1-r} (k-r-1)! (r+1)! C_A C_B}{(r-p+2)!(p+1)! \prod_{j=1}^{l_A} (d_j - a_j)! \prod_{j=l_A+1}^{l_D} (d_j - b_{j-l_A})!},$$

which rewrites as (2.4). This concludes the proof of proposition 2.13. \square

Example 2.15. In this example we set for simplicity $R' := R^{(1)}$ and $R'' := R^{(2)}$. For $k = 6$ we have the expression

$$\begin{aligned}\tilde{\theta}_6(1, \dots, 7) &= -2\Phi_3(1, \dots, 7) \\ &+ C_{2,0} R''(2, 3, 4, R(5, 1, 6), 7) + C_{0,2} R(2, R''(3, 4, 5, 1, 6), 7) \\ &+ C_{1,1} R'(2, 3, R'(4, 5, 1, 6), 7) + C_{0,0,0} R(2, R(3, R(4, 1, 5), 6), 7), \\ C_{2,0} &= 10, \\ C_{0,2} &= 10, \\ C_{1,1} &= 17, \\ C_{0,0,0} &= -\frac{32}{3}.\end{aligned}$$

3 Expliciting the recursive formula for C_D

Lemma 3.1. For any integer $k \geq 3$ and any $D \models_0 k - 2l_D$ with $l_D \geq 1$,

$$C_D = -i \sum_{p \in \text{Supp } D} C_{D-1_p} - k! \sum_{h=1}^{l_D-1} \frac{C_{D_h^-} C_{D_h^+}}{(\|D_h^-\| + 1)! (\|D_h^+\| + 1)!}.$$

Proof. In the case $l_D = 1$ we have $C_{k-2} = -iC_{k-3}$, for $k \geq 3$ by definition. This coincides with what provides the formula in the lemma under consideration. We consider now the case $l_D \geq 2$ and we use the identifications $D \equiv (d_1, \dots, d_{l_D})$, $A \equiv (a_1, \dots, a_{l_D})$. Using the recursive formula (2.4) we write

$$\begin{aligned}&-i \sum_{p \in \text{Supp } D} C_{D-1_p} \\ &= - \sum_{\substack{p \in \text{Supp } D \\ 0 \leq A \leq D-1_p \\ l_A = l_D}} (-i)^{|D|-|A|} \frac{d_p - a_p}{|D| - |A|} \binom{|D| - |A|}{D - A} F_A,\end{aligned}$$

with

$$F_A := \|A\|! \sum_{h=1}^{l_D-1} \frac{C_{A_h^-} C_{A_h^+}}{(\|A_h^-\| + 1)! (\|A_h^+\| + 1)!}.$$

Let $\mathbb{I} \equiv (1, \dots, 1) \in \mathbb{Z}^{l_D}$ and let $S_D := |\text{Supp } D|$. We denote by $\mathbb{I}_D \in \mathbb{Z}^{l_D}$ the vector such that $\text{Supp } \mathbb{I}_D = \text{Supp } D$ and $0 \leq \mathbb{I}_D \leq \mathbb{I}$. Then

$$\begin{aligned}&-i \sum_{p \in \text{Supp } D} C_{D-1_p} \\ &= - \sum_{\substack{0 \leq A \leq D - \mathbb{I}_D \\ l_A = l_D}} (-i)^{|D|-|A|} \left[\sum_{\substack{1 \leq p \leq l_D \\ p \in \text{Supp } D}} \frac{d_p - a_p}{|D| - |A|} \right] \binom{|D| - |A|}{D - A} F_A\end{aligned}$$

$$- \sum_{\substack{p \in \text{Supp } D \\ D - \mathbb{I}_D \leq A \leq D - 1_p \\ l_A = l_D \\ |D| - S_D < |A|}} \frac{(-i)^{|D|-|A|}}{|D|-|A|} \binom{|D|-|A|}{D-A} F_A.$$

We notice indeed that the conditions $D - \mathbb{I}_D \leq A \leq D - 1_p$, $|D| - S_D < |A|$, when they are not empty, imply $a_p = d_p - 1$. We infer

$$\begin{aligned} & -i \sum_{p \in \text{Supp } D} C_{D-1_p} \\ = & - \sum_{\substack{0 \leq A \leq D - \mathbb{I}_D \\ l_A = l_D}} (-i)^{|D|-|A|} \binom{|D|-|A|}{D-A} F_A \\ & - \sum_{\substack{D - \mathbb{I}_D \leq A \leq D \\ l_A = l_D \\ |D| - S_D < |A| < |D|}} |\{p: a_p = d_p - 1\}| \frac{(-i)^{|D|-|A|}}{|D|-|A|} \binom{|D|-|A|}{D-A} F_A \\ = & - \sum_{\substack{0 \leq A \leq D \\ l_A = l_D \\ |A| < |D|}} (-i)^{|D|-|A|} \binom{|D|-|A|}{D-A} F_A, \end{aligned}$$

since $|\{p: a_p = d_p - 1\}| = |D| - |A|$. Then the conclusion follows from the formula (2.4). \square

Definition 3.2. Given $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$ and $A \in \mathbb{Z}_{\geq 0}^p$, $B \in \mathbb{Z}_{\geq 0}^q$, with $p, q \in \mathbb{Z}_{>0}$ we define the concatenation product

$$(t^A s^p) * (t^B s^q) := t_1^{a_1} \dots t_p^{a_p} t_{p+1}^{b_1} \dots t_{p+q}^{b_q} s^{p+q},$$

and we extend it by linearity.

Definition 3.3. Given $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$, $A \in \mathbb{Z}_{\geq 0}^p$ and $j \in \mathbb{Z}_{>0}$ we define the contraction product

$$t_j \lrcorner (t^A s^p) := t^{A+1_j} s^p,$$

where $A + 1_j := A + 1_{p,j}$, if $j \leq p$, with $1_{p,j}$ the vector of length p with vanishing components except the j -th one which has value 1 and where $A + 1_j := A$, if $j > p$. We extend the contraction product by linearity.

Lemma 3.4. Let $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$, and $x \in \mathbb{R}$. Then the function

$$u(x, t, s) := \sum_{D \geq 0} C_D t^D s^{l_D} \frac{x^{\|D\|+1}}{(\|D\|+1)!},$$

is the unique solution of the Riccati-type ODE

$$\partial_x u + \frac{u * u}{x^2} + i \left(\sum_{j \geq 1} t_j \right) \lrcorner u + s x^2 = 0,$$

with the initial condition $u(0, \cdot, \cdot) \equiv 0$.

Proof. We write first

$$\begin{aligned}
\partial_x u(x, t, s) &= \sum_{D \geq 0} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} \\
&= \sum_{\substack{D \geq 0 \\ |D|+2l_D \geq 3}} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} + C_{2,0} s \frac{x^2}{2} \\
&= \sum_{\substack{D \geq 0 \\ |D|+2l_D \geq 3}} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} - s x^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(u * u)(x, t, s) &= \left[\sum_{A \geq 0} C_A t^A s^{l_A} \frac{x^{\|A\|+1}}{(\|A\|+1)!} \right] * \left[\sum_{B \geq 0} C_B t^B s^{l_B} \frac{x^{\|B\|+1}}{(\|B\|+1)!} \right] \\
&= x^2 \sum_{A, B \geq 0} C_A C_B (t^A s^{l_A}) * (t^B s^{l_B}) \frac{x^{\|A\|+\|B\|}}{(\|A\|+1)(\|B\|+1)!} \\
&= x^2 \sum_{D \geq 0} t^D s^{l_D} \sum_{h=1}^{l_D-1} \frac{C_{D_h^-} C_{D_h^+} x^{\|D\|}}{(\|D_h^-\|+1)(\|D_h^+\|+1)!} \\
&= x^2 \sum_{\substack{D \geq 0 \\ |D|+2l_D \geq 3}} t^D s^{l_D} \sum_{h=1}^{l_D-1} \frac{C_{D_h^-} C_{D_h^+} x^{\|D\|}}{(\|D_h^-\|+1)(\|D_h^+\|+1)!},
\end{aligned}$$

and

$$\begin{aligned}
\left[\left(\sum_{j \geq 1} t_j \right) \neg u \right] (x, t, s) &= \sum_{j \geq 1} (t_j \neg u)(x, t, s) \\
&= \sum_{j \geq 1} \sum_{D \geq 0} C_D t^{D+1_j} s^{l_D} \frac{x^{\|D\|+1}}{(\|D\|+1)!} \\
&= \sum_{D \geq 0} \sum_{j=1}^{l_D} C_D t^{D+1_j} s^{l_D} \frac{x^{\|D\|+1}}{(\|D\|+1)!} \\
&= \sum_{D \geq 0} t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} \left(\sum_{p \in \text{Supp } D} C_{D-1_p} \right) \\
&= \sum_{\substack{D \geq 0 \\ |D|+2l_D \geq 3}} t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} \left(\sum_{p \in \text{Supp } D} C_{D-1_p} \right).
\end{aligned}$$

Then lemma (3.1) implies that the function u satisfies the singular Riccati's ODE. The uniqueness statement follows from the following corollary. \square

Corollary 3.5. Let $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$, and $x \in \mathbb{R}$ then the function

$$u(x, t, s) = \sum_{l \geq 1} u_l(x, t_1, \dots, t_l) s^l,$$

$$u_l(x, t_1, \dots, t_l) := \sum_{\substack{D \geq 0 \\ l_D = l}} C_D t^D \frac{x^{\|D\|+1}}{(\|D\|+1)!},$$

satisfies the recursive system of linear ODE's

$$\begin{cases} \partial_x u_1(x, t_1) + i t_1 u_1(x, t_1) + x^2 = 0, \\ \partial_x u_l(x, t_1, \dots, t_l) + i u_l(x, t_1, \dots, t_l) \sum_{j=1}^l t_j + \frac{1}{x^2} \sum_{p=1}^{l-1} u_p(x, t_1, \dots, t_p) u_{l-p}(x, t_{p+1}, \dots, t_l) = 0, \\ \forall l \geq 2. \end{cases} \quad (3.1)$$

with initial conditions $u_l(0, t_1, \dots, t_l) = 0$, for all $l \geq 1$ and all $(t_1, \dots, t_l) \in \mathbb{R}^l$.

Proof. According to the proof of lemma 3.4 we can write

$$\begin{aligned} & (u * u)(x, t, s) \\ &= x^2 \sum_{A, B \geq 0} C_A C_B (t^A s^{l_A}) * (t^B s^{l_B}) \frac{x^{\|A\|+\|B\|}}{(\|A\|+1)! (\|B\|+1)!} \\ &= x^2 \sum_{l \geq 2} s^l \sum_{p=1}^{l-1} \sum_{\substack{A \geq 0 \\ l_A = p}} \sum_{\substack{B \geq 0 \\ l_B = l-p}} \frac{C_A C_B t_1^{a_1} \dots t_p^{a_p} t_{p+1}^{b_1} \dots t_l^{b_{l-p}} x^{\|A\|+\|B\|}}{(\|A\|+1)! (\|B\|+1)!} \\ &= \sum_{l \geq 2} s^l \sum_{p=1}^{l-1} \left[\sum_{\substack{A \geq 0 \\ l_A = p}} \frac{C_A t_1^{a_1} \dots t_p^{a_p} x^{\|A\|+1}}{(\|A\|+1)!} \right] \left[\sum_{\substack{B \geq 0 \\ l_B = l-p}} \frac{C_B t_{p+1}^{b_1} \dots t_l^{b_{l-p}} x^{\|B\|+1}}{(\|B\|+1)!} \right] \\ &= \sum_{l \geq 2} s^l \sum_{p=1}^{l-1} u_p(x, t_1, \dots, t_p) u_{l-p}(x, t_{p+1}, \dots, t_l). \end{aligned}$$

On the other hand still according to the proof of lemma 3.4 we can write

$$\begin{aligned} \left[\left(\sum_{j \geq 1} t_j \right)^{-u} \right] (x, t, s) &= \sum_{D \geq 0} \sum_{j=1}^{l_D} C_D t^{D+1_j} s^{l_D} \frac{x^{\|D\|+1}}{(\|D\|+1)!} \\ &= \sum_{l \geq 1} s^l \sum_{\substack{D \geq 0 \\ l_D = l}} \sum_{j=1}^l C_D t^D t_j \frac{x^{\|D\|+1}}{(\|D\|+1)!} \\ &= \sum_{l \geq 1} s^l \left(\sum_{j=1}^l t_j \right) u_l(x, t_1, \dots, t_l). \end{aligned}$$

The required conclusion follows from the Riccati-type equation in the statement of lemma 3.4. \square

In general we introduce the concatenation product for functions of the type

$$u(x, t, s) = \sum_{l \geq 1} u_l(x, t_1, \dots, t_l) s^l,$$

$$v(x, t, s) = \sum_{l \geq 1} v_l(x, t_1, \dots, t_l) s^l,$$

as

$$(u*v)(x, t, s) = \sum_{l \geq 2} s^l \sum_{p=1}^{l-1} u_p(x, t_1, \dots, t_p) v_{l-p}(x, t_{p+1}, \dots, t_l).$$

The concatenation product is associative.

Lemma 3.6. *The solution u of the system (3.1) is equivalent to the solution*

$$v(x, t, s) = \sum_{l \geq 1} v_l(x, t_1, \dots, t_l) s^l,$$

of the Riccati-type ODE

$$\partial_x v - \frac{i}{x^2} v * v + i x^2 e^{i x t_1} s = 0, \quad (3.2)$$

via the identification

$$u_l(x, t_1, \dots, t_l) = -i e^{-i x \sum_{j=1}^l t_j} v_l(x, t_1, \dots, t_l).$$

Proof. We write

$$\begin{aligned} \partial_x u_l(x, t_1, \dots, t_l) &= -i e^{-i x \sum_{j=1}^l t_j} \partial_x v_l(x, t_1, \dots, t_l) \\ &\quad - e^{-i x \sum_{j=1}^l t_j} v_l(x, t_1, \dots, t_l) \sum_{j=1}^l t_j, \\ i u_l(x, t_1, \dots, t_l) \sum_{j=1}^l t_j &= e^{-i x \sum_{j=1}^l t_j} v_l(x, t_1, \dots, t_l) \sum_{j=1}^l t_j, \end{aligned}$$

and

$$\begin{aligned} &\sum_{p=1}^{l-1} u_p(x, t_1, \dots, t_p) u_{l-p}(x, t_{p+1}, \dots, t_l) \\ &= -e^{-i x \sum_{j=1}^l t_j} \sum_{p=1}^{l-1} v_p(x, t_1, \dots, t_p) v_{l-p}(x, t_{p+1}, \dots, t_l). \end{aligned}$$

We infer that the system (3.1) rewrites as

$$\begin{cases} \partial_x v_1(x, t_1) + i x^2 e^{i x t_1} = 0, \\ \partial_x v_l(x, t_1, \dots, t_l) - \frac{i}{x^2} \sum_{p=1}^{l-1} v_p(x, t_1, \dots, t_p) v_{l-p}(x, t_{p+1}, \dots, t_l) = 0, \\ \forall l \geq 2, \end{cases} \quad (3.3)$$

which is equivalent to the equation (3.2). \square

3.1 General facts about Riccati-type equations

In this subsection we denote for notational simplicity $\dot{y}(x, t, s) \equiv \partial_x y(x, t, s)$.

Lemma 3.7. *A solution*

$$y(x, t, s) = \sum_{l \geq 1} y(x, t_1, \dots, t_l) s^l,$$

of the Riccati-type equation

$$\dot{y} = a_2(x) y * y + a_1(x) y + a_0(x, t, s),$$

with a_2 non zero and once differentiable and with

$$a_0(x, t, s) = \sum_{l \geq 1} a_{0,l}(x, t_1, \dots, t_l) s^l,$$

is equivalent to a solution

$$u(x, t, s) = \sum_{l \geq 1} u(x, t_1, \dots, t_l) s^l,$$

of the linear second order ODE

$$\ddot{u} - R(x) \dot{u} + H * u = 0,$$

$$R := a_1 + \frac{\dot{a}_2}{a_2},$$

$$H := a_2 a_0,$$

via the identification $\dot{u} + a_2 y * u = 0$, with $u_1(x, t_1) \equiv f(t_1)$, for some non vanishing function and

$$y_l(x, t_1, \dots, t_l) = \sum_{\lambda \models l} (-1)^{l_\lambda} \frac{\dot{u}_{\lambda_1+1}(x, t_1, \dots, t_{\lambda_1+1})}{a_2(x) f(t_{\lambda_1+1})} \prod_{j=2}^{l_\lambda} \frac{u_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{f(t_{|\lambda|_j+1})}, \quad (3.4)$$

for all $l \geq 1$, with $|\lambda|_j := \sum_{p=1}^j \lambda_p$.

Proof. The function $v := a_2 y$ satisfies Riccati-type equation

$$\dot{v} = v * v + R(x) v + H(x, t, s).$$

Indeed

$$\begin{aligned}
\dot{v} &= \dot{a}_2 y + a_2 \dot{y} \\
&= \frac{\dot{a}_2}{a_2} v + a_2 [a_0 + a_1 y + a_2 y * y] \\
&= a_0 a_2 + \left(a_1 + \frac{\dot{a}_2}{a_2} \right) v + v * v.
\end{aligned}$$

We now show that if $\dot{u} + v * u = 0$ then the function u satisfies the second order linear ODE in the statement. Indeed

$$\begin{aligned}
\ddot{u} &= -\dot{v} * u - v * \dot{u} \\
&= -\dot{v} * u + v * (v * u) \\
&= (-\dot{v} + v * v) * u \\
&= -[R(x)v + H] * u \\
&= R(x)\dot{u} - H * u.
\end{aligned}$$

We notice now that the identification $\dot{u} + v * u = 0$ implies $\dot{u}_1 \equiv 0$, i.e. $u_1(x, t_1) \equiv f(t_1)$ and

$$y_l(x, t_1, \dots, t_l) = -\frac{\dot{u}_{l+1}(x, t_1, \dots, t_{l+1})}{a_2(x)f(t_{l+1})} - \sum_{p=1}^{l-1} y_p(x, t_1, \dots, t_p) \frac{u_{l+1-p}(x, t_{p+1}, \dots, t_{l+1})}{f(t_{l+1})}, \quad (3.5)$$

for all $l \geq 1$, which implies the explicit formula (3.4). Indeed if we set

$$\begin{aligned}
A(x, t, s) &:= \sum_{l \geq 1} A_l(x, t_1, \dots, t_l) s^l, \\
B(x, t, s) &:= \sum_{l \geq 1} B_l(x, t_1, \dots, t_l) s^l, \\
A_l(x, t_1, \dots, t_l) &:= \frac{u_{l+1}(x, t_1, \dots, t_{l+1})}{f(t_{l+1})}, \\
B_l(x, t_1, \dots, t_l) &:= \frac{\dot{u}_{l+1}(x, t_1, \dots, t_{l+1})}{a_2(x)f(t_{l+1})},
\end{aligned}$$

(notice that the definitions are well posed) then the relation (3.5) writes as $y = -B - y * A$. Thus

$$y = -B * \sum_{k \geq 0} (-1)^k A^{*,k},$$

where we denote by $A^{*,k}$ the k -th power with respect to the concatenation product $*$. (By convention $A^{*,0} = 1$.) In other terms

$$y_l(x, t_1, \dots, t_l) = \sum_{\lambda=l} (-1)^{l_\lambda} B_{\lambda_1}(x, t_1, \dots, t_{\lambda_1}) \prod_{j=2}^{l_\lambda} A_{\lambda_j}(x, t_{|\lambda|_j-1+1}, \dots, t_{|\lambda|_j}),$$

which is precisely formula (3.4). \square

Notice that y_1 is given directly by the solution of the ODE $\dot{y}_1(x, t_1) = a_1(x)y(x, t_1) + a_{0,1}(x, t_1)$.

In this paper we will always consider the case $u(0, t, s) = t_1 s$. In the Riccati-type equation (3.2) we have $a_2(x) = ix^{-2}$, $a_1 = 0$, $a_0(x, t, s) = -ix^2 e^{ixt_1} s$. Therefore the corresponding second order linear ODE writes as

$$\ddot{U} + \frac{2}{x} \dot{U} + (e^{ixt_1} s) * U = 0. \quad (3.6)$$

with $U(0, t, s) = t_1 s$ and $\dot{U}(0, t, s) = 0$. Then according to formula (3.4) we infer the expression

$$v_l(x, t_1, \dots, t_l) = -ix^2 \sum_{\lambda=l} (-1)^{l_\lambda} \frac{\dot{U}_{\lambda+1}(x, t_1, \dots, t_{\lambda+1})}{t_{\lambda+1}} \prod_{j=2}^{l_\lambda} \frac{U_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}}, \quad (3.7)$$

for all $l \geq 1$, for the solution v of the equation (3.2).

In order to compute the solution U we need an other elementary lemma.

Lemma 3.8. *In the set up of the previous lemma, any ODE*

$$\ddot{u} + a_1(x)\dot{u} + a_0 * u = f(x, t, s),$$

is equivalent to the ODE

$$\ddot{w} - \frac{1}{4}(2\dot{a}_1 + a_1^2 - 4a_0) * w = e^{\frac{1}{2} \int a_1} f,$$

via the identification

$$u = e^{-\frac{1}{2} \int a_1} w.$$

Proof. We write $u = A(x)w$ and we observe the elementary equalities $\dot{u} = \dot{A}w + A\dot{w}$,

$$\ddot{u} = \ddot{A}w + 2\dot{A}\dot{w} + A\ddot{w}.$$

Then the ODE on u in the statement is equivalent to the ODE

$$A\ddot{w} + 2\dot{A}\dot{w} + \ddot{A}w + a_1A\dot{w} + a_1\dot{A}w + Aa_0 * w = f,$$

i.e.

$$A\ddot{w} + (2\dot{A} + a_1A)\dot{w} + (\ddot{A} + a_1\dot{A} + Aa_0) * w = f.$$

We seek for A solution of the ODE $2\dot{A} + a_1A = 0$, i.e. $A = e^{-\frac{1}{2} \int a_1}$. We infer in particular the identity

$$\ddot{A} + \frac{1}{2}\dot{a}_1A + \frac{1}{2}a_1\dot{A} = 0.$$

Using this we write

$$\begin{aligned} \ddot{A} + a_1\dot{A} + Aa_0 &= -\frac{1}{2}\dot{a}_1A + \frac{1}{2}a_1\dot{A} + Aa_0 \\ &= -\frac{A}{4}(2\dot{a}_1 + a_1^2 - 4a_0), \end{aligned}$$

which implies the required conclusion. \square

We infer that if we set $U = x^{-1}w$, then the equation (3.6) is equivalent to the equation

$$\ddot{w} + (e^{ixt_1} s) * w = 0, \quad (3.8)$$

with the initial conditions $w(0, t, s) \equiv 0$ and $\dot{w}(0, t, s) = u(0, t, s) = t_1 s$. (The later condition follows deriving the equality $w = xU$). Then formula (3.7) implies the expression

$$\begin{aligned} v_l(x, t_1, \dots, t_l) &= -ix \sum_{\lambda=l} (-1)^{l_\lambda} \frac{\dot{w}_{\lambda_1+1}(x, t_1, \dots, t_{\lambda_1+1})}{t_{\lambda_1+1}} \prod_{j=2}^{l_\lambda} \frac{w_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{x t_{|\lambda|_j+1}} \\ &+ ix \sum_{\lambda=l} (-1)^{l_\lambda} \prod_{j=1}^{l_\lambda} \frac{w_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{x t_{|\lambda|_j+1}}, \end{aligned}$$

for all $l \geq 1$, for the solution v of the equation (3.2).

3.2 Expression for the solution of the equation (3.8)

We notice first the equalities $w(0, t, s) = \ddot{w}(0, t, s) = 0$, $\dot{w}_1(0, t_1) = t_1$, $\ddot{w}_1 \equiv 0$. We set

$$w_{p,l}(t_1, \dots, t_l) := \frac{1}{p!} w_l^{(p)}(0, t_1, \dots, t_l).$$

Differentiating the equation (3.8) with respect to the variable x and evaluating at $x = 0$ we infer for all $l \geq 2$

$$w_{k+2,l}(t_1, \dots, t_l) = -\frac{1}{(k+2)(k+1)} \sum_{r=0}^{k-1} \frac{i^r}{r!} t_1^r w_{k-r,l-1}(t_2, \dots, t_l),$$

which we rewrite as

$$\begin{aligned} w_{p,l}(t_1, \dots, t_l) &= -\frac{1}{p(p-1)} \sum_{r=0}^{p-3} \frac{i^r}{r!} t_1^r w_{p-r-2,l-1}(t_2, \dots, t_l) \\ &= \frac{1}{p(p-1)} \sum_{r=2}^{p-1} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,l-1}(t_2, \dots, t_l). \end{aligned}$$

We deduce

$$w_{p,2}(t_1, t_2) = i^{p-1} \frac{p-2}{p!} t_1^{p-3} t_2,$$

for all $p \geq 3$ and zero otherwise. We deduce also

$$\begin{aligned} w_{p,3}(t_1, t_2, t_3) &= \frac{1}{p(p-1)} \sum_{r=2}^{p-1} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,2}(t_2, t_3) \\ &= \frac{1}{p(p-1)} \sum_{r=2}^{p-3} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,2}(t_2, t_3), \end{aligned}$$

since $w_{p-r,2}=0$ for $p-r \geq 3$. Thus

$$w_{p,3}(t_1, t_2, t_3) = \frac{i^{p-1}}{p(p-1)} \sum_{r=2}^{p-3} (p-r-2) \frac{t_1^{r-2} t_2^{p-r-3}}{(r-2)!(p-r)!} t_3.$$

We compute now

$$\begin{aligned} w_{p,4}(t_1, \dots, t_4) &= \frac{1}{p(p-1)} \sum_{r=2}^{p-1} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,3}(t_2, \dots, t_4) \\ &= \frac{1}{p(p-1)} \sum_{r=2}^{p-5} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,3}(t_2, \dots, t_4), \end{aligned}$$

since $w_{p-r,3}=0$ for $p-r \geq 5$. Thus

$$w_{p,4}(t_1, \dots, t_4) = \frac{i^{p-1}}{p(p-1)} \sum_{r_1=2}^{p-5} \sum_{r_2=2}^{p-r_1-3} \frac{(p-r_1-r_2-2) t_1^{r_1-2} t_2^{r_2-2} t_3^{p-r_1-r_2-3}}{(p-r_1)(p-r_1-1)(r_1-2)!(r_2-2)!(p-r_1-r_2)!} t_4.$$

We compute now

$$\begin{aligned} w_{p,5}(t_1, \dots, t_5) &= \frac{1}{p(p-1)} \sum_{r=2}^{p-1} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,4}(t_2, \dots, t_5) \\ &= \frac{1}{p(p-1)} \sum_{r=2}^{p-7} \frac{i^r}{(r-2)!} t_1^{r-2} w_{p-r,4}(t_2, \dots, t_5), \end{aligned}$$

since $w_{p-r,4}=0$ for $p-r \geq 7$. Thus

$$\begin{aligned} &w_{p,5}(t_1, \dots, t_5) \\ &= \frac{i^{p-1}}{p(p-1)} \sum_{r_1=2}^{p-7} \sum_{r_2=2}^{p-r_1-5} \sum_{r_3=2}^{p-r_1-r_2-3} \frac{p-r_1-r_2-r_3-2}{(p-r_1)(p-r_1-1)(p-r_1-r_2)(p-r_1-r_2-1)} \times \\ &\times \frac{t_1^{r_1-2} t_2^{r_2-2} t_3^{r_3-2} t_4^{p-r_1-r_2-r_3-3}}{(r_1-2)!(r_2-2)!(r_3-2)!(p-r_1-r_2-r_3)!} t_5. \end{aligned}$$

We deduce the general formula

$$\begin{aligned} w_{p,l}(t_1, \dots, t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{\substack{r_j=2 \\ 1 \leq j \leq l-2}}^{p-|r|_{j-1}-2(l-j)+1} (p-|r|_{l-2}-2) \times \\ &\times \left[\prod_{j=1}^{l-3} \frac{1}{(p-|r|_j)(p-|r|_j-1)} \right] \left[\prod_{j=1}^{l-2} \frac{t_j^{r_j-2}}{(r_j-2)!} \right] \frac{t_{l-1}^{p-|r|_{l-2}-3}}{(p-|r|_{l-2})!} t_l, \end{aligned}$$

for all $l \geq 3$. Performing the change of variables $r'_j = r_j - 2$ we infer

$$\begin{aligned} w_{p,l}(t_1, \dots, t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{\substack{r_j=0 \\ 1 \leq j \leq l-2}}^{p-|r|_{j-1}-2l+1} (p-|r|_{l-2}-2l+2) \times \\ &\times \left[\prod_{j=1}^{l-3} \frac{1}{(p-|r|_j-2j)(p-|r|_j-2j-1)} \right] \left[\prod_{j=1}^{l-2} \frac{t_j^{r_j}}{r_j!} \right] \frac{t_{l-1}^{p-|r|_{l-2}-2l+1}}{(p-|r|_{l-2}-2l+4)!} t_l. \end{aligned}$$

If we set $r_{l-1} := p - |r|_{l-2} - 2l + 1$, then we can rewrite the previous sum as

$$\begin{aligned}
w_{p,l}(t_1, \dots, t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{r \models_0^{l-1} p-2l+1} (r_{l-1} + 1) \times \\
&\times \left[\prod_{j=1}^{l-3} \frac{1}{(p - |r|_j - 2j)(p - |r|_j - 2j - 1)} \right] \left[\prod_{j=1}^{l-2} \frac{t_j^{r_j}}{r_j!} \right] \frac{t_{l-1}^{r_{l-1}}}{(r_{l-1} + 3)!} t_l \\
&= \frac{i^{p-1}}{p(p-1)} \sum_{r \models_0^{l-1} p-2l+1} \left[\prod_{j=1}^{l-2} \frac{1}{(p - |r|_j - 2j)(p - |r|_j - 2j - 1)} \right] \left[\prod_{j=1}^{l-1} \frac{t_j^{r_j}}{r_j!} \right] t_l,
\end{aligned}$$

where $r \models_0^{l-1} p - 2l + 1$ denotes the compositions of length $l - 1$ and $|r| = p - 2l + 1$.

We notice that the previous formula hold also for $l = 2$. In conclusion

$$\begin{aligned}
w_1(x, t_1) &= x t_1, \\
w_l(x, t_1, \dots, t_l) &= \sum_{p \geq 2l-1} w_{p,l}(t_1, \dots, t_l) x^p,
\end{aligned}$$

for all $l \geq 2$.

3.3 The explicit expression of the coefficients C_D

By the result of the previous subsection, we infer the expression

$$\frac{w_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} = \sum_{p \geq 2\lambda_j+1} \frac{w_{p,\lambda_j+1}(t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} x^p,$$

with

$$\begin{aligned}
&\frac{w_{p,\lambda_j+1}(t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} \\
&= \frac{i^{p-1}}{p(p-1)} \sum_{r \models_0^{\lambda_j} p-2\lambda_j-1} \left[\prod_{s=1}^{\lambda_j-1} \frac{1}{(p - |r|_s - 2s)(p - |r|_s - 2s - 1)} \right] \left[\prod_{s=1}^{\lambda_j} \frac{t_{|\lambda|_{j-1}+s}^{r_s}}{r_s!} \right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{w_{\lambda_j+1}(x, t_{|\lambda|_{j-1}+1}, \dots, t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} \\
&= \sum_{p \geq 2\lambda_j+1} \frac{i^{p-1} x^p}{p(p-1)} \sum_{r \models_0^{\lambda_j} p-2\lambda_j-1} \left[\prod_{s=1}^{\lambda_j-1} \frac{1}{(p - |r|_s - 2s)(p - |r|_s - 2s - 1)} \right] \left[\prod_{s=1}^{\lambda_j} \frac{t_{|\lambda|_{j-1}+s}^{r_s}}{r_s!} \right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\dot{w}_{\lambda_1+1}(x, t_1, \dots, t_{\lambda_1+1})}{t_{\lambda_1+1}} \\
&= \frac{1}{x} \sum_{p \geq 2\lambda_1+1} \frac{i^{p-1} x^p}{(p-1)} \sum_{r \models_0^{\lambda_1} p-2\lambda_1-1} \left[\prod_{s=1}^{\lambda_1-1} \frac{1}{(p - |r|_s - 2s)(p - |r|_s - 2s - 1)} \right] \left[\prod_{s=1}^{\lambda_1} \frac{t_s^{r_s}}{r_s!} \right].
\end{aligned}$$

We infer in conclusion

$$\begin{aligned}
& v_l(x, t_1, \dots, t_l) \\
&= -i \sum_{\lambda=l} \frac{(-1)^{l_\lambda}}{x^{l_\lambda-1}} \sum_{\substack{p_j \geq 2\lambda_j+1 \\ r_j = \sum_{s=0}^{\lambda_j} p_j - 2\lambda_j - 1 \\ j=1, \dots, l_\lambda}} (p_1 - 1) \prod_{j=1}^{l_\lambda} \frac{i^{p_j-1} x^{p_j}}{p_j(p_j-1)} \times \\
&\times \left[\prod_{j=1}^{l_\lambda} \prod_{s_j=1}^{\lambda_j-1} \frac{1}{(p_j - |r_j|_{s_j} - 2s_j)(p_j - |r_j|_{s_j} - 2s_j - 1)} \right] \left[\prod_{j=1}^{l_\lambda} \prod_{s_j=1}^{\lambda_j} \frac{t^{r_j, s_j}}{r_j, s_j!} \right],
\end{aligned}$$

for all $l \geq 1$. Thus

$$\begin{aligned}
& v_l(x, t_1, \dots, t_l) \\
&= (-1)^{l+1} i \sum_{H \in \mathbb{Z}_{\geq 0}^l} \sum_{\lambda=l} (-1)^{l_\lambda} (|H|_{\lambda_1} + 2\lambda_1) i^{|H|} x^{|H|+2l+1} \times \\
&\times \left[\prod_{j=1}^{l_\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1} \frac{1}{[\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s) + 1][\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s)]} \right] \frac{t^H}{H!},
\end{aligned}$$

for all $l \geq 1$, which we rewrite as

$$\begin{aligned}
& v_l(x, t_1, \dots, t_l) \\
&= (-1)^{l+1} i \sum_{H \in \mathbb{Z}_{\geq 0}^l} i^{|H|} x^{|H|+2l+1} \frac{t^H}{H!} \sum_{\lambda=l} (-1)^{l_\lambda} (|H|_{\lambda_1} + 2\lambda_1) \times \\
&\times \left[\prod_{j=1}^{l_\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1} \frac{1}{[\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s) + 1][\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s)]} \right].
\end{aligned}$$

We now recall that thanks to lemma 3.6 the solution u of the Riccati-type equation in lemma 3.4 is given by

$$u_l(x, t_1, \dots, t_l) = -i e^{-ix \sum_{j=1}^l t_j} v_l(x, t_1, \dots, t_l).$$

Then using the expansion

$$e^{-ix \sum_{j=1}^l t_j} = \sum_{H \in \mathbb{Z}_{\geq 0}^l} (-i)^{|H|} x^{|H|} \frac{t^H}{H!},$$

we deduce the formula

$$\begin{aligned}
& u_l(x, t_1, \dots, t_l) \\
&= (-1)^{l+1} \sum_{D \in \mathbb{Z}_{\geq 0}^l} (-i)^{|D|} x^{|D|+2l+1} t^D \sum_{0 \leq H \leq D} \frac{(-1)^{|H|}}{H! (D-H)!} \sum_{\lambda=l} (-1)^{l_\lambda} (|H|_{\lambda_1} + 2\lambda_1) \times \\
&\times \left[\prod_{j=1}^{l_\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1} \frac{1}{[\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s) + 1][\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s)]} \right],
\end{aligned}$$

for all $l \geq 1$. We conclude the explicit expression

$$C_D = (-1)^{l+1}(-i)^{|D|} (|D| + 2l + 1)! \sum_{0 \leq H \leq D} \frac{(-1)^{|H|}}{H! (D-H)!} \sum_{\lambda=H} (-1)^{l\lambda} (|H|_{\lambda_1} + 2\lambda_1) \times \\ \times \left[\prod_{j=1}^{l\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1} \frac{1}{[\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s) + 1][\sum_{r=s+1}^{|\lambda|_j} h_r + 2(|\lambda|_j - s)]} \right],$$

for all $D \in \mathbb{Z}_{\geq 0}^l$. We write now $C_D = -(-i)^{\|D\|} (\|D\| + 1)! C_D$. We obtain the expression of the coefficients C_D in the statement of theorem 1.6.

4 Expliciting the integrability equations

In this section we provide some basic tools for a general expression of the integrability equations

$$\text{CircSym}_{3,\dots,k+1} \Theta_k^\nabla = 0,$$

for $k \geq 4$. We notice in particular that the equation $\text{CircSym}_{3,4,5} \Theta_4^\nabla = 0$ writes as

$$\text{CircSym}_{3,4,5} [3d_1^\nabla (\nabla R^\nabla)_2 - 2\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla] = 0,$$

with $\tilde{R}^\nabla := \text{Sym}_{2,3} R^\nabla$. Its vanishing has been proved in [Pali] by using a direct computation.

4.1 Expliciting the powers of the 1-differential. Part II

From now on let $\rho_{k,j} := (R \cdot R^{(k+1-j)})^{(j-2)}$ and notice that

$$\rho_{k,j} \in C^\infty(M, T_M^{*, \otimes(j-2)} \otimes_{\mathbb{R}} \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^{*, \otimes(k+1-j)} \otimes_{\mathbb{R}} \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

satisfies the circular identity with respect to its last three entries and in the case $j \leq k$ the tensor $\rho_{k,j} \cdot v$ satisfies the circular identity with respect to its last three entries for any vector $v \in T_M$. We remind the identity

$$R^{(k+1)}(2, \dots, k+3, 1, k+4) = R^{(k+1)}(3, \dots, k+2, 2, k+3, 1, k+4) \\ + \sum_{j=2}^{k+1} \rho_{k,j}(3, \dots, j, 2, j+1, \dots, k+3, 1, k+4),$$

obtained in the proof of the identity (2.3) in proposition 2.5. We rewrite it in the more general form

$$R^{(c-b+1)}(1, b, \dots, c, 2, d) = R^{(c-b+1)}(b, \dots, c-1, 1, c, 2, d) \\ + \sum_{h=b-1}^{c-2} \rho_{c-b, h-b+3}(b, \dots, h, 1, h+1, \dots, c, 2, d),$$

for any $3 \leq b < c$. Differentiating we obtain the general commutation formula

$$R^{(c-a+1)}(a, \dots, b-1, 1, b, \dots, c, 2, d) = R^{(c-a+1)}(a, \dots, c-1, 1, c, 2, d) \\ + \sum_{h=b-1}^{c-2} \rho_{c-a, h-a+3}(a, \dots, h, 1, h+1, \dots, c, 2, d), \quad (4.1)$$

for all $3 \leq a \leq b < c < d$. We now show two key elementary lemmas.

Lemma 4.1. *In the set up of proposition 2.5, for all integers $k \geq 1$,*

$$\text{Circ Sym}_{3,\dots,k+4} T_k = \text{Circ Sym}_{4,\dots,k+4} \hat{T}_k,$$

with

$$\hat{T}_k(1, \dots, k+4) := (-1)^k \sum_{j=2}^{k+1} (j-1) \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4).$$

Proof. We notice first that for any tensor $\theta \equiv \theta(1, \dots, k+4)$, the tensor

$$\tilde{\theta}(1, \dots, k+4) := \sum_{j=3}^{k+4} \theta(1, 2, 4, \dots, j, 3, j+1, \dots, k+4),$$

satisfies the identity

$$\text{Sym}_{3,\dots,k+4} \theta = \text{Sym}_{4,\dots,k+4} \tilde{\theta}.$$

Applying the previous definition to our tensor T_k we obtain the expression

$$\begin{aligned} \tilde{T}_k(1, \dots, k+4) &= (-1)^k \sum_{j=3}^{k+3} R^{(k+1)}(4, \dots, j, 3, j+1, \dots, k+3, 1, 2, k+4) \\ &\quad + (-1)^k R^{(k+1)}(4, \dots, k+4, 1, 2, 3). \end{aligned}$$

Using the commutation identity (4.1) we infer

$$\begin{aligned} &R^{(k+1)}(4, \dots, j, 3, j+1, \dots, k+3, 1, 2, k+4) \\ &= R^{(k+1)}(4, \dots, k+3, 3, 1, 2, k+4) + \sum_{h=j}^{k+2} \rho_{k,h-3}(4, \dots, h, 3, h+1, \dots, k+3, 1, 2, k+4). \end{aligned}$$

We infer the equality

$$\begin{aligned} &\tilde{T}_k(1, \dots, k+4) \\ &= (-1)^k (k+1) R^{(k+1)}(4, \dots, k+3, 3, 1, 2, k+4) + (-1)^k R^{(k+1)}(4, \dots, k+4, 1, 2, 3) \\ &\quad + (-1)^k \sum_{j=3}^{k+2} \sum_{h=j}^{k+2} \rho_{k,h-3}(4, \dots, h, 3, h+1, \dots, k+3, 1, 2, k+4). \end{aligned}$$

Using the elementary equality

$$\sum_{s=a}^b \sum_{h=s}^b C_h = \sum_{h=a}^b (h-a+1) C_h,$$

we obtain the identity

$$\begin{aligned} &\tilde{T}_k(1, \dots, k+4) \\ &= (-1)^k (k+1) R^{(k+1)}(4, \dots, k+3, 3, 1, 2, k+4) + (-1)^k R^{(k+1)}(4, \dots, k+4, 1, 2, 3) \\ &\quad + (-1)^k \sum_{h=3}^{k+2} (h-2) \rho_{k,h-3}(4, \dots, h, 3, h+1, \dots, k+3, 1, 2, k+4). \end{aligned}$$

Using the algebraic and differential Bianchi identities and performing the change of variable $h = j + 1$, we infer the required conclusion. \square

Lemma 4.2. *In the set up of proposition 2.5, for all integers $k \geq 1$,*

$$\text{Circ Sym}_{3, \dots, k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \sum_{j=2}^{k+1} \text{Circ Sym}_{4, \dots, k+4} (\tilde{T}_{k,j} + \tilde{Q}_{k,j} + \tilde{V}_{k,j}),$$

with

$$\begin{aligned} \tilde{T}_{k,j}(1, \dots, k+4) &:= (-1)^k (j-2) \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4), \\ \tilde{Q}_{k,j}(1, \dots, k+4) &:= -(-1)^k \sum_{s=3}^j \rho_{k,j}(4, \dots, s, 3, s+1, \dots, j, 2, j+1, \dots, k+3, 1, k+4) \\ &\quad - (-1)^k \sum_{s=j+1}^{k+3} \rho_{k,j}(4, \dots, j+1, 2, j+2, \dots, s, 3, s+1, \dots, k+3, 1, k+4), \\ \tilde{V}_{k,j}(1, \dots, k+4) &:= (-1)^k \sum_{s=3}^j \rho_{k,j}(4, \dots, s, 3, s+1, \dots, j, 1, j+1, \dots, k+3, 2, k+4) \\ &\quad + (-1)^k \sum_{s=j+1}^{k+3} \rho_{k,j}(4, \dots, j+1, 1, j+2, \dots, s, 3, s+1, \dots, k+3, 2, k+4). \end{aligned}$$

Proof. In the computation that will follow we will denote by $\rho_{k,j}(\dots)_p$ with $p = 0, 1$, the terms that summed-up together annihilate the operator $\text{Circ Sym}_{4, \dots, k+4}$. For all $2 \leq j \leq k+1$, we define

$$\begin{aligned} \hat{T}_{k,j}(1, \dots, k+4) &:= (-1)^k (j-1) \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4), \\ \hat{Q}_{k,j}(1, \dots, k+4) &:= -(-1)^k \sum_{s=3}^j \rho_{k,j}(4, \dots, s, 3, s+1, \dots, j, 2, j+1, \dots, k+3, 1, k+4) \\ &\quad - (-1)^k \sum_{s=j+1}^{k+3} \rho_{k,j}(4, \dots, j+1, 2, j+2, \dots, s, 3, s+1, \dots, k+3, 1, k+4) \\ &\quad - (-1)^k \rho_{k,j}(4, \dots, j+1, 2, j+2, \dots, k+4, 1, 3)_2, \\ \hat{V}_{k,j}(1, \dots, k+4) &:= (-1)^k \sum_{s=3}^j \rho_{k,j}(4, \dots, s, 3, s+1, \dots, j, 1, j+1, \dots, k+3, 2, k+4) \\ &\quad + (-1)^k \sum_{s=j+1}^{k+3} \rho_{k,j}(4, \dots, j+1, 1, j+2, \dots, s, 3, s+1, \dots, k+3, 2, k+4) \\ &= (-1)^k \rho_{k,j}(4, \dots, j+1, 1, j+2, \dots, k+4, 2, 3)_1, \end{aligned}$$

and we notice the equalities

$$\begin{aligned}\mathrm{Sym}_{3,\dots,k+4} Q_k &= \sum_{j=2}^{k+1} \mathrm{Sym}_{4,\dots,k+4} \hat{Q}_{k,j}, \\ \mathrm{Sym}_{3,\dots,k+4} V_k &= \sum_{j=2}^{k+1} \mathrm{Sym}_{4,\dots,k+4} \hat{V}_{k,j}.\end{aligned}$$

Then proposition 2.5 and lemma 4.1 imply

$$\mathrm{Circ} \mathrm{Sym}_{3,\dots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = \sum_{j=2}^{k+1} \mathrm{Circ} \mathrm{Sym}_{4,\dots,k+4} (\hat{T}_{k,j} + \hat{Q}_{k,j} + \hat{V}_{k,j}). \quad (4.2)$$

We write

$$\begin{aligned}\hat{T}_{k,j}(1, \dots, k+4) &= (-1)^k (j-2) \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4) \\ &\quad - (-1)^k \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+4, 1, 2) \\ &\quad - (-1)^k \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 2, k+4, 1) \\ &= (-1)^k (j-2) \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4) \\ &\quad - (-1)^k \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+4, 1, 2)_1 \\ &\quad + (-1)^k \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+4, 2, 1)_2.\end{aligned}$$

We infer the identity

$$\begin{aligned}\mathrm{Circ} \mathrm{Sym}_{4,\dots,k+4} (\hat{T}_{k,j} + \hat{Q}_{k,j} + \hat{V}_{k,j}) \\ = \mathrm{Circ} \mathrm{Sym}_{4,\dots,k+4} (\check{T}_{k,j} + \check{Q}_{k,j} + \check{V}_{k,j}),\end{aligned}$$

which combined with the identity (4.2) implies the required statement. \square

Lemma 4.3. *In the set up of proposition 2.5, for all integers $k \geq 1$,*

$$\mathrm{Circ} \mathrm{Sym}_{3,\dots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2 \right] = (-1)^k \sum_{j=2}^{k+1} \mathrm{Circ} \mathrm{Sym}_{4,\dots,k+4} [(j-2) \mathbb{T}_{k,j} - \mathrm{Alt}_2 \mathbb{Q}_{k,j}],$$

with

$$\begin{aligned}\mathbb{T}_{k,j}(1, \dots, k+4) \\ := \sum_{I \subseteq \{4, \dots, j+1\}} R^{(|I|)}(I, 3, j+2, R^{(k+1-j+|I|)}(\mathbb{C}I, j+3, \dots, k+3, 1, 2, k+4)) \\ - \sum_{\substack{I \subseteq \{4, \dots, j+1\} \\ j+3 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+3, \dots, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 3, j+2, h), \dots, k+3, 1, 2, k+4) \\ - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 3, j+2, 1), 2, k+4)\end{aligned}$$

$$\begin{aligned}
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, 1, R^{(\mathbb{C}I)}(\mathbb{C}I, 3, j+2, 2), k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, 1, 2, R^{(\mathbb{C}I)}(\mathbb{C}I, 3, j+2, k+4)),
\end{aligned}$$

and with

$$\begin{aligned}
& \mathbb{Q}_{k,j}(1, \dots, k+4) \\
:= & \sum_{\substack{3 \leq s \leq j \\ I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}}} R^{(|I|)}(I, 2, j+1, R^{(k+1-j+|\mathbb{C}I|)}(\mathbb{C}I, j+2, \dots, k+3, 1, k+4)) \\
& - \sum_{\substack{3 \leq s \leq j \\ I \subseteq \{4, \dots, s, 3, s+1, \dots, j\} \\ j+2 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+2, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+1, h), \dots, k+3, 1, k+4) \\
& - \sum_{\substack{3 \leq s \leq j \\ I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+1, 1), k+4) \\
& - \sum_{\substack{3 \leq s \leq j \\ I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, 1, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+1, k+4)) \\
& + \sum_{I \subseteq \{4, \dots, j+1\}} R^{(|I|)}(I, 2, 3, R^{(k+1-j+|\mathbb{C}I|)}(\mathbb{C}I, j+2, \dots, k+3, 1, k+4)) \\
& - \sum_{\substack{I \subseteq \{4, \dots, j+1\} \\ j+2 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+2, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, 3, h), \dots, k+3, 1, k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, 1, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, 3, k+4)) \\
& + \sum_{\substack{j+2 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\}}} R^{(|I|)}(I, 2, j+2, R^{(k+1-j+|\mathbb{C}I|)}(\mathbb{C}I, j+3, \dots, s, 3, s+1, \dots, k+3, 1, k+4)) \\
& - \sum_{\substack{j+3 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\} \\ j+3 \leq h \leq s}} R^{(k+1-j+|I|)}(I, j+3, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+2, h), \dots, s, 3, s+1, \dots, k+3, 1, k+4) \\
& - \sum_{\substack{j+2 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\}}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+2, 3), s+1, \dots, k+3, 1, k+4) \\
& - \sum_{\substack{j+2 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\} \\ s+1 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+2, h), \dots, k+3, 1, k+4)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{\substack{j+2 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\}}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, k+3, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+2, 1), k+4) \\
& - \sum_{\substack{j+2 \leq s \leq k+3 \\ I \subseteq \{4, \dots, j+1\}}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, k+3, 1, R^{(\mathbb{C}I)}(\mathbb{C}I, 2, j+2, k+4)),
\end{aligned}$$

where the symbol $I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}$ means a subset of the elements $4, \dots, s, 3, s+1, \dots, j$ written in this order and $\mathbb{C}I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}$ denotes its complementary set respecting the same order.

Proof. We recall first the definition

$$\begin{aligned}
(R.R^{(p)})(1, \dots, p+5) &= R(1, 2, R^{(p)}(3, \dots, p+5)) \\
&= \sum_{h=3}^{p+5} R^{(p)}(3, \dots, R(1, 2, h), \dots, p+5),
\end{aligned}$$

which writes in the case

$$\begin{aligned}
(R.R^{(k+1-j)})(j-1, \dots, k+4) &= R(j-1, j, R^{(k+1-j)}(j+1, \dots, k+4)) \\
&= \sum_{h=j+1}^{k+4} R^{(k+1-j)}(j+1, \dots, R(j-1, j, h), \dots, k+4).
\end{aligned}$$

We infer the expression

$$\begin{aligned}
& \rho_{k,j}(1, \dots, k+4) \\
&= \sum_{I \subseteq \{1, \dots, j-2\}} R^{(|I|)}(I, j-1, j, R^{(k+1-j+|\mathbb{C}I|)}(\mathbb{C}I, j+1, \dots, k+4)) \\
&= \sum_{I \subseteq \{1, \dots, j-2\}} \sum_{h=j+1}^{k+4} R^{(k+1-j+|I|)}(I, j+1, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, j-1, j, h), \dots, k+4).
\end{aligned}$$

Performing the change of variables

$$\begin{array}{cccccccccccc}
1 & \dots & j-2 & j-1 & j & \dots & k+1 & k+2 & k+3 & k+4 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
4 & \dots & j+1 & 3 & j+2 & \dots & k+3 & 1 & 2 & k+4
\end{array},$$

we infer the expression

$$\begin{aligned}
& \rho_{k,j}(4, \dots, j+1, 3, j+2, \dots, k+3, 1, 2, k+4) \\
&= \sum_{I \subseteq \{4, \dots, j+1\}} R^{(|I|)}(I, 3, j+2, R^{(k+1-j+|\mathbb{C}I|)}(\mathbb{C}I, j+3, \dots, k+3, 1, 2, k+4)) \\
&= \sum_{I \subseteq \{4, \dots, j+1\}} \sum_{h=j+3}^{k+3} R^{(k+1-j+|I|)}(I, j+3, \dots, R^{(\mathbb{C}I)}(\mathbb{C}I, 3, j+2, h), \dots, k+3, 1, 2, k+4)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, R^{(\mathbb{G}I)}(\mathbb{C}I, 3, j+2, 1), 2, k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, 1, R^{(\mathbb{G}I)}(\mathbb{C}I, 3, j+2, 2), k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, k+3, 1, 2, R^{(\mathbb{G}I)}(\mathbb{C}I, 3, j+2, k+4)).
\end{aligned}$$

Performing now the change of variables

$$\begin{array}{ccccccccccccccc}
1 & \dots & s-3 & s-2 & s-1 & \dots & j-2 & j-1 & j & \dots & k+2 & k+3 & k+4 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
4 & \dots & s & 3 & s+1 & \dots & j & 2 & j+1 & \dots & k+3 & 1 & k+4
\end{array},$$

we infer the expression

$$\begin{aligned}
& \rho_{k,j}(4, \dots, s, 3, s+1, \dots, j, 2, j+1, \dots, k+3, 1, k+4) \\
& = \sum_{I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}} R^{(|I|)}(I, 2, j+1, R^{(k+1-j+|\mathbb{G}I|)}(\mathbb{C}I, j+2, \dots, k+3, 1, k+4)) \\
& - \sum_{I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}} \sum_{h=j+2}^{k+3} R^{(k+1-j+|I|)}(I, j+2, \dots, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, j+1, h), \dots, k+3, 1, k+4) \\
& - \sum_{I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, j+1, 1), k+4) \\
& - \sum_{I \subseteq \{4, \dots, s, 3, s+1, \dots, j\}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, 1, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, j+1, k+4)).
\end{aligned}$$

We perform finally the change of variables

$$\begin{array}{ccccccccccccccc}
1 & \dots & j-2 & j-1 & j & \dots & s-2 & s-1 & s & \dots & k+2 & k+3 & k+4 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\
4 & \dots & j+1 & 2 & j+2 & \dots & s & 3 & s+1 & \dots & k+3 & 1 & k+4
\end{array}.$$

We distinguish two cases. In the first case, when $s = j+1$, we have

$$\begin{aligned}
& \rho_{k,j}(4, \dots, j+1, 2, 3, j+2, \dots, k+3, 1, k+4) \\
& = \sum_{I \subseteq \{4, \dots, j+1\}} R^{(|I|)}(I, 2, 3, R^{(k+1-j+|\mathbb{G}I|)}(\mathbb{C}I, j+2, \dots, k+3, 1, k+4)) \\
& - \sum_{\substack{I \subseteq \{4, \dots, j+1\} \\ j+2 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+2, \dots, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, 3, h), \dots, k+3, 1, k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, 3, 1), k+4) \\
& - \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+2, \dots, k+3, 1, R^{(\mathbb{G}I)}(\mathbb{C}I, 2, 3, k+4)),
\end{aligned}$$

and in the second case, when $j+2 \leq s \leq k+3$, we have

$$\begin{aligned}
& \rho_{k,j}(4, \dots, j+1, 2, j+2, \dots, s, 3, s+1, \dots, k+3, 1, k+4) \\
&= \sum_{I \subseteq \{4, \dots, j+1\}} R^{(|I|)}(I, 2, j+2, R^{(k+1-j+|I|)}(\mathbb{C}I, j+3, \dots, s, 3, s+1, \dots, k+3, 1, k+4)) \\
&- \sum_{\substack{I \subseteq \{4, \dots, j+1\} \\ j+3 \leq h \leq s}} R^{(k+1-j+|I|)}(I, j+3, \dots, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 2, j+2, h), \dots, s, 3, s+1, \dots, k+3, 1, k+4) \\
&- \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 2, j+2, 3), s+1, \dots, k+3, 1, k+4) \\
&- \sum_{\substack{I \subseteq \{4, \dots, j+1\} \\ s+1 \leq h \leq k+3}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 2, j+2, h), \dots, k+3, 1, k+4) \\
&- \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, k+3, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 2, j+2, 1), k+4) \\
&- \sum_{I \subseteq \{4, \dots, j+1\}} R^{(k+1-j+|I|)}(I, j+3, \dots, s, 3, s+1, \dots, k+3, 1, R^{(|\mathbb{C}I|)}(\mathbb{C}I, 2, j+2, k+4)).
\end{aligned}$$

Then the conclusion follows from lemma 4.2. \square

4.2 Expliciting the 1-exterior product of the lower order term of S_k

Given a subset $S \subset \mathbb{Z}_{\geq 1}$ and $I \subset S$ we denote below by abuse of notation $\mathbb{C}I \subset S$ the complementary set of I inside S . We show now the following key result.

Proposition 4.4. *For all $h \geq 2$, let*

$$\hat{S}_h := \frac{2i^{h+1}}{(h+1)!h!} \text{Sym}_{2, \dots, h+1} \Phi_{h-3}.$$

Then for all $k \geq 1$,

$$\text{Sym}_{3, \dots, k+4} \left[\sum_{r=3}^{k+2} (r+1)! \sum_{p=2}^{r-1} (id_1^\nabla)^{k+2-r} (p\hat{S}_p \wedge_1 \hat{S}_{r-p+1}) \right] = \text{Sym}_{3, \dots, k+4} \mathbb{L}(k),$$

with

$$\mathbb{L}(k) := \sum_{r=3}^{k+2} \sum_{p=2}^{r-1} \frac{4(-i)^{k+1}(r+1)! \mathbb{L}_{k,r,p}}{(r-p+2)!(p+1)!},$$

and with

$$\begin{aligned}
& \mathbb{L}_{k,r,p}(1, \dots, k+4) \\
&:= \sum_{I \subseteq \{1, 3, \dots, k-r+3\}}
\end{aligned}$$

$$\begin{aligned}
& R^{(p+|I|-2)}(I, k_p + 2, \dots, k + 3, R^{(r-p+|\mathbb{C}I|-1)}(\mathbb{C}I, k_r + 1, \dots, k_p, 2, k_p + 1), k_r, k + 4) \\
- & \sum_{I \subseteq \{2, \dots, k-r+3\}} \\
& R^{(p+|I|-2)}(I, k_p + 2, \dots, k + 3, R^{(r-p+|\mathbb{C}I|-1)}(\mathbb{C}I, k_r + 1, \dots, k_p, 1, k_p + 1), k_r, k + 4).
\end{aligned}$$

with $k_p := k - p + 4$, in the case $k + 2 - r \geq 1$ and with

$$\begin{aligned}
& \mathbb{I}_{k, k+2, p}(1, \dots, k + 4) \\
:= & \sum_{j=k_p+1}^{k+3} R^{(p-2)}(k_p + 2, \dots, j, R^{(k-p+1)}(3, \dots, k_p, 2, k_p + 1), j + 1, \dots, k + 3, 1, k + 4) \\
+ & R^{(p-2)}(k_p + 2, \dots, k + 4, 1, R^{(k-p+1)}(3, \dots, k_p, 2, k_p + 1)) \\
- & \sum_{j=k_p+1}^{k+3} R^{(p-2)}(k_p + 2, \dots, j, R^{(k-p+1)}(3, \dots, k_p, 1, k_p + 1), j + 1, \dots, k + 3, 2, k + 4) \\
- & R^{(p-2)}(k_p + 2, \dots, k + 4, 2, R^{(k-p+1)}(3, \dots, k_p, 1, k_p + 1)).
\end{aligned}$$

Proof. We notice that in the case $k + 2 - r \geq 1$,

$$\text{Sym}_{3, \dots, k+4} [(i d_1^\nabla)^{k+2-r} (p \hat{S}_p \wedge_1 \hat{S}_{r-p+1})] = (-i)^{k+2-r} \text{Sym}_{3, \dots, k+4} (\varphi_{k, r, p} - \psi_{k, r, p}),$$

with

$$\begin{aligned}
\varphi_{k, r, p}(1, \dots, k + 4) & := \nabla^{k+2-r} (p \hat{S}_p \wedge_1 \hat{S}_{r-p+1})(2, \dots, k + 3 - r, 1, k + 4 - r, \dots, k + 4), \\
\psi_{k, r, p}(1, \dots, k + 4) & := \nabla^{k+2-r} (p \hat{S}_p \wedge_1 \hat{S}_{r-p+1})(1, 3, \dots, k + 3 - r, 2, k + 4 - r, \dots, k + 4).
\end{aligned}$$

We notice now that

$$\text{Sym}_{3, \dots, k+4} \varphi_{k, r, p} = \text{Sym}_{3, \dots, k+4} \tilde{\varphi}_{k, r, p},$$

$$\text{Sym}_{3, \dots, k+4} \psi_{k, r, p} = \text{Sym}_{3, \dots, k+4} \tilde{\psi}_{k, r, p},$$

with

$$\begin{aligned}
\tilde{\varphi}_{k, r, p}(1, \dots, k + 4) & := \nabla^{k+2-r} \lambda_{r, p}(2, \dots, k + 3 - r, 1, k + 4 - r, \dots, k + 4), \\
\tilde{\psi}_{k, r, p}(1, \dots, k + 4) & := \nabla^{k+2-r} \lambda_{r, p}(1, 3, \dots, k + 3 - r, 2, k + 4 - r, \dots, k + 4),
\end{aligned}$$

with

$$\begin{aligned}
& \lambda_{r, p}(1, k + 4 - r, \dots, k + 4) \\
:= & -p \hat{S}_p(k + 4 - r, \hat{S}_{r-p+1}(1, k + 5 - r, \dots, k + 5 - p), k + 6 - p, \dots, k + 4).
\end{aligned}$$

Therefore

$$\begin{aligned}
& (\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p})(1, \dots, k+4) \\
&= p \sum_{I \subseteq \{1,3,\dots,k-r+3\}} \hat{S}_p^{(|I|)}(I, k_r, \hat{S}_{r-p+1}^{(\mathbb{C}I)}(\mathbb{C}I, 2, k_r+1, \dots, k_p+1), k_p+2, \dots, k+4) \\
&- p \sum_{I \subseteq \{2,\dots,k-r+3\}} \hat{S}_p^{(|I|)}(I, k_r, \hat{S}_{r-p+1}^{(\mathbb{C}I)}(\mathbb{C}I, 1, k_r+1, \dots, k_p+1), k_p+2, \dots, k+4).
\end{aligned}$$

We notice now that for any tensor $\Phi \equiv \Phi(1, \dots, p+1)$, the tensor

$$\tilde{\Phi}(1, \dots, p+1) := \sum_{j=2}^{p+1} \Phi(1, 3, \dots, j, 2, j+1, \dots, p+1),$$

satisfies the identity

$$\text{Sym}_{2,\dots,p+1} \Phi = \text{Sym}_{3,\dots,p+1} \tilde{\Phi}.$$

Thus

$$(\text{Sym}_{2,\dots,p+1} \Phi)^{(h)} = \text{Sym}_{h+3,\dots,h+p+1} \tilde{\Phi}^{(h)}.$$

We infer

$$\begin{aligned}
& (\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p})(1, \dots, k+4) \\
&= \frac{4i^{r+3}}{(p+1)!(p-1)!(r-p+2)!(r-p+1)!} \sum_{I \subseteq \{1,3,\dots,k-r+3\}} \\
& \quad (\text{Sym}_{|I|+3,\dots,|I|+p+1} \tilde{\Phi}_{p-3}^{(|I|)})(I, k_r, (\text{Sym}_{|\mathbb{C}I|+2,\dots,|\mathbb{C}I|+r-p+2} \Phi_{r-p-2}^{(\mathbb{C}I)})(\mathbb{C}I, 2, k_r+1, \dots, k_p+1), \\
& \quad k_p+2, \dots, k+4) \\
&- \frac{4i^{r+3}}{(p+1)!(p-1)!(r-p+2)!(r-p+1)!} \sum_{I \subseteq \{2,\dots,k-r+3\}} \\
& \quad (\text{Sym}_{|I|+3,\dots,|I|+p+1} \tilde{\Phi}_{p-3}^{(|I|)})(I, k_r, (\text{Sym}_{|\mathbb{C}I|+2,\dots,|\mathbb{C}I|+r-p+2} \Phi_{r-p-2}^{(\mathbb{C}I)})(\mathbb{C}I, 1, k_r+1, \dots, k_p+1), \\
& \quad k_p+2, \dots, k+4).
\end{aligned}$$

and thus

$$(-i)^{k+2-r} \text{Sym}_{3,\dots,k+4} (\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p}) = \frac{4(-1)^{k-r} i^{k+1}}{(r-p+2)!(p+1)!} \text{Sym}_{3,\dots,k+4} \tilde{\mathbb{L}}_{k,r,p},$$

with

$$\begin{aligned}
& \tilde{\mathbb{L}}_{k,r,p}(1, \dots, k+4) \\
&:= \sum_{I \subseteq \{1,3,\dots,k-r+3\}} \tilde{\Phi}_{p-3}^{(|I|)}(I, k_r, \Phi_{r-p-2}^{(\mathbb{C}I)}(\mathbb{C}I, 2, k_r+1, \dots, k_p+1), k_p+2, \dots, k+4)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{I \subseteq \{2, \dots, k-r+3\}} \tilde{\Phi}_{p-3}^{(|I|)}(I, k_r, \Phi_{r-p-2}^{(|\mathbb{G}I|)}(\mathbb{C}I, 1, k_r+1, \dots, k_p+1), k_p+2, \dots, k+4) \\
& = (-1)^{r-p-1} \sum_{I \subseteq \{1, 3, \dots, k-r+3\}} \\
& \quad \tilde{\Phi}_{p-3}^{(|I|)}(I, k_r, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 2, k_p+1), k_p+2, \dots, k+4) \\
& - (-1)^{r-p-1} \sum_{I \subseteq \{2, \dots, k-r+3\}} \\
& \quad \tilde{\Phi}_{p-3}^{(|I|)}(I, k_r, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 1, k_p+1), k_p+2, \dots, k+4).
\end{aligned}$$

Using the expression

$$\begin{aligned}
\tilde{\Phi}_{p-3}(1, \dots, p+1) & = (-1)^p \sum_{j=2}^p R^{(p)}(3, \dots, j, 2, j+1, \dots, p, 1, p+1) \\
& + (-1)^p R^{(p)}(3, \dots, p+1, 1, 2), \tag{4.3}
\end{aligned}$$

we deduce the formula

$$(-i)^{k+2-r} \text{Sym}_{3, \dots, k+4}(\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p}) = \frac{4(-i)^{k+1}}{(r-p+2)!(p+1)!} \text{Sym}_{3, \dots, k+4} \hat{\mathbb{L}}_{k,r,p},$$

with

$$\begin{aligned}
& \hat{\mathbb{L}}_{k,r,p}(1, \dots, k+4) \\
& := \sum_{I \subseteq \{1, 3, \dots, k-r+3\}} \sum_{j=k_p+1}^{k+3} \\
& \quad R^{(p+|I|-2)}(I, k_p+2, \dots, j, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 2, k_p+1), j+1, \dots, k+3, k_r, k+4) \\
& + \sum_{I \subseteq \{1, 3, \dots, k-r+3\}} R^{(p+|I|-2)}(I, k_p+2, \dots, k+4, k_r, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 2, k_p+1)) \\
& - \sum_{I \subseteq \{2, \dots, k-r+3\}} \sum_{j=k_p+1}^{k+3} \\
& \quad R^{(p+|I|-2)}(I, k_p+2, \dots, j, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 1, k_p+1), j+1, \dots, k+3, k_r, k+4) \\
& - \sum_{I \subseteq \{2, \dots, k-r+3\}} R^{(p+|I|-2)}(I, k_p+2, \dots, k+4, k_r, R^{(r-p+|\mathbb{G}I|-1)}(\mathbb{C}I, k_r+1, \dots, k_p, 1, k_p+1)).
\end{aligned}$$

The fact that in the case under consideration $k+2-r \geq 1$, i.e $k_r \geq 3$, implies

$$\text{Sym}_{3, \dots, k+4}[(i d_1^\nabla)^{k+2-r}(p \hat{S}_p \wedge_1 \hat{S}_{r-p+1})] = \frac{4(-i)^{k+1}}{(r-p+2)!(p+1)!} \text{Sym}_{3, \dots, k+4} \mathbb{L}_{k,r,p}.$$

We consider now the case $r = k + 2$. We consider the expansion

$$\begin{aligned}
& (p \hat{S}_p \wedge_1 \hat{S}_{k-p+3})(1, \dots, k+4) \\
&= p \hat{S}_p(1, \hat{S}_{k-p+3}(2, \dots, k_p+1), k_p+2, \dots, k+4) \\
&- p \hat{S}_p(2, \hat{S}_{k-p+3}(1, 3, \dots, k_p+1), k_p+2, \dots, k+4) \\
&= \frac{4i^{k+5}}{(p+1)!(p-1)!(k-p+4)!(k-p+3)!} \times \\
&\times (\text{Sym}_{3, \dots, p+1} \tilde{\Phi}_{p-3})(1, (\text{Sym}_{2, \dots, k_p} \Phi_{k-p})(2, \dots, k_p+1), k_p+2, \dots, k+4) \\
&- \frac{4i^{k+5}}{(p+1)!(p-1)!(k-p+4)!(k-p+3)!} \times \\
&\times (\text{Sym}_{3, \dots, p+1} \tilde{\Phi}_{p-3})(2, (\text{Sym}_{2, \dots, k_p} \Phi_{k-p})(1, 3, \dots, k_p+1), k_p+2, \dots, k+4).
\end{aligned}$$

We infer

$$\text{Sym}_{3, \dots, k+4} (p \hat{S}_p \wedge_1 \hat{S}_{r-p+1}) = \frac{4i^{k+5}}{(p+1)!(k-p+4)!} \text{Sym}_{3, \dots, k+4} \tilde{\mathbb{L}}_{k, k+2, p},$$

with

$$\begin{aligned}
& \tilde{\mathbb{L}}_{k, k+2, p}(1, \dots, k+4) \\
&= \tilde{\Phi}_{p-3}(1, \Phi_{k-p}(2, \dots, k_p+1), k_p+2, \dots, k+4) \\
&- \tilde{\Phi}_{p-3}(2, \Phi_{k-p}(1, 3, \dots, k_p+1), k_p+2, \dots, k+4) \\
&= (-1)^{k-p+1} \tilde{\Phi}_{p-3}(1, R^{(k-p+1)}(3, \dots, k_p, 2, k_p+1), k_p+2, \dots, k+4) \\
&- (-1)^{k-p+1} \tilde{\Phi}_{p-3}(2, R^{(k-p+1)}(3, \dots, k_p, 1, k_p+1), k_p+2, \dots, k+4).
\end{aligned}$$

Using (4.4), we obtain

$$\text{Sym}_{3, \dots, k+4} (p \hat{S}_p \wedge_1 \hat{S}_{r-p+1}) = \frac{4(-i)^{k+1}}{(p+1)!(k-p+4)!} \text{Sym}_{3, \dots, k+4} \mathbb{L}_{k, k+2, p},$$

which shows the required formula. \square

5 Vanishing of the integrability conditions in the cases $k = 4, 5$

We recall the notation $R' := R^{(1)}$. In this section we show the identity

$$\text{Circ Sym}_{3, \dots, k+4} \left[2i(id)^k(R')_2 - \sum_{r=3}^{k+2} (r+1)! \sum_{p=2}^{r-1} (id)^{k+2-r} (p \hat{S}_p \wedge \hat{S}_{r-p+1}) \right] = 0,$$

for all integers $k = 1, 2$. We notice that $\hat{S}_p = S_p$ for $p = 2, 3$. This corresponds to the cases $k = 4, 5$ in the main integrability conditions $\text{Circ Sym}_{3, \dots, k+1} \Theta_k^\nabla = 0$.

5.1 Alternative proof of the vanishing of the integrability conditions in the case $k = 4$

We set for simplicity $ab, cde := R(a, b, R(c, d, e))$ and a similar definition for abc, de or a, bcd, e .

We observe first that lemma 4.3 implies the identity

$$\begin{aligned} \text{Circ Sym}_{3,4,5}[d(R')_2] &= \text{Circ Sym}_{4,5} \text{Alt}_2 \mathbb{Q}_{1,2} \\ \mathbb{Q}_{1,2}(1, \dots, 5) &= 23, 415 - 234, 15 - 41, 235 \\ &+ 24, 315 - 243, 15 - 3, 241, 5 - 31, 245. \end{aligned}$$

We notice that $\text{Circ } \mathbb{Q}_{1,2} = \text{Circ } \tilde{\mathbb{Q}}_{1,2}$, with

$$\begin{aligned} \tilde{\mathbb{Q}}_{1,2}(1, \dots, 5) &= 12, 435_1 - 124, 35_2 - 43, 125_3 \\ &+ 34, 125_3 - 142, 35_2 - 3, 241, 5_2 - 12, 345_1 \\ &= -2 \cdot 12, 345 + 2 \cdot 34, 125 - 124, 35 - 142, 35 - 421, 35 \\ &= -2 \cdot 12, 345 + 2 \cdot 34, 125 - 124, 35 + 214, 35 \\ &= -2 \cdot 12, 345 + 2 \cdot 34, 125 - 2 \cdot 124, 35. \end{aligned}$$

(We denote by a subscript $j = 1, 2, 3$, the terms that we combine together). Using the fact that all the operators $\text{Circ}, \text{Sym}_{4,5}, \text{Alt}_2$ commute we infer the formula

$$\begin{aligned} \text{Circ Sym}_{3,4,5}[d(R')_2] &= -4 \text{Circ Sym}_{4,5} \mathbb{Q}_1 \\ \mathbb{Q}_1(1, \dots, 5) &= 12, 345 - 3, 124, 5 - 34, 125. \end{aligned}$$

On the other hand using proposition 4.4, we infer the formula

$$\begin{aligned} S_2 &:= -\frac{i^3}{6} \text{Sym}_{2,3} R, \\ 4!2 \text{Sym}_{3,4,5}(S_2 \wedge S_2) &= -\frac{8}{3} \text{Sym}_{3,4,5} \text{Alt}_2 \mathbb{L}_{1,3,2}, \\ \mathbb{L}_{1,3,2}(1, \dots, 5) &:= 324, 15 + 15, 234. \end{aligned}$$

We show now the identity

$$\text{Circ Sym}_{3,4,5}[2d(R')_2 + 4!2 \text{Sym}_{3,4,5}(S_2 \wedge S_2)] = 0,$$

i.e.

$$12 \text{Circ Sym}_{4,5} \mathbb{Q}_1 - 4 \text{Circ Sym}_{3,4,5} \text{Alt}_2 \mathbb{L}_{1,3,2} = 0. \quad (5.1)$$

Using the identity $[\text{Sym}_{3,4,5}, \text{Alt}_2] = 0$, the equalities

$$\begin{aligned}
\text{Sym}_{3,4,5} \mathbb{L}_{1,3,2} &= \text{Sym}_{4,5} [324, 15 + 423, 15 + 425, 13 + 14, 235 + 13, 245 + 14, 253] \\
&= \text{Sym}_{4,5} [324, 15 + 423, 15 + 425, 13 + 14, 235 + 31, 425 + 14, 253], \\
&\quad \text{Circ Sym}_{4,5} [324, 15 + 423, 15 + 425, 13 + 14, 235 + 23, 415 + 14, 253] \\
&= \text{Circ Sym}_{4,5} [214, 35 + 412, 35 + 435, 21 + 34, 125 + 12, 435 + 34, 152] \\
&= \text{Circ Sym}_{4,5} [214, 35 + 412, 35 - 345, 21 + 34, 125 - 12, 345 + 34, 152], \\
&\quad \text{Alt}_2 [214, 35 + 412, 35 - 345, 21 + 34, 125 - 12, 345 + 34, 152] \\
&= 214, 35 + 412, 35 - 345, 21 + 34, 125 - 12, 345 + 34, 152 \\
&\quad - 124, 35 - 421, 35 + 345, 12 - 34, 215 + 21, 345 - 34, 251 \\
&= -2 \cdot 124, 35 + 214, 35 - 12, 345 + 2 \cdot 34, 125 + 2 \cdot 21, 345 + 34, 125 \\
&= -3 \cdot 124, 35 - 3 \cdot 12, 345 + 3 \cdot 34, 125 \\
&= 3 \cdot 3, 124, 5 - 3 \cdot 12, 345 + 3 \cdot 34, 125.
\end{aligned}$$

We infer

$$\text{Circ Sym}_{3,4,5} \text{Alt}_2 \mathbb{L}_{1,3,2} = -3 \cdot \text{Circ Sym}_{4,5} \mathbf{Q}_1,$$

and thus (5.1).

5.2 Vanishing of the integrability conditions in the case $k = 5$

We want to show the identity

$$\text{Circ Sym}_{3,\dots,6} \left[2id^2(R')_2 + \sum_{r=3}^4 (r+1)! \sum_{p=2}^{r-1} (id)^{4-r} (pS_p \wedge S_{r-p+1}) \right] = 0.$$

We set for simplicity $ab(cdef) := R(a, b, R'(c, d, e, f))$ and a similar definition for $abc(def)$. Using lemma 4.3 we infer the identities

$$\text{Circ Sym}_{3,\dots,6} [d^2(R')_2] = \text{Circ Sym}_{4,5,6} [\mathbb{T}_{2,3} - \text{Alt}_2(\mathbb{Q}_{2,2} + \mathbb{Q}_{2,3})],$$

$$\begin{aligned}
\mathbb{T}_{2,3}(1, \dots, 6) &= 35(4126)_1 + 435(126)_2 \\
&\quad - (4351)26_3 - 4(351)26_4 \\
&\quad + (4352)16_3 + 4(352)16_4
\end{aligned}$$

$$\begin{aligned}
& - 12(4356)_5 - 412(356)_6, \\
\mathbb{Q}_{2,2}(1, \dots, 6) &= 23(4516)_5 - (234)516_7 - 4(235)16_4 - 451(236)_2 \\
& + 24(3516)_8 + 24(5316)_1 - (245)316_9 - (243)516_7 \\
& - 5(243)16_4 - 3(245)16_{10} - 35(241)6_{11} - 53(241)6_4 \\
& - 351(246)_6 - 531(246)_6, \\
\mathbb{Q}_{2,3}(1, \dots, 6) &= 24(3516)_8 + 324(516)_6 - (3245)16_3 - 3(245)16_{10} \\
& - 5(3241)6_{12} - 35(241)6_{11} - 51(3246)_8 - 351(246)_6 \\
& + 23(4516)_5 + 423(516)_6 - (4235)16_3 - 4(235)16_4 \\
& - 51(4236)_8 - 451(236)_2 + 25(4316)_1 + 425(316)_2 \\
& - (4253)16_3 - 4(253)16_4 - 3(4251)6_3 - 43(251)6_4 \\
& - 31(4256)_5 - 431(256)_6,
\end{aligned}$$

where we denote by a subscript $j = 1, \dots, 12$, the terms that we combine together inside the expression $\text{Circ Sym}_{4,\dots,6}[\mathbb{T}_{2,3} - \text{Alt}_2(\mathbb{Q}_{2,2} + \mathbb{Q}_{2,3})]$. ($j = 12$ is combined with itself). Indeed combining:

for $j = 1$,

$$\begin{aligned}
& \text{Sym}_{4,5,6}[35(4126) - 24(5316) + 14(5326) - 25(4316) + 15(4326)] \\
&= \text{Sym}_{4,5,6}[34(5126) - 2 \cdot 24(5316) + 2 \cdot 14(5326)],
\end{aligned}$$

and thus

$$\begin{aligned}
& \text{Circ Sym}_{4,5,6}[35(4126) - 24(5316) + 14(5326) - 25(4316) + 15(4326)] \\
&= \text{Circ Sym}_{4,5,6}[34(5126) - 2 \cdot 34(5126) + 2 \cdot 34(5216)] \\
&= -3 \text{Circ Sym}_{4,5,6}[34(5126)],
\end{aligned}$$

for $j = 2$,

$$\begin{aligned}
& 435(126) + 2 \cdot 451(236) - 2 \cdot 452(136) - 425(316) + 415(326) \\
&= 435(126) + 2 \cdot 451(236) - 2 \cdot 452(136) - 452(136) + 451(236) \\
&= -453(126) + 3 \cdot 451(236) - 3 \cdot 452(136),
\end{aligned}$$

and thus

$$\begin{aligned}
& \text{Circ}[435(126) + 2 \cdot 451(236) - 2 \cdot 452(136) - 425(316) + 415(326)] \\
&= \text{Circ}[-453(126) + 3 \cdot 453(126) - 3 \cdot 453(216)] \\
&= 5 \text{Circ}[453(126)],
\end{aligned}$$

for $j = 3$,

$$\begin{aligned}
& \text{Circ}[(4352)16 - (4351)26 + (3245)16 - (3145)26 \\
&+ (4235)16 - (4135)26 + (4253)16 - (4153)26 + 3(4251)6 - 3(4152)6] \\
&= \text{Circ}[(4352)16 - (4351)26 + (2145)36 - (1245)36 \\
&+ (4235)16 - (4325)16 + (4253)16 + (4513)26 - (4251)36 + (4152)36] \\
&= \text{Circ}[(4352)16 - (4351)26 - (2415)36 - (1245)36 \\
&+ (4235)16 + (4235)16 + (4253)16 + (4532)16 - (4352)16 + (4351)26] \\
&= \text{Circ}[2 \cdot (4235)16 + (4125)36 + (4253)16 + (4532)16] \\
&= \text{Circ}[2 \cdot (4235)16 + (4125)36 - (4325)16] \\
&= \text{Circ}[3(4235)16 + (4125)36] \\
&= 4 \text{Circ}[(4125)36],
\end{aligned}$$

for $j = 4$,

$$\begin{aligned}
& \text{Circ}[4(352)16 - 4(351)26 + 2 \cdot 4(235)16 - 2 \cdot 4(135)26 \\
&+ 4(253)16 - 4(153)26 + 43(251)6 - 43(152)6] \\
&= \text{Circ}[4(251)36 - 4(152)36 + 2 \cdot 4(125)36 - 2 \cdot 4(215)36 \\
&+ 4(152)36 - 4(251)36 - 4(251)36 + 4(152)36] \\
&= \text{Circ}[4 \cdot 4(125)36 + 4(521)36 + 4(152)36] \\
&= \text{Circ}[4 \cdot 4(125)36 - 4(215)36] \\
&= 5 \text{Circ}[4(125)36].
\end{aligned}$$

Moreover for the remaining terms we use the equalities

$$\begin{aligned}
& \text{Sym}_{4,5,6}[5(243)16 - 5(143)26 + 53(241)6 - 53(142)6] \\
&= \text{Sym}_{4,5,6}[5(243)16 - 5(143)26 - 5(241)36 + 5(142)36] \\
&= \text{Sym}_{4,5,6}[4(253)16 - 4(153)26 - 4(251)36 + 4(152)36],
\end{aligned}$$

and

$$\begin{aligned}
& \text{Circ Sym}_{4,5,6}[5(243)16 - 5(143)26 + 53(241)6 - 53(142)6] \\
&= \text{Circ Sym}_{4,5,6}[4(152)36 - 4(251)36 - 4(251)36 + 4(152)36] \\
&= 2 \text{Circ Sym}_{4,5,6}[4(152)36 + 4(521)36] \\
&\quad - 2 \text{Circ Sym}_{4,5,6}[4(215)36] \\
&= 2 \text{Circ Sym}_{4,5,6}[4(125)36].
\end{aligned}$$

Thus for $j = 4$ we have

$$\begin{aligned}
& \text{Circ Sym}_{4,5,6}[4(352)16 - 4(351)26 + 2 \cdot 4(235)16 - 2 \cdot 4(135)26 + 4(253)16 - 4(153)26 \\
&\quad + 43(251)6 - 43(152)6 + 5(243)16 - 5(143)26 + 53(241)6 - 53(142)6] \\
&= 7 \text{Circ Sym}_{4,5,6}[4(125)36].
\end{aligned}$$

For $j = 5$,

$$\begin{aligned}
& \text{Circ}[-12(4356) - 2 \cdot 23(4516) + 2 \cdot 13(4526) + 31(4256) - 32(4156)] \\
&= \text{Circ}[-12(4356) - 2 \cdot 12(4536) + 2 \cdot 21(4536) + 12(4356) - 21(4356)] \\
&= \text{Circ}[-4 \cdot 12(4536) + 12(4356)] \\
&= 5 \text{Circ}[12(4356)].
\end{aligned}$$

For $j = 6$,

$$\begin{aligned}
& \text{Circ}[-412(356) + 351(246) + 531(246) - 324(516) - 423(516) + 431(256) + 351(246)] \\
&= \text{Circ}[-412(356) + 152(346) + 512(346) - 214(536) - 412(536) + 412(356) + 152(346)],
\end{aligned}$$

and

$$\begin{aligned}
& \text{Circ Sym}_{4,5,6}[-412(356) + 351(246) + 531(246) - 324(516) - 423(516) + 431(256) + \\
&\quad 351(246)] \\
&= \text{Circ Sym}_{4,5,6}[-412(356) + 142(356) + 412(356) - 214(536) - 412(536) + \\
&\quad 412(356) + 142(356)]
\end{aligned}$$

$$\begin{aligned}
&= \text{Circ Sym}_{4,5,6}[-412(356) + 2 \cdot 142(356) + 3 \cdot 412(356) + 214(356)] \\
&= \text{Circ Sym}_{4,5,6}[-412(356) + 142(356) - 421(356) + 3 \cdot 412(356)] \\
&= \text{Circ Sym}_{4,5,6}[-412(356) + 142(356) + 4 \cdot 412(356)].
\end{aligned}$$

Moreover

$$\begin{aligned}
&\text{Circ Sym}_{4,5,6}[-412(356) + 142(356) - 241(356) + 4 \cdot 412(356) - 4 \cdot 421(356)] \\
&= \text{Circ Sym}_{4,5,6}[7 \cdot 412(356) + 142(356) + 214(356)] \\
&= \text{Circ Sym}_{4,5,6}[7 \cdot 412(356) - 421(356)] \\
&= 8 \text{Circ Sym}_{4,5,6}[412(356)].
\end{aligned}$$

For $j = 7$,

$$\begin{aligned}
&\text{Circ}[(234)516 - (134)526 + (243)516 - (143)526] \\
&= \text{Circ}[(124)536 - (214)536 + (142)536 - (241)536] \\
&= \text{Circ}[2 \cdot (124)536 - (412)536 - (241)536] \\
&= 3 \text{Circ}[(124)536].
\end{aligned}$$

For $j = 8$,

$$\begin{aligned}
&\text{Sym}_{4,5,6}[-2 \cdot 24(3516) + 2 \cdot 14(3526) + 51(3246) - 52(3146) + 51(4236) - 52(4136)] \\
&= \text{Sym}_{4,5,6}[-2 \cdot 24(3516) + 2 \cdot 14(3526) + 41(3256) - 42(3156) + 41(5236) - 42(5136)] \\
&= \text{Sym}_{4,5,6}[3 \cdot 24(3156) - 3 \cdot 14(3256) - 14(5236) + 24(5136)],
\end{aligned}$$

and

$$\begin{aligned}
&\text{Circ Sym}_{4,5,6}[-2 \cdot 24(3516) + 2 \cdot 14(3526) + 51(3246) - 52(3146) + 51(4236) - 52(4136)] \\
&= \text{Circ Sym}_{4,5,6}[3 \cdot 34(1256) - 3 \cdot 34(2156) - 34(5126) + 34(5216)] \\
&= \text{Circ Sym}_{4,5,6}[3 \cdot 34(1256) + 3 \cdot 34(2516) - 2 \cdot 34(5126)] \\
&= -5 \text{Circ Sym}_{4,5,6}[34(5126)].
\end{aligned}$$

For $j = 9$,

$$\text{Circ}[(245)316] = \text{Circ}[(345)126],$$

and thus

$$\text{Circ}[(245)316 - (145)326] = 2 \text{Circ}[(345)126].$$

For $j = 10$,

$$\begin{aligned} 2 \text{Circ}[3(245)16 - 3(145)26] &= 2 \text{Circ}[1(345)26 - 2(345)16] \\ &= 2 \text{Circ}[1(345)26 + 21(345)6] \\ &= -2 \text{Circ}[(345)216] \\ &= 2 \text{Circ}[(345)126]. \end{aligned}$$

For $j = 11$,

$$\begin{aligned} 2 \cdot 35(241)6 - 2 \cdot 35(142)6 &= 2 \cdot 35(241)6 + 2 \cdot 35(412)6 \\ &= -2 \cdot 35(124)6. \end{aligned}$$

For $j = 12$,

$$\begin{aligned} 5(3241)6 - 5(3142)6 &= 5(3241)6 + 5(3412)6 \\ &= -5(3124)6. \end{aligned}$$

We infer the formula

$$\begin{aligned} \text{Circ Sym}_{3,\dots,6} \left[d^2(R')_2 \right] &= \text{Circ Sym}_{4,5,6} \mathbb{T}_2, \\ \mathbb{T}_2(1, \dots, 6) &= -3 \cdot 34(5126)_1 + 5 \cdot 453(126) + 4 \cdot (4125)36 \\ &\quad + 7 \cdot 4(125)36 + 5 \cdot 12(4356) + 8 \cdot 412(356) \\ &\quad + 3 \cdot (124)536 - 5 \cdot 34(5126)_1 + 4 \cdot (345)126 \\ &\quad - 2 \cdot 35(124)6 - 5(3124)6. \end{aligned}$$

Moreover

$$\begin{aligned} \text{Circ Sym}_{4,5,6} \mathbb{T}_2 &= \text{Circ Sym}_{4,5,6} \tilde{\mathbb{T}}_2, \\ \tilde{\mathbb{T}}_2(1, \dots, 6) &= -8 \cdot 34(5126) + 5 \cdot 453(126) + 4 \cdot (4125)36 \\ &\quad + 7 \cdot 5(124)36_1 + 5 \cdot 12(4356) + 8 \cdot 412(356) \\ &\quad + 3 \cdot (124)536_1 + 4 \cdot (345)126 - 2 \cdot 35(124)6_1. \end{aligned}$$

For $j = 1$, we have

$$\begin{aligned} 7 \cdot 5(124)36 - 3 \cdot (124)356 - 2 \cdot 35(124)6 &= 7 \cdot 5(124)36 - (124)356 + 2 \cdot 5(124)36 \\ &= 9 \cdot 5(124)36 - (124)356. \end{aligned}$$

We conclude

$$\begin{aligned} \text{Circ Sym}_{3,\dots,6} \left[d^2 (R')_2 \right] &= \text{Circ Sym}_{4,5,6} \mathbf{T}_2, \\ \mathbf{T}_2(1, \dots, 6) &= -8 \cdot 34(5126) + 5 \cdot 453(126) + 4 \cdot (4125)36 \\ &\quad + 9 \cdot 4(125)36 + 5 \cdot 12(4356) + 8 \cdot 412(356) \\ &\quad - (124)356 + 4 \cdot (345)126. \end{aligned}$$

On the other hand using proposition 4.4, we infer the identity

$$\begin{aligned} \text{Sym}_{3,\dots,6} \left[\sum_{r=3}^4 (r+1)! \sum_{p=2}^{r-1} (id)^{4-r} (pS_p \wedge S_{r-p+1}) \right] \\ = \frac{2i}{3} \text{Sym}_{3,\dots,6} \text{Alt}_2(4\mathbb{L}_{232} + 5\mathbb{L}_{242} + 5\mathbb{L}_{243}), \end{aligned}$$

$$\mathbb{L}_{232}(1, \dots, 6) = (1425)36 + 1(425)36,$$

$$\mathbb{L}_{242}(1, \dots, 6) = (3425)16 + 61(3425),$$

$$\mathbb{L}_{243}(1, \dots, 6) = (324)516 + 5(324)16 + 561(324).$$

We write

$$\begin{aligned} \text{Sym}_{3,\dots,6} \text{Alt}_2(4\mathbb{L}_{232} + 5\mathbb{L}_{242} + 5\mathbb{L}_{243}) &= \text{Sym}_{4,5,6} \text{Alt}_2 \mathbb{L}_2, \\ \mathbb{L}_2(1, \dots, 6) &= 4 \cdot (1425)36 + 4 \cdot (1325)46 + 4 \cdot (1423)56 + 4 \cdot (1425)63 \\ &\quad + 4 \cdot 1(425)36 + 4 \cdot 1(325)46 + 4 \cdot 1(423)56 + 4 \cdot 1(425)63 \\ &\quad + 5 \cdot (3425)16 + 5 \cdot (4325)16 + 5 \cdot (5423)16 + 5 \cdot (6425)13 \\ &\quad + 5 \cdot 61(3425) + 5 \cdot 61(4325) + 5 \cdot 61(5423) + 5 \cdot 31(6425) \\ &\quad + 5 \cdot (324)516 + 5 \cdot (423)516 + 5 \cdot (524)316 + 5 \cdot (624)513 \\ &\quad + 5 \cdot 5(324)16 + 5 \cdot 5(423)16 + 5 \cdot 3(524)16 + 5 \cdot 5(624)13 \\ &\quad + 5 \cdot 561(324) + 5 \cdot 561(423) + 5 \cdot 361(524) + 5 \cdot 531(624). \end{aligned}$$

We notice that $\text{Circ } \mathbb{L}_2 = \text{Circ } \tilde{\mathbb{L}}_2$, with

$$\begin{aligned}
\tilde{\mathbb{L}}_2(1, \dots, 6) = & 4 \cdot (1425)36 + 4 \cdot (1423)56 + 4 \cdot (1425)63 \\
& + 4 \cdot 2(435)16 + 4 \cdot 3(215)46 + 4 \cdot 3(412)56 + 4 \cdot 2(435)61 \\
& + 5 \cdot (2415)36 + 5 \cdot (4215)36 + 5 \cdot (5412)36 + 5 \cdot (6435)21 \\
& + 5 \cdot 63(2415) + 5 \cdot 63(4215) + 5 \cdot 63(5412) + 5 \cdot 12(6435) \\
& + 5 \cdot (214)536 + 5 \cdot (412)536 + 5 \cdot (534)126 + 5 \cdot (634)521 \\
& + 5 \cdot 5(214)36 + 5 \cdot 5(412)36 + 5 \cdot 1(534)26 + 5 \cdot 5(634)21 \\
& + 5 \cdot 563(214) + 5 \cdot 563(412) + 5 \cdot 162(534) + 5 \cdot 512(634),
\end{aligned}$$

and $\text{Sym}_{4,5,6} \tilde{\mathbb{L}}_2 = \text{Sym}_{4,5,6} \mathbf{L}_2$, with

$$\begin{aligned}
\mathbf{L}_2(1, \dots, 6) = & 4 \cdot (1425)36 + 4 \cdot (1423)56 + 4 \cdot (1425)63 \\
& + 4 \cdot 2(435)16 + 4 \cdot 3(214)56 + 4 \cdot 3(412)56 + 4 \cdot 2(435)61 \\
& + 5 \cdot (2415)36 + 5 \cdot (4215)36 + 5 \cdot (4512)36 + 5 \cdot (6435)21 \\
& + 5 \cdot 63(2415) + 5 \cdot 63(4215) + 5 \cdot 63(4512) + 5 \cdot 12(6435) \\
& + 5 \cdot (214)536 + 5 \cdot (412)536 + 5 \cdot (435)126 + 5 \cdot (435)621 \\
& + 5 \cdot 5(214)36 + 5 \cdot 5(412)36 + 5 \cdot 1(435)26 + 5 \cdot 5(435)21 \\
& + 5 \cdot 563(214) + 5 \cdot 563(412) + 5 \cdot 162(435) + 5 \cdot 612(435).
\end{aligned}$$

We write now

$$\begin{aligned}
(\text{Alt}_2 \mathbf{L}_2)(1, \dots, 6) = & 4 \cdot (1425)36_1 + 4 \cdot (1423)56_3 + 4 \cdot (1425)63_4 \\
& - 4 \cdot (2415)36_1 - 4 \cdot (2413)56_3 - 4 \cdot (2415)63_4 \\
& + 4 \cdot 2(435)16_5 + 4 \cdot 3(214)56_6 + 4 \cdot 3(412)56_6 + 4 \cdot 2(435)61_7 \\
& - 4 \cdot 1(435)26_5 - 4 \cdot 3(124)56_6 - 4 \cdot 3(421)56_6 - 4 \cdot 1(435)62_7 \\
& + 5 \cdot (2415)36_1 + 5 \cdot (4215)36_8 + 5 \cdot (4512)36_8 + 5 \cdot (6435)21_9 \\
& - 5 \cdot (1425)36_1 - 5 \cdot (4125)36_8 - 5 \cdot (4521)36_8 - 5 \cdot (6435)12_9 \\
& + 5 \cdot 63(2415)_{10} + 5 \cdot 63(4215)_{10} + 5 \cdot 63(4512)_{10} + 5 \cdot 12(6435)_9
\end{aligned}$$

$$\begin{aligned}
& - 5 \cdot 63(1425)_{10} - 5 \cdot 63(4125)_{10} - 5 \cdot 63(4521)_{10} - 5 \cdot 21(6435)_9 \\
& + 5 \cdot (214)536_{11} + 5 \cdot (412)536_{11} + 5 \cdot (435)126_5 + 5 \cdot (435)621_7 \\
& - 5 \cdot (124)536_{11} - 5 \cdot (421)536_{11} - 5 \cdot (435)216_5 - 5 \cdot (435)612_7 \\
& + 5 \cdot 5(214)36_{12} + 5 \cdot 5(412)36_{12} + 5 \cdot 1(435)26_5 + 5 \cdot 6(435)21_7 \\
& - 5 \cdot 5(124)36_{12} - 5 \cdot 5(421)36_{12} - 5 \cdot 2(435)16_5 - 5 \cdot 6(435)12_7 \\
& + 5 \cdot 563(214)_{13} + 5 \cdot 563(412)_{13} + 5 \cdot 162(435)_7 + 5 \cdot 612(435)_7 \\
& - 5 \cdot 563(124)_{13} - 5 \cdot 563(421)_{13} - 5 \cdot 261(435)_7 - 5 \cdot 621(435)_7.
\end{aligned}$$

We need to explain the details for the case $j=7$. In this case we sum

$$\begin{aligned}
& 4 \cdot 2(435)61 + 5 \cdot (435)621 + 5 \cdot 6(435)21_0 + 5 \cdot 162(435)_a + 5 \cdot 612(435)_a \\
& - 4 \cdot 1(435)62 - 5 \cdot (435)612 - 5 \cdot 6(435)12_0 - 5 \cdot 261(435)_a - 5 \cdot 621(435)_a \\
& = 4 \cdot 2(435)61_1 + 4 \cdot (435)621_1 + (435)621_3 + 5 \cdot 612(435)_0 + 15 \cdot 612(435)_a \\
& - 4 \cdot 1(435)62_2 - 4 \cdot (435)612_2 - (435)612_3 \\
& = -4 \cdot 62(435)1_1 + 4 \cdot 61(435)2_2 + (435)126_3 + 20 \cdot 612(435) \\
& = 24 \cdot 612(435) + (435)126.
\end{aligned}$$

The result is (we keep the indices according to the sums on $j=1, \dots, 13$.)

$$\begin{aligned}
(\text{Alt}_2 \mathbf{L}_2)(1, \dots, 6) & = -(4125)36_1 + 4 \cdot (4123)56_3 + 4 \cdot (4125)63_4 \\
& + 11 \cdot (435)126_5 - 12 \cdot 3(124)56_6 + 24 \cdot 612(435)_7 + (435)126_7 \\
& - 15 \cdot (4125)36_8 + 15 \cdot 12(6435)_9 - 20 \cdot 63(4125)_{10} - 15 \cdot (124)536_{11} \\
& - 15 \cdot 5(124)36_{12} - 15 \cdot 563(124)_{13}.
\end{aligned}$$

We recombine now the terms of the previous sum under the form

$$\begin{aligned}
(\text{Alt}_2 \mathbf{L}_2)(1, \dots, 6) & = -16 \cdot (4125)36_1 + 4 \cdot (4123)56 + 4 \cdot (4125)63_1 \\
& + 12 \cdot (435)126 - 12 \cdot 3(124)56_2 + 24 \cdot 612(435) \\
& + 15 \cdot 12(6435) - 20 \cdot 63(4125)_1 - 15 \cdot (124)536_2 \\
& - 15 \cdot 5(124)36_2 - 15 \cdot 563(124).
\end{aligned}$$

Indeed for $j = 1$,

$$\begin{aligned}
& -16 \cdot (4125)36 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
&= -12 \cdot (4125)36 - 4 \cdot (4125)36 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
&= -12 \cdot (4125)36 + 4 \cdot 3(4125)6 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
&= -12 \cdot (4125)36 - 24 \cdot 63(4125),
\end{aligned}$$

and for $j = 2$,

$$\begin{aligned}
& -12 \cdot 3(124)56 - 15 \cdot (124)536 - 15 \cdot 5(124)36 \\
&= 12 \cdot 53(124)5 - 3 \cdot (124)536 - 15 \cdot 5(124)36 \\
&= -27 \cdot 5(124)36 - 3 \cdot (124)536.
\end{aligned}$$

In conclusion

$$\text{Circ Sym}_{3,\dots,6} \left[\sum_{r=3}^4 (r+1)! \sum_{p=2}^{r-1} (id)^{4-r} (pS_p \wedge S_{r-p+1}) \right] = \frac{2i}{3} \text{Circ Sym}_{4,5,6} \tilde{\Lambda}_2,$$

with

$$\begin{aligned}
\tilde{\Lambda}_2(1, \dots, 6) &= -12 \cdot (4125)36 + 24 \cdot 36(4125) + 12 \cdot (435)126 + 24 \cdot 612(435) \\
&+ 15 \cdot 12(6435) - 27 \cdot 5(124)36 - 3 \cdot (124)536 - 15 \cdot 563(124).
\end{aligned}$$

We notice that $\text{Sym}_{4,5,6} \tilde{\Lambda}_2 = \text{Sym}_{4,5,6} \Lambda_2$, with

$$\begin{aligned}
\Lambda_2(1, \dots, 6) &= -12 \cdot (4125)36 + 24 \cdot 34(5126) + 12 \cdot (435)126 + 24 \cdot 412(536) \\
&+ 15 \cdot 12(4536) - 27 \cdot 4(125)36 - 3 \cdot (124)536 - 15 \cdot 453(126).
\end{aligned}$$

We deduce $\Lambda_3 = -3\mathbf{T}_2$, which implies the required conclusion in the case $k = 2$.

6 Appendix

We recall first the following very elementary lemmas. The arguments below can be found in [Pali].

Lemma 6.1. *Let M be a smooth manifold, let also ∇ be a covariant derivative operator acting on the smooth sections of T_M and consider the vector field over T_M*

$$\zeta_\eta^\nabla := H_\eta^\nabla \cdot \eta, \tag{6.1}$$

$\eta \in T_M$. We denote by Φ_t^∇ the corresponding 1-parameter subgroup of transformations of T_M . Then for any $\eta \in T_M$ the curve $c_t := \pi_{T_M} \circ \Phi_t^\nabla(\eta)$ is the ∇ -geodesic with initial speed $\dot{c}_0 = \eta$ and $\dot{c}_t = \Phi_t^\nabla(\eta)$.

Proof. The flow line $\eta_t := \Phi_t^\nabla(\eta)$ satisfies the identity

$$\dot{\eta}_t = H_{\eta_t}^\nabla \cdot \eta_t. \quad (6.2)$$

We deduce

$$\begin{aligned} \dot{c}_t &= d_{\eta_t} \pi_{T_M} \cdot \dot{\eta}_t \\ &= d_{\eta_t} \pi_{T_M} \cdot H_{\eta_t}^\nabla \cdot \eta_t \\ &= \eta_t, \end{aligned}$$

and $\ddot{c}_t = H_{\dot{c}_t}^\nabla \cdot \dot{c}_t$, which is the ∇ -geodesic equation. \square

Corollary 6.2. *Let M be a smooth manifold, let also ∇ be a covariant derivative operator acting on the smooth sections of T_M and let U be an open neighborhood U of M inside T_M . A complex structure J over U satisfies the conditions*

$$J|_M = J^{\text{can}}, \quad (6.3)$$

$$J_\eta H_\eta^\nabla \cdot \eta = T_\eta \cdot \eta, \quad (6.4)$$

for any $\eta \in U$ if and only if for any $\eta \in U$, the complex curve $\psi_\eta : t + is \mapsto s\Phi_t^\nabla(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J -holomorphic.

Proof. We observe first that the differential of the maps ψ_η is given by

$$d_{t_0+is_0} \psi_\eta \left(a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right) = ad(s_0 \mathbb{I}_{T_M}) \dot{\Phi}_{t_0}^\nabla(\eta) + bT_{s_0 \Phi_{t_0}^\nabla(\eta)} \Phi_{t_0}^\nabla(\eta).$$

But

$$\begin{aligned} \dot{\Phi}_{t_0}^\nabla(\eta) &= \zeta^\nabla \circ \Phi_{t_0}^\nabla(\eta) \\ &= H_{\Phi_{t_0}^\nabla(\eta)}^\nabla \cdot \Phi_{t_0}^\nabla(\eta), \end{aligned}$$

thanks to (6.1). Then using the property, $H_{\lambda\eta}^\nabla = d_\eta(\lambda \mathbb{I}_{T_M}) \cdot H_\eta^\nabla$, of the linear connection ∇ (see the identity 8.5 in [Pali]) we infer

$$d_{t_0+is_0} \psi_\eta \left(a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right) = (aH_{s_0 \Phi_{t_0}^\nabla(\eta)} + bT_{s_0 \Phi_{t_0}^\nabla(\eta)}) \cdot \Phi_{t_0}^\nabla(\eta). \quad (6.5)$$

The complex curve ψ_η is J -holomorphic if and only if

$$d_{t_0+is_0} \psi_\eta \left(-b \frac{\partial}{\partial t} + a \frac{\partial}{\partial s} \right) = J d_{t_0+is_0} \psi_\eta \left(a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right),$$

thus, if and only if

$$(-bH_{s_0 \Phi_{t_0}^\nabla(\eta)} + aT_{s_0 \Phi_{t_0}^\nabla(\eta)}) \cdot \Phi_{t_0}^\nabla(\eta) = J(aH_{s_0 \Phi_{t_0}^\nabla(\eta)} + bT_{s_0 \Phi_{t_0}^\nabla(\eta)}) \cdot \Phi_{t_0}^\nabla(\eta).$$

For $s_0 \neq 0$ this is equivalent to (6.4). For $s_0 = 0$ this is equivalent to (6.3). We deduce the required conclusion. \square

The condition (6.3) implies that J is an M -totally real complex structure. We provide now the proof of corollary 1.3.

Proof. If we write $A = \alpha + iTB$ and $\alpha = H^\nabla - T\Gamma$, then $S := T^{-1}(H^\nabla - \bar{A}) = \Gamma + iB$. We set $S_k = \Gamma_k + iB_k$. From the proof of corollary 6.2 we know that in the case J is integrable over U , the curve ψ_η is J -holomorphic if and only if hold (6.4). The later rewrites as

$$H_\eta^\nabla \cdot \eta = -J_\eta T_\eta \cdot \eta.$$

Using the property

$$J_{\eta|Ker d_\eta \pi} = -\alpha_\eta B_\eta^{-1} T_\eta^{-1},$$

(see the identity 1.7 in [Pali]), we infer that the previous identity is equivalent to

$$H_\eta^\nabla \cdot \eta = \alpha_\eta B_\eta^{-1} \cdot \eta. \quad (6.6)$$

Taking $d_\eta \pi$ on both sides of (6.6) we deduce $\eta = B_\eta^{-1} \cdot \eta$. Therefore (6.6) is equivalent to the system

$$\begin{cases} B_\eta \cdot \eta = \eta, \\ H_\eta^\nabla \cdot \eta = \alpha_\eta \cdot \eta. \end{cases} \quad (6.7)$$

Then the system (6.7) rewrites as

$$\begin{cases} \sum_{k \geq 1} B_k(\eta^{k+1}) = 0, \\ \sum_{k \geq 1} \Gamma_k(\eta^{k+1}) = 0. \end{cases}$$

and thus as $S_k(\eta^{k+1}) = 0$ for all $k \geq 1$. We remind now that, according to the main theorem in [Pali], the integrability of the structure J implies the condition $S_1 \in C^\infty(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C} T_M)$. We infer $S_1 = 0$.

We notice that, with the notations of the statement of the main theorem in [Pali] the equation $\text{Circ } \beta_k = 0$ hold for all $k \geq 1$. This combined with the identity $[\text{Circ}, \text{Sym}_{2, \dots, k+2}] = 0$, implies

$$\text{Circ } \text{Sym}_{2, \dots, k+2} \beta_k = 0, \quad (6.8)$$

for all $k \geq 1$. So if we apply the Circ operator to both sides of the definition of S_2 in the statement of the main theorem in [Pali], we infer $\text{Circ } S_2 = \text{Circ } \sigma_2 = 3\sigma_2$. If we evaluate this equality to η^3 we infer $S_2(\eta^3) = \sigma_2(\eta^3)$, which implies $\sigma_2 = 0$. We show now by induction that $\sigma_k = 0$ for all $k \geq 2$. Indeed by the inductive assumption

$$S_{k+1} = \frac{i}{(k+2)!} \text{Sym}_{2, \dots, k+2} \beta_k + \sigma_{k+1}.$$

Applying the Circ operator to both sides of this identity and using the equation (6.8), we infer $\text{Circ } S_{k+1} = \text{Circ } \sigma_{k+1} = 3\sigma_{k+1}$, which evaluated at η^{k+2} gives $S_{k+1}(\eta^{k+2}) = \sigma_{k+1}(\eta^{k+2})$. We deduce $\sigma_{k+1} = 0$. Using the identity

$$\text{Sym}_{2, \dots, k+1} \text{Sym}_{3, \dots, k+1} = (k-1)! \text{Sym}_{2, \dots, k+1}, \quad (6.9)$$

we deduce from the statement of the main theorem in [Pali] and with the notations there

$$S_k = \frac{i}{(k+1)!k!} \text{Sym}_{2,\dots,k+1} \theta_{k-1},$$

for $k \geq 2$, with $\theta_1 := 2R^\nabla$ and

$$\begin{aligned} \theta_k &:= -2i (i d_1^\nabla)^{k-2} (\nabla R^\nabla)_2 \\ &+ \sum_{r=3}^k (r+1)! \sum_{p=2}^{r-1} (i d_1^\nabla)^{k-r} (p S_p \wedge_1 S_{r-p+1}), \end{aligned}$$

for all $k \geq 2$. Moreover the equation $\text{Circ } \beta_k = 0$, $k \geq 3$ rewrites as $\text{Circ } \text{Sym}_{3,\dots,k+2} \theta_k = 0$. If we set $\Theta_k^\nabla := \theta_{k-1}$, for all $k \geq 2$ we obtain the required expansion.

On the other hand if the expansion in the statement of the lemma under consideration hold then J is integrable thanks to the main theorem in [Pali] and $\text{Circ } S_k = 0$ for all $k \geq 2$ ($S_1 = 0$). Indeed for $k = 2, 3$ this equality follows from the identities $\text{Circ } \Theta_k^\nabla = 0$ and

$$[\text{Circ}, \text{Sym}_{2,\dots,k+1}] = 0. \quad (6.10)$$

For $k \geq 4$ we use the identities (6.10), (6.9) and the integrability equations satisfied by the covariant derivative operator ∇ . We deduce $S_k(\eta^{k+1}) = 0$, for all $k \geq 1$, which is equivalent to (6.6) and so to the fact that the curves ψ_η are J -holomorphic. \square

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