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# Supplementary Material to “A hypothesis test for the domain of attraction of a random variable”

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October 31, 2023

## Abstract

In the paper [14] we address theoretical aspects of testing whether a sampled probability distribution of a random variable  $V$  belongs to the domain of attraction of the Normal law or in the domain of attraction of a stable law with index smaller than 2.

In this supplementary paper we present and discuss numerical results which allow us to illustrate satisfying properties of the proposed test.

## 1 Introduction

In the paper [14] we address theoretical aspects of testing whether a sampled probability distribution of a random variable belongs to the domain of attraction of the Normal law or in the domain of attraction of a stable law with index smaller than 2.

More precisely, we develop an hypothesis test for which the null and alternative hypotheses respectively are:

$$\mathbf{H}_0 : X \in \text{DA}(2)$$

and

$$\mathbf{H}_1 : \exists 0 < \alpha < 2, \quad X \in \text{DA}(\alpha).$$

Here is our main result.

**Theorem 1.1.** *Assume that  $X$  belongs to some domain of attraction. Consider an i.i.d. sample  $X_1, \dots, X_m$  of  $X$ , and the statistic*

$$\widehat{\mathcal{F}}_n^m = \frac{\sum_{i=1}^{n-1} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \bar{X}_m) \right| \left| \sum_{j=\lfloor \frac{mi}{n} \rfloor + 1}^{\lfloor \frac{m(i+1)}{n} \rfloor} (X_j - \bar{X}_m) \right|}{\sum_{i=1}^n \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \bar{X}_m) \right|^2},$$

where  $\bar{X}_m$  stands for the sample mean. Let  $z_q$  denote the  $q$ -quantile of a standard normal random variable and let  $\sigma_\pi^2 := 1 + \frac{4}{\pi} - \frac{20}{\pi^2}$ . The rejection region

$$C_{n,m} := \left\{ \left| \widehat{\mathcal{F}}_n^m - \frac{2}{\pi} \right| > z_{1-q/2} \sqrt{\frac{\sigma_\pi^2}{n}} \right\}$$

satisfies:

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1.  $\limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} \mathbb{P}(C_{n,m} | \mathbf{H}_0) \leq q.$
2.  $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{P}(C_{n,m} | \mathbf{H}_1) = 1.$

Our construction of an effective hypothesis test is original. Unexpectedly, it is based on fine properties of bivariate semimartingales and a test for jumps which allows one to discriminate between discontinuous stable processes and Brownian motions.

**Plan of the paper:** The plan of the paper is as follows.

In Section 2 we recall some known results about stable laws.

In Section 3 we illustrate by numerical examples that classical limit theorems are not well suited to test  $\mathbf{H}_0$  against  $\mathbf{H}_1$ .

In Section 4 we present our hypothesis test and summarize our theoretical results in [14].

In Section 5 we discuss many numerical experiments. In particular, we examine the sensitivity of our test to its two key parameters: the number  $n$  of discretization times and the sample size  $m$ .

In Section 6 we present numerical results which illustrate the robustness of our proposed test to our standing and classical hypothesis that the sampled probability distribution belongs to the domain of attraction of a stable law.

## 2 A few reminders on stable laws and domains of attraction

Let  $X, X_1, X_2, \dots$  be a sequence of non-degenerate i.i.d. random variables and let

$$S_m := \sum_{j=1}^m X_j, \quad m = 1, 2, \dots$$

One says that  $X$ , or the law of  $X$ , belongs to the domain of attraction of a given law  $\mathcal{L}$  if there exist centering constants  $\mu_m$  and positive normalizing constants  $c_m$  such that  $(S_m - \mu_m)/c_m$  converges in distribution to  $\mathcal{L}$ .

The only probability laws which have a non-empty domain of attraction are the stable laws. Any stable law has a characteristic function of the form

$$\exp(i a \lambda - b |\lambda|^\alpha (1 + i \beta \operatorname{sign}(\lambda) w(\lambda, \alpha)))$$

for some parameters  $0 < \alpha \leq 2$  and  $-1 \leq \beta \leq 1$ , where

$$w(\lambda, \alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} \log(|\lambda|) & \text{if } \alpha = 1. \end{cases}$$

The parameter  $\alpha$  is called the characteristic exponent or the index of the stable law. Given  $0 < \alpha \leq 2$ , every stable law with index  $\alpha$  is called an  $\alpha$ -stable distribution and denoted by  $\mathcal{L}_\alpha$ . Its domain of attraction is denoted by  $D(\alpha)$ .

Tail properties of stable laws imply the following moment properties (see e.g. Embrechts et al. [3, Cor.2.2.10]).

**Theorem 2.1.** *If the random variable  $X$  belongs to  $D(\alpha)$  then*

$$\begin{aligned} \mathbb{E}(|X|^\delta) &< \infty \quad \text{for } 0 < \delta < \alpha, \\ \mathbb{E}(|X|^\delta) &= \infty \quad \text{for } \delta > \alpha \quad \text{and} \quad \alpha < 2. \end{aligned}$$

*In particular,  $\mathbb{E}(X^2) = \infty$  for  $\alpha < 2$ .*

In [14] we explain why often one cannot get a priori theoretical tail estimates on the probability distribution of particles with complex dynamics. Consequently, even if one a priori knows that this probability distribution belongs to some domain of attraction, to determine the parameter  $\alpha$  is a hard statistical question. That was a motivation to build a specific hypothesis test which is based on the following limit theorems.

We start with Donsker's invariance principle for i.i.d. random variables. See e.g. Whitt [18, Thms 4.5.2-4.5.3].

**Theorem 2.2.** *Let  $X, X_1, X_2, \dots$  be a sequence of non-degenerate i.i.d. random variables such that  $X \in \text{DA}(\alpha)$ . Then there exist centering constants  $\mu_m$  and normalizing constants  $c_m$  such that*

$$L^m := \left( \frac{S_{\lfloor mt \rfloor} - \mu_m t}{c_m}, t \geq 0 \right) \xrightarrow[m \rightarrow \infty]{\mathcal{D}} L,$$

where  $L$  is a standard  $\alpha$ -stable Lévy process if  $\alpha < 2$ , whereas for  $\alpha = 2$   $L$  is a Brownian motion.

For  $\alpha < 2$ , the trajectories of  $\alpha$ -stable processes are a.s. discontinuous, whereas for  $\alpha = 2$  the trajectories of the Brownian motion are a.s. continuous. In addition, one expects that for  $m$  large enough, the trajectories of  $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$  resemble the trajectories of the limit process (see Fig. 1). Therefore, testing for jumps in the trajectories of  $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$  should allow to discriminate between  $X \in \text{DA}(2)$  and  $X \in \text{DA}(\alpha)$ . This is illustrated by Fig. 1. Simulations have been run with  $m = 10000$ . For  $r = 0.2$  we are under  $\mathbf{H}_0$  and we can compute the explicit value of  $\mathbb{E}(X)$  and  $\mathbb{E}(X^2)$ . For  $r = 0.8$  we know that  $|G|^{-r}$  is in the normal domain of attraction of stable distribution with index  $1/r$ . Hence, the normalizing constant is  $c_m = m^{1/r}$ . Indeed, respectively denoting by  $\Phi$  and  $\phi$  the cumulative distribution function and the density function of a standard normal distribution we have

$$\mathbb{P}(|G|^{-r} > x) = 2\Phi\left(\frac{1}{x^{1/r}}\right) - 1.$$

From L'Hôpital's Theorem it follows that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(|G|^{-r} > x)}{1/x^{1/r}} = \lim_{x \rightarrow \infty} \frac{2\Phi(1/x^{1/r}) - 1}{1/x^{1/r}} = \lim_{x \rightarrow \infty} \frac{2\phi(1/x^{1/r})(1/x^{1/r})'}{(1/x^{1/r})'} = \sqrt{\frac{2}{\pi}}.$$

We thus deduce the desired result from [18, Thm.4.5.2].

This heuristic approach has a severe drawback: we do not know the values of  $\mu_m$  and  $c_m$ , in particular because we do not know  $\alpha$ . Nevertheless, as Corollary 2.4 below shows that one can use the Mapping Theorem to bypass the fact that  $\mu_m$  is unknown.

**Remark 2.3.** *The conclusion of Theorem 2.2 still holds true in some cases when the  $X_i$ 's are weakly dependent. Sufficient conditions to respectively obtain Brownian or stable Lévy limits can be found in Shao [15] or Tyran-Kamiska [17].*

One can easily prove the following

**Proposition 2.4.** *Let*

$$Z^m := \left( \frac{S_{\lfloor mt \rfloor} - tS_m}{c_m}, t \geq 0 \right). \quad (2.1)$$

*If  $X \in \text{DA}(\alpha)$ , then*

$$Z^m \xrightarrow[m \rightarrow \infty]{\mathcal{D}} Z := \Psi(L),$$

where  $L$  is a standard  $\alpha$ -stable Lévy process for  $\alpha < 2$  (and therefore  $\Psi(L)$  is a discontinuous process), whereas for  $\alpha = 2$   $L$  is a Brownian motion (and therefore  $\Psi(L)$  is a Brownian bridge).

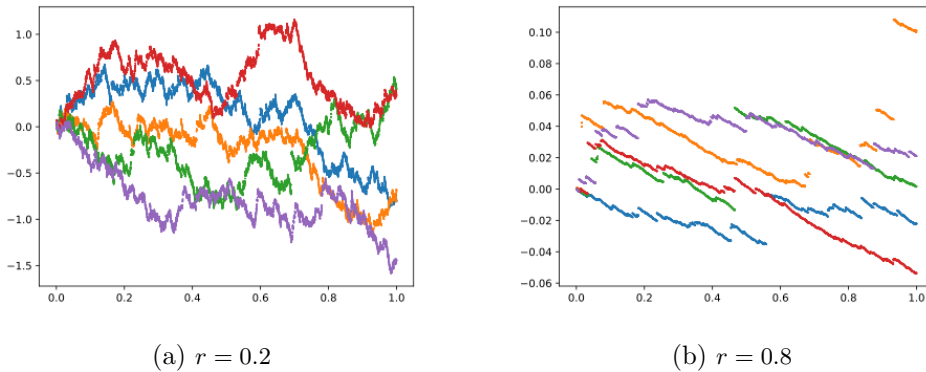


Figure 1: Trajectories of  $L^m$  when the subjacent random variable  $X \sim |G|^{-r}$ , where  $G \sim \mathcal{N}(0, 1)$ . In the left column  $r = 0.2$ , therefore the limit of  $L^m$  is a Brownian motion and the trajectories of  $L^m$  seem to be continuous. In the right column,  $r = 0.8$ , hence the limit of  $L^m$  is a Lévy process and the trajectories of  $L^m$  seem to have jumps.

To conclude, testing for jumps the trajectories of the limit process of  $Z^m$  defined in (2.1) should allow one to discriminate between  $X \in \text{DA}(2)$  and  $X \in \text{DA}(\alpha)$ ,  $0 < \alpha < 2$ . It now remains to construct an hypothesis test for the continuity of  $\Psi(L)$  which does not suppose that  $c_m$  is known.

Before describing and analyzing our test, let us present numerical experiments which show that naive statistical procedures to accept or reject  $\mathbf{H}_0$  are unsatisfying.

### 3 Ineffectiveness of tests based on limit theorems for I.I.D. sequences: An empirical evidence

In this section we empirically study several heuristic approaches based on limit theorems for I.I.D. sequences to determine the heaviness of the tails of a distribution function. Our numerical experiments below tend to show that such approaches are ineffective.

Given  $(X, X_1, X_2, \dots)$  a sequence of non-degenerate i.i.d. random variables, we respectively denote the sample sum, the sample mean and the sum of the squares of the sample by

$$S_m := \sum_{j=1}^m X_j, \quad \bar{X}_m := \frac{S_m}{m} \quad \text{and} \quad V_m^2 = \sum_{j=1}^m X_j^2.$$

#### 3.1 A test based on fluctuations of running maxima

Numerical experiments discussed in Section 5.2 show that empirical means do not suffice to confidently decide whether expectations are finite or infinite. The next proposition, which follows from results in O'Brien [13, Thm.1] and Kesten [9], shows a strong discrepancy in the behavior of the running maxima of i.i.d. sequences depending on whether the probability distribution of the random variables has finite or infinite expectation.

**Proposition 3.1.** *Let  $(X_n)$  be a sequence of i.i.d. positive random variables and let  $M_n := \max\{X_1, \dots, X_n\}$  be the running maximum of the sequence.*

1. *If  $\mathbb{E}(X) = \infty$  then  $\limsup_n \frac{M_n}{n} = \infty$  a.s.*
2. *If  $\mathbb{E}(X) < \infty$  then  $\lim_n \frac{M_n}{n} = 0$  a.s.*

In particular,

$$\mathbb{E}(X) < \infty \iff \lim_n \frac{M_n}{n} = 0 \quad a.s.$$

The previous proposition suggests to use the behavior of the running maxima as an indicator of the finiteness of the mean of a positive random variable. One could also use the ratio of the running maximum and the partial sum: See the analysis of this ratio and of other statistics for heavy-tailed data in Embrechts et al. [3, Sec.6.2.6]. Notice that such an approach does not contradict Hawkins' result [6] (see the Introduction in [14]) since the results compiled in [3, Sec.6.2.6] are asymptotic.

To numerically study the effectiveness of such an approach we consider a sample of  $V = X^2$ , where  $X = |G|^{-r}$  with  $G$  a standard normal random variable, For  $r < 0.5$  one has  $\mathbb{E}(V) < \infty$  whereas for  $r \geq 0.5$  one has  $\mathbb{E}(V) = \infty$ .

In Fig. 2a and 2b we draw ten trajectories of  $\frac{M_n}{n}$  for  $r = 0.3$  in the left column and, respectively, for  $r = 0.425$  in the right column. As expected, the trajectories tend to zero.

However, in accordance with Hawkins' result [6], this approach is too naive to detect that the unknown expectation is infinite. Actually, Fig. 3a and 3b illustrate that, when  $\mathbb{E}(V)$  is infinite, very big values are too rare in the sample to lead to significantly excessive values of  $\frac{M_n}{n}$ , even when  $n$  is very large. Notice that, the larger is  $n$ , the larger needs to be  $V_n$  to make  $\frac{M_n}{n}$  significantly larger than  $\frac{M_{n-1}}{n-1}$ . Therefore, the trajectories of  $\frac{M_n}{n}$  are piecewise decreasing as in the finite expectation case. Jumps occur at times  $T_n$  where new upper record values appear. As for any i.i.d. sequence the probability law of  $T_{n+1} - T_n$  does not depend on the law of the subjacent random variable and has infinite expectation (see Nevzorov [Thms.4.1-4.2] in [12]) it seems impossible to elaborate an effective rule for choosing a time  $n$  at which one could confidently decide that  $\mathbb{E}(V) = \infty$ .

### 3.2 A test based on the Central Limit Theorem

For random variables  $X$  with  $\mathbb{E}(X) = 0$ , it holds that  $X \in \text{DA}(2)$  if and only if  $S_m/V_m$  converges in distribution to the standard normal law (See Giné et al. [4]). Therefore, to test if a random variable belongs to  $\text{DA}(2)$  it seems natural to apply a normality test to a sample of  $S_m/V_m$  (obtained by using the observed sample of  $X$ ). This approach suffers from the two following drawbacks. First, one needs to center the random variable while, in our setting, the mean of  $X$  is unknown. Second, the asymptotic behavior of  $S_m/V_m$  for  $X \notin \text{DA}(2)$  is unknown. Additionally, for poorly chosen simulation parameters we can have an empirical rejection rate equal to 1 when  $X \in \text{DA}(2)$ . For example, we have followed this strategy for the random variable  $X = |G|^{-0.4}$ , with  $G \sim \mathcal{N}(0, 1)$ . We have fixed  $m = 1000$  and we have generated a sample of  $S_{1000}/V_{1000}$  with size 1000 and then we have performed the Shapiro-Wilk normality test. Repeating this

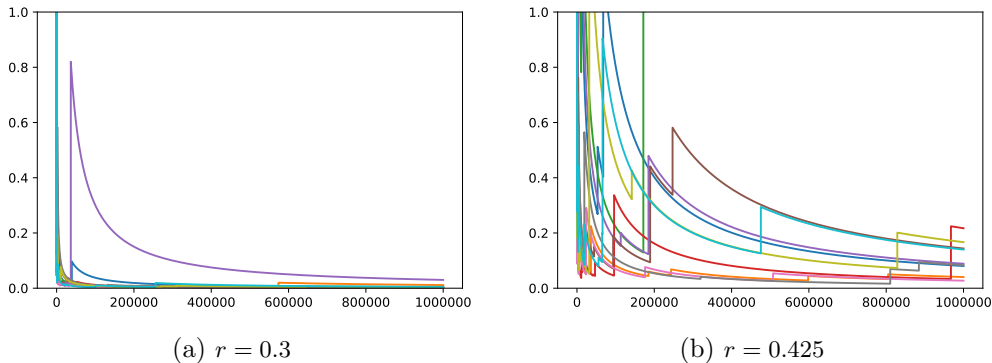


Figure 2: Trajectories of  $\frac{M_n}{n}$  in the finite expectation of  $V$  case.

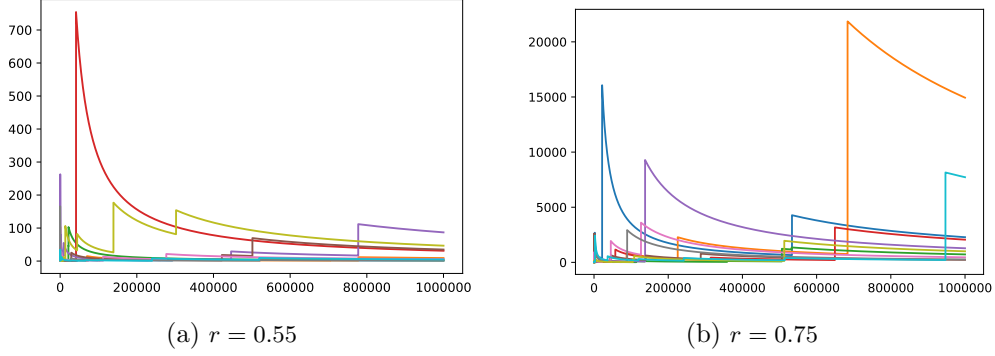


Figure 3: Trajectories of  $\frac{M_n}{n}$  in the infinite expectation of  $V$  case.

procedure 1000 times we have obtained an empirical rejection rate equal to 1. That is, each one of our samples of  $S_{1000}/V_{1000}$  leads to the rejection of the hypothesis that the probability distribution is Gaussian. In Fig. 4 we show the  $p$ -values of the Shapiro-Wilk test obtained in 200 of our experiments. We omit the 800 other results for the sake of a good visualisation.

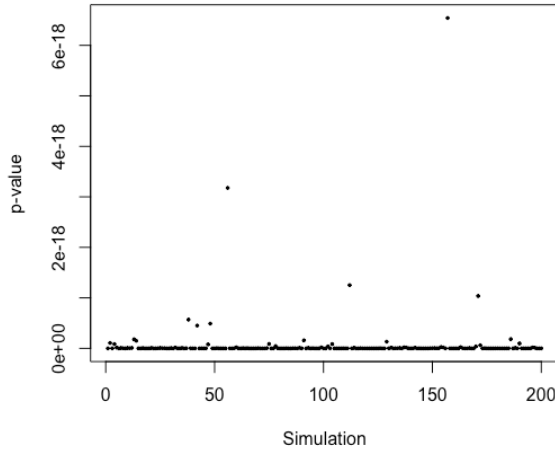


Figure 4:  $p$ -values of the Shapiro-Wilk normality test for 200 of our experiment (See details in Section 3.2). The empirical rejection rate is equal to 1.

### 3.3 A test based on the Self-normalized Iterated Logarithm Law

Griffin and Kuelbs [5, Thm.1] have shown that if  $X$  belongs to  $DA(2)$  and  $\mathbb{E}(X) = 0$ , then

$$\limsup_{m \rightarrow \infty} \frac{S_m}{V_m \sqrt{2 \log \log(m)}} = 1 \quad \text{a.s.}$$

Disregarding the fact that  $\mathbb{E}(X)$  is unknown, it seems appealing to use this result to decide whether the probability law of  $X$  belongs to  $DA(2)$ .

Let us consider  $X = |G|^{-r}$ , where  $G$  is a standard normal random variable. We know that for  $r \leq 0.5$ ,  $X \in DA(2)$ , whereas for  $r > 0.5$ ,  $X \in DA(1/r)$ . In Figs. 5 and 6 we show the trajectories of  $S_m/V_m$ , and we compare them with  $\sqrt{2 \log \log(m)}$ . As expected, for  $r \leq 1/2$ , the paths of  $S_m/V_m$  starting at a large enough value of  $m$  do not cross the curves of the maps  $-\sqrt{2 \log \log(m)}$  and  $\sqrt{2 \log \log(m)}$  (Fig. 5). For  $r > 1/2$  we do not observe, at least

qualitatively, a very different behavior (Fig. 6). In fact, most of the trajectories remain confined between the curves of the maps  $-\sqrt{2 \log \log(m)}$  and  $\sqrt{2 \log \log(m)}$ .

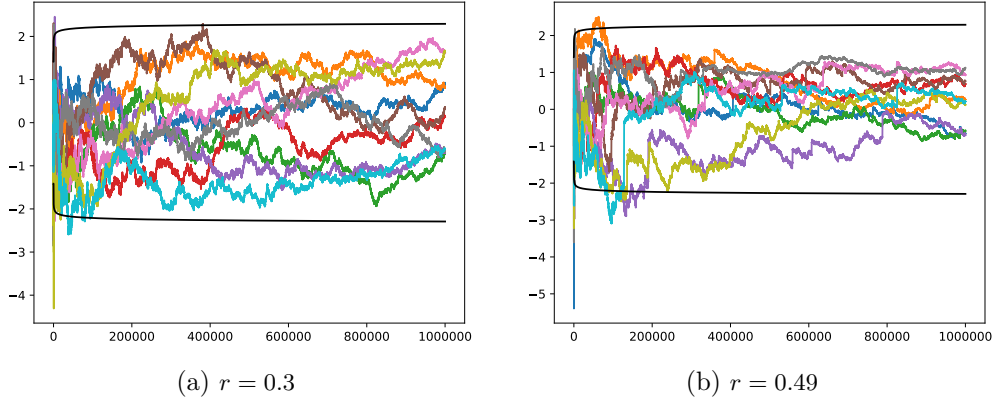


Figure 5: Trajectories of the self-normalized empirical mean for laws in DA(2)

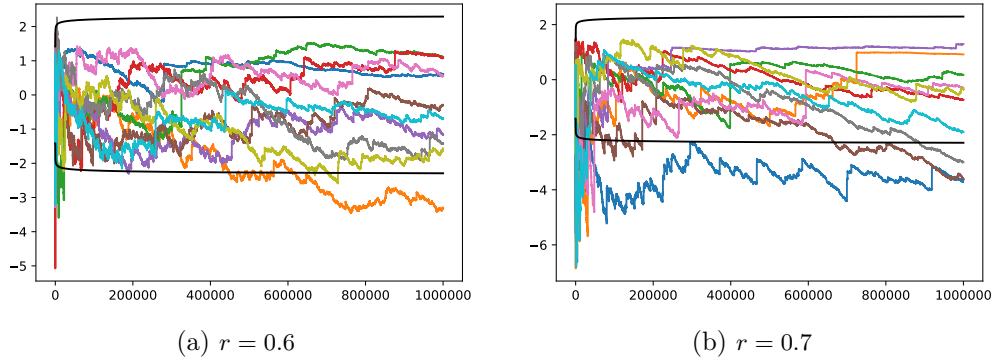


Figure 6: Trajectories of the Self-normalized empirical mean for laws NOT in DA(2)

### 3.4 Estimation of the tail index: The Meerschaert-Scheffler estimator

To determine the index  $\alpha$  of the domain of attraction of a sampled random variable  $X$  one can use the Meerschaert-Scheffler estimator [11] defined as

$$\hat{\gamma}_m := \frac{\max\{\log(\sum_{i=1}^m (X_i - \bar{X}_m)^2), 0\}}{2 \log(m)}.$$

This estimator is an alternative to the classical Hill estimator introduced in [7]. Meerschaert and Scheffler proved that the estimator  $\hat{\gamma}_n$  is asymptotically consistent when the data belong to some  $DA(\alpha)$ . More precisely,

**Theorem 3.2** (Theorem 1 and 2 in [11]). *Let  $X, X_1, X_2, \dots$  be a sequence of non-degenerate i.i.d. random variables such that  $X \in DA(\alpha)$ .*

1. If  $0 < \alpha < 2$ :

$$\hat{\gamma}_m \xrightarrow[m \rightarrow \infty]{\mathbb{P}, L^1, L^2} \frac{1}{\alpha}$$

and for some sequence  $(c_m)$  tending to 0 and some  $\frac{\alpha}{2}$ -stable r.v.  $Y$ ,

$$2 \log(m) (\hat{\gamma}_m - \frac{1}{\alpha} - c_m) \Rightarrow \log(Y).$$



2. If  $\alpha = 2$ :

$$\hat{\gamma}_m \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \frac{1}{\alpha}.$$

Additionally:

(a) if the variance  $\sigma^2$  of  $X$  is finite, then

$$2 \log(m) (\hat{\gamma}_m - \frac{1}{2}) \xrightarrow[m \rightarrow \infty]{\mathbb{P}} \log(\sigma^2).$$

(b) if the variance of  $X$  is infinite, then there exists some sequence  $(c_m)$  tending to 0 in probability and some 1-stable r.v.  $Y$ , such that

$$2 \log(m) (\hat{\gamma}_m - \frac{1}{2} - c_m) \Rightarrow \log(|Y|).$$

The convergence rate of  $\hat{\gamma}_m$  is low but the limit distribution of the suitably normalized error is narrow. Meerschaert and Scheffler also proved that  $\hat{\gamma}_m$  converges under weaker assumptions, namely, when the sample is in the domain of semistable attraction of a semistable law [11, Thm.3].

The table 1 below illustrates the behaviour of  $\hat{\gamma}_m$ . As expected, the convergence towards the limit value is slow. However, we notice that  $\hat{\gamma}_m$  takes acceptable finite values for any large enough  $m$ . We also notice that  $\hat{\gamma}_m$  seems to be biased.

In table 2 we present the results obtained when  $X \sim |G|^{-0.45}$ , where  $G \sim N(0,1)$ . In this case, the second moment of  $X$  is finite. Thus,  $X$  belongs to the domain of attraction of the Normal distribution. We compute  $\gamma_m$  for 10000 simulated samples of size  $m \in \{10^5, 10^6, 10^7, 10^8, 10^9\}$ . Table 2 shows that  $\gamma_m$  is always larger than 0.5. This could lead one to conclude that  $X$  does not belong to DA(2). Hence, because of its slow convergence, one cannot use this estimator to test  $\mathbf{H}_0$  vs.  $\mathbf{H}_1$ .

Random Variable	$\alpha$	$m = 10^5$	$m = 10^6$	$m = 10^7$	$m = 10^8$
$ G ^{-0.6}, G \sim \mathcal{N}(0, 1)$	0.6	0.67540532	0.66530621	0.65674121	0.65042646
$ G ^{-0.7}, G \sim \mathcal{N}(0, 1)$	0.7	0.76454733	0.75509289	0.74727714	0.74128
$ G ^{-0.8}, G \sim \mathcal{N}(0, 1)$	0.8	0.85825199	0.84867939	0.84254414	0.83792346

Table 1: Mean value of the Meerschaert-Scheffler estimator of the tail index for different random variables over 10000 simulations of samples with size  $m$ .

$m$	Empirical mean of $\hat{\gamma}$	Empirical sd of $\hat{\gamma}$	Empirical min of $\hat{\gamma}$
$10^5$	0.55740888	0.0169447807	0.5331192
$10^6$	0.55160767	0.0113382256	0.53850901
$10^7$	0.54669192	0.00862631795	0.53852758
$10^8$	0.54221338	0.00650946694	0.53628999
$10^9$	0.53842022	0.00491848875	0.53440811

Table 2: Descriptive statistics of the Meerschaert-Scheffler estimator of the tail index of  $X = |G|^{-0.45}$  over 10000 simulations of samples with different sizes.

## 4 Our hypothesis test for domains of attraction of stable laws

Determining whether a stochastic process  $Y$  is continuous or not from the observation of one single path at discrete times, is an important modelling issue in many fields, notably in eco-

nomics and financial mathematics. It has been addressed by several authors in the last years. See for example Ait-Sahalia and Jacod [1] and the references therein.

Our situation is somehow different since we are not observing a trajectory of a given process. We rather are constructing one discrete time path by means of a normalization procedure of our (observed or simulated) data and this constructed trajectory is always discontinuous. We therefore aim to construct a statistic whose asymptotic properties will allow us to apply detection of jumps of semimartingale methods.

Following Barndorff-Nielsen and Shephard [2], for any stochastic process  $(Y_t)_{0 \leq t \leq 1}$  we set  $\Delta_i^n Y = Y_{i/n} - Y_{(i-1)/n}$  and consider the realized bivariate variation, the realized quadratic variation and the normalized bivariate variation respectively defined by

$$\widehat{B}(Y, n) := \sum_{i=1}^{n-1} |\Delta_i^n Y| |\Delta_{i+1}^n Y|, \quad \widehat{Q}(Y, n) := \sum_{i=1}^n |\Delta_i^n Y|^2, \quad \widehat{\mathcal{F}}_n(Y) := \frac{\widehat{B}(Y, n)}{\widehat{Q}(Y, n)}. \quad (4.1)$$

The hypotheses made by Barndorff-Nielsen and Shephard to justify their test (see [2, Thm.1] and our discussion in [14]) do not apply in our context. This led us to introduce the following new statistic which is adapted to our specific situation.

**Definition 4.1.** For any  $m, n \in \mathbb{N}$  consider the functional  $\widehat{\mathcal{F}}_n^m : D[0, 1] \rightarrow \mathbb{R}$  defined as

$$\widehat{\mathcal{F}}_n^m(z) := \frac{\sum_{i=1}^{n-1} |z(i/n) - z((i-1)/n)| |z((i+1)/n) - z(i/n)|}{\sum_{i=1}^n |z(i/n) - z((i-1)/n)|^2}. \quad (4.2)$$

Given an i.i.d. sample of  $X : X_1, \dots, X_m$  and the corresponding process  $Z^m$  as in (2.1) we define our statistic  $\widehat{\mathcal{F}}_n^m$  as the normalized bivariate variation of  $Z^m$ , that is,

$$\widehat{\mathcal{F}}_n^m := \widehat{\mathcal{F}}_n(Z^m). \quad (4.3)$$

The next result provides a key property of the statistic  $\widehat{\mathcal{F}}_n^m$ .

**Proposition 4.2.** The statistic  $\widehat{\mathcal{F}}_n^m$  is scale-free and satisfies

$$\widehat{\mathcal{F}}_n^m = \frac{\sum_{i=1}^{n-1} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \bar{X}_m) \right| \left| \sum_{j=\lfloor \frac{mi}{n} \rfloor + 1}^{\lfloor \frac{m(i+1)}{n} \rfloor} (X_j - \bar{X}_m) \right|}{\sum_{i=1}^n \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \bar{X}_m) \right|^2}.$$

**Remark 4.3.** Notice that computing  $\widehat{\mathcal{F}}_n^m$  only needs the values of the sample. In particular, one does not need the unknown centering and normalizing factors  $\mu_m$  and  $c_m$  in Theorem 2.2.

Our hypothesis test for  $\mathbf{H}_0$  against  $\mathbf{H}_1$  is based on the following theorem:

**Theorem 4.4.** Assume that  $X$  belongs to some domain of attraction. Consider and i.i.d. sample  $X_1, \dots, X_m$  of a r.v.  $X$ . We consider the test hypotheses

$\mathbf{H}_0$ :  $X \in \text{DA}(2)$

and

$\mathbf{H}_1$ :  $\exists 0 < \alpha < 2, X \in \text{DA}(\alpha)$ .

Let  $z_q$  denote the  $q$ -quantile of a standard normal random variable and  $\sigma_\pi^2 = 1 + \frac{4}{\pi} - \frac{20}{\pi^2}$ . The rejection region

$$C_{n,m} := \left\{ \left| \widehat{\mathcal{F}}_n^m - \frac{2}{\pi} \right| > z_{1-q/2} \sqrt{\frac{\sigma_\pi^2}{n}} \right\}$$

satisfies:

1.  $\limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} \mathbb{P}(C_{n,m} | \mathbf{H}_0) \leq q$ .
2.  $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{P}(C_{n,m} | \mathbf{H}_1) = 1$ .

We also state a central limit theorem for our statistic under  $\mathbf{H}_0$ .

**Proposition 4.5.** *If the subjacent random variable  $X$  belongs to the domain of attraction of the normal law, for any bounded and continuous function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  we have*

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{E} \left[ \varphi \left( \frac{\sqrt{n}}{\sigma_\pi} \left( \widehat{\mathcal{F}}_n^m - \frac{2}{\pi} \right) \right) \right] = \mathbb{E}[\varphi(\mathcal{N})],$$

where  $\sigma_\pi^2 = 1 + \frac{4}{\pi} - \frac{20}{\pi^2}$  and  $\mathcal{N}$  is a standard Gaussian random variable.

**Remark 4.6. Other tests in the literature related to ours:** In [8] Jurečkov and Picek develop a statistical test for the heaviness of the tail of a distribution function  $F$  assuming that  $F$  is absolutely continuous and strictly increasing on the set  $\{x : F(x) > 0\}$ . In their case, for  $m_0$  given, the null hypothesis is

$$H_{m_0} : x^{m_0}(1 - F(x)) \geq 1, \quad \forall x > x_0 \text{ for some } x_0 \geq 0,$$

whereas the alternative hypothesis is

$$K_{m_0} : \limsup_{x \rightarrow \infty} x^{m_0}(1 - F(x)) < 1.$$

Notice that for  $m_0 = 2$  the non-rejection  $H_{m_0}$  implies that  $F$  has infinite second moment. Although this test has a very good behavior even for small samples, it is only applicable to absolutely continuous distributions whereas our test does not need such a condition, which is essential for applications to samples produced by complex simulations of random processes. Moreover, Jurečkov and Picek's test applies to I.I.D. random variables, whereas our methodology can be extended to some weak dependence cases (See Section 7.3 in [14]). This potentially allow applications to interacting particles which propagate chaos. Finally, we refer to [10] for other nonparametric techniques in the context of heavy tailed distributions.

## 5 Numerical analysis of our test

In this section we present numerical experiments which illustrate our main theorem 4.4 and its limitations when applied to synthetic data.

In this section we consider the r.v.  $X = |G|^{-r}$  with  $G \sim \mathcal{N}(0, 1)$  and  $r > 0$ . Notice that  $X$  has finite moments of order smaller than  $1/r$  and that  $X \in \text{DA}(2)$  when  $r \in (0, 1/2]$ , whereas  $X \in \text{DA}(1/r)$  when  $r > 1/2$ .

### 5.1 Experiments under $\mathbf{H}_0$

In Tables 3a- 7b we report empirical rejection rates. The levels of confidence are either  $q = 0.1$  or  $q = 0.05$  and the parameter  $r$  takes values in  $\{0.1, 0.2, 0.3, 0.4, 0.45\}$ . For each value of  $r$ , we have generated  $10^4$  independent samples with sizes from  $m = 10^5$  up to  $m = 10^8$ . In the cases  $r = 0.4$  and  $r = 0.45$  we have also considered  $m = 10^9$ . To study the respective effects of  $m$  and  $n$ , for each sample size  $m$  we have made the number of time steps  $n$  vary from  $n = 10^1$  up to  $n = 10^4$ .

According to Theorem 4.4 one expects that the larger  $m$  and  $n$  are, the closer to  $q$  the empirical rejection rate is. The tables below show that increasing  $m$  improves the approximation of the empirical rejection rates to their expected values whereas increasing  $n$  does not have the same effect. One actually observes that the empirical rejection rates become close to 1 when  $n$

becomes too large. Moreover, it seems that  $m$  being fixed the optimal choice of  $n$  depends on the value of  $r$ . The closer  $r$  is to the critical value  $\frac{1}{2}$ , the smaller  $n$  should be chosen.

To summarize, even if our theoretical result is valid when  $n$  goes to infinity, in practice, we should use  $n$  significantly smaller than  $m$ . In addition, as  $m$  increases, the convergence of the process  $Z^m$  can be very slow. Therefore, if  $n$  is chosen too large, one is zooming up too much and sees the discontinuities that  $Z^m$  has by construction. See Subsection 5.4 below for more comments on that issue.

The figures 7, 8, 9, 10 and 11 compare the histograms of the empirical distribution of the standardized  $\widehat{\mathcal{F}}_n^m$  derived from our simulations and of the asymptotic distribution under  $\mathbf{H}_0$  provided by Proposition 4.5. We observe that the asymptotic normality regime is detected for  $m = 10^5$  and larger if  $n$  is not too large compared to  $m$  (less than  $10^2$ , say). When  $n$  increases, the empirical distribution is clearly different from the limit one. For example, for  $r = 0.40$ , when  $m = 10^6$  and  $n = 10^4$  the empirical distribution is even not centered at 0. This behavior becomes worse and worse as  $r$  approaches to the critical value 0.5 (see Fig. 11).

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.095	0.098	0.117	0.997
$10^6$	0.094	0.093	0.096	0.170
$10^7$	0.103	0.099	0.097	0.103
$10^8$	0.100	0.100	0.095	0.101

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.051	0.047	0.063	0.991
$10^6$	0.053	0.048	0.051	0.101
$10^7$	0.055	0.051	0.048	0.052
$10^8$	0.054	0.049	0.048	0.050

(b)  $q = 0.05$ .

Table 3: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.1$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.097	0.107	0.224	1.000
$10^6$	0.093	0.095	0.106	0.714
$10^7$	0.100	0.102	0.104	0.122
$10^8$	0.098	0.098	0.100	0.099

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.053	0.056	0.147	1.000
$10^6$	0.050	0.048	0.058	0.617
$10^7$	0.053	0.052	0.052	0.066
$10^8$	0.056	0.048	0.047	0.050

(b)  $q = 0.05$ .

Table 4: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.2$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.101	0.136	0.705	1.000
$10^6$	0.106	0.106	0.240	1.000
$10^7$	0.100	0.101	0.126	0.776
$10^8$	0.097	0.101	0.104	0.199

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.056	0.082	0.630	1.000
$10^6$	0.058	0.056	0.168	1.000
$10^7$	0.056	0.053	0.069	0.691
$10^8$	0.050	0.048	0.053	0.128

(b)  $q = 0.05$ .

Table 5: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.3$ , different values for  $m$  and  $n$ , different confidence levels and simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.118	0.302	0.991	1.000
$10^6$	0.103	0.202	0.860	1.000
$10^7$	0.101	0.151	0.571	1.000
$10^8$	0.101	0.126	0.356	0.999
$10^9$	0.096	0.118	0.229	0.947

(a)  $q = 0.1$ .

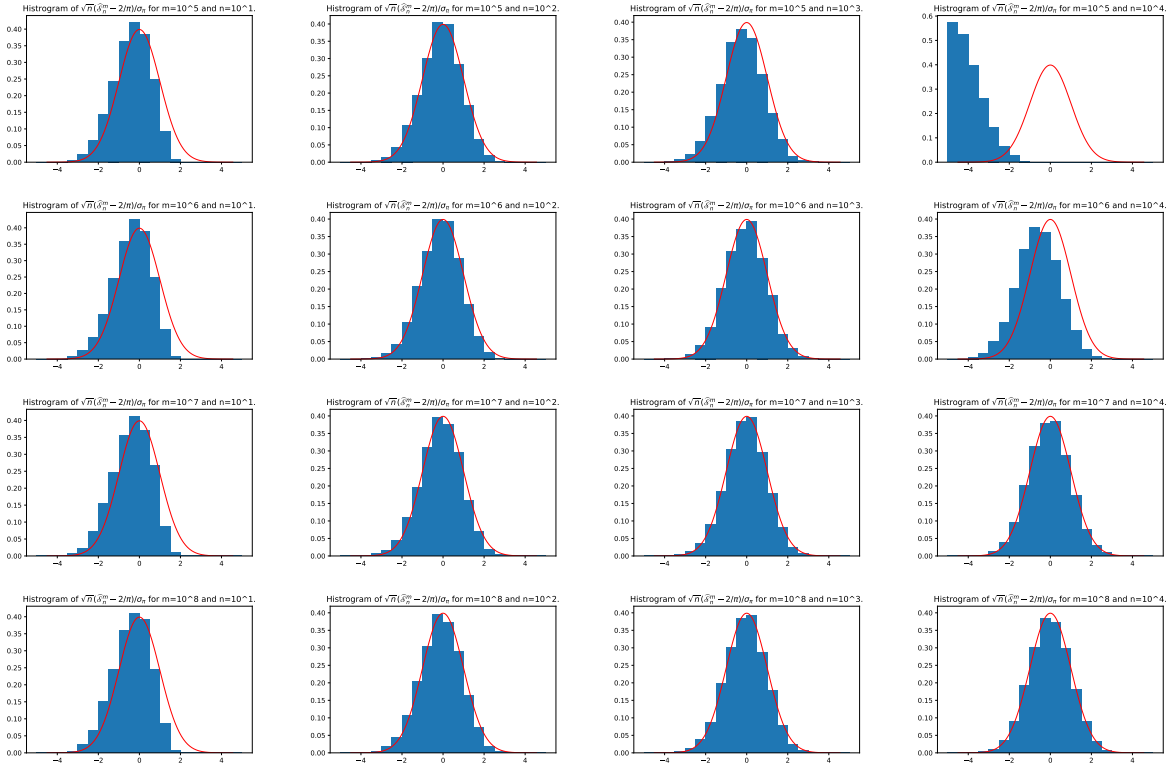
$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.071	0.234	0.985	1.000
$10^6$	0.057	0.141	0.810	1.000
$10^7$	0.056	0.097	0.494	1.000
$10^8$	0.053	0.077	0.284	0.999
$10^9$	0.055	0.066	0.165	0.917

(b)  $q = 0.05$ .Table 6: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.4$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.123	0.436	1.000	1.000
$10^6$	0.114	0.332	0.985	1.000
$10^7$	0.118	0.262	0.909	1.000
$10^8$	0.109	0.220	0.773	1.000
$10^9$	0.109	0.190	0.623	1.000

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.075	0.367	0.999	1.000
$10^6$	0.069	0.265	0.976	1.000
$10^7$	0.067	0.197	0.878	1.000
$10^8$	0.061	0.160	0.716	1.000
$10^9$	0.064	0.130	0.550	1.000

(b)  $q = 0.05$ .Table 7: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.45$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.Figure 7: Empirical distribution of standardized  $\widehat{\mathcal{F}}_n^m$  for  $r = 0.1$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^8$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.

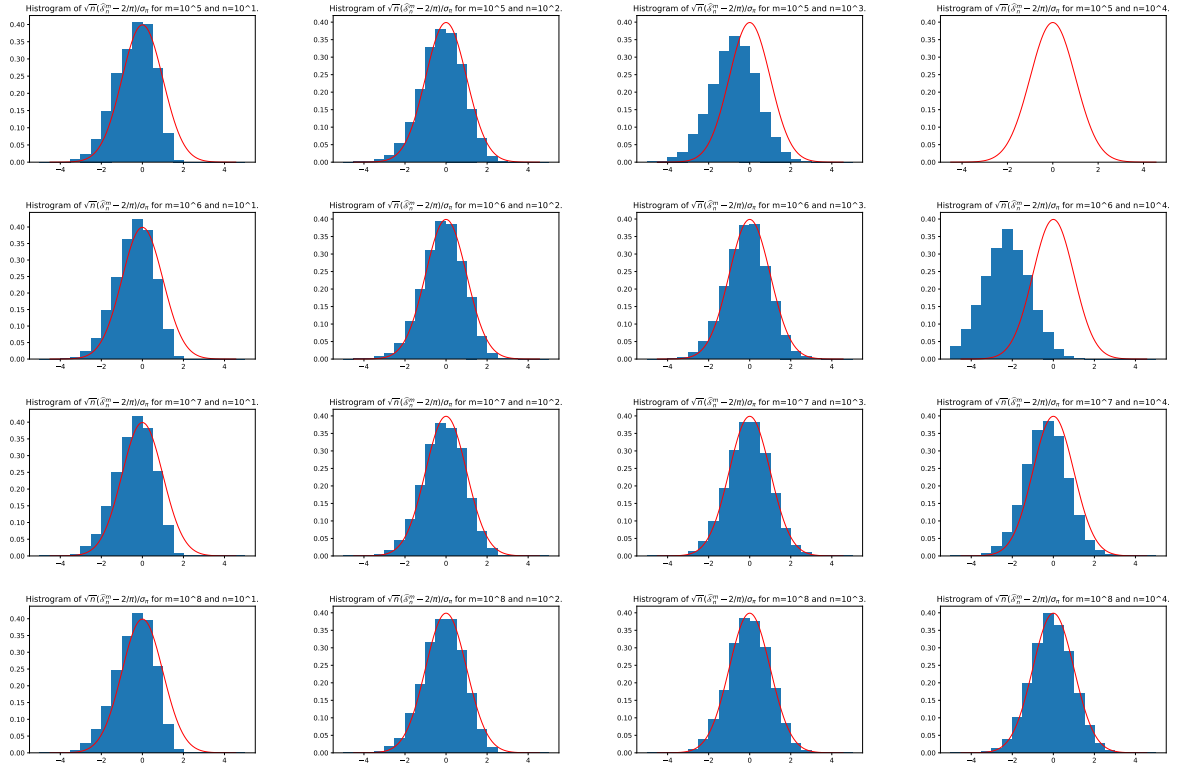


Figure 8: Empirical distribution of standardized  $\widehat{\mathcal{F}}_n^m$  for  $r = 0.2$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^8$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.

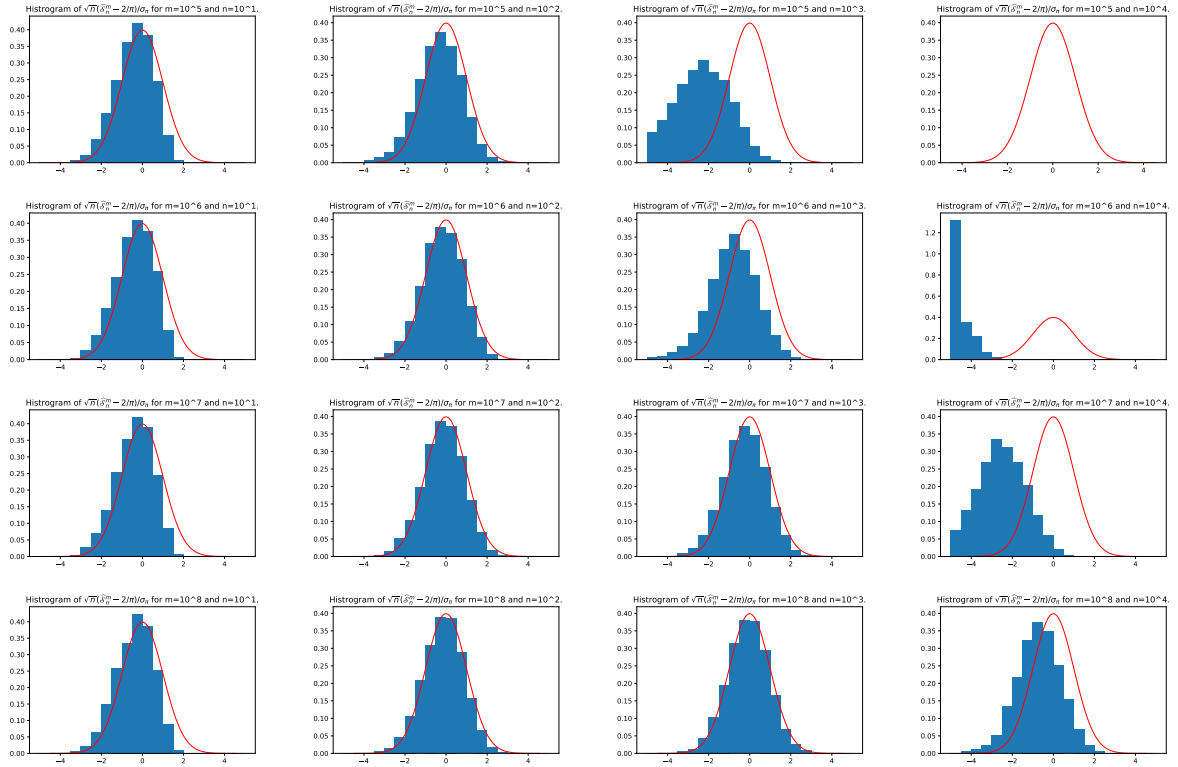


Figure 9: Empirical distribution of standardized  $\widehat{\mathcal{F}}_n^m$  for  $r = 0.3$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^8$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.

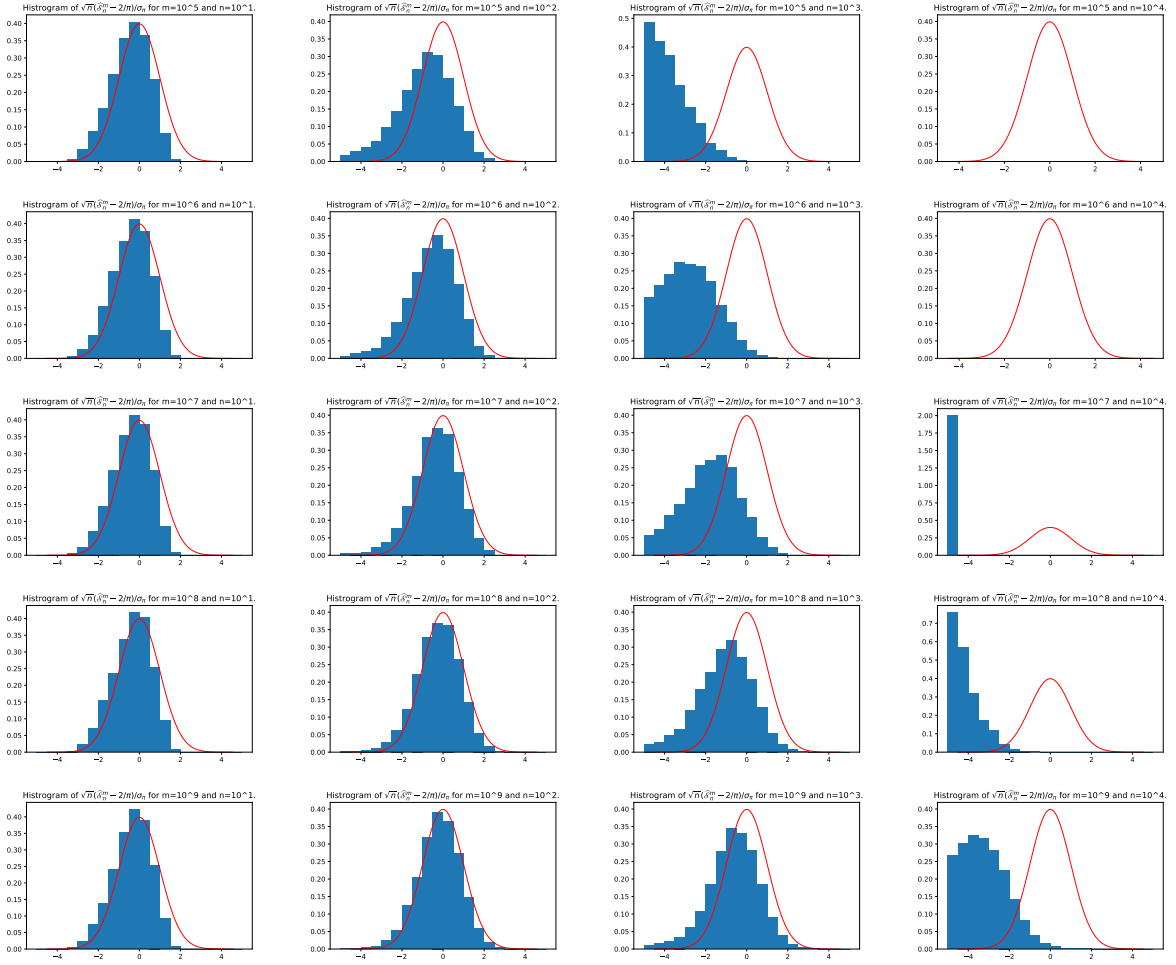


Figure 10: Empirical distribution of standardized  $\widehat{\mathcal{F}}_n^m$  for  $r = 0.4$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^9$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.

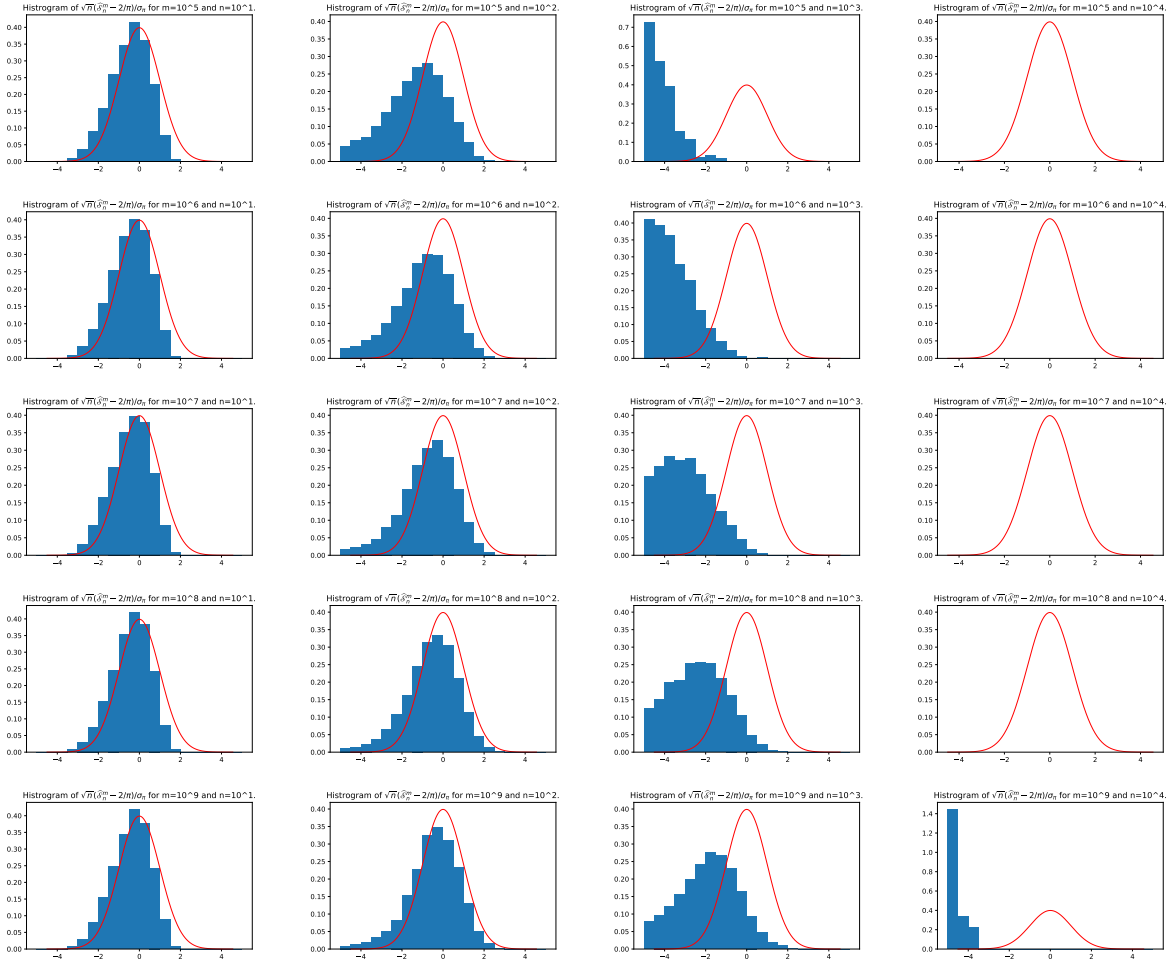


Figure 11: Empirical distribution of standardized  $\hat{\mathcal{F}}_n^m$  for  $r = 0.45$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^9$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.



## 5.2 Experiments under $\mathbf{H}_1$

In Tables 8a- 10b we report empirical rejection rates. The levels of confidence are either  $q = 0.1$  or  $q = 0.05$  and the parameter  $r$  takes values in  $r \in \{0.6, 0.75, 0.9\}$ . For these values of  $r$  one expects to observe empirical rejection rates close to 1. It is actually the case for  $r = 0.75$  and  $r = 0.9$  if  $n > 10$ . When  $r$  is close to the limit case 0.5 we observe a bigger ‘type 2’ error. As above, we have let  $m$  and  $n$  vary.

Our simulations seem to show that under  $\mathbf{H}_1$  choosing a too small value of  $n$  leads to poor empirical rejection rates. Notice also that, for small values of  $n$ , the empirical rejection rate does not seem to be monotonically increasing with  $m$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.190	0.821	1.000	1.000
$10^6$	0.199	0.789	1.000	1.000
$10^7$	0.198	0.765	1.000	1.000
$10^8$	0.194	0.761	1.000	1.000

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.125	0.780	1.000	1.000
$10^6$	0.133	0.744	1.000	1.000
$10^7$	0.136	0.719	1.000	1.000
$10^8$	0.123	0.712	1.000	1.000

(b)  $q = 0.05$ .

Table 8: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.6$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.267	0.956	1.000	1.000
$10^6$	0.276	0.954	1.000	1.000
$10^7$	0.264	0.951	1.000	1.000
$10^8$	0.268	0.950	1.000	1.000

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.191	0.941	1.000	1.000
$10^6$	0.198	0.940	1.000	1.000
$10^7$	0.186	0.936	1.000	1.000
$10^8$	0.189	0.934	1.000	1.000

(b)  $q = 0.05$ .

Table 9: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.75$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.332	0.988	1.000	1.000
$10^6$	0.332	0.985	1.000	1.000
$10^7$	0.331	0.987	1.000	1.000
$10^8$	0.340	0.987	1.000	1.000

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.248	0.982	1.000	1.000
$10^6$	0.245	0.979	1.000	1.000
$10^7$	0.246	0.980	1.000	1.000
$10^8$	0.251	0.984	1.000	1.000

(b)  $q = 0.05$ .

Table 10: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test with  $r = 0.9$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

As a complement to Section 3.1 and to Tables 8a- 10b, Tables 11, 12 and 13 show values taken by the empirical second moment of  $X$ . Since we are under  $\mathbf{H}_1$ , we know that  $\mathbb{E}(X^2) = \infty$ . Actually, Tables 12 and 13 show big values of the empirical second moment. However, it is not so obvious that this information could suffice to decide that the second moment is infinite. More striking, in the case of Table 11 the empirical second moment is not a reliable indicator of the infiniteness of  $\mathbb{E}(X^2)$ , regardless the size of the sample. However, even when  $r = 0.6$ ,

the empirical rejection rates of our test are close to 1 under  $\mathbf{H}_1$ . This seems to illustrate the relevance of our test to detect infinite expectations.

Statistic	$m$			
	$10^5$	$10^6$	$10^7$	$10^8$
Mean of $\hat{M}_2$	$3.07 \times 10^2$	$5.51 \times 10^2$	$5.36 \times 10^2$	$8.54 \times 10^2$
Std. Deviation of $\hat{M}_2$	$1.04 \times 10^4$	$1.23 \times 10^4$	$6.48 \times 10^3$	$1.31 \times 10^4$
Min of $\hat{M}_2$	$2.00 \times 10^1$	$3.28 \times 10^1$	$5.30 \times 10^1$	$9.05 \times 10^1$
Max of $\hat{M}_2$	$9.76 \times 10^5$	$7.68 \times 10^5$	$4.44 \times 10^5$	$9.70 \times 10^5$
Quantile(0.95) of $\hat{M}_2$	$3.09 \times 10^2$	$5.68 \times 10^2$	$8.72 \times 10^2$	$1.31 \times 10^3$
Quantile(0.99) of $\hat{M}_2$	$1.80 \times 10^3$	$3.44 \times 10^3$	$5.03 \times 10^3$	$7.67 \times 10^3$

Table 11: Statistics of the empirical second moment computed in 10000 scenarios for  $r = 0.6$ . The sample sizes vary from  $10^5$  to  $10^8$ .

Statistic	$m$			
	$10^5$	$10^6$	$10^7$	$10^8$
Mean of $\hat{M}_2$	$2.76 \times 10^5$	$2.20 \times 10^5$	$6.05 \times 10^5$	$2.80 \times 10^6$
Std. Deviation of $\hat{M}_2$	$2.13 \times 10^7$	$7.21 \times 10^6$	$2.75 \times 10^7$	$1.37 \times 10^8$
Min of $\hat{M}_2$	$1.36 \times 10^2$	$4.27 \times 10^2$	$1.31 \times 10^3$	$4.30 \times 10^3$
Max of $\hat{M}_2$	$2.10 \times 10^9$	$5.30 \times 10^8$	$2.24 \times 10^9$	$1.22 \times 10^{10}$
Quantile(0.95) of $\hat{M}_2$	$1.96 \times 10^4$	$7.38 \times 10^4$	$1.94 \times 10^5$	$6.12 \times 10^5$
Quantile(0.99) of $\hat{M}_2$	$2.29 \times 10^5$	$6.79 \times 10^5$	$1.94 \times 10^6$	$5.61 \times 10^6$

Table 12: Statistics of the empirical second moment computed in 10000 scenarios for  $r = 0.75$ . The sample sizes vary from  $10^5$  to  $10^8$ .

Statistic	$m$			
	$10^5$	$10^6$	$10^7$	$10^8$
Mean of $\hat{M}_2$	$8.54 \times 10^6$	$3.66 \times 10^8$	$6.88 \times 10^8$	$1.33 \times 10^9$
Std. Deviation of $\hat{M}_2$	$3.82 \times 10^8$	$2.53 \times 10^{10}$	$4.08 \times 10^{10}$	$4.11 \times 10^{10}$
Min of $\hat{M}_2$	$1.21 \times 10^3$	$7.92 \times 10^3$	$5.35 \times 10^4$	$2.43 \times 10^5$
Max of $\hat{M}_2$	$3.67 \times 10^{10}$	$2.39 \times 10^{12}$	$3.90 \times 10^{12}$	$2.99 \times 10^{12}$
Quantile(0.95) of $\hat{M}_2$	$1.53 \times 10^6$	$9.58 \times 10^6$	$6.68 \times 10^7$	$3.58 \times 10^8$
Quantile(0.99) of $\hat{M}_2$	$3.20 \times 10^7$	$1.87 \times 10^8$	$1.08 \times 10^9$	$6.42 \times 10^9$

Table 13: Statistics of the empirical second moment computed in 10000 scenarios for  $r = 0.9$ . The sample sizes vary from  $10^5$  to  $10^8$ .

### 5.3 Numerical experiments under weak dependence

In [14] we have shown that Theorem 4.4 still holds if we assume that the sample is stationary,  $r$ -dependent and satisfies the two following conditions:

$$\sup \{ |\text{Corr}(f, g)| : \text{real } f \in L^2(\sigma(X_1)), \text{real } g \in L^2(\sigma(X_2, X_3, \dots)) \} < 1 \quad (5.1)$$

and

$$(\forall \epsilon > 0, \forall j = 2, \dots, k) \quad \lim_{m \rightarrow \infty} \mathbb{P}(|X_j| > \epsilon c_m / |X_1| > \epsilon c_m) = 0. \quad (5.2)$$

To illustrate the behavior of the test for non-independent random variables, we consider a sequence  $Y_2, Y_3, \dots, Y_m$  of 1-dependent random variables constructed as follows. First, we generate a sample  $X_1, \dots, X_m$  of independent copies of  $|G|^{-r}$  where  $G \sim \mathcal{N}(0, 1)$ , and then

$$Y_k = \frac{X_{k-1}}{X_{k-1} + 1} X_k,$$

Second, we apply our methodology to  $Y_2, \dots, Y_m$  for  $r = 0.3$  and  $r = 0.75$ . In Tables 14a and 14b we report the empirical rejection rates for two different levels of confidence. As  $m$  increases the empirical rates tend to the theoretical ones. As in the i.i.d. case we also observe that the performance of the methodology decreases when  $n$  becomes too large. In Fig. 12 illustrates the asymptotic normality of the statistic  $\widehat{\mathcal{F}}_n^m$  as in the i.i.d. case. Finally, in Tables 15a and 15b, we report the results derived from our simulations for  $r = 0.75$ , that is, under  $\mathbf{H}_1$ . Again, as in the i.i.d. case, smaller values of  $n$  leads to bigger ‘type 2’ error. Nevertheless, for  $n = 10^2$  we already observe empirical rejection rates close to 1.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.096	0.123	0.527	1.000
$10^6$	0.096	0.105	0.175	0.999
$10^7$	0.100	0.102	0.111	0.480
$10^8$	0.096	0.098	0.098	0.133

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.054	0.070	0.433	1.000
$10^6$	0.053	0.054	0.112	0.998
$10^7$	0.057	0.051	0.057	0.370
$10^8$	0.055	0.050	0.050	0.078

(b)  $q = 0.05$ .

Table 14: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a 1-dependent sequence of random variables for  $r = 0.3$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.262	0.953	1.000	1.000
$10^6$	0.263	0.955	1.000	1.000
$10^7$	0.271	0.952	1.000	1.000
$10^8$	0.270	0.951	1.000	1.000

(a)  $q = 0.1$ .

$m$	$n$			
	$10^1$	$10^2$	$10^3$	$10^4$
$10^5$	0.186	0.937	1.000	1.000
$10^6$	0.188	0.942	1.000	1.000
$10^7$	0.193	0.938	1.000	1.000
$10^8$	0.193	0.935	1.000	1.000

(b)  $q = 0.05$ .

Table 15: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a 1-dependent sequence of random variables for  $\alpha = 0.75$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

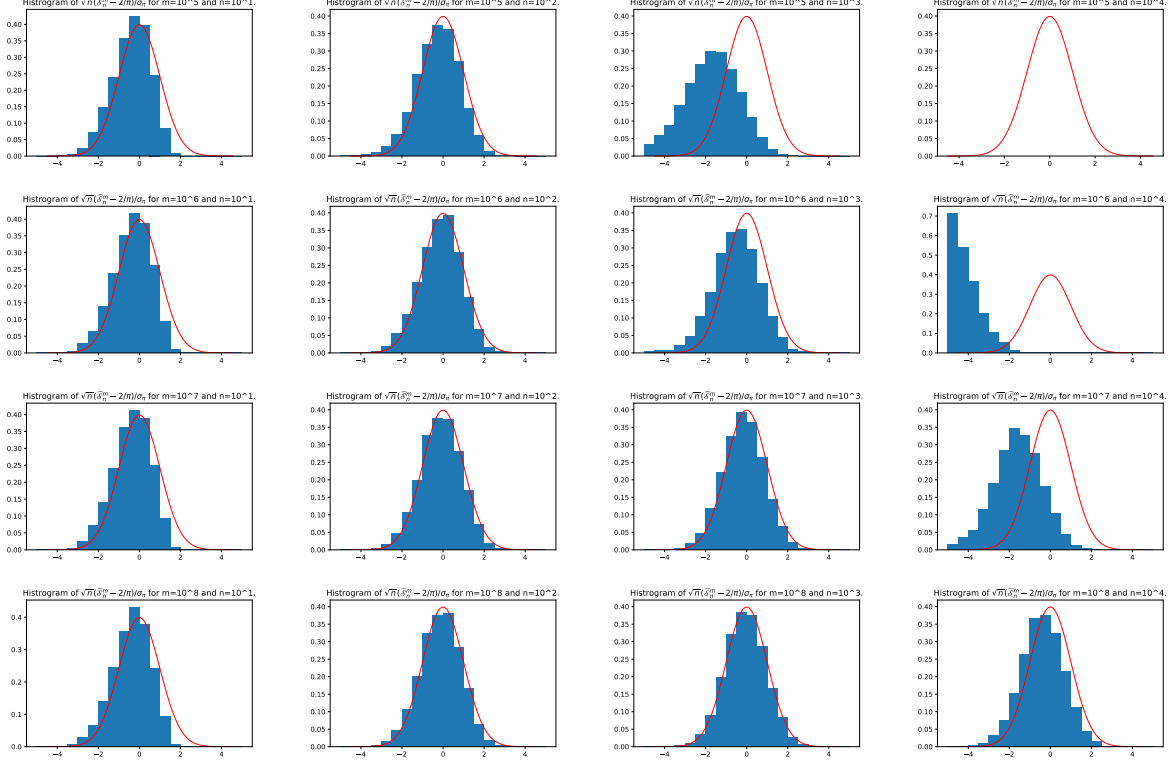


Figure 12: Empirical distribution of standardized  $\widehat{\mathcal{F}}_n^m$  for a 1-dependent sequence of random variables when  $r = 0.3$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^8$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^4$  in the right column.

#### 5.4 Choosing $m$ in terms of $n$

According to Theorem 4.4, we expect that under  $\mathbf{H}_0$ , as  $m$  and  $n$  increase, the empirical rejection rate should tend to  $q$ , and that is what we have observed numerically. However, we have also observed that the empirical rejection rate does not seem to be monotone with respect to  $n$ . This is not totally unexpected. On one hand, the order in which the limits are taken in Theorem 4.4 cannot be inverted. On the other hand, our test is based on discretized trajectories of  $Z^m$  which by construction are discontinuous. Therefore, when one increases the number  $n$  of discretization times, one detects the small jumps of  $Z^m$  at these discretization times, especially if the sample size  $m$  is not large enough.

From the previous considerations, it follows that the choice of  $n$  cannot be independent of the choice of  $m$ . In Tables 16a and 16b we report the experimental values of  $m$  and  $n$  for which, under  $\mathbf{H}_0$ , the best empirical rejection rate is attained. Notice that the best value for  $n$  decreases when the parameter  $r$  tends to the critical value  $r = 0.5$  (in other words, when the subjacent random variable has lower and lower finite moments). In particular, it does not seem to exist an optimal selection for the parameters  $m$  and  $n$  which is satisfying for any random variable in the domain of attraction of the Gaussian law.

Under  $\mathbf{H}_1$  the empirical rejection rate is expected to be close to 1. For  $n = 10^3$  it is exactly what our numerical experiments show. Nevertheless, under  $\mathbf{H}_0$ , the choice  $n = 10^3$  led to very poor results for  $r \sim 0.5$ . In Tables 17a and 17b we report the empirical rejection rates for different values of  $r$  and  $q$  when  $n < 10^3$ . Notice the change of regime between  $n = 10^1$  and  $n = 10^2$ .

$r$	$m$	$n$	ERR
0.1	$10^8$	$10^1, 10^2$	0.100
0.2	$10^8$	$10^3$	0.100
0.3	$10^7$	$10^1$	0.100
0.4	$10^7, 10^8$	$10^1$	0.101
0.45	$10^8, 10^9$	$10^1$	0.109

(a)  $q = 0.1$ .

$r$	$m$	$n$	ERR
0.1	$10^8$	$10^4$	0.050
0.2	$10^8$	$10^4$	0.050
0.3	$10^8$	$10^1$	0.050
0.4	$10^8$	$10^1$	0.053
0.45	$10^8$	$10^1$	0.061

(b)  $q = 0.05$ .Table 16: Values of the parameters  $m$  and  $n$  for which the best empirical rejection rate (ERR) of the  $\widehat{\mathcal{F}}_n^m$ -test is attained under  $\mathbf{H}_0$  for different values of  $r$  and  $q$ .

$r$	$n = 10^1$	$n = 10^2$
0.6	0.194	0.761
0.75	0.268	0.950
0.9	0.340	0.987

(a)  $q = 0.1$ .

$r$	$n = 10^1$	$n = 10^2$
0.6	0.123	0.712
0.75	0.189	0.934
0.9	0.251	0.984

(b)  $q = 0.05$ .Table 17: Values of the empirical rejection rate (ERR) of the  $\widehat{\mathcal{F}}_n^m$ -test for  $m = 10^8$  for different values of  $r$  under  $\mathbf{H}_1$  and different values of  $q$ .

## 6 When the sampled probability distribution does not belong to any $D(\alpha)$

As in many other works we have here supposed that the sampled probability distribution belongs to some  $D(\alpha)$ . This might be difficult to check a priori, notably when the sample is produced by complex numerical methods. For example, usually it is hard to check that the characterizations of stable laws are satisfied by the probability distribution of particle systems with singular interaction kernels (See the Introduction in [14]).

In this section we present numerical results which tend to show that our test nicely rejects  $\mathbf{H}_0$  when the sampled probability distribution does not belong to any  $D(\alpha)$ .

### 6.1 The case of the St Petersburg probability distribution

The integer valued St Petersburg random variables  $Y$  satisfy

$$\mathbb{P}(Y = 2^k) = 2^{-k}, \quad k \geq 1.$$

It is known that such a r.v.  $Y$  does not belong to the domain of attraction of any stable distribution, However, it belongs to the domain of semi-stable attraction of a semi-stable distribution. Moreover,

$$\mathbb{E}(Y^p) = \sum_{k=1}^{\infty} (2^k)^p \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{(2^{1-p})^k} < +\infty, \quad \text{if } p < 1.$$

Therefore, for  $X = Y^r$  one has  $\mathbb{E}(X^q) < \infty$  if  $q < 1/r$ . Thus, when  $r < 0.5$  the r.v.  $X$  has finite variance and belongs to the domain of attraction of the normal law.

In the sequel, we will call the probability distribution of  $X^r$  as a “ $r$ -powered St Petersburg probability distribution”.

Tables 18a and 18b show empirical rejection rates for a  $r$ -powered St-Petersburg probability distribution with  $r = 0.25$ . In this case,  $X \in \text{DA}(2)$  and we observe empirical rejection rates close to the theoretical ones. In Fig. 13 we show empirical distributions of the standardized  $\widehat{\mathcal{F}}_n^m$ . As expected, the histograms resemble the limit distribution for  $m$  large enough.

Tables 19a to 21b show the empirical rejection rate for  $r$ -powered St-Petersburg probability distributions for  $r \in \{1, 2, 4\}$ . In these cases the subjacent random variable is not in  $DA(\alpha)$  for any  $\alpha$ . However, our test behaves consistently with the fact that the null hypothesis  $\mathbf{H}_0$  is false.

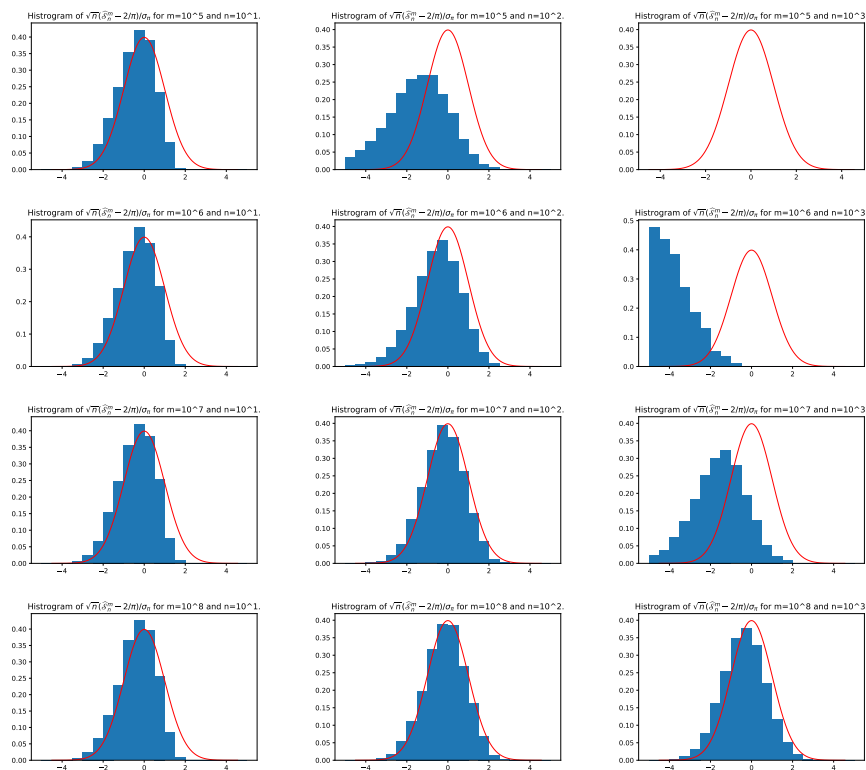


Figure 13: Histograms of standardized  $\widehat{\mathcal{F}}_n^m$  for  $r$ -powered St Petersburg probability distribution with  $r = 0.25$ . The value of  $m$  increases from  $10^5$  on the top row up to  $10^8$  in the bottom row. The value of  $n$  increases from  $10^1$  on the left column up to  $10^3$  in the right column.

	$n$		
$m$	$10^1$	$10^2$	$10^3$
$10^5$	0.107	0.461	1.000
$10^6$	0.103	0.181	0.988
$10^7$	0.097	0.114	0.470
$10^8$	0.095	0.106	0.141

(a)  $q = 0.1$ .

	$n$		
$m$	$10^1$	$10^2$	$10^3$
$10^5$	0.058	0.377	1.000
$10^6$	0.057	0.119	0.981
$10^7$	0.054	0.058	0.373
$10^8$	0.053	0.052	0.080

(b)  $q = 0.05$ .

Table 18: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a  $r$ -powered St Petersburg probability distribution with  $r = 0.25$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.372	0.995	1.000
$10^6$	0.367	0.995	1.000
$10^7$	0.362	0.995	1.000
$10^8$	0.372	0.994	1.000

(a)  $q = 0.1$ .

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.276	0.992	1.000
$10^6$	0.274	0.993	1.000
$10^7$	0.271	0.992	1.000
$10^8$	0.284	0.991	1.000

(b)  $q = 0.05$ .

Table 19: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a  $r$ -powered St Petersburg probability distribution with  $r = 1$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.617	1.000	1.000
$10^6$	0.631	1.000	1.000
$10^7$	0.630	1.000	1.000
$10^8$	0.635	0.999	1.000

(a)  $q = 0.1$ .

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.495	0.999	1.000
$10^6$	0.502	0.999	1.000
$10^7$	0.495	1.000	1.000
$10^8$	0.501	0.999	1.000

(b)  $q = 0.05$ .

Table 20: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a  $r$ -powered St Petersburg probability distribution with  $r = 2$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.750	1.000	1.000
$10^6$	0.741	1.000	1.000
$10^7$	0.750	1.000	1.000
$10^8$	0.757	1.000	1.000

(a)  $q = 0.1$ .

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.739	0.999	1.000
$10^6$	0.729	1.000	1.000
$10^7$	0.738	1.000	1.000
$10^8$	0.743	1.000	1.000

(b)  $q = 0.05$ .

Table 21: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a  $r$ -powered St Petersburg probability distribution with  $r = 4$ , different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

## 6.2 The case of a log-tailed distribution

We say that a random variable  $Y$  has a log-tailed distribution if

$$\mathbb{P}(Y \leq t) = \begin{cases} 0 & \text{for } t < \sqrt{e-1}, \\ 1 - \frac{1}{\log(1+t^2)} & \text{for } t \geq \sqrt{e-1}. \end{cases}$$

In view of Theorem 2.1 the random variable  $Y$  cannot belong to the domain of attraction of a stable distribution. Notice that  $E(Y^r) = +\infty$  for any  $r > 0$ . Even more,  $Y$  does not belong to the domain of partial attraction of any semi stable distribution (cf. Shimizu [16, Thm.1]).

The tables 22a and 22b show the empirical rejection rates of our hypothesis test. Notice that, as soon as  $n > 10$ , the empirical rejection rates are very close to 1, which is consistent with the fact that  $\mathbf{H}_0$  is false.

All the examples in this section seem to indicate that our test is robust with respect to the subjacent hypothesis that  $X$  belongs to some  $\text{DA}(\alpha)$  and correctly rejects the null hypothesis

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.987	1.000	1.000
$10^6$	0.956	1.000	1.000
$10^7$	0.683	1.000	1.000
$10^8$	0.330	0.912	1.000

(a)  $q = 0.1$ .

$m$	$n$		
	$10^1$	$10^2$	$10^3$
$10^5$	0.983	1.000	1.000
$10^6$	0.939	1.000	1.000
$10^7$	0.601	1.000	1.000
$10^8$	0.318	0.911	1.000

(b)  $q = 0.05$ .

Table 22: Empirical rejection rates for the  $\widehat{\mathcal{F}}_n^m$ -test for a log-tailed distribution with different values for  $m$  and  $n$ , different confidence levels and  $10^4$  simulations.

when  $X \notin \text{DA}(\alpha)$  for any  $\alpha$ .

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