Explicit maximal totally real embeddings

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Abstract

This article deals with an explicit canonical construction of a maximal totally real embedding for real analytic manifolds equipped with a covariant derivative operator acting on the real analytic sections of its tangent bundle or of its complexified tangent bundle. The existence of maximal totally real embeddings for real analytic manifolds is known from previous celebrated works by Bruhat-Whitney [Br-Wh] and Grauert [Gra]. Their construction is based on the use of analytic continuation of local frames and local coordinates that are far from being canonical or explicit. As a consequence, the form of the corresponding complex structure has been a mystery since the very beginning. A quite simple recursive expression for such complex structures has been provided in the first author's work "On maximal totally real embeddings" [Pali2]. In our series of articles we focus on the case of torsion free connections. In the present article we give a fiberwise Taylor expansion of the canonical complex structure which is expressed in terms of symmetrization of curvature monomials and a rather simple and explicit expression of the coefficients of the expansion. We explain also a rather simple geometric characterization of such canonical complex structures. Our main result and argument can be useful for the study of open questions in the theory of the embeddings in consideration such as their moduli space.

Contents

1 Introduction and statement of the main result

1.1 Previous works and motivation

We first recall notation and definitions from previous work [Pali2].

Over a smooth manifold M, the vector bundle $T_{T_M|M} \simeq T_M \oplus T_M$ is a complex one with the canonical complex structure $J^{\rm can} : (u, v) \longmapsto (-v, u)$ acting on the fibers. Any almost complex structure which is a continuous extension of $J^{\rm can}$ in a neighborhood of M inside T_M makes M an (almost complex) maximally totally real sub-manifold of T_M. Let $\pi : T_M \longrightarrow M$ be the canonical projection. We denote by

$$T \quad \in \quad C^\infty(T_M, \pi^* T_M^* \otimes_{\mathbb{R}} T_{T_M})$$

the canonical section which, at the point $\eta \in T_M$, takes the value $T_\eta : T_{M,p} \longrightarrow T_{T_{M,p},\eta}$, which is the canonical isomorphism map.

Definition 1 *Let M be a smooth manifold. An M-*totally real almost complex structure *over an open neighborhood $U \subseteq T_M$ of the image of the zero section 0_M is a couple (α, B) with $\alpha \in C^\infty(U, \pi^* T_M^* \otimes_{\mathbb{R}} T_{T_M})$ and $B \in C^\infty(U, \pi^* \,{\rm GL}(T_M))$ such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^* T_M}$ over U and such that $\alpha_{0_p} = d_p 0_M$, $B_{0_p} = \mathbb{I}_{T_{M,p}}$, for all $p \in M$. With $A := \alpha + iTB$, the almost complex structure J_A associated to (α, B) is the one which satisfies $T^{0,1}_{T_M, J_A, \eta} = A_\eta(\mathbb{C}T_{M,\pi(\eta)}) \subset \mathbb{C}T_{T_M,\eta}$ for all $\eta \in U \subseteq T_M$.*

Every almost complex smooth extension of the canonical complex structure $J^{\rm can}$ of $T_{T_M|M}$ over a neighborhood of M inside T_M can be expressed, over a sufficiently small neighborhood $U \subseteq T_M$ of M, as the almost complex structure associated to a unique M-totally real almost complex structure over U.

Over a real analytic manifold X, Bruhat-Whitney [Br-Wh] and Grauert [Gra] have shown the existence of an M-totally real complex structure J, over an open neighborhood $U \subseteq T_M$ of the image of the zero section. The construction in these works uses local analytic continuation of local frames constructed by means of local coordinates defined over the base manifold.

The authors [Br-Wh, Gra] show that the construction is independent of the choice of the local frame. Their local construction is not intrinsic.

The nature of their local construction shows that it is not possible to to obtain an explicit expression of the complex structure. As a consequence it is not possible to determine by using the local expression in the articles [Br-Wh, Gra] if a general "canonical" global section defined over a neighborhood of the base manifold inside the tangent bundle is holomorphic or not. The term *"canonical" global section* means here an intrinsic section on a neighborhood of the base manifold inside the tangent bundle obtained in an explicit way from the choice of a linear connection operator on the base manifold.

The main motivation of the present article is to provide a rather *simple and explicit global expression of the complex structure* J from [Br-Wh, Gra]. This is important for applications to analytic micro local analysis over manifolds. It allows indeed an explicit global construction of the complex extension of a given global Fourier integral operator defined on a real analytic manifold. Applications in the theory of polarized complex deformations with real analytic regularity are part of a work in progress.

It has been 64 years since the existence of complex structures on Grauert Tubes was proven for the first time by Bruhat-Whitney [Br-Wh]. Still, up to now, the explicit form of the Taylor expansion has remained quite mysterious to the community of experts in the field. This is finally clarified in the main theorem 1 below. Indeed, in the main theorem 1, the expression of the complex structure is given in terms of fiberwise Taylor expansion constructed by means of global intrinsic sections (symmetrization of compositions of covariant derivatives of the curvature) over the base manifold. If we compare our Taylor expansion with a standard one on functions we can observe that the sections S_k correspond to the polynomials of total degree k in the standard Taylor expansion. The total degree k in our article is measured by the pseudo-norm $\|D\|$ of D (see the definition of pseudo-norm in notation 1 below) in the statement of the main theorem 1.

Polynomials are given by coefficients and monomials of multi degree D. In our article the monomials correspond to symmetrization of compositions of covariant derivatives of the curvature. The compositions of covariant derivatives of the curvature (denoted by R^D in the statement of the main theorem) are encoded by the following quite simple and elementary object.

We take a sequence of sequential increasing numbers from 2 to $d := \|D\| + 1$ and we start by opening parenthesis which are collocated at a "distance" $d_j + 1$ until we reach the central parentheses which is of the form $(...k, 1, k+1)$. Then we start closing parenthesis one by one under the form

$$(...(...k, 1, k+1), k+2)k+3)...d).$$

The expression of the coefficients $\boldsymbol{C}_D$ (see the expressions 1, 2) in the statement of the main theorem 1 is rather simple if compared with standard formulas in formal calculus. We can not expect to have a trivial and elementary expression for the coefficients of the Taylor expansion of a sophisticated object such as an integrable complex structure. The meaning of the expression 2 for the coefficient $C(H)$ is the following. We take a sum of arbitrary "partitions" λ of the sub-compositions H of D, we compute the pseudo-norm of each "piece" included between the "middle" s of any part of the partition and the end of that part. Call x that pseudo norm, compute the function $1/(x(x+1))$ and take the product over all indices.

The expression in the main theorem 1 allows also to perform useful explicit global intrinsic operator computations in the sense of [Pali1]. In more explicit terms, given a global intrinsic section over the Grauert tube, an explicit formula for the complex structure such as the one in in the main theorem 1, allows to determine if the section is holomorphic or not. This is due to the fact that the same linear connection operator is used to define, via a global explicit expression, both the section and the complex structure.

A first step towards an explicit global expression was done in work of the first author [Pali2], where a recursive formula for the coefficients of the fiber-wise Taylor expansion at the origin was provided. We recall this result in Corollary 1 below. We start with a few definitions from [Pali2].

Definition 2 *We denote by* Circ *the circular operator*

$$(\mathrm{Circ}\,\theta)(v_1, v_2, v_3, \bullet) \quad = \quad \theta(v_1, v_2, v_3, \bullet) + \theta(v_2, v_3, v_1, \bullet) + \theta(v_3, v_1, v_2, \bullet),$$

acting on the first three entries of any q-tensor θ, with $q \geqslant 3$. We also define the permutation operation $\theta_2(v_1, v_2, \bullet) := \theta(v_2, v_1, \bullet)$.

For any complex covariant derivative operator ∇ acting on the smooth sections of $\mathbb{C}T_M$ we denote by H^∇ the associated horizontal map (see lemmas 14, 16 and definition 5 in subsection 9.1 of the appendix of the article [Pali2] for precise definitions and properties of the horizontal map H^∇). We define the operator

$$d_1^\nabla : C^\infty(M, T_M^{*,\otimes k} \otimes_{\mathbb{R}} \mathbb{C}T_M) \quad \longrightarrow \quad C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^{*,\otimes(k-1)} \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

with $k \geqslant 1$ as follows

$$d_1^\nabla A(\xi_1, \xi_2, \mu) \quad := \quad \nabla_{\xi_1} A(\xi_2, \mu) - \nabla_{\xi_2} A(\xi_1, \mu),$$

with $\xi_1, \xi_2 \in T_M$ and with $\mu \in T_M^{\oplus(k-1)}$.

For any two sections $A \in C^\infty(M, T_M^{*,\otimes(k+1)} \otimes_{\mathbb{R}} \mathbb{C}T_M)$ and $B \in C^\infty(M, T_M^{*,\otimes(l+1)} \otimes_{\mathbb{R}} \mathbb{C}T_M)$ we define the exterior product

$$A \wedge_1 B \quad \in \quad C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^{*,\otimes(k+l-1)} \otimes_{\mathbb{R}} \mathbb{C}T_M)$$

as

$$(A \wedge_1 B)(\xi_1, \xi_2, \eta, \mu) \quad := \quad A(\xi_1, B(\xi_2, \eta), \mu) - A(\xi_2, B(\xi_1, \eta), \mu),$$

with $\xi_1, \xi_2 \in T_M$, $\eta \in T_M^{\oplus l}$ and $\mu \in T_M^{\oplus(k-1)}$. We denote by $\mathrm{Sym}_{r_1,...,r_s}$ the symmetrizing operator (without normalizing coefficient!) acting on the entries $r_1, \ldots, r_s$ of a multi-linear form. We use the convention that a sum and a product running over an empty set are equal respectively to 0 and 1.

For any complex covariant derivative operator ∇ acting on the smooth sections of T_M, we denote by $(\eta, t) \longmapsto \Phi_t^{\nabla}(\eta) \in T_M$, the geodesic flow of ∇ starting at any point $\eta \in T_M$.

We observe that the argument given in the proof of Corollary 1 in [Pali2] applies without modifications in the case of a general torsion free covariant derivative operator acting on the smooth sections of the tangent bundle. For the convenience of the reader, the argument in this more general setting is given in Section 6 below. We therefore have the following statement.

Corollary 1 *Let M be a smooth manifold equipped with a torsion free covariant derivative operator ∇ acting on the smooth sections of the tangent bundle T_M, let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers, let $J \equiv J_A$ be an M-totally real almost complex structure over U, real analytic along the fibers of U and consider the fiber-wise Taylor expansion at the origin*

$$T_\eta^{-1}\big(H^{\nabla} - \overline{A}\big)_\eta \cdot \xi \quad = \quad i\xi + \sum_{k \geqslant 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a sufficiently small neighborhood of the image of the zero section, with $\xi \in T_{M,\pi(\eta)}$ arbitrary with $S_k \in C^\infty\left(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ and with $\eta^k := \eta^{\times k} \in T^{\oplus k}_{M,\pi(\eta)}$.

Then the statements (a) and (b) below are equivalent.

(a) The almost complex structure J is integrable over U and for any $\eta \in U$, the smooth map $\psi_\eta : t + is \longmapsto s\Phi_t^{\nabla}(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J-holomorphic.

(b) The components S_k satisfy $S_1 = 0$,

$$S_k \quad = \quad \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\ldots,k+1}\,\Theta_k^{\nabla}, \qquad k \geq 2,$$

with $\Theta_2^{\nabla} := 2R^{\nabla}$ and

$$\Theta_k^{\nabla} \quad := \quad -2i(id_1^{\nabla})^{k-3}(\nabla R^{\nabla})_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2} (id_1^{\nabla})^{k-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 3$ and the equations $\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+1}\,\Theta_k^{\nabla} = 0$ are satisfied for all $k \geqslant 4$.

In particular, the famous Grauert tube [Gra] of a compact real analytic Riemannian manifold has a complex structure which, paired with the Levi-Civita connection, satisfies the statement (a) above. Therefore the vanishing of the tensors $I_k := \mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+1}\,\Theta_k^{\nabla}$, for all $k \geqslant 4$, is certainly true in the compact real analytic case considered in [Br-Wh, Gra]. We feel however that a proof independent of their celebrated work provides a more general statement useful for the applications aimed to the global complex extension of a global integral Fourier operator defined via a torsion free covariant derivative operator acting on the smooth sections of the complexified tangent bundle. Indeed, in many analysis applications, these operators are very often defined by this type of more general connections.

The first author has shown [Pali2] that the equation $I_k = 0$ is satisfied for $k = 4$ even in the more general case of a covariant derivative operator acting on the smooth sections of the complexified tangent bundle. In this more general case it is not possible to have a statement similar to Corollary 1, simply because there are no geodesics in M associated to the complex covariant derivative operator.

In this article we build a general formalism that allows us to give a more compact proof of the vanishing of the tensor I_k in the case $k = 4$, and which allows us to provide also a proof in the case $k = 5$. Using computer algebra (see sections 2 and 5 in [Pal-Sal]) we can show the vanishing of I_k for $k = 4, ...7$. In section 5 in [Pal-Sal] we use the explicit expression in theorems 1, 2 below and we observe that in the case $k = 7$, the computer perform the computation in approximately one second, but we expect that the case $k = 8$ would take a computation of approximately two weeks. We feel confident at this point to formulate the following conjecture.

Conjecture 1 *Let M be a smooth manifold and let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$. Then the sequence of tensors $S_k \in C^\infty\left(M, T^*_M \otimes_{\mathbb{R}} S^k T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$, $k \geqslant 2$, defined by the inductive rule*

$$S_k \quad := \quad \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\ldots,k+1}\,\Theta^\nabla_k,$$

with $\Theta^\nabla_2 := 2R^\nabla$ and with

$$\Theta^\nabla_k \quad := \quad -2i(id^\nabla_1)^{k-3}(\nabla R^\nabla)_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2}(id^\nabla_1)^{k-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 3$, satisfies the identities

$$I_k := \mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+1}\,\Theta^\nabla_k = 0,$$

for all $k \geqslant 4$.

A general mathematical proof for the vanishing of all the integrability conditions I_k is part of a long and difficult work in progress.

1.2 Statement of the main result

We provide now more notations useful for the statement of our main Theorem.

Notation 1 *We denote by $D \vDash_0 p$ any element $D \equiv (d_1, \ldots, d_N) \in \mathbb{Z}^N_{\geqslant 0}$ with $N \in \mathbb{Z}_{>0}$ arbitrary such that $|D| := \sum_{j=1}^N d_j = p$. We set $l_D := N$. We denote also $D! := d_1! \cdots d_N!$,*

$$\begin{aligned} |D|_j \quad &:= \quad \sum_{s=1}^{j} d_s, \\ \|D\|_j \quad &:= \quad |D|_j + j, \end{aligned}$$

for any $1 \leqslant j \leqslant N$, and we set $\|D\| := \|D\|_N + N$. For any $l = 1, \ldots, N$, we define $D^-_l := (d_1, \ldots, d_l)$, $D^+_l := (d_{l+1}, \ldots, d_N)$ and

$$D'_{l,h} \quad := \quad (D^-_h)^+_l = (d_{l+1}, \ldots, d_h),$$

for any $l, h \in \mathbb{Z}_{\geqslant 0}$ with $l \leqslant h \leqslant N$. In all this article, for notation simplicity, we use the identification $(v_1, \ldots, v_p) \equiv (1, \ldots, p)$, for a given element $(v_1, \ldots, v_p) \in T^{\oplus p}_M$.

For any integer $a \geqslant 0$, let $R^{(a)} := \nabla^a R^\nabla$. We denote by $R^{(a)}(1, \ldots, a+1, \bullet, a+2)$ the endomorphism

$$v \quad \longmapsto \quad R^{(a)}(1, \ldots, a+1, v, a+2).$$

Given a family of endomorphisms $(T_j)_{j=1}^p$ we denote by

$$\prod_{j=1}^{p} T_j \quad := \quad T_1 \circ \cdots \circ T_p.$$

We also define the curvature monomial of multi-degree D as

$$R^D(1, \ldots, \|D\|+1) \quad := \quad \left[\prod_{j=1}^{l_D} R^{(d_j)}\left(\|D\|_{j-1}+2, \ldots, \|D\|_j+1, \bullet, \|D\|-j+2\right)\right] \cdot 1,$$

i.e. the curvature multinomial R^D writes as

$$\begin{aligned}
&R^D(1,\dots,\|D\|+1)\\
&\quad= R^{(d_1)}\Big(2,\dots,d_1+2,R^{(d_2)}\Big(d_1+3,\dots,d_1+d_2+3,\\
&\qquad R^{(d_3)}\Big(d_1+d_2+4,\dots,d_1+d_2+d_3+4,R^{(d_4)}\Big(\dots\\
&\qquad\dots,R^{(d_{l_D})}\Big(\|D\|_{l_D-1}+2,\dots,\|D\|_{l_D}+1,1,\|D\|_{l_D}+2\Big),\|D\|_{l_D}+3\Big),\dots\\
&\qquad\qquad\dots\Big),\|D\|+1\Big).
\end{aligned}$$

We notice that the data of a curvature monomial R^D is encoded in a multi-parenthesis as follows

$$\begin{aligned}
&R^D(1,\dots,\|D\|+1)\\
&\quad\equiv(2,\dots,d_1+2,(\,d_1+3,\dots,d_1+d_2+3,(d_1+d_2+4,\dots,d_1+d_2+d_3+4,(\dots\\
&\qquad\dots,(\|D\|_{l_D-1}+2,\dots,\|D\|_{l_D}+1,1,\|D\|_{l_D}+2)\,,\|D\|_{l_D}+3)\,,\dots),\|D\|+1)\,,
\end{aligned}$$

since the distance of the opening parenthesis determine the degrees d_j.

We denote by $\lambda\vDash l$ any $\lambda\in\mathbb{Z}^p_{\geqslant 1}$ such that $|\lambda|=l$ and we define for any $D\in\mathbb{Z}^l_{\geqslant 0}$ the coefficient

$$\boldsymbol{C}_D \;:=\; \sum_{0\leqslant H\leqslant A}\frac{(-1)^{|H|}C(H)}{H!(D-H)!},\quad \textit{with}\quad l_H=l, \tag{1}$$

$$C(H) \;:=\; \sum_{\lambda\vDash l_H}(-1)^{l_\lambda}\|H^-_{\lambda_1}\|\prod_{j=1}^{l_\lambda}\prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1}\frac{1}{\|H'_{s,|\lambda|_j}\|(\|H'_{s,|\lambda|_j}\|+1)}. \tag{2}$$

With these notations we can state our main Theorem.

Theorem 1 (***Main Theorem: Explicit maximal totally real embeddings***)
With the same notation and hypothesis as in Corollary 1, statement (b) can be replaced by:
(b) The components S_k satisfy $S_1=0$,

$$S_k \;=\; \frac{1}{i^{k+1}k!}\sum_{\|D\|=k}\boldsymbol{C}_D\,\mathrm{Sym}_{2,\dots,k+1}R^D,\quad \textit{with}\quad D\geqslant 0,$$

for all $k\geqslant 2$ and the equations $\mathrm{Circ}\,\mathrm{Sym}_{3,\dots,k+1}\Theta^\nabla_k=0$, with

$$\Theta^\nabla_k \;:=\; -2i(id^\nabla_1)^{k-3}(\nabla R^\nabla)_2+\sum_{r=4}^{k}r!\sum_{p=2}^{r-2}(id^\nabla_1)^{k-r}(pS_p\wedge_1 S_{r-p}),$$

are satisfied for all $k\geqslant 6$.

The main Theorem 1 is a direct consequence of corollary 1, combined with the following general result (as well as the vanishing for $k=4,5$ that we provide below).

Theorem 2 *Let M be a smooth manifold and let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$. Then for all $k\geqslant 2$, a section $S_k\in C^\infty\left(M,T^*_M\otimes_{\mathbb{R}}S^kT^*_M\otimes_{\mathbb{R}}\mathbb{C}T_M\right)$ satisfies*

$$S_k \;=\; \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\dots,k+1}\Theta^\nabla_k,$$

with $\Theta^\nabla_2:=2R^\nabla$ and with

$$\Theta^\nabla_k \;:=\; -2i(id^\nabla_1)^{k-3}(\nabla R^\nabla)_2+\sum_{r=4}^{k}r!\sum_{p=2}^{r-2}(id^\nabla_1)^{k-r}(pS_p\wedge_1 S_{r-p}),$$

for all $k \geqslant 3$, if and only if

$$S_k \quad = \quad \frac{1}{i^{k+1}k!} \sum_{\|D\|=k} \boldsymbol{C}_D \operatorname{Sym}_{2,\ldots,k+1} R^D, \quad \textit{with} \quad D \geqslant 0,$$

for all $k \geqslant 2$.

A corollary of the solution of the conjecture 1, of the main Theorem in [Pali2] and of theorem 2 will be the following striking result which allow canonical construction of maximal totally real embeddings.

Corollary 2 (***Canonical maximal totally real embeddings***).
Let M be a real analytic manifold and let ∇ be a torsion free complex covariant derivative operator acting on the real analytic sections of the complexified tangent bundle $\mathbb{C}T_M$. Then there exists an open neighborhood $U \subseteq T_M$ of the zero section with connected fibers and a fiberwise real-analytic section S of $\pi^* \operatorname{End}(T_M)$ over U with fiberwise Taylor expansion at the origin

$$S_\eta \cdot \xi \quad = \quad \sum_{k \geqslant 2} S_k \left(\xi, \eta^k\right),$$

for any $\eta \in U$ and any $\xi \in T_{M,\pi(\eta)}$, with $S_k \in C^\infty\left(M, T^*_M \otimes_{\mathbb{R}} S^k T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ for all $k \geqslant 2$, (we denote by $\eta^k := \eta^{\times k} \in T^{\oplus k}_{M,\pi(\eta)}$) given by the explicit formula

$$S_k \quad = \quad \frac{1}{i^{k+1}k!} \sum_{\|D\|=k} \boldsymbol{C}_D \operatorname{Sym}_{2,\ldots,k+1} R^D, \quad \textit{with} \quad D \geqslant 0,$$

for all $k \geqslant 2$, such that J_A with

$$\overline{A} \quad := \quad -iT\mathbb{I}_{T_M} + H^\nabla - TS,$$

is an M-totally real complex structure over U which is real-analytic over U.

We notice that the notation H^∇ in the above definition of the section A is slightly abusive. We mean there by H^∇ the restriction to T_M of the horizontal map over $\mathbb{C}T_M$ associated to the complex covariant derivative operator ∇. We must observe here the obvious inclusion $T_{\mathbb{C}T_M|T_M} \subset \mathbb{C}T_{T_M}$.

Definition 3 *The M-totally real complex structure J in the statement of Corollary 2 is called the canonical M-totally real complex structure associated to ∇ over U.*

We introduce now an important vector field which allows to characterize the canonical M-totally real complex structure.

Definition 4 *Let M be a smooth manifold equipped with a covariant derivative operator ∇ acting on the smooth sections of $\mathbb{C}T_M$. We define the canonical vector field $\xi^\nabla \in C^\infty(T_M, \mathbb{C}T_{T_M})$ associated to ∇ by*

$$\eta \longmapsto \xi^\nabla_\eta := H^\nabla_\eta \cdot \eta - iT_\eta \cdot \eta \equiv [(H^\nabla - iT)\mathbb{I}_{T_M}]_\eta.$$

(We think here the identity operator $\mathbb{I}_{T_M}$ as a section in $C^\infty(T_M, \pi^* T_M)$). With the above notation we have the following characterization.

Corollary 3 *Let M be a real analytic manifold equipped with a torsion free covariant derivative operator ∇ acting on the real analytic sections of the complexified tangent bundle $\mathbb{C}T_M$, let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers and let $J \equiv J_A$ be an M-totally real almost complex structure over U which is real analytic along the fibers of U. Then the almost complex structure J is integrable over U and the vector field ξ^∇ is of type $(1,0)$ for J over U, if and only if J is the canonical M-totally real complex structure associated to ∇ over U.*

We recall that in the general setting of a torsion free complex covariant derivative operator ∇ acting on the sections of the complexified tangent bundle $\mathbb{C}T_M$ there are no geodesics associated to ∇. (Cauchy's existence theorem does not apply).

We wish also to point out that in mathematics and in theoretical physics there are many important natural complex differential operators that are defined via complex connections as above.

We learned from Professor Jöel Merker that the results and techniques developed in this article and in [Pali2] can be applied to the articles by Foo-Merker-Ta [Fo-Mer-Ta1, Fo-Mer-Ta2], Merker-Nurowski [Mer-Nur1, Mer-Nur2], Hill-Merker-Nie-Nurowski [Hill-Mer-Nie-Nur1, Hill-Mer-Nie-Nur2], Merker [Mer] as well as Merker-Sabzevari [Mer-Sab], in order to obtain even more explicit results.

In the auxiliary preprint [Pal-Sal], we provide a Maple program verifying the main equivalence in the proof of theorem 2 for the first values of k. The result for $k \leqslant 20$ is automatically verified in 6 minutes.

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2 Proof of theorem 2

In all the article, with the exception of the appendix, we assume ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$.

2.1 Expliciting the powers of the 1-differential. Part I

Notation 2 *For any $A \in T^{*,\otimes p}_M \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}T_M)$ and for any $\theta \in T^{*,\otimes q}_M \otimes \mathbb{C}T_M$, the product operations of tensors $A \cdot \theta, A\neg\theta \in T^{*,\otimes(p+q)}_M \otimes \mathbb{C}T_M$ are defined by*

$$\begin{aligned}(A \cdot \theta)(u_1, \ldots, u_p, v_1, \ldots, v_q) &:= A(u_1, \ldots, u_p) \cdot \theta(v_1, \ldots, v_q),\\ (A\neg\theta)(u_1, \ldots, u_p, v_1, \ldots, v_q) &:= \sum_{j=1}^{q} \theta(v_1, \ldots, A(u_1, \ldots, u_p) \cdot v_j, \ldots, v_q).\end{aligned}$$

We denote $R^\nabla.\theta := R^\nabla \cdot \theta - R^\nabla\neg\theta$.

We denote by Alt_2, the alternating operator (without normalizing coefficient!) acting on the first two entries of any tensor. We recall the following well known fact (see [Pali2]).

Lemma 1 *For any complex covariant derivative operator ∇ acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$ and for any tensor $\theta \in C^\infty\left(M, T^{*,\otimes k}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$, holds the commutation identity*

$$\mathrm{Alt}_2 \nabla^2\theta = R^\nabla.\theta. \tag{3}$$

We recall that the notation $a, \ldots, b$ for integers $a < b$ denotes the increasing by one sequence from the left to the right hand side. We ignore this notation when $a > b$. We observe now two other elementary lemmas.

Lemma 2 *For any p-tensor θ,*

$$(d_1^\nabla)^k\theta(1, \ldots, k+p) = \sum_{\sigma \in S^*_{k+1}} \varepsilon_\sigma \nabla^k\theta(\sigma_1, \ldots, \sigma_{k+1}, k+2, \ldots, k+p),$$

where S^*_{k+1} is the set of permutations σ of the set $\{1, \ldots, k+1\}$ such that if $1 \leqslant j < k < l \leqslant k+1$ then $\sigma_j < \sigma_k$ or $\sigma_j < \sigma_l$.

Proof Note that if $\sigma \in S^*_{k+1}$, then necessarily $\sigma_1 \in \{1,2\}$. Let $\phi_+ : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ be defined by $\phi_+(i) = i+1$ for all i and let $\phi_- : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ be defined by $\phi_-(i) = i+1$ for all $i \geq 2$ and $\phi_-(1) = 1$. For each $\sigma \in S^*_{k+1}$ one obtains two permutations in S^*_{k+2} as

$$\pi^+(\sigma) = (1, \phi_+(\sigma_1), \ldots, \phi_+(\sigma_{k+1})), \quad \pi^-(\sigma) = (2, \phi_-(\sigma_1), \ldots, \phi_-(\sigma_{k+1})),$$

with $\epsilon(\pi^\pm) = \pm\epsilon(\sigma)$. Conversely, all elements of S^*_{k+2} can be constructed this way , see [Sim-Sch].

For $k = 1$, Lemma 2 follows from the definition of d_1. Then by induction it holds for all values of k. Indeed we assume true the formula in the statement of the lemma for k and we show that it hold true for $k+1$. For this purpose we consider the equality

$$(d_1^\nabla)^{k+1}\theta(1, \ldots, k+p+1) = \nabla((d_1^\nabla)^k\theta)(1, \ldots, k+p+1) - \nabla((d_1^\nabla)^k\theta)(2, 1, 3, \ldots, k+p+1). \quad (4)$$

Notice now the identity

$$(d_1^\nabla)^k\theta(1, 3, \ldots, k+p+1) = \sum_{\sigma \in S^*_{k+1}} \varepsilon_\sigma \nabla^k\theta(\phi^-(\sigma_1), \ldots, \phi^-(\sigma_{k+1}), k+3, \ldots, k+p+1).$$

We deduce

$$\begin{aligned} & (d_1^\nabla)^{k+1}\theta(1, \ldots, k+p+1) \\ = & \sum_{\sigma \in S^*_{k+1}} \varepsilon_\sigma \nabla^{k+1}\theta(\pi_1^+(\sigma), \ldots, \pi_{k+2}^+(\sigma), k+3, \ldots, k+p+1) \\ - & \sum_{\sigma \in S^*_{k+1}} \varepsilon_\sigma \nabla^{k+1}\theta(\pi_1^-(\sigma), \ldots, \pi_{k+2}^-(\sigma), k+3, \ldots, k+p+1), \end{aligned}$$

which implies the required conclusion. □

We can also write in operator terminology

$$(d_1^\nabla)^k = \sum_{\sigma \in S^*_{k+1}} \varepsilon_\sigma \sigma \neg \nabla^k.$$

The previous formulas write in the following more explicit way

Lemma 3 *For any p-tensor θ,*

$$(d_1^\nabla)^k\theta(1, \ldots, k+p) = \sum_{L \in \mathcal{P}_*[k+1]} \varepsilon_{\sigma^L} \nabla^k\theta(\sigma_1^L, \ldots, \sigma_{k+1}^L, k+2, \ldots, k+p), \quad (5)$$

where $\mathcal{P}_*[k+1]$ is the set of parts L of $[k+1] := \{1, \ldots, k+1\}$ of even cardinality $\aleph_L$, and if in the case $\nu := \aleph_L/2 > 0$, we use the identification $L \equiv (l_1, \ldots, l_{2\nu})$, then $\sigma^L \in S_{k+1}$ denotes the permutation

$$\begin{aligned} \sigma^L := \; & (1, \ldots, \widehat{l_1}, \ldots, l_2, l_1, l_2+1, \ldots, \widehat{l_3}, \ldots, l_4, l_3, l_4+1, \ldots \\ & \ldots, \widehat{l_{2r-3}}, \ldots, l_{2r-2}, l_{2r-3}, l_{2r-2}+1, \ldots, \widehat{l_{2r-1}}, \ldots, l_{2r}, l_{2r-1}, l_{2r}+1, \ldots, \\ & \ldots, \widehat{l_{2\nu-3}}, \ldots, l_{2\nu-2}, l_{2\nu-3}, l_{2\nu-2}+1, \ldots, \widehat{l_{2\nu-1}}, \ldots, l_{2\nu}, l_{2\nu-1}, l_{2\nu}+1, \ldots, k+1), \end{aligned}$$

In the case $\nu = 0$, we set $\sigma^\emptyset := \mathrm{Id}$.

All elements of S^*_{k+1} can be described in this way. (The elementary proof is left to the reader). We notice the equality

$$\varepsilon_{\sigma^L} \quad = \quad (-1)^{\sum_{r=1}^{\nu}(l_{2r} - l_{2r-1})} = (-1)^{|L|}.$$

We give a proof of Lemma 3 independent of the statement of Lemma 2.

Proof For $k = 1$, Lemma 3 follows from the definition of d_1^∇. We show by induction that it holds for all values of k. Indeed we assume true the formula (5) for a certain k. We define the

set $\mathcal{P}_*(A)$, for any subset $A = \{a_0\} + \{a_1, \ldots, a_2\} \subset \mathbb{Z}_{>0}$, $0 < a_0 \leqslant a_1 < a_2$, by replacing $[k+1]$ with A in the definition of $\mathcal{P}_*[k+1]$. For any $L \in \mathcal{P}_*(A)$, we define also by $\sigma^L \in S_A := \{\text{the set of permutations of } A\}$ as we did for $[k+1]$. With these notations we have the expressions

$$\begin{aligned}
&(d_1^\nabla)^k \theta(2, \ldots, k+p+1) \\
= &\sum_{L \in \mathcal{P}_*([k+1]+1)} \varepsilon_{\sigma^L} \nabla^k \theta(\sigma_1^L, \ldots, \sigma_{k+1}^L, k+3, \ldots, k+p+1), \\
&(d_1^\nabla)^k \theta(1, 3, \ldots, k+p+1) \\
= &\sum_{L \in \mathcal{P}_*([k+2]-\{2\})} \varepsilon_{\sigma^L} \nabla^k \theta(\sigma_1^L, \ldots, \sigma_{k+1}^L, k+3, \ldots, k+p+1).
\end{aligned}$$

Using the equality (4), we deduce

$$\begin{aligned}
&(d_1^\nabla)^{k+1} \theta(1, \ldots, k+p+1) \\
= &\sum_{L \in \mathcal{P}_*([k+1]+1)} \varepsilon_{\sigma^L} \nabla^{k+1} \theta(1, \sigma_1^L, \ldots, \sigma_{k+1}^L, k+3, \ldots, k+p+1) \\
- &\sum_{L \in \mathcal{P}_*([k+2]-\{2\})} \varepsilon_{\sigma^L} \nabla^{k+1} \theta(2, \sigma_1^L, \ldots, \sigma_{k+1}^L, k+3, \ldots, k+p+1).
\end{aligned}$$

For any element $L \in \mathcal{P}_*[k+2]$, we distinguish two cases. In the case $l_1 > 1$ we have the expression $\sigma^L = (1, \sigma^{L'})$, with $L' \in \mathcal{P}_*([k+1]+1)$ given by $L' := L$. In this case we have also the obvious identity $\varepsilon_{\sigma^L} = \varepsilon_{\sigma^{L'}}$. In the case $l_1 = 1$ we have $\sigma^L = (2, \sigma^{L''})$, with $L'' \in \mathcal{P}_*([k+2]-\{2\})$ given by $L'' := L$ in the sub-case $l_2 > 2$ and by $L'' := (l_3, \ldots, l_{2\nu})$, in the sub-case $l_2 = 2$. (We notice that $L'' = \emptyset$, if $\nu = 1$). In both sub cases $\varepsilon_{\sigma^L} = -\varepsilon_{\sigma^{L''}}$, which implies the required conclusion of the induction. □

We show now the following fundamental Proposition.

Proposition 1 *Let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of $\mathbb{C}T_M$ with curvature operator $R^\nabla(\cdot,\cdot)\cdot \equiv R^\nabla(\cdot,\cdot,\cdot)$. Then for all integers $k \geqslant 1$,*

$$\mathrm{Sym}_{2,\ldots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2\right] = \mathrm{Sym}_{2,\ldots,k+4} \Phi_k, \tag{6}$$

$$\mathrm{Sym}_{3,\ldots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2\right] = (-1)^k \mathrm{Sym}_{3,\ldots,k+4} (T_k - \mathrm{Alt}_2 Q_k), \tag{7}$$

with

$$\begin{aligned}
\Phi_k(1, \ldots, k+4) &:= -(-1)^k \nabla^{k+1} R^\nabla(2, \ldots, k+3, 1, k+4), \\
T_k(1, \ldots, k+4) &:= \nabla^{k+1} R^\nabla(3, \ldots, k+3, 1, 2, k+4), \\
Q_k(1, \ldots, k+4) &:= \sum_{j=2}^{k+1} \nabla^{j-2} \left(R^\nabla . \nabla^{k+1-j} R^\nabla\right)(3, \ldots, j, 2, j+1, \ldots, k+3, 1, k+4).
\end{aligned}$$

Proof We notice first that the symmetrization of a tensor with two alternating entries vanishes. Applying this fact to the covariant derivatives of R^∇ in the expression (5) with $\theta := (\nabla R^\nabla)_2$, we infer the identities

$$\begin{aligned}
\mathrm{Sym}_{2,\ldots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2\right] &= (-1)^k \mathrm{Sym}_{2,\ldots,k+4} \varphi_k, \\
\mathrm{Sym}_{3,\ldots,k+4} \left[(d_1^\nabla)^k (\nabla R^\nabla)_2\right] &= (-1)^k \mathrm{Sym}_{3,\ldots,k+4} \mathrm{Alt}_2 \varphi_k,
\end{aligned}$$

with

$$\varphi_k(1, \ldots, k+4) := \nabla^k (\nabla R^\nabla)_2(2, \ldots, k+1, 1, k+2, k+3, k+4),$$

which we can rewrite as

$$\begin{aligned}
\varphi_k(1, \ldots, k+4) &= \nabla^{k+1} R^\nabla(2, \ldots, k+2, 1, k+3, k+4) \\
&= -\nabla^{k+1} R^\nabla(2, \ldots, k+3, 1, k+4),
\end{aligned}$$

thanks to the alternating property of R^∇. We deduce in particular the identity (6). Using the identity (3) we infer

$$\begin{aligned}\varphi_k(1,\dots,k+4) &= -\nabla^{k+1}R^\nabla(3,2,4,\dots,k+3,1,k+4)\\ &- \left(R^\nabla.\nabla^{k-1}R^\nabla\right)(2,\dots,k+3,1,k+4).\end{aligned}$$

We now show by finite induction on p, the identity

$$\begin{aligned}\varphi_k(1,\dots,k+4) &= -\nabla^{k+1}R^\nabla(3,\dots,p+1,2,p+2,\dots,k+3,1,k+4)\\ &- \sum_{j=2}^{p}\nabla^{j-2}\left(R^\nabla.\nabla^{k+1-j}R^\nabla\right)(3,\dots,j,2,j+1,\dots,k+3,1,k+4),\end{aligned}$$

for any integer p, with $2 \leqslant p \leqslant k+1$. We assume them true for $p < k+1$. Applying the covariant derivative $\nabla^{p-1}_{3,\dots,p+1}$ to the identity (3), with $\theta := \nabla^{k+2-p}R^\nabla$, we infer the conclusion of the induction. If we set $p := k+1$ in the previous identities we obtain

$$\begin{aligned}\mathrm{Alt}_2\,\varphi_k(1,\dots,k+4) &= -\nabla^{k+1}R^\nabla(3,\dots,k+2,2,k+3,1,k+4)\\ &- \nabla^{k+1}R^\nabla(3,\dots,k+2,1,2,k+3,k+4)\\ &- \mathrm{Alt}_2\,Q_k(1,\dots,k+4),\end{aligned}$$

thanks to the alternating property of R^∇. Then the identity (7) follows from the differential Bianchi identity. □

2.2 Equivalent definitions of the tensor S_k

We start by noticing a few elementary equivalent definitions of the tensors S_k introduced in the statement of corollary 1. We assume more in general here that ∇ acts on the smooth sections of $\mathbb{C}T_M$. For notational simplicity we use the identification $\nabla^k S \equiv S^{(k)}$ for any tensor S. We notice that the tensor Φ_k in the statement of proposition 1, is well defined for all $k \geqslant -1$. Using the identity (6) we infer the following equivalent definition for S_k.

Lemma 4 *The tensors S_k defined in the statement of corollary 1 satisfy for all $k \geqslant 2$,*

$$\begin{aligned}S_k &= \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\dots,k+1}\,\theta_k,\\ \theta_k &:= 2i^k\Phi_{k-3} + k!\sum_{p=2}^{k-2}(pS_p\wedge S_{k-p}) + \sum_{r=3}^{k-2}(r+1)!\sum_{p=2}^{r-1}\rho_{k,r,p},\\ \rho_{k,r,p}(1,\dots,k+1) &:= (-i)^{k-1-r}(pS_p\wedge S_{r-p+1})^{(k-1-r)}(2,\dots,k-r,1,k-r+1,\dots,k+1).\end{aligned}$$

Remark 1 Notice that

$$\theta_4 = 2\Phi_1 + 4!2S_2\wedge S_2,$$

and

$$\begin{aligned}\mathrm{Sym}_{2,3,4,5}(S_2\wedge S_2) &= \mathrm{Sym}_{2,3,4,5}\,\mu_4,\\ \mu_4(1,\dots,5) &:= -S_2(2,S_2(1,3,4),5).\end{aligned}$$

This implies

$$\begin{aligned}\mathrm{Sym}_{2,3,4,5}(S_2\wedge S_2) &= \frac{1}{18}\,\mathrm{Sym}_{2,3,4,5}\,\tilde\mu_4,\\ \tilde\mu_4(1,\dots,5) &= R(2,R(1,3,4),5).\end{aligned}$$

Remark 2 Notice that

$$\begin{aligned}\theta_5 &= 2i\Phi_2 + 5!2S_2 \wedge S_3 + 5!3S_3 \wedge S_2 + 4!\rho_{5,3,2},\\ \rho_{5,3,2}(1,\dots,6) &= -2i(S_2 \wedge S_2)'(2,1,3,4,5,6),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sym}_{2,3,4,5,6}\,\theta_5 &= \mathrm{Sym}_{2,3,4,5,6}(2i\Phi_2 + \mu_5),\\ \mu_5(1,\dots,6) &:= -5!2S_2(2,S_3(1,3,4,5),6) - 5!3S_3(2,S_2(1,3,4),5,6)\\ &+ 2i4!S_2'(2,3,S_2(1,4,5),6) + 2i4!S_2(3,S_2'(2,1,4,5),6).\end{aligned}$$

$$\begin{aligned}\mathrm{Sym}_{2,3,4,5,6}\,\mu_5 &= \mathrm{Sym}_{2,3,4,5,6}\,\tilde{\mu}_5\\ \tilde{\mu}_5(1,\dots,6) &:= -6iR(2,R'(3,1,4,5),6) - 6iR'(2,3,R(1,4,5),6).\end{aligned}$$

Remark 3 We observe the equality

$$\begin{aligned}&(S_p \wedge_1 S_{r-p+1})(1,k-r+1,\dots,k+1)\\ :=\;& S_p(1,S_{r-p+1}(k-r+1,\dots,k-p+2),k-p+3,\dots,k+1)\\ -\;& S_p(k-r+1,S_{r-p+1}(1,k-r+2,\dots,k-p+2),k-p+3,\dots,k+1).\end{aligned}$$

Now, the fact that the tensors θ_k have at least one couple of alternating entries combined with the fact that the total symmetrization of such tensors vanishes implies the following equivalent definition of S_k.

Lemma 5 *The tensors S_k defined in the statement of corollary 1 satisfy for all $k \geqslant 2$,*

$$\begin{aligned}S_k &= \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\dots,k+1}\,\tilde{\theta}_k,\\ \tilde{\theta}_k &:= 2i^k\Phi_{k-3} + \mu_k\\ \mu_k &:= -\sum_{r=3}^{k-1}(r+1)!(-i)^{k-1-r}\sum_{p=2}^{r-1}p\sum_{I\subseteq\{2,\dots,k-r\}}\rho^I_{k,r,p},\end{aligned}$$

with

$$\begin{aligned}&\rho^I_{k,r,p}(1,\dots,k+1)\\ :=\;& S_p^{(|I|)}(I,k-r+1,S_{r-p+1}^{(|\complement I|)}(\complement I,1,k-r+2,\dots,k-p+2),k-p+3,\dots,k+1).\end{aligned}$$

(Notice that in the case $r = k-1$ the set I is empty). Using the elementary properties of the symmetrization operators we infer the following equivalent definition of S_k.

Lemma 6 *The tensors S_k defined in the statement of corollary 1 satisfy for all $k \geqslant 2$,*

$$\begin{aligned}S_k &= \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\dots,k+1}\,\boldsymbol{\theta}_k,\\ \boldsymbol{\theta}_k &:= 2i^k\Phi_{k-3} + \boldsymbol{\mu}_k,\\ \boldsymbol{\mu}_k &:= \sum_{r=3}^{k-1}\sum_{p=2}^{r-1}\frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!}\sum_{I\subseteq\{2,\dots,k-r\}}\sum_{j=k-p+2}^{k+1}\rho^I_{k,r,p,j},\end{aligned}$$

with

$$\begin{aligned}&\rho^I_{k,r,p,j}(1,\dots,k+1)\\ :=\;& \boldsymbol{\theta}_p^{(|I|)}(I,k-r+1,k-p+3,\dots,j,\boldsymbol{\theta}_{r-p+1}^{(|\complement I|)}(\complement I,1,k-r+2,\dots,k-p+2),j+1,\dots,k+1).\end{aligned}$$

For any $A \equiv (a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geqslant 0}$, we define the multinomial

$$\binom{|A|}{A} \quad := \quad \frac{|A|!}{A!}.$$

Proposition 2 *For any integer $k \geqslant 2$,*

$$S_k \quad = \quad \frac{i}{(k+1)!k!} \operatorname{Sym}_{2,\ldots,k+1} \widetilde{\boldsymbol{\theta}}_k,$$

with

$$\widetilde{\boldsymbol{\theta}}_k \quad = \quad \sum_{D \vDash_0 k-2l_D} C_D R^D,$$

where the coefficients C_D are given by $C_D := 2(-i)^k$ for $l_D = 1$ and by the recursive formula

$$C_D = - \sum_{\substack{1 \leqslant h \leqslant l_D - 1 \\ 0 \leqslant A \leqslant D \\ l_A = l_D}} (-i)^{|D-A|} \binom{|D-A|}{D-A} \frac{\|A\|! C_{A_h^-} C_{A_h^+}}{(\|A_h^-\|+1)!(\|A_h^+\|+1)!}, \tag{8}$$

for $l_D \geqslant 2$.

Proof We assume by induction on $k \geqslant 3$ that

$$S_p \quad = \quad \frac{i}{(p+1)!p!} \operatorname{Sym}_{2,\ldots,p+1} \tilde{\boldsymbol{\theta}}_p,$$

for any $p = 2, \ldots, k-1$, with $\widetilde{\boldsymbol{\theta}}_p$ under the form

$$\tilde{\boldsymbol{\theta}}_p \quad = \quad \sum_{A \vDash_0 p-2l_A} C_A R^A,$$

for some coefficients C_A, with obviously $\tilde{\boldsymbol{\theta}}_2 = 2R$ and $C_A = 2(-i)^p$ in the case $l_A = 1$, thanks to the equivalent definition of S_p given in lemma (6). We show that $\widetilde{\boldsymbol{\theta}}_k$ writes under the form claimed in the statement of the proposition 2 with the coefficients C_D given by the recursive formula (8). For this purpose we notice first that the straightforward argument showing the equivalent definition (6) of S_k implies for all $k \geqslant 3$ the identity

$$S_k \quad = \quad \frac{i}{(k+1)!k!} \operatorname{Sym}_{2,\ldots,k+1} \hat{\boldsymbol{\theta}}_k,$$

with

$$\begin{aligned} \hat{\boldsymbol{\theta}}_k \quad &:= \quad 2i^k \Phi_{k-3} + \widehat{\boldsymbol{\mu}}_k, \\ \hat{\boldsymbol{\mu}}_k \quad &:= \quad \sum_{r=3}^{k-1} \sum_{p=2}^{r-1} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{I \subseteq \{2,\ldots,k-r\}} \sum_{j=k-p+2}^{k+1} \hat{\rho}^I_{k,r,p,j}, \end{aligned}$$

and with

$$\begin{aligned} & \hat{\rho}^I_{k,r,p,j}(1, \ldots, k+1) \\ := \quad & \tilde{\boldsymbol{\theta}}_p^{(|I|)}(I, k-r+1, k-p+3, \ldots, j, \tilde{\boldsymbol{\theta}}_{r-p+1}^{(|\complement I|)}(\complement I, 1, k-r+2, \ldots, k-p+2), j+1, \ldots, k+1). \end{aligned}$$

For any integer $a \geqslant 1$, we denote by $[a] := \{1, \ldots, a\}$ and we set $[0] := \emptyset$. For any integer $b \geqslant 1$, we denote by $\operatorname{Map}(a, b)$ the set of maps $f : [a] \longrightarrow [b]$.

We observe now that for any $A \vDash_0 p - 2l_A$ and any integer $q \geqslant 0$, the Leibniz identity implies

$$(R^A)^{(q)}(1,\dots,q+p+1) \quad = \quad \sum_{f\in \mathrm{Map}(q,l_A)} \left[\prod_{j=1}^{l_A} R^{A,q}_{f,j}\right]\cdot(q+1),$$

with

$$R^{A,q}_{f,j} \quad := \quad R^{(a_j+|f^{-1}(j)|)}\left(f^{-1}(j), \|A\|_{j-1}+q+2,\dots,\|A\|_j+q+1,\bullet,p-j+q+2\right).$$

We write now

$$\tilde{\boldsymbol{\theta}}^{(|I|)}_p \quad = \quad \sum_{A\vDash_0 p-2l_A} C_A (R^A)^{(|I|)},$$

and we notice that the shape of $(R^A)^{(|I|)}$ shows that $\hat{\rho}^I_{k,r,p,k-h+1}$, with $h := l_A$, are the only terms with non vanishing symmetrization of the variables $2,\dots,k+1$. Indeed we consider the factor

$$R^{(a_h+|f^{-1}(h)|)}(f^{-1}(h), p-h+1-a_h,\dots,p-h+1,1,p-h+2), \tag{9}$$

with $f\in \mathrm{Map}(I,h)$ in the expression of $(R^A)^{(|I|)}(I,1,\dots,p+1)$ and we perform the change of variables

$$\begin{array}{ccccccccc} 1 & & 2 & \dots & j-k+p-1 & & j-k+p & \dots & p+1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k-r+1 & & k-p+3 & \dots & j & \widetilde{\boldsymbol{\theta}}^{(|\complement I|)}_{r-p+1}(\complement I,1,k-r+2,\dots,k-p+2) & j+1 & \dots & k+1 \end{array}$$

in the factor (9) with $j\in\{k-p+2,\dots,k+1\}$. (We recall that we ignore the standard increasing notation $a,\dots,b$ when $a>b$.) The only case when the symmetrization of the variables $2,\dots,k+1$ does not annihilate the factor (9), is when the index $j\in\{k-p+2,\dots,k+1\}$ satisfies the equality

$$j-k+p \quad = \quad p-h+1,$$

i.e. only when $j=k-h+1$. This shows the required statement about $\hat{\rho}^I_{k,r,p,k-h+1}$.

We infer that for all $k\geqslant 3$ the equality hold

$$S_k \quad = \quad \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\dots,k+1}\,\check{\boldsymbol{\theta}}_k,$$

with

$$\check{\boldsymbol{\theta}}_k \quad := \quad 2i^k\Phi_{k-3} + \check{\boldsymbol{\mu}}_k,$$

with

$$\check{\boldsymbol{\mu}}_k \quad := \quad \sum_{\substack{3\leqslant r\leqslant k-1\\ 2\leqslant p\leqslant r-1}} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A\vDash_0 p-2l_A\\ B\vDash_0 r-p+1-2l_B\\ I\subseteq\{2,\dots,k-r\}}} C_A C_B \check{\rho}^{I,A,B}_{k,r,p},$$

with

$$\begin{array}{rl} & \check{\rho}^{I,A,B}_{k,r,p}(1,\dots,k+1) \\ := & (R^A)^{(|I|)}(I,k_r+1,k_p+3,\dots,k_{l_A}+1,(R^B)^{(|\complement I|)}(\complement I,1,k_r+2,\dots,k_p+2),k_{l_A}+2,\dots,k+1), \end{array}$$

and with $k_s := k-s$. Using the expression

$$R^A(x,y,\dots,p+y-1) \quad = \quad \left[\prod_{j=1}^{l_A} R^{(a_j)}\left(\|A\|_{j-1}+y,\dots,\|A\|_j+y-1,\bullet,p-j+y\right)\right]\cdot x,$$

for any integers $x, y > 0$ and using the change of variables

$$\begin{array}{ccccccc} k_p+3 & \dots & k_{l_A}+1 & & k_{l_A}+2 & k_{l_A}+3 & \dots & k+2 \\ \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow \\ k_p+3 & \dots & k_{l_A}+1 & R^B(1,k_r+2,\dots,k_p+2) & & k_{l_A}+2 & \dots & k+1 \end{array}$$

we infer the equality

$$\begin{aligned} & R^A(k_r+1,k_p+3,\dots,k_{l_A}+1,R^B(1,k_r+2,\dots,k_p+2),k_{l_A}+2,\dots,k+1) \\ = & \left[\prod_{j=1}^{l_A-1} R^{(a_j)}\left(\|A\|_{j-1}+k_p+3,\dots,\|A\|_j+k_p+2,\bullet,k_j+2\right)\right]\cdot \\ \cdot & R^{(a_{l_A})}\left(k_{l_A}+2-a_{l_A},\dots,k_{l_A}+1,R^B\left(1,k_r+2,\dots,k_p+2\right),k_r+1,k_{l_A}+2\right). \end{aligned}$$

Indeed notice that $|A|+l_A = p-l_A = k_{l_A}+2-(k_p+2)$ and $p-j+(k_p+3)-1 = k_j+2$. Expanding $R^B\left(1,k_r+2,\dots,k_p+2\right)$ we obtain

$$\begin{aligned} & R^A(k_r+1,k_p+3,\dots,k_{l_A}+1,R^B(1,k_r+2,\dots,k_p+2),k_{l_A}+2,\dots,k+1) \\ = & \left[\prod_{j=1}^{l_A-1} R^{(a_j)}\left(\|A\|_{j-1}+k_p+3,\dots,\|A\|_j+k_p+2,\bullet,k_j+2\right)\right]\cdot \\ \cdot & R^{(a_{l_A})}\left(k_{l_A}+2-a_{l_A},\dots,k_{l_A}+1,\bullet,k_r+1,k_{l_A}+2\right)\cdot \\ \cdot & \left[\prod_{t=1}^{l_B} R^{(b_t)}\left(\|B\|_{t-1}+k_r+2,\dots,\|B\|_t+k_r+1,\bullet,k_p-t+3\right)\right]\cdot 1, \end{aligned}$$

since $\|B\|-t+k_r+2 = k_p-t+3$.

For any integer $a \geqslant 1$, we denote by $[a]_1 := \{2,\dots,a+1\}$ and we set $[0]_1 := \emptyset$. For any integer $b \geqslant 1$, we denote by $\mathrm{Map}([a]_1,b)$ the set of maps $f : [a]_1 \longrightarrow [b]$. Using the Leibniz formula and the alternating property of R we infer the identity

$$\check{\boldsymbol{\mu}}_k = \sum_{\substack{3\leqslant r\leqslant k-1\\ 2\leqslant p\leqslant r-1}} \frac{(r+1)!(-i)^{k-1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A\vDash_0 p-2l_A\\ B\vDash_0 r-p+1-2l_B\\ f\in\mathrm{Map}([k_r-1]_1,l_A+l_B)}} C_A C_B \check{\rho}_{k,r,p}^{f,A,B}, \tag{10}$$

with

$$\begin{aligned} - \quad & \check{\rho}_{k,r,p}^{f,A,B}(1,\dots,k+1) \\ := \quad & \left[\prod_{j=1}^{l_A-1} R^{(a_j+|f^{-1}(j)|)}(f^{-1}(j),\|A\|_{j-1}+k_p+3,\dots,\|A\|_j+k_p+2,\bullet,k_j+2)\right]\cdot \\ \cdot \quad & R^{(a_{l_A}+|f^{-1}(l_A)|)}\left(f^{-1}(l_A),k_{l_A}+2-a_{l_A},\dots,k_{l_A}+1,k_r+1,\bullet,k_{l_A}+2\right)\cdot \\ \cdot \quad & \left[\prod_{t=1}^{l_B} R^{(b_t+|f^{-1}(l_A+t)|)}\left(f^{-1}(l_A+t),\|B\|_{t-1}+k_r+2,\dots,\|B\|_t+k_r+1,\bullet,k_p-t+3\right)\right]\cdot 1. \end{aligned}$$

We observe the equality

$$-\,\mathrm{Sym}_{2,\dots,k+1}\,\check{\rho}_{k,r,p}^{f,A,B} = \mathrm{Sym}_{2,\dots,k+1}\,R^D, \tag{11}$$

with $l_D := l_A + l_B$ and with $D \vDash_0 k-2l_D$ given by

$$d_j := \begin{cases} a_j + |f^{-1}(j)|, & j = 1,\dots,l_A, \\ b_{j-l_A} + |f^{-1}(j)|\,, & j = l_A+1,\dots,l_D. \end{cases} \tag{12}$$

We deduce that $\widetilde{\boldsymbol{\theta}}_k$ writes under the form claimed in the statement of the proposition 2.

We denote by abuse of notations $D-A-B \geqslant 0$ when $d_j - a_j \geqslant 0$ for all $j = 0, \ldots, l_A$ and when $d_j - b_{j-l_A} \geqslant 0$ for all $j = l_A+1, \ldots, l_D$.

Given A, B, D as before we denote by $\mathrm{Map}^D_{A,B}([k-r-1]_1, l_D)$ the sub-set of $\mathrm{Map}([k-r-1]_1, l_D)$ given by the elements f which satisfy condition (12). We denote also by

$$M^D_{A,B,k,r} \quad := \quad |\,\mathrm{Map}^D_{A,B}([k-r-1]_1, l_D)|.$$

Using (10) and (11), we infer the recursive formula

$$C_D \quad = \quad \sum_{\substack{3 \leqslant r \leqslant k-1 \\ 2 \leqslant p \leqslant r-1}} \frac{(r+1)!(-i)^{k+1-r}}{(r-p+2)!(p+1)!} \sum_{\substack{A \vDash_0 p-2l_A \\ B \vDash_0 r-p+1-2l_B \\ l_A+l_B = l_D \\ D-A-B \geqslant 0}} C_A C_B M^D_{A,B,k,r},$$

We remind that in the case $l_D = 1$ hold the formula $C_D = 2(-i)^k$ for all $k \geqslant 2$.

Remark 4 Let $A \vDash_0 a$ with $l_A = q$ and consider the sub-set $\mathrm{Map}_A(a,q)$ of $\mathrm{Map}(a,q)$ given by the elements f such that $|f^{-1}(j)| = a_j$, for all $j = 1, \ldots, q$. Then

$$|\,\mathrm{Map}_A(a,q)| \quad = \quad \binom{a}{A}.$$

The fact that in our set-up $|D| - |A| - |B| = k-r-1$, allows to apply the previous remark to the set $\mathrm{Map}^D_{A,B}([k-r-1]_1, l_D)$. We infer the formula

$$M^D_{A,B,k,r} \quad = \quad (k-r-1)! \prod_{j=1}^{l_A} \frac{1}{(d_j - a_j)!} \prod_{j=l_A+1}^{l_D} \frac{1}{(d_j - b_{j-l_A})!}.$$

We conclude the explicit recursive formula

$$C_D \quad = \quad \sum_{\substack{3 \leqslant r \leqslant k-1 \\ 2 \leqslant p \leqslant r-1 \\ A \vDash_0 p-2l_A \\ B \vDash_0 r-p+1-2l_B \\ l_A+l_B = l_D \\ D-A-B \geqslant 0}} \frac{(-i)^{k+1-r}(k-r-1)!(r+1)!C_A C_B}{(r-p+2)!(p+1)! \prod_{j=1}^{l_A}(d_j-a_j)! \prod_{j=l_A+1}^{l_D}(d_j - b_{j-l_A})!},$$

which rewrites as (8). This concludes the proof of proposition 2. □

Example 1 In this example we set for simplicity $R' := R^{(1)}$ and $R'' := R^{(2)}$. For $k = 6$ we have the expression

$$\begin{aligned}
\tilde{\boldsymbol{\theta}}_6(1,\ldots,7) &= -2\Phi_3(1,\ldots,7) \\
&+ C_{2,0}R''(2,3,4,R(5,1,6),7) + C_{0,2}R(2,R''(3,4,5,1,6),7) \\
&+ C_{1,1}R'(2,3,R'(4,5,1,6),7) + C_{0,0,0}R(2,R(3,R(4,1,5),6),7), \\
C_{2,0} &= 10, \\
C_{0,2} &= 10, \\
C_{1,1} &= 17, \\
C_{0,0,0} &= -\frac{32}{3}.
\end{aligned}$$

3 Expliciting the recursive formula for C_D

Lemma 7 *For any integer $k \geqslant 3$ and any $D \vDash_0 k - 2l_D$ with $l_D \geqslant 1$,*

$$C_D \quad = \quad -i \sum_{p \,\in\, \mathrm{Supp}\, D} C_{D-1_p} - k! \sum_{h=1}^{l_D-1} \frac{C_{D_h^-} C_{D_h^+}}{(\|D_h^-\|+1)!(\|D_h^+\|+1)!}.$$

Proof In the case $l_D = 1$ we have $C_{k-2} = -iC_{k-3}$, for $k \geqslant 3$ by definition. This coincides with what provides the formula in the lemma under consideration. We consider now the case $l_D \geqslant 2$ and we use the identifications $D \equiv (d_1, \ldots, d_{l_D}), A \equiv (a_1, \ldots, a_{l_D})$. Using the recursive formula (8) we write

$$\begin{aligned} & -i \sum_{p \in \operatorname{Supp} D} C_{D-1_p} \\ = & - \sum_{\substack{p \in \operatorname{Supp} D \\ 0 \leqslant A \leqslant D-1_p \\ l_A = l_D}} (-i)^{|D|-|A|} \frac{d_p - a_p}{|D|-|A|} \binom{|D|-|A|}{D-A} F_A, \end{aligned}$$

with

$$F_A := \|A\|! \sum_{h=1}^{l_D-1} \frac{C_{A_h^-} C_{A_h^+}}{(\|A_h^-\|+1)!(\|A_h^+\|+1)!}.$$

Let $\mathbb{I} \equiv (1, \ldots, 1) \in \mathbb{Z}^{l_D}$ and let $S_D := |\operatorname{Supp} D|$. We denote by $\mathbb{I}_D \in \mathbb{Z}^{l_D}$ the vector such that $\operatorname{Supp} \mathbb{I}_D = \operatorname{Supp} D$ and $0 \leqslant \mathbb{I}_D \leqslant \mathbb{I}$. Then

$$\begin{aligned} & -i \sum_{p \in \operatorname{Supp} D} C_{D-1_p} \\ = & - \sum_{\substack{0 \leqslant A \leqslant D - \mathbb{I}_D \\ l_A = l_D}} (-i)^{|D|-|A|} \left[\sum_{\substack{1 \leqslant p \leqslant l_D \\ p \in \operatorname{Supp} D}} \frac{d_p - a_p}{|D|-|A|} \right] \binom{|D|-|A|}{D-A} F_A \\ & - \sum_{\substack{p \in \operatorname{Supp} D \\ D - \mathbb{I}_D \leqslant A \leqslant D-1_p \\ l_A = l_D \\ |D| - S_D < |A|}} \frac{(-i)^{|D|-|A|}}{|D|-|A|} \binom{|D|-|A|}{D-A} F_A. \end{aligned}$$

We notice indeed that the conditions $D - \mathbb{I}_D \leqslant A \leqslant D - 1_p$, $|D| - S_D < |A|$, when they are not empty, imply $a_p = d_p - 1$. We infer

$$\begin{aligned} & -i \sum_{p \in \operatorname{Supp} D} C_{D-1_p} \\ = & - \sum_{\substack{0 \leqslant A \leqslant D - \mathbb{I}_D \\ l_A = l_D}} (-i)^{|D|-|A|} \binom{|D|-|A|}{D-A} F_A \\ & - \sum_{\substack{D - \mathbb{I}_D \leqslant A \leqslant D \\ l_A = l_D \\ |D| - S_D < |A| < |D|}} |\{p : a_p = d_p - 1\}| \frac{(-i)^{|D|-|A|}}{|D|-|A|} \binom{|D|-|A|}{D-A} F_A \\ = & - \sum_{\substack{0 \leqslant A \leqslant D \\ l_A = l_D \\ |A| < |D|}} (-i)^{|D|-|A|} \binom{|D|-|A|}{D-A} F_A, \end{aligned}$$

since $|\{p : a_p = d_p - 1\}| = |D| - |A|$. Then the conclusion follows from the formula (8). □

Definition 5 *Given $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$ and $A \in \mathbb{Z}^p_{\geqslant 0}, B \in \mathbb{Z}^q_{\geqslant 0}$, with $p, q \in \mathbb{Z}_{>0}$ we define the concatenation product*

$$(t^A s^p) * (t^B s^q) := t_1^{a_1} \cdots t_p^{a_p} t_{p+1}^{b_1} \cdots t_{p+q}^{b_q} s^{p+q},$$

and we extend it by linearity.

Definition 6 *Given $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$, $A \in \mathbb{Z}^p_{\geqslant 0}$ and $j \in \mathbb{Z}_{>0}$ we define the contraction product*

$$t_j \neg (t^A s^p) \quad := \quad t^{A+1_j} s^p,$$

where $A + 1_j := A + 1_{p,j}$, if $j \leqslant p$, with $1_{p,j}$ the vector of length p with vanishing components except the j-th one which has value 1 and where $A + 1_j := A$, if $j > p$. We extend the contraction product by linearity.

Lemma 8 *Let $t \in \mathbb{R}^{\mathbb{Z}_{>0}}$, $s \in \mathbb{R}$, and $x \in \mathbb{R}$. Then the function*

$$u(x,t,s) \quad := \quad \sum_{D \geqslant 0} C_D t^D s^{l_D} \frac{x^{\|D\|+1}}{(\|D\|+1)!},$$

is the unique solution of the Riccati-type ODE

$$\partial_x u + \frac{u * u}{x^2} + i \left(\sum_{j \geqslant 1} t_j \right) \neg u + s x^2 \quad = \quad 0,$$

with the initial condition $u(0,\cdot,\cdot) \equiv 0$.

Proof We write first

$$\begin{aligned}
\partial_x u(x,t,s) &= \sum_{D \geqslant 0} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} \\
&= \sum_{\substack{D \geqslant 0 \\ |D| + 2l_D \geqslant 3}} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} + C_{2,\mathbf{0}} s \frac{x^2}{2} \\
&= \sum_{\substack{D \geqslant 0 \\ |D| + 2l_D \geqslant 3}} C_D t^D s^{l_D} \frac{x^{\|D\|}}{\|D\|!} - s x^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(u * u)(x,t,s) &= \left[\sum_{A \geqslant 0} C_A t^A s^{l_A} \frac{x^{\|A\|+1}}{(\|A\|+1)!} \right] * \left[\sum_{B \geqslant 0} C_B t^B s^{l_B} \frac{x^{\|B\|+1}}{(\|B\|+1)!} \right] \\
&= x^2 \sum_{A,B \geqslant 0} C_A C_B (t^A s^{l_A}) * (t^B s^{l_B}) \frac{x^{\|A\|+\|B\|}}{(\|A\|+1)!(\|B\|+1)!} \\
&= x^2 \sum_{D \geqslant 0} t^D s^{l_D} \sum_{h=1}^{l_D - 1} \frac{C_{D_h^-} C_{D_h^+} x^{\|D\|}}{(\|D_h^-\|+1)!(\|D_h^+\|+1)!} \\
&= x^2 \sum_{\substack{D \geqslant 0 \\ |D| + 2l_D \geqslant 3}} t^D s^{l_D} \sum_{h=1}^{l_D - 1} \frac{C_{D_h^-} C_{D_h^+} x^{\|D\|}}{(\|D_h^-\|+1)!(\|D_h^+\|+1)!},
\end{aligned}$$

and

$$\begin{aligned}
\left[\left(\sum_{j\geqslant 1} t_j\right)\neg u\right](x,t,s) &= \sum_{j\geqslant 1}(t_j\neg u)(x,t,s)\\
&= \sum_{j\geqslant 1}\sum_{D\geqslant 0} C_D t^{D+1_j} s^{l_D}\frac{x^{\|D\|+1}}{(\|D\|+1)!}\\
&= \sum_{D\geqslant 0}\sum_{j=1}^{l_D} C_D t^{D+1_j} s^{l_D}\frac{x^{\|D\|+1}}{(\|D\|+1)!}\\
&= \sum_{D\geqslant 0} t^D s^{l_D}\frac{x^{\|D\|}}{\|D\|!}\left(\sum_{p\,\in\,\mathrm{Supp}\,D} C_{D-1_p}\right)\\
&= \sum_{\substack{D\geqslant 0\\ |D|+2l_D\geqslant 3}} t^D s^{l_D}\frac{x^{\|D\|}}{\|D\|!}\left(\sum_{p\,\in\,\mathrm{Supp}\,D} C_{D-1_p}\right).
\end{aligned}$$

Then lemma (7) implies that the function u satisfies the singular Riccati's ODE. The uniqueness statement follows from the following corollary. □

Corollary 4 *Let $t\in\mathbb{R}^{\mathbb{Z}_{>0}}$, $s\in\mathbb{R}$, and $x\in\mathbb{R}$ then the function*

$$\begin{aligned}
u(x,t,s) &= \sum_{l\geqslant 1} u_l(x,t_1,\ldots,t_l)s^l,\\
u_l(x,t_1,\ldots,t_l) &:= \sum_{\substack{D\geqslant 0\\ l_D=l}} C_D t^D\frac{x^{\|D\|+1}}{(\|D\|+1)!},
\end{aligned}$$

satisfies the recursive system of linear ODE's

$$\left\{\begin{array}{l}
\partial_x u_1(x,t_1)+it_1u_1(x,t_1)+x^2=0,\\
\partial_x u_l(x,t_1,\ldots,t_l)+iu_l(x,t_1,\ldots,t_l)\sum_{j=1}^{l}t_j+\frac{1}{x^2}\sum_{p=1}^{l-1}u_p(x,t_1,\ldots,t_p)u_{l-p}(x,t_{p+1},\ldots,t_l) = 0,\\
\forall l\geqslant 2.
\end{array}\right. \tag{13}$$

with initial conditions $u_l(0,t_1,\ldots,t_l)=0$, for all $l\geqslant 1$ and all $(t_1,\ldots,t_l)\in\mathbb{R}^l$.

Proof According to the proof of lemma 8 we can write

$$\begin{aligned}
&(u*u)(x,t,s)\\
=\;& x^2\sum_{A,B\geqslant 0} C_AC_B(t^As^{l_A})*(t^Bs^{l_B})\frac{x^{\|A\|+\|B\|}}{(\|A\|+1)!(\|B\|+1)!}\\
=\;& x^2\sum_{l\geqslant 2}s^l\sum_{p=1}^{l-1}\sum_{\substack{A\geqslant 0\\ l_A=p}}\sum_{\substack{B\geqslant 0\\ l_B=l-p}}\frac{C_AC_Bt_1^{a_1}\cdots t_p^{a_p}t_{p+1}^{b_1}\cdots t_l^{b_{l-p}}x^{\|A\|+\|B\|}}{(\|A\|+1)!(\|B\|+1)!}\\
=\;& \sum_{l\geqslant 2}s^l\sum_{p=1}^{l-1}\left[\sum_{\substack{A\geqslant 0\\ l_A=p}}\frac{C_At_1^{a_1}\cdots t_p^{a_p}x^{\|A\|+1}}{(\|A\|+1)!}\right]\left[\sum_{\substack{B\geqslant 0\\ l_B=l-p}}\frac{C_Bt_{p+1}^{b_1}\cdots t_l^{b_{l-p}}x^{\|B\|+1}}{(\|B\|+1)!}\right]\\
=\;& \sum_{l\geqslant 2}s^l\sum_{p=1}^{l-1}u_p(x,t_1,\ldots,t_p)u_{l-p}(x,t_{p+1},\ldots,t_l).
\end{aligned}$$

On the other hand still according to the proof of lemma 8 we can write

$$\begin{aligned}\left[\left(\sum_{j\geqslant 1} t_j\right)\neg u\right](x,t,s) &= \sum_{D\geqslant 0}\sum_{j=1}^{l_D} C_D t^{D+1_j} s^{l_D}\frac{x^{\|D\|+1}}{(\|D\|+1)!}\\ &= \sum_{l\geqslant 1} s^l \sum_{\substack{D\geqslant 0\\ l_D=l}}\sum_{j=1}^{l} C_D t^D t_j \frac{x^{\|D\|+1}}{(\|D\|+1)!}\\ &= \sum_{l\geqslant 1} s^l\left(\sum_{j=1}^{l} t_j\right) u_l(x,t_1,\ldots,t_l).\end{aligned}$$

The required conclusion follows from the Riccati-type equation in the statement of lemma 8. □

In general we introduce the concatenation product for functions of the type

$$\begin{aligned}u(x,t,s) &= \sum_{l\geqslant 1} u_l(x,t_1,\ldots,t_l)s^l,\\ v(x,t,s) &= \sum_{l\geqslant 1} v_l(x,t_1,\ldots,t_l)s^l,\end{aligned}$$

as

$$(u*v)(x,t,s) = \sum_{l\geqslant 2} s^l \sum_{p=1}^{l-1} u_p(x,t_1,\ldots,t_p)v_{l-p}(x,t_{p+1},\ldots,t_l).$$

The concatenation product is associative.

Lemma 9 *The solution u of the system* (13) *is equivalent to the solution*

$$v(x,t,s) = \sum_{l\geqslant 1} v_l(x,t_1,\ldots,t_l)s^l,$$

of the Riccati-type ODE

$$\partial_x v - \frac{i}{x^2} v*v + ix^2 e^{ixt_1}s = 0, \tag{14}$$

via the identification

$$u_l(x,t_1,\ldots,t_l) = -ie^{-ix\sum_{j=1}^{l} t_j} v_l(x,t_1,\ldots,t_l).$$

Proof We write

$$\begin{aligned}\partial_x u_l(x,t_1,\ldots,t_l) &= -ie^{-ix\sum_{j=1}^{l} t_j}\partial_x v_l(x,t_1,\ldots,t_l)\\ &- e^{-ix\sum_{j=1}^{l} t_j} v_l(x,t_1,\ldots,t_l)\sum_{j=1}^{l} t_j,\end{aligned}$$

$$iu_l(x,t_1,\ldots,t_l)\sum_{j=1}^{l} t_j = e^{-ix\sum_{j=1}^{l} t_j} v_l(x,t_1,\ldots,t_l)\sum_{j=1}^{l} t_j,$$

and

$$\begin{aligned}&\sum_{p=1}^{l-1} u_p(x,t_1,\ldots,t_p)u_{l-p}(x,t_{p+1},\ldots,t_l)\\ =\ & -e^{-ix\sum_{j=1}^{l} t_j}\sum_{p=1}^{l-1} v_p(x,t_1,\ldots,t_p)v_{l-p}(x,t_{p+1},\ldots,t_l).\end{aligned}$$

We infer that the system (13) rewrites as

$$\left\{\begin{array}{l} \partial_x v_1(x,t_1) + ix^2 e^{ixt_1} = 0, \\ \\ \partial_x v_l(x,t_1,\ldots,t_l) - \frac{i}{x^2}\sum_{p=1}^{l-1} v_p(x,t_1,\ldots,t_p) v_{l-p}(x,t_{p+1},\ldots,t_l) \quad = 0, \\ \\ \forall l \geqslant 2, \end{array}\right. \tag{15}$$

which is equivalent to the equation (14). □

3.1 General facts about Riccati-type equations

In this subsection we denote for notational simplicity $\dot{y}(x,t,s) \equiv \partial_x y(x,t,s)$.

Lemma 10 *A solution*

$$y(x,t,s) \quad = \quad \sum_{l\geqslant 1} y(x,t_1,\ldots,t_l)s^l,$$

of the Ricati-type equation

$$\dot{y} \quad = \quad a_2(x) y * y + a_1(x) y + a_0(x,t,s),$$

with a_2 non zero and once differentiable and with

$$a_0(x,t,s) \quad = \quad \sum_{l\geqslant 1} a_{0,l}(x,t_1,\ldots,t_l)s^l,$$

is equivalent to a solution

$$u(x,t,s) \quad = \quad \sum_{l\geqslant 1} u(x,t_1,\ldots,t_l)s^l,$$

of the linear second order ODE

$$\begin{array}{rcl} \ddot{u} - R(x)\dot{u} + H * u & = & 0, \\ R & := & a_1 + \dfrac{\dot{a}_2}{a_2}, \\ H & := & a_2 a_0, \end{array}$$

via the identification $\dot{u} + a_2 y * u = 0$, with $u_1(x,t_1) \equiv f(t_1)$, for some non vanishing function and

$$y_l(x,t_1,\ldots,t_l) = \sum_{\lambda\vDash l} (-1)^{l_\lambda} \frac{\dot{u}_{\lambda_1+1}(x,t_1,\ldots,t_{\lambda_1+1})}{a_2(x) f(t_{\lambda_1+1})} \prod_{j=2}^{l_\lambda} \frac{u_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\ldots,t_{|\lambda|_j+1})}{f(t_{|\lambda|_j+1})}, \tag{16}$$

for all $l \geqslant 1$, with $|\lambda|_j := \sum_{p=1}^{j} \lambda_p$.

Proof The function $v := a_2 y$ satisfies Riccati-type equation

$$\dot{v} \quad = \quad v * v + R(x) v + H(x,t,s).$$

Indeed

$$\begin{array}{rcl} \dot{v} & = & \dot{a}_2 y + a_2 \dot{y} \\ & = & \dfrac{\dot{a}_2}{a_2} v + a_2 \left[a_0 + a_1 y + a_2 y * y\right] \\ & = & a_0 a_2 + \left(a_1 + \dfrac{\dot{a}_2}{a_2}\right) v + v * v. \end{array}$$

We now show that if $\dot{u}+v*u=0$ then the function u satisfies the second order linear ODE in the statement. Indeed

$$\begin{aligned}\ddot{u} &= -\dot{v}*u-v*\dot{u}\\ &= -\dot{v}*u+v*(v*u)\\ &= (-\dot{v}+v*v)*u\\ &= -[R(x)v+H]*u\\ &= R(x)\dot{u}-H*u.\end{aligned}$$

We notice now that the identification $\dot{u}+v*u=0$ implies $\dot{u}_1\equiv 0$, i.e. $u_1(x,t_1)\equiv f(t_1)$ and

$$y_l(x,t_1,\dots,t_l)=-\frac{\dot{u}_{l+1}(x,t_1,\dots,t_{l+1})}{a_2(x)f(t_{l+1})}-\sum_{p=1}^{l-1}y_p(x,t_1,\dots,t_p)\frac{u_{l+1-p}(x,t_{p+1},\dots,t_{l+1})}{f(t_{l+1})},\qquad(17)$$

for all $l\geqslant 1$, which implies the explicit formula (16). Indeed if we set

$$\begin{aligned}A(x,t,s) &:= \sum_{l\geqslant 1}A_l(x,t_1,\dots,t_l)s^l,\\ B(x,t,s) &:= \sum_{l\geqslant 1}B_l(x,t_1,\dots,t_l)s^l,\\ A_l(x,t_1,\dots,t_l) &:= \frac{u_{l+1}(x,t_1,\dots,t_{l+1})}{f(t_{l+1})},\\ B_l(x,t_1,\dots,t_l) &:= \frac{\dot{u}_{l+1}(x,t_1,\dots,t_{l+1})}{a_2(x)f(t_{l+1})},\end{aligned}$$

(notice that the definitions are well posed) then the relation (17) writes as $y=-B-y*A$. Thus

$$y = -B*\sum_{k\geqslant 0}(-1)^k A^{*,k},$$

where we denote by $A^{*,k}$ the k-th power with respect to the concatenation product $*$. (By convention $A^{*,0}:=1$.) In other terms

$$y_l(x,t_1,\dots,t_l) = \sum_{\lambda\vDash l}(-1)^{l_\lambda}B_{\lambda_1}(x,t_1,\dots,t_{\lambda_1})\prod_{j=2}^{l_\lambda}A_{\lambda_j}(x,t_{|\lambda|_{j-1}+1},\dots,t_{|\lambda|_j}),$$

which is precisely formula (16). □

Notice that y_1 is given directly by the solution of the ODE $\dot{y}_1(x,t_1)=a_1(x)y(x,t_1)+a_{0,1}(x,t_1)$.

In this paper we will always consider the case $u(0,t,s)=t_1 s$. In the Riccati-type equation (14) we have $a_2(x)=ix^{-2}$, $a_1=0$, $a_0(x,t,s)=-ix^2e^{ixt_1}s$. Therefore the corresponding second order linear ODE writes as

$$\ddot{U}+\frac{2}{x}\dot{U}+(e^{ixt_1}s)*U=0.\qquad(18)$$

with $U(0,t,s)=t_1 s$ and $\dot{U}(0,t,s)=0$. Then according to formula (16) we infer the expression

$$v_l(x,t_1,\dots,t_l)=-ix^2\sum_{\lambda\vDash l}(-1)^{l_\lambda}\frac{\dot{U}_{\lambda_1+1}(x,t_1,\dots,t_{\lambda_1+1})}{t_{\lambda_1+1}}\prod_{j=2}^{l_\lambda}\frac{U_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\dots,t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}},\qquad(19)$$

for all $l\geqslant 1$, for the solution v of the equation (14).

In order to compute the solution U we need an other elementary lemma.

Lemma 11 *In the set up of the previous lemma, any ODE*

$$\ddot{u}+a_1(x)\dot{u}+a_0*u = f(x,t,s),$$

is equivalent to the ODE

$$\ddot{w} - \frac{1}{4}(2\dot{a}_1 + a_1^2 - 4a_0) * w \quad = \quad e^{\frac{1}{2}\int a_1} f,$$

via the identification

$$u \quad = \quad e^{-\frac{1}{2}\int a_1} w.$$

Proof We write $u = A(x)w$ and we observe the elementary equalities $\dot{u} = \dot{A}w + A\dot{w}$,

$$\ddot{u} \quad = \quad \ddot{A}w + 2\dot{A}\dot{w} + A\ddot{w}.$$

Then the ODE on u in the statement is equivalent to the ODE

$$A\ddot{w} + 2\dot{A}\dot{w} + \ddot{A}w + a_1 A\dot{w} + a_1\dot{A}w + Aa_0 * w \quad = \quad f,$$

i.e.

$$A\ddot{w} + (2\dot{A} + a_1 A)\dot{w} + (\ddot{A} + a_1\dot{A} + Aa_0) * w \quad = \quad f.$$

We seek for A solution of the ODE $2\dot{A} + a_1 A = 0$, i.e. $A = e^{-\frac{1}{2}\int a_1}$. We infer in particular the identity

$$\ddot{A} + \frac{1}{2}\dot{a}_1 A + \frac{1}{2}a_1\dot{A} \quad = \quad 0.$$

Using this we write

$$\begin{aligned} \ddot{A} + a_1\dot{A} + Aa_0 \quad &= \quad -\frac{1}{2}\dot{a}_1 A + \frac{1}{2}a_1\dot{A} + Aa_0 \\ &= \quad -\frac{A}{4}(2\dot{a}_1 + a_1^2 - 4a_0), \end{aligned}$$

which implies the required conclusion. □

We infer that if we set $U = x^{-1}w$, then the equation (18) is equivalent to the equation

$$\ddot{w} + (e^{ixt_1} s) * w = 0, \tag{20}$$

with the initial conditions $w(0,t,s) \equiv 0$ and $\dot{w}(0,t,s) = u(0,t,s) = t_1 s$. (The later condition follows deriving the equality $w = xU$). Then formula (19) implies the expression

$$\begin{aligned} v_l(x,t_1,\ldots,t_l) \quad &= \quad -ix\sum_{\lambda\vDash l}(-1)^{l_\lambda}\frac{\dot{w}_{\lambda_1+1}(x,t_1,\ldots,t_{\lambda_1+1})}{t_{\lambda_1+1}}\prod_{j=2}^{l_\lambda}\frac{w_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\ldots,t_{|\lambda|_j+1})}{xt_{|\lambda|_j+1}} \\ &+ \quad ix\sum_{\lambda\vDash l}(-1)^{l_\lambda}\prod_{j=1}^{l_\lambda}\frac{w_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\ldots,t_{|\lambda|_j+1})}{xt_{|\lambda|_j+1}}, \end{aligned}$$

for all $l \geqslant 1$, for the solution v of the equation (14).

3.2 Expression for the solution of the equation (20)

We notice first the equalities $w(0,t,s) = \ddot{w}(0,t,s) = 0$, $\dot{w}_1(0,t_1) = t_1$, $\ddot{w}_1 \equiv 0$. We set

$$w_{p,l}(t_1,\ldots,t_l) \quad := \quad \frac{1}{p!}w_l^{(p)}(0,t_1,\ldots,t_l).$$

Differentiating the equation (20) with respect to the variable x and evaluating at $x = 0$ we infer for all $l \geqslant 2$

$$w_{k+2,l}(t_1,\ldots,t_l) \quad = \quad -\frac{1}{(k+2)(k+1)}\sum_{r=0}^{k-1}\frac{i^r}{r!}t_1^r w_{k-r,l-1}(t_2,\ldots,t_l),$$

which we rewrite as

$$\begin{aligned} w_{p,l}(t_1,\dots,t_l) &= -\frac{1}{p(p-1)}\sum_{r=0}^{p-3}\frac{i^r}{r!}t_1^r w_{p-r-2,l-1}(t_2,\dots,t_l) \\ &= \frac{1}{p(p-1)}\sum_{r=2}^{p-1}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,l-1}(t_2,\dots,t_l). \end{aligned}$$

We deduce

$$w_{p,2}(t_1,t_2) = i^{p-1}\frac{p-2}{p!}t_1^{p-3}t_2,$$

for all $p \geqslant 3$ and zero otherwise. We deduce also

$$\begin{aligned} w_{p,3}(t_1,t_2,t_3) &= \frac{1}{p(p-1)}\sum_{r=2}^{p-1}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,2}(t_2,t_3) \\ &= \frac{1}{p(p-1)}\sum_{r=2}^{p-3}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,2}(t_2,t_3), \end{aligned}$$

since $w_{p-r,2}=0$ for $p-r \geqslant 3$. Thus

$$w_{p,3}(t_1,t_2,t_3) = \frac{i^{p-1}}{p(p-1)}\sum_{r=2}^{p-3}(p-r-2)\frac{t_1^{r-2}t_2^{p-r-3}}{(r-2)!(p-r)!}t_3.$$

We compute now

$$\begin{aligned} w_{p,4}(t_1,\dots,t_4) &= \frac{1}{p(p-1)}\sum_{r=2}^{p-1}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,3}(t_2,\dots,t_4) \\ &= \frac{1}{p(p-1)}\sum_{r=2}^{p-5}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,3}(t_2,\dots,t_4), \end{aligned}$$

since $w_{p-r,3}=0$ for $p-r \geqslant 5$. Thus

$$w_{p,4}(t_1,\dots,t_4) = \frac{i^{p-1}}{p(p-1)}\sum_{r_1=2}^{p-5}\sum_{r_2=2}^{p-r_1-3}\frac{(p-r_1-r_2-2)t_1^{r_1-2}t_2^{r_2-2}t_3^{p-r_1-r_2-3}}{(p-r_1)(p-r_1-1)(r_1-2)!(r_2-2)!(p-r_1-r_2)!}t_4.$$

We compute now

$$\begin{aligned} w_{p,5}(t_1,\dots,t_5) &= \frac{1}{p(p-1)}\sum_{r=2}^{p-1}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,4}(t_2,\dots,t_5) \\ &= \frac{1}{p(p-1)}\sum_{r=2}^{p-7}\frac{i^r}{(r-2)!}t_1^{r-2}w_{p-r,4}(t_2,\dots,t_5), \end{aligned}$$

since $w_{p-r,4}=0$ for $p-r \geqslant 7$. Thus

$$\begin{aligned} & w_{p,5}(t_1,\dots,t_5) \\ = \; & \frac{i^{p-1}}{p(p-1)}\sum_{r_1=2}^{p-7}\sum_{r_2=2}^{p-r_1-5}\sum_{r_3=2}^{p-r_1-r_2-3}\frac{p-r_1-r_2-r_3-2}{(p-r_1)(p-r_1-1)(p-r_1-r_2)(p-r_1-r_2-1)}\times \\ \times \; & \frac{t_1^{r_1-2}t_2^{r_2-2}t_3^{r_3-2}t_4^{p-r_1-r_2-r_3-3}}{(r_1-2)!(r_2-2)!(r_3-2)!(p-r_1-r_2-r_3)!}t_5. \end{aligned}$$

We deduce the general formula

$$\begin{aligned} w_{p,l}(t_1,\dots,t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{\substack{r_j=2 \\ 1\leqslant j\leqslant l-2}}^{p-|r|_{j-1}-2(l-j)+1} (p-|r|_{l-2}-2)\times \\ &\times \left[\prod_{j=1}^{l-3}\frac{1}{(p-|r|_j)\,(p-|r|_j-1)}\right]\left[\prod_{j=1}^{l-2}\frac{t_j^{r_j-2}}{(r_j-2)!}\right]\frac{t_{l-1}^{p-|r|_{l-2}-3}}{(p-|r|_{l-2})!}t_l, \end{aligned}$$

for all $l\geqslant 3$. Performing the change of variables $r'_j=r_j-2$ we infer

$$\begin{aligned} w_{p,l}(t_1,\dots,t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{\substack{r_j=0 \\ 1\leqslant j\leqslant l-2}}^{p-|r|_{j-1}-2l+1} (p-|r|_{l-2}-2l+2)\times \\ &\times \left[\prod_{j=1}^{l-3}\frac{1}{(p-|r|_j-2j)\,(p-|r|_j-2j-1)}\right]\left[\prod_{j=1}^{l-2}\frac{t_j^{r_j}}{r_j!}\right]\frac{t_{l-1}^{p-|r|_{l-2}-2l+1}}{(p-|r|_{l-2}-2l+4)!}t_l. \end{aligned}$$

If we set $r_{l-1}:=p-|r|_{l-2}-2l+1$, then we can rewrite the previous sum as

$$\begin{aligned} w_{p,l}(t_1,\dots,t_l) &= \frac{i^{p-1}}{p(p-1)} \sum_{r\vDash_0^{l-1}p-2l+1} (r_{l-1}+1)\times \\ &\times \left[\prod_{j=1}^{l-3}\frac{1}{(p-|r|_j-2j)\,(p-|r|_j-2j-1)}\right]\left[\prod_{j=1}^{l-2}\frac{t_j^{r_j}}{r_j!}\right]\frac{t_{l-1}^{r_{l-1}}}{(r_{l-1}+3)!}t_l \\ &= \frac{i^{p-1}}{p(p-1)} \sum_{r\vDash_0^{l-1}p-2l+1} \left[\prod_{j=1}^{l-2}\frac{1}{(p-|r|_j-2j)\,(p-|r|_j-2j-1)}\right]\left[\prod_{j=1}^{l-1}\frac{t_j^{r_j}}{r_j!}\right]t_l, \end{aligned}$$

where $r\vDash_0^{l-1}p-2l+1$ denotes the compositions of length $l-1$ and $|r|=p-2l+1$.

We notice that the previous formula hold also for $l=2$. In conclusion

$$\begin{aligned} w_1(x,t_1) &= xt_1, \\ w_l(x,t_1,\dots,t_l) &= \sum_{p\geqslant 2l-1} w_{p,l}(t_1,\dots,t_l)x^p, \end{aligned}$$

for all $l\geqslant 2$.

3.3 The explicit expression of the coefficients C_D

By the result of the previous subsection, we infer the expression

$$\frac{w_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\dots,t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} = \sum_{p\geqslant 2\lambda_j+1}\frac{w_{p,\lambda_j+1}(t_{|\lambda|_{j-1}+1},\dots,t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}}x^p,$$

with

$$\begin{aligned} &\frac{w_{p,\lambda_j+1}(t_{|\lambda|_{j-1}+1},\dots,t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}} \\ &= \frac{i^{p-1}}{p(p-1)} \sum_{r\vDash_0^{\lambda_j}p-2\lambda_j-1} \left[\prod_{s=1}^{\lambda_j-1}\frac{1}{(p-|r|_s-2s)\,(p-|r|_s-2s-1)}\right]\left[\prod_{s=1}^{\lambda_j}\frac{t_{|\lambda|_{j-1}+s}^{r_s}}{r_s!}\right]. \end{aligned}$$

Thus

$$\begin{aligned}
&\frac{w_{\lambda_j+1}(x,t_{|\lambda|_{j-1}+1},\ldots,t_{|\lambda|_j+1})}{t_{|\lambda|_j+1}}\\
&= \sum_{p\geqslant 2\lambda_j+1}\frac{i^{p-1}x^p}{p(p-1)}\sum_{r\vDash_0^{\lambda_j}p-2\lambda_j-1}\left[\prod_{s=1}^{\lambda_j-1}\frac{1}{(p-|r|_s-2s)\,(p-|r|_s-2s-1)}\right]\left[\prod_{s=1}^{\lambda_j}\frac{t_{|\lambda|_{j-1}+s}^{r_s}}{r_s!}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\dot{w}_{\lambda_1+1}(x,t_1,\ldots,t_{\lambda_1+1})}{t_{\lambda_1+1}}\\
&= \frac{1}{x}\sum_{p\geqslant 2\lambda_1+1}\frac{i^{p-1}x^p}{(p-1)}\sum_{r\vDash_0^{\lambda_1}p-2\lambda_1-1}\left[\prod_{s=1}^{\lambda_1-1}\frac{1}{(p-|r|_s-2s)\,(p-|r|_s-2s-1)}\right]\left[\prod_{s=1}^{\lambda_1}\frac{t_s^{r_s}}{r_s!}\right].
\end{aligned}$$

We infer in conclusion

$$\begin{aligned}
&v_l(x,t_1,\ldots,t_l)\\
&= -i\sum_{\lambda\vDash l}\frac{(-1)^{l_\lambda}}{x^{l_\lambda-1}}\sum_{\substack{p_j\geqslant 2\lambda_j+1\\ r_j\vDash_0^{\lambda_j}p_j-2\lambda_j-1\\ j=1,\ldots l_\lambda}}(p_1-1)\prod_{j=1}^{l_\lambda}\frac{i^{p_j-1}x^{p_j}}{p_j(p_j-1)}\times\\
&\times\left[\prod_{j=1}^{l_\lambda}\prod_{s_j=1}^{\lambda_j-1}\frac{1}{(p_j-|r_j|_{s_j}-2s_j)\,(p_j-|r_j|_{s_j}-2s_j-1)}\right]\left[\prod_{j=1}^{l_\lambda}\prod_{s_j=1}^{\lambda_j}\frac{t_{|\lambda|_{j-1}+s_j}^{r_{j,s_j}}}{r_{j,s_j}!}\right],
\end{aligned}$$

for all $l\geqslant 1$. Thus

$$\begin{aligned}
&v_l(x,t_1,\ldots,t_l)\\
&= (-1)^{l+1}i\sum_{H\in\mathbb{Z}_{\geqslant 0}^l}\sum_{\lambda\vDash l}(-1)^{l_\lambda}(|H|_{\lambda_1}+2\lambda_1)i^{|H|}x^{|H|+2l+1}\times\\
&\times\left[\prod_{j=1}^{l_\lambda}\prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1}\frac{1}{\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)+1\right]\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)\right]}\right]\frac{t^H}{H!},
\end{aligned}$$

for all $l\geqslant 1$, which we rewrite as

$$\begin{aligned}
&v_l(x,t_1,\ldots,t_l)\\
&= (-1)^{l+1}i\sum_{H\in\mathbb{Z}_{\geqslant 0}^l}i^{|H|}x^{|H|+2l+1}\frac{t^H}{H!}\sum_{\lambda\vDash l}(-1)^{l_\lambda}(|H|_{\lambda_1}+2\lambda_1)\times\\
&\times\left[\prod_{j=1}^{l_\lambda}\prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1}\frac{1}{\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)+1\right]\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)\right]}\right].
\end{aligned}$$

We now recall that thanks to lemma 9 the solution u of the Riccati-type equation in lemma 8 is given by

$$u_l(x,t_1,\ldots,t_l) = -ie^{-ix\sum_{j=1}^l t_j}v_l(x,t_1,\ldots,t_l).$$

Then using the expansion

$$e^{-ix\sum_{j=1}^l t_j} = \sum_{H\in\mathbb{Z}_{\geqslant 0}^l}(-i)^{|H|}x^{|H|}\frac{t^H}{H!},$$

we deduce the formula

$$\begin{aligned} & u_l(x,t_1,\ldots,t_l) \\ = \; & (-1)^{l+1}\sum_{D\in\mathbb{Z}^l_{\geqslant 0}}(-i)^{|D|}x^{|D|+2l+1}t^D\sum_{0\leqslant H\leqslant D}\frac{(-1)^{|H|}}{H!(D-H)!}\sum_{\lambda\vDash l}(-1)^{l_\lambda}(|H|_{\lambda_1}+2\lambda_1)\times \\ \times \; & \left[\prod_{j=1}^{l_\lambda}\prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1}\frac{1}{\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)+1\right]\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)\right]}\right], \end{aligned}$$

for all $l\geqslant 1$. We conclude the explicit expression

$$\begin{aligned} C_D \; = \; & (-1)^{l+1}(-i)^{|D|}(|D|+2l+1)!\sum_{0\leqslant H\leqslant D}\frac{(-1)^{|H|}}{H!(D-H)!}\sum_{\lambda\vDash l}(-1)^{l_\lambda}(|H|_{\lambda_1}+2\lambda_1)\times \\ \times \; & \left[\prod_{j=1}^{l_\lambda}\prod_{s=|\lambda|_{j-1}}^{|\lambda|_j-1}\frac{1}{\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)+1\right]\left[\sum_{r=s+1}^{|\lambda|_j}h_r+2(|\lambda|_j-s)\right]}\right], \end{aligned}$$

for all $D\in\mathbb{Z}^l_{\geqslant 0}$. We write now $C_D=-(-i)^{\|D\|}(\|D\|+1)!\boldsymbol{C}_D$. We obtain the expression of the coefficients $\boldsymbol{C}_D$ in the statement of theorem 2.

4 Expliciting the integrability equations

In this section we provide some basic tools for a general expression of the integrability equations

$$\operatorname{Circ}\operatorname{Sym}_{3,\ldots,k+1}\Theta^\nabla_k \;=\; 0,$$

for $k\geqslant 4$. We notice in particular that the equation $\operatorname{Circ}\operatorname{Sym}_{3,4,5}\Theta^\nabla_4=0$ writes as

$$\operatorname{Circ}\operatorname{Sym}_{3,4,5}\left[3d^\nabla_1(\nabla R^\nabla)_2-2\tilde{R}^\nabla\wedge_1\tilde{R}^\nabla\right]=0,$$

with $\tilde{R}^\nabla:=\operatorname{Sym}_{2,3}R^\nabla$. Its vanishing has been proved in [Pali2] by using a direct computation.

4.1 Expliciting the powers of the 1-differential. Part II

From now on let $\rho_{k,j}:=\left(R.R^{(k+1-j)}\right)^{(j-2)}$ and notice that

$$\rho_{k,j} \;\in\; C^\infty\left(M,T^{*,\otimes(j-2)}_M\otimes_{\mathbb{R}}\Lambda^2T^*_M\otimes_{\mathbb{R}}T^{*,\otimes(k+1-j)}_M\otimes_{\mathbb{R}}\Lambda^2T^*_M\otimes_{\mathbb{R}}T^*_M\otimes_{\mathbb{R}}\mathbb{C}T_M\right),$$

satisfies the circular identity with respect to its last three entries and in the case $j\leqslant k$ the tensor $\rho_{k,j}\cdot v$ satisfies the circular identity with respect to its last three entries for any vector $v\in T_M$. We remind the identity

$$\begin{aligned} R^{(k+1)}(2,\ldots,k+3,1,k+4) \; = \; & R^{(k+1)}(3,\ldots,k+2,2,k+3,1,k+4) \\ + \; & \sum_{j=2}^{k+1}\rho_{k,j}(3,\ldots,j,2,j+1,\ldots,k+3,1,k+4), \end{aligned}$$

obtained in the proof of the identity (7) in proposition 1. We rewrite it in the more general form

$$\begin{aligned} R^{(c-b+1)}(1,b,\ldots,c,2,d) \; = \; & R^{(c-b+1)}(b,\ldots,c-1,1,c,2,d) \\ + \; & \sum_{h=b-1}^{c-2}\rho_{c-b,h-b+3}(b,\ldots,h,1,h+1,\ldots,c,2,d), \end{aligned}$$

for any $3 \leqslant b < c$. Differentiating we obtain the general commutation formula

$$\begin{aligned} R^{(c-a+1)}(a,\dots,b-1,1,b,\dots,c,2,d) &= R^{(c-a+1)}(a,\dots,c-1,1,c,2,d) \\ &+ \sum_{h=b-1}^{c-2} \rho_{c-a,h-a+3}(a,\dots,h,1,h+1,\dots,c,2,d), \quad (21) \end{aligned}$$

for all $3 \leqslant a \leqslant b < c < d$. We now show two key elementary lemmas.

Lemma 12 *In the set up of proposition 1, for all integers $k \geqslant 1$,*

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\dots,k+4} T_k = \mathrm{Circ}\,\mathrm{Sym}_{4,\dots,k+4} \widehat{T}_k,$$

with

$$\widehat{T}_k(1,\dots,k+4) := \sum_{j=2}^{k+1} (j-1)\rho_{k,j}(4,\dots,j+1,3,j+2,\dots,k+3,1,2,k+4).$$

Proof We notice first that for any tensor $\theta \equiv \theta(1,\dots,k+4)$, the tensor

$$\widetilde{\theta}(1,\dots,k+4) := \sum_{j=3}^{k+4} \theta(1,2,4,\dots,j,3,j+1,\dots,k+4),$$

satisfies the identity

$$\mathrm{Sym}_{3,\dots,k+4}\,\theta = \mathrm{Sym}_{4,\dots,k+4}\,\widetilde{\theta}.$$

Applying the previous definition to our tensor T_k we obtain the expression

$$\begin{aligned} \widetilde{T}_k(1,\dots,k+4) &= \sum_{j=3}^{k+3} R^{(k+1)}(4,\dots,j,3,j+1,\dots,k+3,1,2,k+4) \\ &+ R^{(k+1)}(4,\dots,k+4,1,2,3). \end{aligned}$$

Using the commutation identity (21) we infer

$$\begin{aligned} & R^{(k+1)}(4,\dots,j,3,j+1,\dots,k+3,1,2,k+4) \\ = \; & R^{(k+1)}(4,\dots,k+3,3,1,2,k+4) + \sum_{h=j}^{k+2} \rho_{k,h-3}(4,\dots,h,3,h+1,\dots,k+3,1,2,k+4). \end{aligned}$$

We infer the equality

$$\begin{aligned} & \widetilde{T}_k(1,\dots,k+4) \\ = \; & (k+1)R^{(k+1)}(4,\dots,k+3,3,1,2,k+4) + (-1)^k R^{(k+1)}(4,\dots,k+4,1,2,3) \\ + \; & \sum_{j=3}^{k+2}\sum_{h=j}^{k+2} \rho_{k,h-3}(4,\dots,h,3,h+1,\dots,k+3,1,2,k+4). \end{aligned}$$

Using the elementary equality

$$\sum_{s=a}^{b}\sum_{h=s}^{b} C_h = \sum_{h=a}^{b} (h-a+1)C_h,$$

we obtain the identity

$$\begin{aligned} & \widetilde{T}_k(1,\dots,k+4) \\ = \; & (k+1)R^{(k+1)}(4,\dots,k+3,3,1,2,k+4) + (-1)^k R^{(k+1)}(4,\dots,k+4,1,2,3) \\ + \; & \sum_{h=3}^{k+2} (h-2)\rho_{k,h-3}(4,\dots,h,3,h+1,\dots,k+3,1,2,k+4). \end{aligned}$$

Using the algebraic and differential Bianchi identities and performing the change of variable $h = j+1$, we infer the required conclusion. □

Lemma 13 *In the set up of proposition 1, for all integers $k \geqslant 1$,*

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+4}\left[(d_1^\nabla)^k(\nabla R^\nabla)_2\right] \;=\; (-1)^k\sum_{j=2}^{k+1}\mathrm{Circ}\,\mathrm{Sym}_{4,\ldots,k+4}(\check{T}_{k,j}-\mathrm{Alt}_2\,\check{Q}_{k,j}),$$

with

$$\begin{aligned}
\check{T}_{k,j}(1,\ldots,k+4) &:= (j-2)\rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+3,1,2,k+4),\\
\check{Q}_{k,j}(1,\ldots,k+4) &:= \sum_{s=3}^{j}\rho_{k,j}(4,\ldots,s,3,s+1,\ldots,j,2,j+1,\ldots,k+3,1,k+4)\\
&+ \sum_{s=j+1}^{k+3}\rho_{k,j}(4,\ldots,j+1,2,j+2,\ldots,s,3,s+1,\ldots,k+3,1,k+4).
\end{aligned}$$

Proof In the computation that will follow we will denote by $\rho_{k,j}(\ldots)_{_p}$ with $p=0,1$, the terms that summed-up together annihilate the operator $\mathrm{Circ}\,\mathrm{Sym}_{4,\ldots,k+4}$. For all $2\leqslant j\leqslant k+1$, we define

$$\begin{aligned}
\widehat{T}_{k,j}(1,\ldots,k+4) &:= (j-1)\rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+3,1,2,k+4),\\
\hat{Q}_{k,j}(1,\ldots,k+4) &:= \sum_{s=3}^{j}\rho_{k,j}(4,\ldots,s,3,s+1,\ldots,j,2,j+1,\ldots,k+3,1,k+4)\\
&+ \sum_{s=j+1}^{k+3}\rho_{k,j}(4,\ldots,j+1,2,j+2,\ldots,s,3,s+1,\ldots,k+3,1,k+4)\\
&+ \rho_{k,j}(4,\ldots,j+1,2,j+2,\ldots,k+4,1,3)_{_2},\\
\widehat{V}_{k,j}(1,\ldots,k+4) &:= -\sum_{s=3}^{j}\rho_{k,j}(4,\ldots,s,3,s+1,\ldots,j,1,j+1,\ldots,k+3,2,k+4)\\
&- \sum_{s=j+1}^{k+3}\rho_{k,j}(4,\ldots,j+1,1,j+2,\ldots,s,3,s+1,\ldots,k+3,2,k+4)\\
&- \rho_{k,j}(4,\ldots,j+1,1,j+2,\ldots,k+4,2,3)_{_1},
\end{aligned}$$

and we notice the equalities

$$\mathrm{Sym}_{3,\ldots,k+4}\,\mathrm{Alt}_2\,Q_k \;=\; \sum_{j=2}^{k+1}\mathrm{Sym}_{4,\ldots,k+4}(\hat{Q}_{k,j}+\widehat{V}_{k,j}),$$

Then proposition 1 and lemma 12 imply

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+4}\left[(d_1^\nabla)^k(\nabla R^\nabla)_2\right] = (-1)^k\sum_{j=2}^{k+1}\mathrm{Circ}\,\mathrm{Sym}_{4,\ldots,k+4}(\widehat{T}_{k,j}-\hat{Q}_{k,j}-\widehat{V}_{k,j}). \qquad (22)$$

We write

$$\begin{aligned}
\widehat{T}_{k,j}(1,\ldots,k+4) &= (j-2)\rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+3,1,2,k+4)\\
&- \rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+4,1,2)\\
&- \rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+3,2,k+4,1)\\
&= (j-2)\rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+3,1,2,k+4)\\
&- \rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+4,1,2)_{_1}\\
&+ \rho_{k,j}(4,\ldots,j+1,3,j+2,\ldots,k+4,2,1)_{_2}.
\end{aligned}$$

We infer the identity

$$\begin{aligned} & \operatorname{Circ}\operatorname{Sym}_{4,\dots,k+4}(\widehat{T}_{k,j} - \hat{Q}_{k,j} - \widehat{V}_{k,j}) \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,\dots,k+4}(\check{T}_{k,j} - \operatorname{Alt}_2 \check{Q}_{k,j}), \end{aligned}$$

which combined with the identity (22) implies the required statement. □

Lemma 14 *In the set up of proposition 1, for all integers $k \geqslant 1$,*

$$\operatorname{Circ}\operatorname{Sym}_{3,\dots,k+4}\left[(d_1^\nabla)^k(\nabla R^\nabla)_2\right] \quad = \quad (-1)^k \sum_{j=2}^{k+1} \operatorname{Circ}\operatorname{Sym}_{4,\dots,k+4}\left[(j-2)\mathbb{T}_{k,j} - \operatorname{Alt}_2 \mathbb{Q}_{k,j}\right],$$

with

$$\begin{aligned} & \mathbb{T}_{k,j}(1,\dots,k+4) \\ := \quad & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(|I|)}(I,3,j+2,R^{(k+1-j+|\complement I|)}(\complement I,j+3,\dots,k+3,1,2,k+4)) \\ - \quad & \sum_{\substack{I\subseteq\{4,\dots,j+1\} \\ j+3\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+3,\dots,R^{(|\complement I|)}(\complement I,3,j+2,h),\dots,k+3,1,2,k+4) \\ - \quad & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,R^{(|\complement I|)}(\complement I,3,j+2,1),2,k+4) \\ - \quad & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,1,R^{(|\complement I|)}(\complement I,3,j+2,2),k+4) \\ - \quad & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,1,2,R^{(|\complement I|)}(\complement I,3,j+2,k+4)), \end{aligned}$$

and with

$$
\begin{aligned}
& \mathbb{Q}_{k,j}(1,\ldots,k+4) \\
:= & \sum_{\substack{3\leqslant s\leqslant j \\ I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\}}} R^{(|I|)}(I,2,j+1,R^{(k+1-j+|\complement I|)}(\complement I,j+2,\ldots,k+3,1,k+4)) \\
- & \sum_{\substack{3\leqslant s\leqslant j \\ I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\} \\ j+2\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+2,\ldots,R^{(|\complement I|)}(\complement I,2,j+1,h),\ldots,k+3,1,k+4) \\
- & \sum_{\substack{3\leqslant s\leqslant j \\ I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\}}} R^{(k+1-j+|I|)}(I,j+2,\ldots,k+3,R^{(|\complement I|)}(\complement I,2,j+1,1),k+4) \\
- & \sum_{\substack{3\leqslant s\leqslant j \\ I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\}}} R^{(k+1-j+|I|)}(I,j+2,\ldots,k+3,1,R^{(|\complement I|)}(\complement I,2,j+1,k+4)) \\
+ & \sum_{I\subseteq\{4,\ldots,j+1\}} R^{(|I|)}(I,2,3,R^{(k+1-j+|\complement I|)}(\complement I,j+2,\ldots,k+3,1,k+4)) \\
- & \sum_{\substack{I\subseteq\{4,\ldots,j+1\} \\ j+2\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+2,\ldots,R^{(|\complement I|)}(\complement I,2,3,h),\ldots,k+3,1,k+4) \\
- & \sum_{I\subseteq\{4,\ldots,j+1\}} R^{(k+1-j+|I|)}(I,j+2,\ldots,k+3,1,R^{(|\complement I|)}(\complement I,2,3,k+4)) \\
+ & \sum_{\substack{j+2\leqslant s\leqslant k+3 \\ I\subseteq\{4,\ldots,j+1\}}} R^{(|I|)}(I,2,j+2,R^{(k+1-j+|\complement I|)}(\complement I,j+3,\ldots,s,3,s+1,\ldots,k+3,1,k+4)) \\
- & \sum_{\substack{j+3\leqslant s\leqslant k+3 \\ I\subseteq\{4,\ldots,j+1\} \\ j+3\leqslant h\leqslant s}} R^{(k+1-j+|I|)}(I,j+3,\ldots,R^{(|\complement I|)}(\complement I,2,j+2,h),\ldots,s,3,s+1,\ldots,k+3,1,k+4) \\
- & \sum_{\substack{j+2\leqslant s\leqslant k+3 \\ I\subseteq\{4,\ldots,j+1\}}} R^{(k+1-j+|I|)}(I,j+3,\ldots,s,R^{(|\complement I|)}(\complement I,2,j+2,3),s+1,\ldots,k+3,1,k+4) \\
- & \sum_{\substack{j+2\leqslant s\leqslant k+2 \\ I\subseteq\{4,\ldots,j+1\} \\ s+1\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+3,\ldots,s,3,s+1,\ldots,R^{(|\complement I|)}(\complement I,2,j+2,h),\ldots,k+3,1,k+4) \\
- & \sum_{\substack{j+2\leqslant s\leqslant k+3 \\ I\subseteq\{4,\ldots,j+1\}}} R^{(k+1-j+|I|)}(I,j+3,\ldots,s,3,s+1,\ldots,k+3,R^{(|\complement I|)}(\complement I,2,j+2,1),k+4) \\
- & \sum_{\substack{j+2\leqslant s\leqslant k+3 \\ I\subseteq\{4,\ldots,j+1\}}} R^{(k+1-j+|I|)}(I,j+3,\ldots,s,3,s+1,\ldots,k+3,1,R^{(|\complement I|)}(\complement I,2,j+2,k+4)),
\end{aligned}
$$

where the symbol $I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\}$ means a subset of the elements $4,\ldots,s,3,s+1,\ldots,j$ written in this order and $\complement I\subseteq\{4,\ldots,s,3,s+1,\ldots,j\}$ denotes its complementary set respecting the same order.

Proof We recall first the definition

$$
\begin{aligned}
(R.R^{(p)})(1,\ldots,p+5) &= R(1,2,R^{(p)}(3,\ldots,p+5)) \\
&- \sum_{h=3}^{p+5} R^{(p)}(3,\ldots,R(1,2,h),\ldots,p+5),
\end{aligned}
$$

which writes in the case

$$\begin{aligned}(R.R^{(k+1-j)})(j-1,\dots,k+4) &= R(j-1,j,R^{(k+1-j)}(j+1,\dots,k+4))\\ &- \sum_{h=j+1}^{k+4} R^{(k+1-j)}(j+1,\dots,R(j-1,j,h),\dots,k+4).\end{aligned}$$

We infer the expression

$$\begin{aligned}&\rho_{k,j}(1,\dots,k+4)\\ =& \sum_{I\subseteq\{1,\dots,j-2\}} R^{(|I|)}(I,j-1,j,R^{(k+1-j+|\complement I|)}(\complement I,j+1,\dots,k+4))\\ -& \sum_{I\subseteq\{1,\dots,j-2\}}\sum_{h=j+1}^{k+4} R^{(k+1-j+|I|)}(I,j+1,\dots,R^{(|\complement I|)}(\complement I,j-1,j,h),\dots,k+4).\end{aligned}$$

Performing the change of variables

$$\begin{array}{ccccccccccc} 1 & \dots & j-2 & j-1 & j & \dots & k+1 & k+2 & k+3 & k+4\\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow\\ 4 & \dots & j+1 & 3 & j+2 & \dots & k+3 & 1 & 2 & k+4 \end{array},$$

we infer the expression

$$\begin{aligned}&\rho_{k,j}(4,\dots,j+1,3,j+2,\dots,k+3,1,2,k+4)\\ =& \sum_{I\subseteq\{4,\dots,j+1\}} R^{(|I|)}(I,3,j+2,R^{(k+1-j+|\complement I|)}(\complement I,j+3,\dots,k+3,1,2,k+4))\\ -& \sum_{I\subseteq\{4,\dots,j+1\}}\sum_{h=j+3}^{k+3} R^{(k+1-j+|I|)}(I,j+3,\dots,R^{(|\complement I|)}(\complement I,3,j+2,h),\dots,k+3,1,2,k+4)\\ -& \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,R^{(|\complement I|)}(\complement I,3,j+2,1),2,k+4)\\ -& \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,1,R^{(|\complement I|)}(\complement I,3,j+2,2),k+4)\\ -& \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,k+3,1,2,R^{(|\complement I|)}(\complement I,3,j+2,k+4)).\end{aligned}$$

Performing now the change of variables

$$\begin{array}{ccccccccccccc} 1 & \dots & s-3 & s-2 & s-1 & \dots & j-2 & j-1 & j & \dots & k+2 & k+3 & k+4\\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow\\ 4 & \dots & s & 3 & s+1 & \dots & j & 2 & j+1 & \dots & k+3 & 1 & k+4 \end{array},$$

we infer the expression

$$\begin{aligned}&\rho_{k,j}(4,\dots,s,3,s+1,\dots,j,2,j+1,\dots,k+3,1,k+4)\\ =& \sum_{I\subseteq\{4,\dots,s,3,s+1,\dots,j\}} R^{(|I|)}(I,2,j+1,R^{(k+1-j+|\complement I|)}(\complement I,j+2,\dots,k+3,1,k+4))\\ -& \sum_{I\subseteq\{4,\dots,s,3,s+1,\dots,j\}}\sum_{h=j+2}^{k+3} R^{(k+1-j+|I|)}(I,j+2,\dots,R^{(|\complement I|)}(\complement I,2,j+1,h),\dots,k+3,1,k+4)\\ -& \sum_{I\subseteq\{4,\dots,s,3,s+1,\dots,j\}} R^{(k+1-j+|I|)}(I,j+2,\dots,k+3,R^{(|\complement I|)}(\complement I,2,j+1,1),k+4)\\ -& \sum_{I\subseteq\{4,\dots,s,3,s+1,\dots,j\}} R^{(k+1-j+|I|)}(I,j+2,\dots,k+3,1,R^{(|\complement I|)}(\complement I,2,j+1,k+4)).\end{aligned}$$

We perform finally the change of variables

$$\begin{array}{ccccccccccccc} 1 & \dots & j-2 & j-1 & j & \dots & s-2 & s-1 & s & \dots & k+2 & k+3 & k+4 \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ 4 & \dots & j+1 & 2 & j+2 & \dots & s & 3 & s+1 & \dots & k+3 & 1 & k+4 \end{array}.$$

We distinguish two cases. In the first case, when $s = j+1$, we have

$$\begin{aligned} & \rho_{k,j}(4,\dots,j+1,2,3,j+2,\dots,k+3,1,k+4) \\ = & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(|I|)}(I,2,3,R^{(k+1-j+|\complement I|)}(\complement I,j+2,\dots,k+3,1,k+4)) \\ - & \sum_{\substack{I\subseteq\{4,\dots,j+1\}\\ j+2\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+2,\dots,R^{(|\complement I|)}(\complement I,2,3,h),\dots,k+3,1,k+4) \\ - & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+2,\dots,k+3,R^{(|\complement I|)}(\complement I,2,3,1),k+4) \\ - & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+2,\dots,k+3,1,R^{(|\complement I|)}(\complement I,2,3,k+4)), \end{aligned}$$

and in the second case, when $j+2\leqslant s\leqslant k+3$, we have

$$\begin{aligned} & \rho_{k,j}(4,\dots,j+1,2,j+2,\dots,s,3,s+1,\dots,k+3,1,k+4) \\ = & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(|I|)}(I,2,j+2,R^{(k+1-j+|\complement I|)}(\complement I,j+3,\dots,s,3,s+1,\dots,k+3,1,k+4)) \\ - & \sum_{\substack{I\subseteq\{4,\dots,j+1\}\\ j+3\leqslant h\leqslant s}} R^{(k+1-j+|I|)}(I,j+3,\dots,R^{(|\complement I|)}(\complement I,2,j+2,h),\dots,s,3,s+1,\dots,k+3,1,k+4) \\ - & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,s,R^{(|\complement I|)}(\complement I,2,j+2,3),s+1,\dots,k+3,1,k+4) \\ - & \sum_{\substack{I\subseteq\{4,\dots,j+1\}\\ s+1\leqslant h\leqslant k+3}} R^{(k+1-j+|I|)}(I,j+3,\dots,s,3,s+1,\dots,R^{(|\complement I|)}(\complement I,2,j+2,h),\dots,k+3,1,k+4) \\ - & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,s,3,s+1,\dots,k+3,R^{(|\complement I|)}(\complement I,2,j+2,1),k+4) \\ - & \sum_{I\subseteq\{4,\dots,j+1\}} R^{(k+1-j+|I|)}(I,j+3,\dots,s,3,s+1,\dots,k+3,1,R^{(|\complement I|)}(\complement I,2,j+2,k+4)). \end{aligned}$$

Then the conclusion follows from lemma 13. □

4.2 Expliciting the 1-exterior product of the lower order term of S_k

Given a subset $S\subset\mathbb{Z}_{\geqslant 1}$ and $I\subset S$ we denote below by abuse of notation $\complement I\subset S$ the complementary set of I inside S. We show now the following key result.

Proposition 3 *For all $h\geqslant 2$, let*

$$\hat{S}_h \;:=\; \frac{2i^{h+1}}{(h+1)!h!}\,\mathrm{Sym}_{2,\dots,h+1}\,\Phi_{h-3}.$$

Then for all $k\geqslant 1$,

$$\mathrm{Sym}_{3,\dots,k+4}\left[\sum_{r=3}^{k+2}(r+1)!\sum_{p=2}^{r-1}(id_1^{\nabla})^{k+2-r}(p\hat{S}_p\wedge_1\hat{S}_{r-p+1})\right] \;=\; \mathrm{Sym}_{3,\dots,k+4}\,\mathbb{L}(k),$$

with

$$\mathbb{L}(k) \quad := \quad \sum_{r=3}^{k+2}\sum_{p=2}^{r-1}\frac{4(-i)^{k+1}(r+1)!\mathbb{L}_{k,r,p}}{(r-p+2)!(p+1)!},$$

and with

$$\begin{aligned}
&\mathbb{L}_{k,r,p}(1,\ldots,k+4)\\
:=\ &\sum_{I\subseteq\{1,3,\ldots,k-r+3\}}\\
&R^{(p+|I|-2)}(I,k_p+2,\ldots,k+3,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\ldots,k_p,2,k_p+1),k_r,k+4)\\
-\ &\sum_{I\subseteq\{2,\ldots,k-r+3\}}\\
&R^{(p+|I|-2)}(I,k_p+2,\ldots,k+3,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\ldots,k_p,1,k_p+1),k_r,k+4).
\end{aligned}$$

with $k_p := k-p+4$, *in the case* $k+2-r \geqslant 1$ *and with*

$$\begin{aligned}
&\mathbb{L}_{k,k+2,p}(1,\ldots,k+4)\\
:=\ &\sum_{j=k_p+1}^{k+3}R^{(p-2)}(k_p+2,\ldots,j,R^{(k-p+1)}(3,\ldots,k_p,2,k_p+1),j+1,\ldots,k+3,1,k+4)\\
+\ &R^{(p-2)}(k_p+2,\ldots,k+4,1,R^{(k-p+1)}(3,\ldots,k_p,2,k_p+1))\\
-\ &\sum_{j=k_p+1}^{k+3}R^{(p-2)}(k_p+2,\ldots,j,R^{(k-p+1)}(3,\ldots,k_p,1,k_p+1),j+1,\ldots,k+3,2,k+4)\\
-\ &R^{(p-2)}(k_p+2,\ldots,k+4,2,R^{(k-p+1)}(3,\ldots,k_p,1,k_p+1)).
\end{aligned}$$

Proof We notice that in the case $k+2-r \geqslant 1$,

$$\mathrm{Sym}_{3,\ldots,k+4}\left[(id_1^{\nabla})^{k+2-r}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1})\right] \quad = \quad (-i)^{k+2-r}\,\mathrm{Sym}_{3,\ldots,k+4}(\varphi_{k,r,p}-\psi_{k,r,p}),$$

with

$$\begin{aligned}
\varphi_{k,r,p}(1,\ldots,k+4) \quad &:= \quad \nabla^{k+2-r}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1})(2,\ldots,k+3-r,1,k+4-r,\ldots,k+4),\\
\psi_{k,r,p}(1,\ldots,k+4) \quad &:= \quad \nabla^{k+2-r}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1})(1,3,\ldots,k+3-r,2,k+4-r,\ldots,k+4).
\end{aligned}$$

We notice now that

$$\begin{aligned}
\mathrm{Sym}_{3,\ldots,k+4}\,\varphi_{k,r,p} \quad &= \quad \mathrm{Sym}_{3,\ldots,k+4}\,\tilde{\varphi}_{k,r,p},\\
\mathrm{Sym}_{3,\ldots,k+4}\,\psi_{k,r,p} \quad &= \quad \mathrm{Sym}_{3,\ldots,k+4}\,\tilde{\psi}_{k,r,p},
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\varphi}_{k,r,p}(1,\ldots,k+4) \quad &:= \quad \nabla^{k+2-r}\lambda_{r,p}(2,\ldots,k+3-r,1,k+4-r,\ldots,k+4),\\
\tilde{\psi}_{k,r,p}(1,\ldots,k+4) \quad &:= \quad \nabla^{k+2-r}\lambda_{r,p}(1,3,\ldots,k+3-r,2,k+4-r,\ldots,k+4),
\end{aligned}$$

with

$$\begin{aligned}
&\lambda_{r,p}(1,k+4-r,\ldots,k+4)\\
:=\ &-p\hat{S}_p(k+4-r,\hat{S}_{r-p+1}(1,k+5-r,\ldots,k+5-p),k+6-p,\ldots,k+4).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\tilde{\varphi}_{k,r,p}-\tilde{\psi}_{k,r,p})(1,\ldots,k+4)\\
=\ &p\sum_{I\subseteq\{1,3,\ldots,k-r+3\}}\hat{S}_p^{(|I|)}(I,k_r,\hat{S}_{r-p+1}^{(|\complement I|)}(\complement I,2,k_r+1,\ldots,k_p+1),k_p+2,\ldots,k+4)\\
-\ &p\sum_{I\subseteq\{2,\ldots,k-r+3\}}\hat{S}_p^{(|I|)}(I,k_r,\hat{S}_{r-p+1}^{(|\complement I|)}(\complement I,1,k_r+1,\ldots,k_p+1),k_p+2,\ldots,k+4).
\end{aligned}$$

We notice now that for any tensor $\Phi \equiv \Phi(1,\dots,p+1)$, the tensor

$$\tilde{\Phi}(1,\dots,p+1) \quad := \quad \sum_{j=2}^{p+1} \Phi(1,3,\dots,j,2,j+1,\dots,p+1),$$

satisfies the identity

$$\mathrm{Sym}_{2,\dots,p+1}\,\Phi \quad = \quad \mathrm{Sym}_{3,\dots,p+1}\,\tilde{\Phi}.$$

Thus

$$(\mathrm{Sym}_{2,\dots,p+1}\,\Phi)^{(h)} \quad = \quad \mathrm{Sym}_{h+3,\dots,h+p+1}\,\tilde{\Phi}^{(h)}.$$

We infer

$$\begin{aligned}
& (\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p})(1,\dots,k+4) \\
= \quad & \frac{4i^{r+3}}{(p+1)!(p-1)!(r-p+2)!(r-p+1)!} \sum_{I\subseteq\{1,3,\dots,k-r+3\}} \\
& (\mathrm{Sym}_{|I|+3,\dots,|I|+p+1}\,\tilde{\Phi}^{(|I|)}_{p-3})(I,k_r,(\mathrm{Sym}_{|\complement I|+2,\dots,|\complement I|+r-p+2}\,\Phi^{(|\complement I|)}_{r-p-2})(\complement I,2,k_r+1,\dots,k_p+1),k_p+2,\dots,k+4) \\
- \quad & \frac{4i^{r+3}}{(p+1)!(p-1)!(r-p+2)!(r-p+1)!} \sum_{I\subseteq\{2,\dots,k-r+3\}} \\
& (\mathrm{Sym}_{|I|+3,\dots,|I|+p+1}\,\tilde{\Phi}^{(|I|)}_{p-3})(I,k_r,(\mathrm{Sym}_{|\complement I|+2,\dots,|\complement I|+r-p+2}\,\Phi^{(|\complement I|)}_{r-p-2})(\complement I,1,k_r+1,\dots,k_p+1),k_p+2,\dots,k+4).
\end{aligned}$$

and thus

$$(-i)^{k+2-r}\,\mathrm{Sym}_{3,\dots,k+4}(\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p}) \quad = \quad \frac{4(-1)^{k-r}i^{k+1}}{(r-p+2)!(p+1)!}\,\mathrm{Sym}_{3,\dots,k+4}\,\widetilde{\mathbb{L}}_{k,r,p},$$

with

$$\begin{aligned}
& \widetilde{\mathbb{L}}_{k,r,p}(1,\dots,k+4) \\
:= \quad & \sum_{I\subseteq\{1,3,\dots,k-r+3\}} \tilde{\Phi}^{(|I|)}_{p-3}(I,k_r,\Phi^{(|\complement I|)}_{r-p-2}(\complement I,2,k_r+1,\dots,k_p+1),k_p+2,\dots,k+4) \\
- \quad & \sum_{I\subseteq\{2,\dots,k-r+3\}} \tilde{\Phi}^{(|I|)}_{p-3}(I,k_r,\Phi^{(|\complement I|)}_{r-p-2}(\complement I,1,k_r+1,\dots,k_p+1),k_p+2,\dots,k+4) \\
= \quad & (-1)^{r-p-1} \sum_{I\subseteq\{1,3,\dots,k-r+3\}} \\
& \tilde{\Phi}^{(|I|)}_{p-3}(I,k_r,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,2,k_p+1),k_p+2,\dots,k+4) \\
- \quad & (-1)^{r-p-1} \sum_{I\subseteq\{2,\dots,k-r+3\}} \\
& \tilde{\Phi}^{(|I|)}_{p-3}(I,k_r,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,1,k_p+1),k_p+2,\dots,k+4).
\end{aligned}$$

Using the expression

$$\begin{aligned}
\tilde{\Phi}_{p-3}(1,\dots,p+1) \quad &= \quad (-1)^p \sum_{j=2}^{p} R^{(p)}(3,\dots,j,2,j+1,\dots,p,1,p+1) \\
&+ \quad (-1)^p R^{(p)}(3,\dots,p+1,1,2), \qquad (23)
\end{aligned}$$

we deduce the formula

$$(-i)^{k+2-r}\,\mathrm{Sym}_{3,\dots,k+4}(\tilde{\varphi}_{k,r,p} - \tilde{\psi}_{k,r,p}) \quad = \quad \frac{4(-i)^{k+1}}{(r-p+2)!(p+1)!}\,\mathrm{Sym}_{3,\dots,k+4}\,\widehat{\mathbb{L}}_{k,r,p},$$

with

$$
\begin{aligned}
& \widehat{\mathbb{L}}_{k,r,p}(1,\dots,k+4) \\
:= & \sum_{I\subseteq\{1,3,\dots,k-r+3\}} \sum_{j=k_p+1}^{k+3} \\
& R^{(p+|I|-2)}(I,k_p+2,\dots,j,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,2,k_p+1),j+1,\dots,k+3,k_r,k+4) \\
+ & \sum_{I\subseteq\{1,3,\dots,k-r+3\}} R^{(p+|I|-2)}(I,k_p+2,\dots,k+4,k_r,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,2,k_p+1)) \\
- & \sum_{I\subseteq\{2,\dots,k-r+3\}} \sum_{j=k_p+1}^{k+3} \\
& R^{(p+|I|-2)}(I,k_p+2,\dots,j,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,1,k_p+1),j+1,\dots,k+3,k_r,k+4) \\
- & \sum_{I\subseteq\{2,\dots,k-r+3\}} R^{(p+|I|-2)}(I,k_p+2,\dots,k+4,k_r,R^{(r-p+|\complement I|-1)}(\complement I,k_r+1,\dots,k_p,1,k_p+1)).
\end{aligned}
$$

The fact that in the case under consideration $k+2-r \geqslant 1$, i.e $k_r \geqslant 3$, implies

$$
\mathrm{Sym}_{3,\dots,k+4}\left[(id_1^{\nabla})^{k+2-r}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1})\right] = \frac{4(-i)^{k+1}}{(r-p+2)!(p+1)!}\,\mathrm{Sym}_{3,\dots,k+4}\,\mathbb{L}_{k,r,p}.
$$

We consider now the case $r=k+2$. We consider the expansion

$$
\begin{aligned}
& (p\hat{S}_p \wedge_1 \hat{S}_{k-p+3})(1,\dots,k+4) \\
= \; & p\hat{S}_p(1,\hat{S}_{k-p+3}(2,\dots,k_p+1),k_p+2,\dots,k+4) \\
- \; & p\hat{S}_p(2,\hat{S}_{k-p+3}(1,3,\dots,k_p+1),k_p+2,\dots,k+4) \\
= \; & \frac{4i^{k+5}}{(p+1)!(p-1)!(k-p+4)!(k-p+3)!}\times \\
\times \; & (\mathrm{Sym}_{3,\dots,p+1}\,\tilde{\Phi}_{p-3})(1,(\mathrm{Sym}_{2,\dots,k_p}\,\Phi_{k-p})(2,\dots,k_p+1),k_p+2,\dots,k+4) \\
- \; & \frac{4i^{k+5}}{(p+1)!(p-1)!(k-p+4)!(k-p+3)!}\times \\
\times \; & (\mathrm{Sym}_{3,\dots,p+1}\,\tilde{\Phi}_{p-3})(2,(\mathrm{Sym}_{2,\dots,k_p}\,\Phi_{k-p})(1,3,\dots,k_p+1),k_p+2,\dots,k+4).
\end{aligned}
$$

We infer

$$
\mathrm{Sym}_{3,\dots,k+4}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1}) = \frac{4i^{k+5}}{(p+1)!(k-p+4)!}\,\mathrm{Sym}_{3,\dots,k+4}\,\widetilde{\mathbb{L}}_{k,k+2,p},
$$

with

$$
\begin{aligned}
& \widetilde{\mathbb{L}}_{k,k+2,p}(1,\dots,k+4) \\
= \; & \tilde{\Phi}_{p-3}(1,\Phi_{k-p}(2,\dots,k_p+1),k_p+2,\dots,k+4) \\
- \; & \tilde{\Phi}_{p-3}(2,\Phi_{k-p}(1,3,\dots,k_p+1),k_p+2,\dots,k+4) \\
= \; & (-1)^{k-p+1}\tilde{\Phi}_{p-3}(1,R^{(k-p+1)}(3,\dots,k_p,2,k_p+1),k_p+2,\dots,k+4) \\
- \; & (-1)^{k-p+1}\tilde{\Phi}_{p-3}(2,R^{(k-p+1)}(3,\dots,k_p,1,k_p+1),k_p+2,\dots,k+4).
\end{aligned}
$$

Using (23), we obtain

$$
\mathrm{Sym}_{3,\dots,k+4}(p\hat{S}_p \wedge_1 \hat{S}_{r-p+1}) = \frac{4(-i)^{k+1}}{(p+1)!(k-p+4)!}\,\mathrm{Sym}_{3,\dots,k+4}\,\mathbb{L}_{k,k+2,p},
$$

which shows the required formula. □

5 Vanishing of the integrability conditions in the cases $k = 4, 5$

We recall the notation $R' := R^{(1)}$. In this section we show the identity

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+4}\left[2i(id)^k(R^{'})_2 - \sum_{r=3}^{k+2}(r+1)!\sum_{p=2}^{r-1}(id)^{k+2-r}(p\hat{S}_p \wedge \hat{S}_{r-p+1})\right] \quad = \quad 0,$$

for all integers $k = 1, 2$. We notice that $\hat{S}_p = S_p$ for $p = 2, 3$. This corresponds to the cases $k = 4, 5$ in the main integrability conditions $\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+1}\,\Theta_k^{\nabla} = 0$.

5.1 Alternative proof of the vanishing of the integrability conditions in the case $k = 4$

We set for simplicity $ab, cde := R(a, b, R(c, d, e))$ and a similar definition for abc, de or a, bcd, e.

We observe first that lemma 14 implies the identity

$$\begin{aligned}
\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}[d(R^{'})_2] &= \mathrm{Circ}\,\mathrm{Sym}_{4,5}\,\mathrm{Alt}_2\,\mathbb{Q}_{1,2}\\
\mathbb{Q}_{1,2}(1,\ldots,5) &= 23,415 - 234,15 - 41,235\\
&+ \quad 24,315 - 243,15 - 3,241,5 - 31,245.
\end{aligned}$$

We notice that $\mathrm{Circ}\,\mathbb{Q}_{1,2} = \mathrm{Circ}\,\tilde{\mathbb{Q}}_{1,2}$, with

$$\begin{aligned}
\tilde{\mathbb{Q}}_{1,2}(1,\ldots,5) &= 12,435_{_1} - 124,35_{_2} - 43,125_{_3}\\
&+ \quad 34,125_{_3} - 142,35_{_2} - 3,241,5_{_2} - 12,345_{_1}\\
&= -2\cdot 12,345 + 2\cdot 34,125 - 124,35 - 142,35 - 421,35\\
&= -2\cdot 12,345 + 2\cdot 34,125 - 124,35 + 214,35\\
&= -2\cdot 12,345 + 2\cdot 34,125 - 2\cdot 124,35.
\end{aligned}$$

(We denote by a subscript $j = 1, 2, 3$, the terms that we combine together). Using the fact that all the operators $\mathrm{Circ}, \mathrm{Sym}_{4,5}, \mathrm{Alt}_2$ commute we infer the formula

$$\begin{aligned}
\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}[d(R^{'})_2] &= -4\,\mathrm{Circ}\,\mathrm{Sym}_{4,5}\,\boldsymbol{Q}_1\\
\boldsymbol{Q}_1(1,\ldots,5) &= 12,345 - 3,124,5 - 34,125.
\end{aligned}$$

On the other hand using proposition 3, we infer the formula

$$\begin{aligned}
S_2 &:= -\frac{i^3}{6}\,\mathrm{Sym}_{2,3}\,R,\\
4!2\,\mathrm{Sym}_{3,4,5}(S_2 \wedge S_2) &= -\frac{8}{3}\,\mathrm{Sym}_{3,4,5}\,\mathrm{Alt}_2\,\mathbb{L}_{1,3,2},\\
\mathbb{L}_{1,3,2}(1,\ldots,5) &:= 324,15 + 15,234.
\end{aligned}$$

We show now the identity

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\left[2d(R^{'})_2 + 4!2\,\mathrm{Sym}_{3,4,5}(S_2 \wedge S_2)\right] \quad = \quad 0,$$

i.e.

$$12\,\mathrm{Circ}\,\mathrm{Sym}_{4,5}\,\boldsymbol{Q}_1 - 4\,\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\mathrm{Alt}_2\,\mathbb{L}_{1,3,2} = 0. \tag{24}$$

Using the identity $[\mathrm{Sym}_{3,4,5}, \mathrm{Alt}_2] = 0$, the equalities

$$\begin{aligned}
\mathrm{Sym}_{3,4,5}\,\mathbb{L}_{1,3,2} &= \mathrm{Sym}_{4,5}\,[324, 15+423, 15+425, 13+14, 235+13, 245+14, 253] \\
&= \mathrm{Sym}_{4,5}\,[324, 15+423, 15+425, 13+14, 235+31, 425+14, 253]\,, \\
&\quad \mathrm{Circ}\,\mathrm{Sym}_{4,5}\,[324, 15+423, 15+425, 13+14, 235+23, 415+14, 253] \\
&= \mathrm{Circ}\,\mathrm{Sym}_{4,5}\,[214, 35+412, 35+435, 21+34, 125+12, 435+34, 152] \\
&= \mathrm{Circ}\,\mathrm{Sym}_{4,5}\,[214, 35+412, 35-345, 21+34, 125-12, 345+34, 152]\,, \\
&\quad \mathrm{Alt}_2\,[214, 35+412, 35-345, 21+34, 125-12, 345+34, 152] \\
&= 214, 35+412, 35-345, 21+34, 125-12, 345+34, 152 \\
&- 124, 35-421, 35+345, 12-34, 215+21, 345-34, 251 \\
&= -2\cdot 124, 35+214, 35-12, 345+2\cdot 34, 125+2\cdot 21, 345+34, 125 \\
&= -3\cdot 124, 35-3\cdot 12, 345+3\cdot 34, 125 \\
&= 3\cdot 3, 124, 5-3\cdot 12, 345+3\cdot 34, 125.
\end{aligned}$$

We infer

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\mathrm{Alt}_2\,\mathbb{L}_{1,3,2} = -3\cdot \mathrm{Circ}\,\mathrm{Sym}_{4,5}\,\boldsymbol{Q}_1,$$

and thus (24).

5.2 Vanishing of the integrability conditions in the case $k = 5$

We want to show the identity

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,6}\left[2id^2(R^{'})_2 + \sum_{r=3}^{4}(r+1)!\sum_{p=2}^{r-1}(id)^{4-r}(pS_p \wedge S_{r-p+1})\right] = 0.$$

We set for simplicity $ab(cdef) := R(a, b, R'(c, d, e, f))$ and a similar definition for $abc(def)$. Using lemma 14 we infer the identities

$$\begin{aligned}
\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,6}\left[d^2(R')_2\right] &= \mathrm{Circ}\,\mathrm{Sym}_{4,5,6}\,[\mathbb{T}_{2,3} - \mathrm{Alt}_2(\mathbb{Q}_{2,2}+\mathbb{Q}_{2,3})]\,, \\
\mathbb{T}_{2,3}(1,\ldots,6) &= 35(4126)_{_1} + 435(126)_{_2} \\
&- (4351)26_{_3} - 4(351)26_{_4} \\
&+ (4352)16_{_3} + 4(352)16_{_4} \\
&- 12(4356)_{_5} - 412(356)_{_6}, \\
\mathbb{Q}_{2,2}(1,\ldots,6) &= 23(4516)_{_5} - (234)516_{_7} - 4(235)16_{_4} - 451(236)_{_2} \\
&+ 24(3516)_{_8} + 24(5316)_{_1} - (245)316_{_9} - (243)516_{_7} \\
&- 5(243)16_{_4} - 3(245)16_{_{10}} - 35(241)6_{_{11}} - 53(241)6_{_4} \\
&- 351(246)_{_6} - 531(246)_{_6}, \\
\mathbb{Q}_{2,3}(1,\ldots,6) &= 24(3516)_{_8} + 324(516)_{_6} - (3245)16_{_3} - 3(245)16_{_{10}} \\
&- 5(3241)6_{_{12}} - 35(241)6_{_{11}} - 51(3246)_{_8} - 351(246)_{_6} \\
&+ 23(4516)_{_5} + 423(516)_{_6} - (4235)16_{_3} - 4(235)16_{_4} \\
&- 51(4236)_{_8} - 451(236)_{_2} + 25(4316)_{_1} + 425(316)_{_2} \\
&- (4253)16_{_3} - 4(253)16_{_4} - 3(4251)6_{_3} - 43(251)6_{_4} \\
&- 31(4256)_{_5} - 431(256)_{_6},
\end{aligned}$$

where we denote by a subscript $j = 1, \ldots, 12$, the terms that we combine together inside the expression $\mathrm{Circ}\,\mathrm{Sym}_{4,\ldots,6}[\mathbb{T}_{2,3} - \mathrm{Alt}_2(\mathbb{Q}_{2,2}+\mathbb{Q}_{2,3})]$. ($j = 12$ is combined with itself). Indeed combining:

for $j = 1$,

$$\begin{aligned}
&\mathrm{Sym}_{4,5,6}\,[35(4126) - 24(5316) + 14(5326) - 25(4316) + 15(4326)] \\
= \;&\mathrm{Sym}_{4,5,6}\,[34(5126) - 2\cdot 24(5316) + 2\cdot 14(5326)]\,,
\end{aligned}$$

and thus

$$\begin{aligned}
& \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[35(4126)-24(5316)+14(5326)-25(4316)+15(4326)\right] \\
= \;& \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[34(5126)-2\cdot 34(5126)+2\cdot 34(5216)\right] \\
= \;& -3\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[34(5126)\right],
\end{aligned}$$

for $j=2$,

$$\begin{aligned}
& 435(126)+2\cdot 451(236)-2\cdot 452(136)-425(316)+415(326) \\
= \;& 435(126)+2\cdot 451(236)-2\cdot 452(136)-452(136)+451(236) \\
= \;& -453(126)+3\cdot 451(236)-3\cdot 452(136),
\end{aligned}$$

and thus

$$\begin{aligned}
& \operatorname{Circ}\left[435(126)+2\cdot 451(236)-2\cdot 452(136)-425(316)+415(326)\right] \\
= \;& \operatorname{Circ}\left[-453(126)+3\cdot 453(126)-3\cdot 453(216)\right] \\
= \;& 5\operatorname{Circ}\left[453(126)\right],
\end{aligned}$$

for $j=3$,

$$\begin{aligned}
& \operatorname{Circ}\left[(4352)16-(4351)26+(3245)16-(3145)26\right. \\
+ \;& \left.(4235)16-(4135)26+(4253)16-(4153)26+3(4251)6-3(4152)6\right] \\
= \;& \operatorname{Circ}\left[(4352)16-(4351)26+(2145)36-(1245)36\right. \\
+ \;& \left.(4235)16-(4325)16+(4253)16+(4513)26-(4251)36+(4152)36\right] \\
= \;& \operatorname{Circ}\left[(4352)16-(4351)26-(2415)36-(1245)36\right. \\
+ \;& \left.(4235)16+(4235)16+(4253)16+(4532)16-(4352)16+(4351)26\right] \\
= \;& \operatorname{Circ}\left[2\cdot(4235)16+(4125)36+(4253)16+(4532)16\right] \\
= \;& \operatorname{Circ}\left[2\cdot(4235)16+(4125)36-(4325)16\right] \\
= \;& \operatorname{Circ}\left[3(4235)16+(4125)36\right] \\
= \;& 4\operatorname{Circ}\left[(4125)36\right],
\end{aligned}$$

for $j=4$,

$$\begin{aligned}
& \operatorname{Circ}\left[4(352)16-4(351)26+2\cdot 4(235)16-2\cdot 4(135)26\right. \\
+ \;& \left.4(253)16-4(153)26+43(251)6-43(152)6\right] \\
= \;& \operatorname{Circ}\left[4(251)36-4(152)36+2\cdot 4(125)36-2\cdot 4(215)36\right. \\
+ \;& \left.4(152)36-4(251)36-4(251)36+4(152)36\right] \\
= \;& \operatorname{Circ}\left[4\cdot 4(125)36+4(521)36+4(152)36\right] \\
= \;& \operatorname{Circ}\left[4\cdot 4(125)36-4(215)36\right] \\
= \;& 5\operatorname{Circ}\left[4(125)36\right].
\end{aligned}$$

Moreover for the remaining terms we use the equalities

$$\begin{aligned}
& \operatorname{Sym}_{4,5,6}\left[5(243)16-5(143)26+53(241)6-53(142)6\right] \\
= \;& \operatorname{Sym}_{4,5,6}\left[5(243)16-5(143)26-5(241)36+5(142)36\right] \\
= \;& \operatorname{Sym}_{4,5,6}\left[4(253)16-4(153)26-4(251)36+4(152)36\right],
\end{aligned}$$

and

$$\begin{aligned}
& \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[5(243)16-5(143)26+53(241)6-53(142)6\right] \\
= \;& \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(152)36-4(251)36-4(251)36+4(152)36\right] \\
= \;& 2\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(152)36+4(521)36\right] \\
- \;& 2\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(215)36\right] \\
= \;& 2\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(125)36\right].
\end{aligned}$$

Thus for $j = 4$ we have

$$\begin{aligned} & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(352)16 - 4(351)26 + 2\cdot 4(235)16 - 2\cdot 4(135)26 + 4(253)16 - 4(153)26\right. \\ + \quad & \left. 43(251)6 - 43(152)6 + 5(243)16 - 5(143)26 + 53(241)6 - 53(142)6\right] \\ = \quad & 7\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[4(125)36\right]. \end{aligned}$$

For $j = 5$,

$$\begin{aligned} & \operatorname{Circ}\left[-12(4356) - 2\cdot 23(4516) + 2\cdot 13(4526) + 31(4256) - 32(4156)\right] \\ = \quad & \operatorname{Circ}\left[-12(4356) - 2\cdot 12(4536) + 2\cdot 21(4536) + 12(4356) - 21(4356)\right] \\ = \quad & \operatorname{Circ}\left[-4\cdot 12(4536) + 12(4356)\right] \\ = \quad & 5\operatorname{Circ}\left[12(4356)\right]. \end{aligned}$$

For $j = 6$,

$$\begin{aligned} & \operatorname{Circ}\left[-412(356) + 351(246) + 531(246) - 324(516) - 423(516) + 431(256) + 351(246)\right] \\ = \quad & \operatorname{Circ}\left[-412(356) + 152(346) + 512(346) - 214(536) - 412(536) + 412(356) + 152(346)\right], \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 351(246) + 531(246) - 324(516) - 423(516) + 431(256) + 351(246)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 142(356) + 412(356) - 214(536) - 412(536) + 412(356) + 142(356)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 2\cdot 142(356) + 3\cdot 412(356) + 214(356)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 142(356) - 421(356) + 3\cdot 412(356)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 142(356) + 4\cdot 412(356)\right]. \end{aligned}$$

Moreover

$$\begin{aligned} & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-412(356) + 142(356) - 241(356) + 4\cdot 412(356) - 4\cdot 421(356)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[7\cdot 412(356) + 142(356) + 214(356)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[7\cdot 412(356) - 421(356)\right] \\ = \quad & 8\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[412(356)\right]. \end{aligned}$$

For $j = 7$,

$$\begin{aligned} & \operatorname{Circ}\left[(234)516 - (134)526 + (243)516 - (143)526\right] \\ = \quad & \operatorname{Circ}\left[(124)536 - (214)536 + (142)536 - (241)536\right] \\ = \quad & \operatorname{Circ}\left[2\cdot(124)536 - (412)536 - (241)536\right] \\ = \quad & 3\operatorname{Circ}\left[(124)536\right]. \end{aligned}$$

For $j = 8$,

$$\begin{aligned} & \operatorname{Sym}_{4,5,6}\left[-2\cdot 24(3516) + 2\cdot 14(3526) + 51(3246) - 52(3146) + 51(4236) - 52(4136)\right] \\ = \quad & \operatorname{Sym}_{4,5,6}\left[-2\cdot 24(3516) + 2\cdot 14(3526) + 41(3256) - 42(3156) + 41(5236) - 42(5136)\right] \\ = \quad & \operatorname{Sym}_{4,5,6}\left[3\cdot 24(3156) - 3\cdot 14(3256) - 14(5236) + 24(5136)\right], \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[-2\cdot 24(3516) + 2\cdot 14(3526) + 51(3246) - 52(3146) + 51(4236) - 52(4136)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[3\cdot 34(1256) - 3\cdot 34(2156) - 34(5126) + 34(5216)\right] \\ = \quad & \operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[3\cdot 34(1256) + 3\cdot 34(2516) - 2\cdot 34(5126)\right] \\ = \quad & -5\operatorname{Circ}\operatorname{Sym}_{4,5,6}\left[34(5126)\right]. \end{aligned}$$

For $j = 9$,

$$\operatorname{Circ}\left[(245)316\right] \quad = \quad \operatorname{Circ}\left[(345)126\right],$$

and thus

$$\operatorname{Circ}\left[(245)316-(145)326\right] \;=\; 2\operatorname{Circ}\left[(345)126\right].$$

For $j=10$,

$$\begin{aligned}
2\operatorname{Circ}\left[3(245)16-3(145)26\right] &= 2\operatorname{Circ}\left[1(345)26-2(345)16\right]\\
&= 2\operatorname{Circ}\left[1(345)26+21(345)6\right]\\
&= -2\operatorname{Circ}\left[(345)216\right]\\
&= 2\operatorname{Circ}\left[(345)126\right].
\end{aligned}$$

For $j=11$,

$$\begin{aligned}
2\cdot 35(241)6-2\cdot 35(142)6 &= 2\cdot 35(241)6+2\cdot 35(412)6\\
&= -2\cdot 35(124)6.
\end{aligned}$$

For $j=12$,

$$\begin{aligned}
5(3241)6-5(3142)6 &= 5(3241)6+5(3412)6\\
&= -5(3124)6.
\end{aligned}$$

We infer the formula

$$\begin{aligned}
\operatorname{Circ}\operatorname{Sym}_{3,\ldots,6}\left[d^2(R')_2\right] &= \operatorname{Circ}\operatorname{Sym}_{4,5,6}\mathbb{T}_2,\\
\mathbb{T}_2(1,\ldots,6) &= -3\cdot 34(5126)_{_1}+5\cdot 453(126)+4\cdot(4125)36\\
&+ 7\cdot 4(125)36+5\cdot 12(4356)+8\cdot 412(356)\\
&+ 3\cdot(124)536-5\cdot 34(5126)_{_1}+4\cdot(345)126\\
&- 2\cdot 35(124)6-5(3124)6.
\end{aligned}$$

Moreover

$$\begin{aligned}
\operatorname{Circ}\operatorname{Sym}_{4,5,6}\mathbb{T}_2 &= \operatorname{Circ}\operatorname{Sym}_{4,5,6}\tilde{\mathbb{T}}_2,\\
\tilde{\mathbb{T}}_2(1,\ldots,6) &= -8\cdot 34(5126)+5\cdot 453(126)+4\cdot(4125)36\\
&+ 7\cdot 5(124)36_{_1}+5\cdot 12(4356)+8\cdot 412(356)\\
&+ 3\cdot(124)536_{_1}+4\cdot(345)126-2\cdot 35(124)6_{_1}.
\end{aligned}$$

For $j=1$, we have

$$\begin{aligned}
7\cdot 5(124)36-3\cdot(124)356-2\cdot 35(124)6 &= 7\cdot 5(124)36-(124)356+2\cdot 5(124)36\\
&= 9\cdot 5(124)36-(124)356.
\end{aligned}$$

We conclude

$$\begin{aligned}
\operatorname{Circ}\operatorname{Sym}_{3,\ldots,6}\left[d^2(R')_2\right] &= \operatorname{Circ}\operatorname{Sym}_{4,5,6}\boldsymbol{T}_2,\\
\boldsymbol{T}_2(1,\ldots,6) &= -8\cdot 34(5126)+5\cdot 453(126)+4\cdot(4125)36\\
&+ 9\cdot 4(125)36+5\cdot 12(4356)+8\cdot 412(356)\\
&- (124)356+4\cdot(345)126.
\end{aligned}$$

On the other hand using proposition 3, we infer the identity

$$\begin{aligned}
&\operatorname{Sym}_{3,\ldots,6}\left[\sum_{r=3}^{4}(r+1)!\sum_{p=2}^{r-1}(id)^{4-r}(pS_p\wedge S_{r-p+1})\right]\\
&= \frac{2i}{3}\operatorname{Sym}_{3,\ldots,6}\operatorname{Alt}_2\left(4\mathbb{L}_{232}+5\mathbb{L}_{242}+5\mathbb{L}_{243}\right),\\
\mathbb{L}_{232}(1,\ldots,6) &= (1425)36+1(425)36,\\
\mathbb{L}_{242}(1,\ldots,6) &= (3425)16+61(3425),\\
\mathbb{L}_{243}(1,\ldots,6) &= (324)516+5(324)16+561(324).
\end{aligned}$$

We write

$$\mathrm{Sym}_{3,\ldots,6}\,\mathrm{Alt}_2\,(4\mathbb{L}_{232}+5\mathbb{L}_{242}+5\mathbb{L}_{243}) \quad = \quad \mathrm{Sym}_{4,5,6}\,\mathrm{Alt}_2\,\mathbb{L}_2,$$

$$\begin{aligned}
\mathbb{L}_2(1,\ldots,6) &= 4\cdot(1425)36+4\cdot(1325)46+4\cdot(1423)56+4\cdot(1425)63\\
&+ 4\cdot 1(425)36+4\cdot 1(325)46+4\cdot 1(423)56+4\cdot 1(425)63\\
&+ 5\cdot(3425)16+5\cdot(4325)16+5\cdot(5423)16+5\cdot(6425)13\\
&+ 5\cdot 61(3425)+5\cdot 61(4325)+5\cdot 61(5423)+5\cdot 31(6425)\\
&+ 5\cdot(324)516+5\cdot(423)516+5\cdot(524)316+5\cdot(624)513\\
&+ 5\cdot 5(324)16+5\cdot 5(423)16+5\cdot 3(524)16+5\cdot 5(624)13\\
&+ 5\cdot 561(324)+5\cdot 561(423)+5\cdot 361(524)+5\cdot 531(624).
\end{aligned}$$

We notice that $\mathrm{Circ}\,\mathbb{L}_2 = \mathrm{Circ}\,\widetilde{\mathbb{L}}_2$, with

$$\begin{aligned}
\tilde{\mathbb{L}}_2(1,\ldots,6) &= 4\cdot(1425)36+4\cdot(1423)56+4\cdot(1425)63\\
&+ 4\cdot 2(435)16+4\cdot 3(215)46+4\cdot 3(412)56+4\cdot 2(435)61\\
&+ 5\cdot(2415)36+5\cdot(4215)36+5\cdot(5412)36+5\cdot(6435)21\\
&+ 5\cdot 63(2415)+5\cdot 63(4215)+5\cdot 63(5412)+5\cdot 12(6435)\\
&+ 5\cdot(214)536+5\cdot(412)536+5\cdot(534)126+5\cdot(634)521\\
&+ 5\cdot 5(214)36+5\cdot 5(412)36+5\cdot 1(534)26+5\cdot 5(634)21\\
&+ 5\cdot 563(214)+5\cdot 563(412)+5\cdot 162(534)+5\cdot 512(634),
\end{aligned}$$

and $\mathrm{Sym}_{4,5,6}\,\widetilde{\mathbb{L}}_2 = \mathrm{Sym}_{4,5,6}\,\boldsymbol{L}_2$, with

$$\begin{aligned}
\boldsymbol{L}_2(1,\ldots,6) &= 4\cdot(1425)36+4\cdot(1423)56+4\cdot(1425)63\\
&+ 4\cdot 2(435)16+4\cdot 3(214)56+4\cdot 3(412)56+4\cdot 2(435)61\\
&+ 5\cdot(2415)36+5\cdot(4215)36+5\cdot(4512)36+5\cdot(6435)21\\
&+ 5\cdot 63(2415)+5\cdot 63(4215)+5\cdot 63(4512)+5\cdot 12(6435)\\
&+ 5\cdot(214)536+5\cdot(412)536+5\cdot(435)126+5\cdot(435)621\\
&+ 5\cdot 5(214)36+5\cdot 5(412)36+5\cdot 1(435)26+5\cdot 5(435)21\\
&+ 5\cdot 563(214)+5\cdot 563(412)+5\cdot 162(435)+5\cdot 612(435).
\end{aligned}$$

We write now

$$\begin{aligned}
(\mathrm{Alt}_2\,\boldsymbol{L}_2)(1,\ldots,6) &= 4\cdot(1425)36_{_1}+4\cdot(1423)56_{_3}+4\cdot(1425)63_{_4}\\
&- 4\cdot(2415)36_{_1}-4\cdot(2413)56_{_3}-4\cdot(2415)63_{_4}\\
&+ 4\cdot 2(435)16_{_5}+4\cdot 3(214)56_{_6}+4\cdot 3(412)56_{_6}+4\cdot 2(435)61_{_7}\\
&- 4\cdot 1(435)26_{_5}-4\cdot 3(124)56_{_6}-4\cdot 3(421)56_{_6}-4\cdot 1(435)62_{_7}\\
&+ 5\cdot(2415)36_{_1}+5\cdot(4215)36_{_8}+5\cdot(4512)36_{_8}+5\cdot(6435)21_{_9}\\
&- 5\cdot(1425)36_{_1}-5\cdot(4125)36_{_8}-5\cdot(4521)36_{_8}-5\cdot(6435)12_{_9}\\
&+ 5\cdot 63(2415)_{_{10}}+5\cdot 63(4215)_{_{10}}+5\cdot 63(4512)_{_{10}}+5\cdot 12(6435)_{_9}\\
&- 5\cdot 63(1425)_{_{10}}-5\cdot 63(4125)_{_{10}}-5\cdot 63(4521)_{_{10}}-5\cdot 21(6435)_{_9}\\
&+ 5\cdot(214)536_{_{11}}+5\cdot(412)536_{_{11}}+5\cdot(435)126_{_5}+5\cdot(435)621_{_7}\\
&- 5\cdot(124)536_{_{11}}-5\cdot(421)536_{_{11}}-5\cdot(435)216_{_5}-5\cdot(435)612_{_7}\\
&+ 5\cdot 5(214)36_{_{12}}+5\cdot 5(412)36_{_{12}}+5\cdot 1(435)26_{_5}+5\cdot 6(435)21_{_7}\\
&- 5\cdot 5(124)36_{_{12}}-5\cdot 5(421)36_{_{12}}-5\cdot 2(435)16_{_5}-5\cdot 6(435)12_{_7}\\
&+ 5\cdot 563(214)_{_{13}}+5\cdot 563(412)_{_{13}}+5\cdot 162(435)_{_7}+5\cdot 612(435)_{_7}\\
&- 5\cdot 563(124)_{_{13}}-5\cdot 563(421)_{_{13}}-5\cdot 261(435)_{_7}-5\cdot 621(435)_{_7}.
\end{aligned}$$

We need to explain the details for the case $j = 7$. In this case we sum

$$\begin{aligned}
& 4 \cdot 2(435)61 + 5 \cdot (435)621 + 5 \cdot 6(435)21_{_0} + 5 \cdot 162(435)_{_a} + 5 \cdot 612(435)_{_a} \\
- \; & 4 \cdot 1(435)62 - 5 \cdot (435)612 - 5 \cdot 6(435)12_{_0} - 5 \cdot 261(435)_{_a} - 5 \cdot 621(435)_{_a} \\
= \; & 4 \cdot 2(435)61_{_1} + 4 \cdot (435)621_{_1} + (435)621_{_3} + 5 \cdot 612(435)_{_0} + 15 \cdot 612(435)_{_a} \\
- \; & 4 \cdot 1(435)62_{_2} - 4 \cdot (435)612_{_2} - (435)612_{_3} \\
= \; & -4 \cdot 62(435)1_{_1} + 4 \cdot 61(435)2_{_2} + (435)126_{_3} + 20 \cdot 612(435) \\
= \; & 24 \cdot 612(435) + (435)126.
\end{aligned}$$

The result is (we keep the indices according to the sums on $j = 1, \ldots, 13$.)

$$\begin{aligned}
(\mathrm{Alt}_2 \boldsymbol{L}_2)(1, \ldots, 6) \; = \; & -(4125)36_{_1} + 4 \cdot (4123)56_{_3} + 4 \cdot (4125)63_{_4} \\
+ \; & 11 \cdot (435)126_{_5} - 12 \cdot 3(124)56_{_6} + 24 \cdot 612(435)_{_7} + (435)126_{_7} \\
- \; & 15 \cdot (4125)36_{_8} + 15 \cdot 12(6435)_{_9} - 20 \cdot 63(4125)_{_{10}} - 15 \cdot (124)536_{_{11}} \\
- \; & 15 \cdot 5(124)36_{_{12}} - 15 \cdot 563(124)_{_{13}}.
\end{aligned}$$

We recombine now the terms of the previous sum under the form

$$\begin{aligned}
(\mathrm{Alt}_2 \boldsymbol{L}_2)(1, \ldots, 6) \; = \; & -16 \cdot (4125)36_{_1} + 4 \cdot (4123)56 + 4 \cdot (4125)63_{_1} \\
+ \; & 12 \cdot (435)126 - 12 \cdot 3(124)56_{_2} + 24 \cdot 612(435) \\
+ \; & 15 \cdot 12(6435) - 20 \cdot 63(4125)_{_1} - 15 \cdot (124)536_{_2} \\
- \; & 15 \cdot 5(124)36_{_2} - 15 \cdot 563(124).
\end{aligned}$$

Indeed for $j = 1$,

$$\begin{aligned}
& -16 \cdot (4125)36 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
= \; & -12 \cdot (4125)36 - 4 \cdot (4125)36 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
= \; & -12 \cdot (4125)36 + 4 \cdot 3(4125)6 + 4 \cdot (4125)63 - 20 \cdot 63(4125) \\
= \; & -12 \cdot (4125)36 - 24 \cdot 63(4125),
\end{aligned}$$

and for $j = 2$,

$$\begin{aligned}
& -12 \cdot 3(124)56 - 15 \cdot (124)536 - 15 \cdot 5(124)36 \\
= \; & 12 \cdot 53(124)5 - 3 \cdot (124)536 - 15 \cdot 5(124)36 \\
= \; & -27 \cdot 5(124)36 - 3 \cdot (124)536.
\end{aligned}$$

In conclusion

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,6} \left[\sum_{r=3}^{4} (r+1)! \sum_{p=2}^{r-1} (id)^{4-r} (pS_p \wedge S_{r-p+1}) \right] \; = \; \frac{2i}{3} \,\mathrm{Circ}\,\mathrm{Sym}_{4,5,6}\, \tilde{\boldsymbol{\Lambda}}_2,$$

with

$$\begin{aligned}
\tilde{\boldsymbol{\Lambda}}_2(1, \ldots, 6) \; = \; & -12 \cdot (4125)36 + 24 \cdot 36(4125) + 12 \cdot (435)126 + 24 \cdot 612(435) \\
+ \; & 15 \cdot 12(6435) - 27 \cdot 5(124)36 - 3 \cdot (124)536 - 15 \cdot 563(124).
\end{aligned}$$

We notice that $\mathrm{Sym}_{4,5,6}\, \tilde{\boldsymbol{\Lambda}}_2 = \mathrm{Sym}_{4,5,6}\, \boldsymbol{\Lambda}_2$, with

$$\begin{aligned}
\boldsymbol{\Lambda}_2(1, \ldots, 6) \; = \; & -12 \cdot (4125)36 + 24 \cdot 34(5126) + 12 \cdot (435)126 + 24 \cdot 412(536) \\
+ \; & 15 \cdot 12(4536) - 27 \cdot 4(125)36 - 3 \cdot (124)536 - 15 \cdot 453(126).
\end{aligned}$$

We deduce $\boldsymbol{\Lambda}_3 = -3\boldsymbol{T}_2$, which implies the required conclusion in the case $k = 2$.

6 Proof of Corollaries 1, 2 and 3

We recall first the following very elementary lemmas. The arguments below can be found in [Pali2].

Lemma 15 *Let M be a smooth manifold, let also ∇ be a covariant derivative operator acting on the smooth sections of T_M and consider the vector field over T_M*

$$\zeta^{\nabla}_{\eta} := H^{\nabla}_{\eta} \cdot \eta, \tag{25}$$

$\eta \in T_M$. We denote by Φ^{∇}_t the corresponding 1-parameter subgroup of transformations of T_M. Then for any $\eta \in T_M$ the curve $c_t := \pi_{T_M} \circ \Phi^{\nabla}_t(\eta)$ is the ∇-geodesic with initial speed $\dot{c}_0 = \eta$ and $\dot{c}_t = \Phi^{\nabla}_t(\eta)$.

Proof The flow line $\eta_t := \Phi^{\nabla}_t(\eta)$ satisfies the identity

$$\dot{\eta}_t = H^{\nabla}_{\eta_t} \cdot \eta_t. \tag{26}$$

We deduce

$$\begin{aligned} \dot{c}_t &= d_{\eta_t}\pi_{T_M} \cdot \dot{\eta}_t \\ &= d_{\eta_t}\pi_{T_M} \cdot H^{\nabla}_{\eta_t} \cdot \eta_t \\ &= \eta_t, \end{aligned}$$

and $\ddot{c}_t = H^{\nabla}_{\dot{c}_t} \cdot \dot{c}_t$, which is the ∇-geodesic equation. □

Corollary 5 *Let M be a smooth manifold, let also ∇ be a covariant derivative operator acting on the smooth sections of T_M and let U be an open neighborhood U of M inside T_M. A complex structure J over U satisfies the conditions*

$$J_{|M} = J^{\mathrm{can}}, \tag{27}$$

$$J_{\eta} H^{\nabla}_{\eta} \cdot \eta = T_{\eta} \cdot \eta, \tag{28}$$

for any $\eta \in U$ if and only if for any $\eta \in U$, the smooth map $\psi_\eta : t + is \longmapsto s\Phi^{\nabla}_t(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J-holomorphic.

Proof We observe first that the differential of the maps ψ_η is given by

$$d_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} \right) = ad(s_0 \mathbb{I}_{T_M})\dot{\Phi}^{\nabla}_{t_0}(\eta) + bT_{s_0\Phi^{\nabla}_{t_0}(\eta)}\Phi^{\nabla}_{t_0}(\eta).$$

But

$$\begin{aligned} \dot{\Phi}^{\nabla}_{t_0}(\eta) &= \zeta^{\nabla} \circ \Phi^{\nabla}_{t_0}(\eta) \\ &= H^{\nabla}_{\Phi^{\nabla}_{t_0}(\eta)} \cdot \Phi^{\nabla}_{t_0}(\eta), \end{aligned}$$

thanks to (25). Then using the property, $H^{\nabla}_{\lambda\eta} = d_\eta(\lambda \mathbb{I}_{T_M}) \cdot H^{\nabla}_{\eta}$, of the linear connection ∇ (see the identity 8.5 in [Pali2]), we infer

$$d_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} \right) = \left(aH^{\nabla}_{s_0\Phi^{\nabla}_{t_0}(\eta)} + bT_{s_0\Phi^{\nabla}_{t_0}(\eta)} \right) \cdot \Phi^{\nabla}_{t_0}(\eta). \tag{29}$$

The smooth map ψ_η is J-holomorphic if and only if

$$d_{t_0+is_0}\psi_\eta \left(-b\frac{\partial}{\partial t} + a\frac{\partial}{\partial s} \right) = Jd_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s} \right),$$

thus, if and only if

$$\left(-bH^{\nabla}_{s_0\Phi^{\nabla}_{t_0}(\eta)} + aT_{s_0\Phi^{\nabla}_{t_0}(\eta)} \right) \cdot \Phi^{\nabla}_{t_0}(\eta) = J \left(aH^{\nabla}_{s_0\Phi^{\nabla}_{t_0}(\eta)} + bT_{s_0\Phi^{\nabla}_{t_0}(\eta)} \right) \cdot \Phi^{\nabla}_{t_0}(\eta).$$

For $s_0 \neq 0$ this is equivalent to (28). For $s_0 = 0$ this is equivalent to (27). We deduce the required conclusion. □

The condition (27) implies that J is an M-totally real complex structure. We provide now the proof of corollary 1.

Proof If we write $A = \alpha + iTB$ and $\alpha = H^\nabla - T\Gamma$, then $S := T^{-1}\left(H^\nabla - \overline{A}\right) = \Gamma + iB$. We set $S_k = \Gamma_k + iB_k$. From the proof of corollary 5 we know that in the case J is integrable over U, the smooth map ψ_η is J-holomorphic if and only if hold (28). The later rewrites as

$$H^\nabla_\eta \cdot \eta \quad = \quad -J_\eta T_\eta \cdot \eta.$$

Using the property

$$J_{\eta|\mathrm{Ker}\, d_\eta\pi} = -\alpha_\eta B_\eta^{-1} T_\eta^{-1},$$

(see the identity 1.7 in [Pali2]), we infer that the previous identity is equivalent to

$$H^\nabla_\eta \cdot \eta = \alpha_\eta B_\eta^{-1} \cdot \eta. \tag{30}$$

Taking $d_\eta\pi$ on both sides of (30) we deduce $\eta = B_\eta^{-1} \cdot \eta$. Therefore (30) is equivalent to the system

$$\left\{ \begin{array}{lll} B_\eta \cdot \eta & = \eta\,, \\ \\ H^\nabla_\eta \cdot \eta & = & \alpha_\eta \cdot \eta\,. \end{array} \right. \tag{31}$$

Then the system (31) rewrites as

$$\left\{ \begin{array}{ll} \sum_{k\geqslant 1} B_k(\eta^{k+1}) & = 0\,, \\ \\ \sum_{k\geqslant 1} \Gamma_k(\eta^{k+1}) & = \quad 0\,. \end{array} \right.$$

and thus as $S_k(\eta^{k+1}) \quad = 0$ for all $k \geqslant 1$. We remind now that, according to the main theorem in [Pali2], the integrability of the structure J implies the condition $S_1 \in C^\infty\left(M, S^2T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$. We infer $S_1 = 0$.

We notice that, with the notations of the statement of the main theorem in [Pali2] the equation $\mathrm{Circ}\,\beta_k = 0$ hold for all $k \geqslant 1$. This combined with the identity $[\mathrm{Circ}, \mathrm{Sym}_{2,\ldots,k+2}] = 0$, implies

$$\mathrm{Circ}\,\mathrm{Sym}_{2,\ldots,k+2}\,\beta_k = 0, \tag{32}$$

for all $k \geqslant 1$. So if we apply the Circ operator to both sides of the definition of S_2 in the statement of the main theorem in [Pali2], we infer $\mathrm{Circ}\,S_2 = \mathrm{Circ}\,\sigma_2 = 3\sigma_2$. If we evaluate this equality to η^3 we infer $S_2(\eta^3) = \sigma_2(\eta^3)$, which implies $\sigma_2 = 0$. We show now by induction that $\sigma_k = 0$ for all $k \geqslant 2$. Indeed by the inductive assumption

$$S_{k+1} \quad = \quad \frac{i}{(k+2)!}\,\mathrm{Sym}_{2,\ldots,k+2}\,\beta_k + \sigma_{k+1}.$$

Applying the Circ operator to both sides of this identity and using the equation (32), we infer $\mathrm{Circ}\,S_{k+1} = \mathrm{Circ}\,\sigma_{k+1} = 3\sigma_{k+1}$, which evaluated at η^{k+2} gives $S_{k+1}(\eta^{k+2}) = \sigma_{k+1}(\eta^{k+2})$. We deduce $\sigma_{k+1} = 0$. Using the identity

$$\mathrm{Sym}_{2,\ldots,k+1}\,\mathrm{Sym}_{3,\ldots,k+1} = (k-1)!\,\mathrm{Sym}_{2,\ldots,k+1}, \tag{33}$$

we deduce from the statement of the main theorem in [Pali2] and with the notations there

$$S_k \quad = \quad \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\ldots,k+1}\,\theta_{k-1},$$

for $k \geqslant 2$, with $\theta_1 := 2R^\nabla$ and

$$\theta_k \quad := \quad -2i(id_1^\nabla)^{k-2}(\nabla R^\nabla)_2 + \sum_{r=4}^{k+1} r! \sum_{p=2}^{r-2} (id_1^\nabla)^{k+1-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 2$. Moreover the equation $\operatorname{Circ} \beta_k = 0$, $k \geqslant 3$ rewrites as $\operatorname{Circ} \operatorname{Sym}_{3,\ldots,k+2} \theta_k = 0$. If we set $\Theta_k^{\nabla} := \theta_{k-1}$, for all $k \geqslant 2$ we obtain the required expansion.

On the other hand if the expansion in the statement of the lemma under consideration hold then J is integrable thanks to the main theorem in [Pali2] and $\operatorname{Circ} S_k = 0$ for all $k \geqslant 2$ ($S_1 = 0$). Indeed for $k = 2, 3$ this equality follows from the identities $\operatorname{Circ} \Theta_k^{\nabla} = 0$ and

$$[\operatorname{Circ}, \operatorname{Sym}_{2,\ldots,k+1}] = 0. \tag{34}$$

For $k \geqslant 4$ we use the identities (34), (33) and the integrability equations satisfied by the covariant derivative operator ∇. We deduce $S_k(\eta^{k+1}) \quad = 0$, for all $k \geqslant 1$, which is equivalent to (30) and so to the fact that the curves ψ_η are J-holomorphic. □

We provide now the proof of Corollary 2.

Proof We use first the notations in the main Theorem of [Pali2]. Then we have

$$\begin{aligned} \overline{A} &= H^{\nabla} - TS, \\ H^{\nabla^{S_1}} &= H^{\nabla} - TS_1, \end{aligned}$$

Replacing in the statement of the main Theorem in [Pali2], the torsion free covariant derivative operator ∇^{S_1} with the arbitrary torsion free complex covariant derivative ∇ in the statements of Corollary 2 and Conjecture 1, replacing the sum $\sum_{k \geqslant 2} S_k$ with S and setting $\sigma_k \equiv 0$, for all $k \geqslant 2$, we deduce the expression of S_k in the statement of Conjecture 1. (We use here the identity (33)). Then the integrability equations in the statement of the main theorem [Pali2] reduce to the identities $I_k = 0$, for all $k \geqslant 4$ in the statement of Conjecture 1. On the other hand by Theorem 2, an M-totally real structure has a fiberwise Taylor expansion with S_k under the form in the statement of Conjecture 1 if and only if S_k is the one in the statement of Corollary 2.

We show now the convergence of the Taylor expansion in the statement of Corollary 2. We fix a Riemannian metric g over M. Then the analytic assumption on the covariant derivative operator ∇ implies that for any relatively compact open set $W \subset X$ there exist a constant $K_W > 0$ such that

$$\|R^{(h)}\|_g \leqslant h! K_W^h.$$

Then using the explicit expression of S_k in the statement of Corollary 2 we obtain the inequality

$$\begin{aligned} \|S_k(\eta^{k+1})\|_g &\leqslant P_k(K_W) \|\eta\|_g^{k+1}, \\ P_k(x) &:= \frac{1}{(k+1)!} \sum_{\|D\|=k} |C_D| D! x^{|D|}. \end{aligned}$$

We recall here the identity $C_D = -(-i)^{\|D\|}(\|D\| + 1)! \boldsymbol{C}_D$, observed at the very end of section 3. We have the estimates (the technical details will appear elsewhere)

$$|P_k(x)| \leqslant \lfloor k/2 \rfloor C_{\max}(P_k) |x|^{k-2},$$

where the maximal coefficient $C_{\max}(P_k)$ of the polynomial P_k, satisfies the asymptotic expansion

$$C_{\max}(P_k) = \frac{2}{k^3} + \frac{6(2\gamma - 3) + 12 \ln(k)}{k^4} + o\left(\frac{1}{k^4}\right),$$

where γ is Euler's constant. We infer the convergence of the fiberwise Taylor expansion over a sufficiently small neighborhood of the image of the zero section inside T_W. □

It is straightforward to see that the condition (28) is equivalent to say that the vector ξ_η^{∇} is of type $(1, 0)$ for J. System (31) shows that (28) is equivalent to the identity

$$\left(H^{\nabla} - \overline{A}\right)_\eta \cdot \eta = i T_\eta \cdot \eta.$$

which rewrites also as $\xi^{\nabla}_{\eta} = \overline{A}_{\eta} \cdot \eta$. Using the identity (29) we can see that the vector field ξ^{∇} is related to the J-holomorphic curves ψ_{η} by the equation

$$\frac{\partial_J \psi_{\eta}}{\partial z} = \frac{i}{z - \bar{z}} \xi^{\nabla} \circ \psi_{\eta},$$

with

$$\frac{\partial_J \psi_{\eta}}{\partial z} := \left[d_z \psi_{\eta} \cdot \frac{\partial}{\partial z} \right]^{1,0}_{J} = d_z \psi_{\eta} \cdot \frac{\partial}{\partial z},$$

since ψ_{η} is J-holomorphic.

In general we have the following result.

Lemma 16 *Let M be a smooth manifold equipped with a covariant derivative operator ∇ acting on the smooth sections of the complexified tangent bundle $\mathbb{C}T_M$ and let $J \equiv J_A$ be an M-totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the image of the zero section. Then the vector field ξ^{∇} is of type $(1,0)$ for J over U if and only if $\xi^{\nabla}_{\eta} = \overline{A}_{\eta} \cdot \eta$, for all points $\eta \in U$.*

Proof The "if" direction is obvious. We show the "only if" direction. We assume thus $\xi \equiv \xi^{\nabla}$ is of type $(1,0)$ for J over U. This is equivalent to the equality $\xi = \xi^{1,0}_{J}$ over U, which writes as

$$\xi = -iJ\xi. \tag{35}$$

We decompose

$$H^{\nabla} = H - iTK,$$

with $H \in C^{\infty}(T_M, \pi^* T^*_M \otimes_{\mathbb{R}} T_{T_M})$ and $K \in C^{\infty}(T_M, \pi^* \operatorname{End}(T_M))$ such that $d\pi \cdot H = \mathbb{I}_{\pi^* T_M}$. Here as usual we denote $(TK \cdot v)_{\eta} = T_{\eta} K(v, \eta)$. Then (35) writes as

$$\left[H - iT(K + \mathbb{I}_{T_M}) \right]_{\eta} \cdot \eta = - \left[iJH + JT(K + \mathbb{I}_{T_M}) \right]_{\eta} \cdot \eta.$$

In its turn this identity is equivalent to

$$H_{\eta} \cdot \eta = - \left[JT(K + \mathbb{I}_{T_M}) \right]_{\eta} \cdot \eta,$$

which thanks to the identity 1.7 in [Pal], is equivalent to

$$H_{\eta} \cdot \eta = \left[\alpha B^{-1} (K + \mathbb{I}_{T_M}) \right]_{\eta} \cdot \eta, \tag{36}$$

where $A = \alpha + iTB$. By taking $d_{\eta}\pi$ on both sides of (36) we deduce the identity

$$\eta = \left[B^{-1} (\mathbb{I}_{T_M} + K) \right]_{\eta} \cdot \eta.$$

We deduce that (36) is equivalent to the system

$$\begin{cases} (B - K)_{\eta} \cdot \eta &= \eta, \\ H_{\eta} \cdot \eta &= \alpha_{\eta} \cdot \eta \,, \end{cases}$$

which in its turn is equivalent to the identity

$$\left(H^{\nabla} - \overline{A} \right)_{\eta} \cdot \eta = iT_{\eta} \cdot \eta, \tag{37}$$

for all points $\eta \in U$. We deduce the required conclusion. □

We can show now Corollary 3.

Proof For any M-totally real almost complex structure over U, real analytic along the fibers of U, we consider the fiber-wise Taylor expansion at the origin

$$T_\eta^{-1}\big(H^\nabla - \overline{A}\big)_\eta \cdot \xi \quad = \quad i\xi + \sum_{k\geqslant 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a sufficiently small neighborhood of the zero section, with $\xi \in T_{M,\pi(\eta)}$ arbitrary with

$$S_k \in C^\infty\left(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right),$$

and with $\eta^k := \eta^{\times k} \in T^{\oplus k}_{M,\pi(\eta)}$. Then the identity (37) hold over U if and only if $S_k(\eta^{k+1}) = 0$, for all $k \geqslant 1$ and all η in an arbitrarily small neighborhood of the image of the zero section of T_M inside U. (This is because A is real analytic along the fibers of U). Thus by lemma 16, the vector field ξ^∇ is of type $(1,0)$ for J over U if and only if the above vanishing conditions are satisfied.

By the argument in the proof of Corollary 1 (see also the proofs of Corollary 1 in [Pali]) we see that J is integrable over U and the above vanishing conditions are satisfied if and only if J has a fiberwise Taylor expansion with S_k under the form in the statement of Conjecture 1. By Theorem 2 this hold true if and only if J is the canonical M-totally real complex structure associated to ∇ over U. □

References

[Br-Wh] Bruhat, F., Whitney, H., *Quelque propriétés fondamentales des ensembles analytiques-réels*, comment. Math. Helv. 33, (1959), 132-160.

[Fo-Mer-Ta1] Foo, W., Merker, J., Ta, T., *Nonvanishing of Cartan CR curvature on boundaries of Grauert tubes around hyperbolic surfaces*, arXiv:1904.10203, (2019).

[Fo-Mer-Ta2] Foo, W., Merker, J., Ta, T., *On convergent Poincaré-Moser reduction for Levi degenerate embedded 5-dimensional CR manifolds*, New York J. Math. 28, 250-336 (2022).

[Gra] Grauert, H. *On Levi's problem and the imbedding of real analytic manifolds*, Ann. of Math. (2), 68, (1958), 460-472.

[Mer-Nur1] Merker, J., Nurowski, P., *Homogeneous CR and Para-CR structures in dimensions 5 and 3*, J. Geom. Anal. 34, No. 1, Paper No. 27, 50 p. (2024).

[Mer-Nur2] Merker, J., Nurowski, P., *Five-dimensional para-CR manifolds and contact projective geometry in dimension three*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 24, No. 1, 519-549 (2023).

[Mer] Merker, J., *Equivalences of PDE systems associated to degenerate para-CR structures: foundational aspects*, SN Partial Differ. Equ. Appl. 3, No. 1, Paper No. 4, 57 p. (2022).

[Hill-Mer-Nie-Nur1] Hill, C.D., Merker, J., Nie, Z., Nurowski, P., *Infinitesimal CR Symmetries of Accidental CR Structures*, arXiv:2302.06467, (2023).

[Hill-Mer-Nie-Nur2] Hill, C.D., Merker, J., Nie, Z., Nurowski, P., *Accidental CR structures*, arXiv:2302.03119, (2023).

[Mer-Sab] Merker, J., Sabzevari, M., *Cartan equivalence problem for 5-dimensional bracket-generating CR manifolds in C4*, J. Geom. Anal. 26, No. 4, 3194-3251 (2016).

[Pali1] Pali, N. *Exact Fourier inversion formula over manifolds*, Journal of Pseudo-Differential Operators and Applications, 8 (2017), no. 4, 623-628.

[Pali2] Pali, N., *On maximal totally real embeddings*, to appear in Complex Manifolds (2024), available at HAL server, https ://hal.science/hal-04242579, (2023), 57 pp. and arXiv:1904.09220 (2023).

[Pal-Sal] Pali, N., Salvy, B., *Computer verifications of the claims in the article "Explicit maximal totally real embeddings"*, https ://hal.science/hal-04518407, (2024), 34 pp. (with an appendix by F. Guenard).

[Sim-Sch] Simion, R., Schmidt, F. W., *Restricted permutations*, European J.Combin. 6 (1985), no. 4, 383–406.