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# A $\delta$ -persistently-exciting formation controller for non-holonomic systems over directed graphs

Maitreyee Dutta   Antonio Loría   Elena Panteley   Srikant Sukumar   Emmanuel Nuño

**Abstract**— We study a formation control problem of non-holonomic vehicles which consists in making a group of them gather around a given rendezvous set-point and acquire a common orientation. This task may be regarded as part of a more complex maneuver, *e.g.*, requiring the robots to advance in a scouting mission on a path composed of straight lines and occasional turns. The control approach relies on addressing separately the problems of *stabilization* (on the plane) and of orientation *consensus*. For the former we use individual controllers involving smooth time-varying terms and for the latter we use distributed consensus control, under the assumption that the robots form a directed graph that contains a spanning tree.

**Index Terms**— Formation consensus, non-holonomic constraints, Lyapunov stability

## I. INTRODUCTION

FORMATION control of nonholonomic vehicles is a well-studied problem in multi-disciplinary literature, but one which has different interpretations and motivations. One pertains to *formation-consensus* in which case it is required for the networked-interconnected vehicles to reach a consensual, non-predefined posture. This is inherently a *set-point* stabilization problem (with the well-known technical difficulties that this implies for nonholonomic systems [1], [2]) that may involve full consensus in position and orientation [3]–[6] or partial, either in orientation [7] or in position [1], [8], [9]. The term formation control, however, may also refer to the case in which a group of robots are required to follow a leader, real or virtual, while keeping a formation [10]–[16]. We may generically refer to this scenario as that of leader-follower formation-tracking control.

In the case of formation-consensus the equilibrium is not pre-defined, but it depends on the topology of the network, as well as on the initial postures and the nonlinearities of the systems. In leader-follower-based formation-control problems the vehicles are meant to track a leader, so both problems are fundamentally different, are motivated by distinct practical applications and, from a control viewpoint, demand very different approaches. A well-known fact, *e.g.*, is that smooth

controllers must rely on persistency of excitation [17] both in stabilization and tracking control tasks [2]. However, the two problems are not solvable simultaneously via the same controller [2].

Now, besides the difficulties imposed by the nonholonomic constraints, controlling *networked* autonomous vehicles imposes challenges related to the network's topology. Even though the literature on consensus is very rich and covers a wide range of graph topologies, this is so mostly for linear systems. Alternatively, in the context of formation and consensus of multiple vehicles, most works are restricted to undirected-graph topologies—see *e.g.*, [4], [6], [7], [9], [11], [14], [15], [18]–[20]. Articles covering directed-graph topologies are relatively scarce—see *e.g.*, [3], [21] for formation consensus and formation-tracking control, respectively. However, in both cases knowledge of the Laplacian is needed. Leader-follower-based schemes are often restricted to graphs that consist in (do not simply contain) a spanning tree [5], [12], [22], [23].

In this letter we study a problem of formation control that consists in all the robots converging to a static formation with a pre-imposed reference position on the Cartesian plane and acquiring a non-predefined orientation. This is a task that involves both stabilization and orientation consensus, so, as in [1], [7]–[9] we solve a problem of partial consensus. Our main contribution resides in that we consider that the systems are interconnected over a generic directed graph, *i.e.*, that it *contains* a spanning tree. Hence, from a network-topology viewpoint, the leader-follower scenarios are covered. Relative to [3], [21] we stress that our controller is fully distributed. Formation-tracking control is out of scope, but the problem that we address may be regarded as part of a more complex maneuver. For instance, as a byproduct of our main statement, we show that our controller applies also in simple tasks that require a group of vehicles to advance in formation; not following a target or a leader, as in [10], [13], [21], but advancing with constant forward speed, along paths composed of straight-lines—cf. [12] and predefined turns.

## II. PROBLEM FORMULATION

Let us consider a group of  $N$  nonholonomic vehicles modeled by the kinematic equations

$$\dot{x}_i = \cos(\theta_i)v_i \quad (1a)$$

$$\dot{y}_i = \sin(\theta_i)v_i \quad (1b)$$

$$\dot{\theta}_i = \omega_i, \quad i \leq N, \quad (1c)$$

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where  $(x_i, y_i) := p_i$  denote the Cartesian coordinates of the center of mass of the  $i$ th robot on the plane,  $\theta_i$  denotes its orientation relative to the abscissae; these variables are measured and are to be controlled. The forward velocity  $v_i$  and the angular velocity  $\omega_i$  constitute the control inputs.

It is assumed that each system has access to its own coordinates  $p_i$  and  $\theta_i$ , as well as the orientation of neighbor vehicles,  $\theta_j$ , where  $j \in \mathcal{N}_i$  and  $\mathcal{N}_i$  is the set of indexes corresponding to the neighbors of the  $i$ th vehicle. In that regard, we assume that through their interconnections the systems form a directed graph that contains a spanning tree. We stress that such condition is necessary for consensus and, therefore, cannot be relaxed. In *particular* it is satisfied in leader-follower configurations [5], [12], [22] in which the graph *consists* in a spanning tree and cycles are excluded. For these systems, we address the following problem.

**Formation-stabilization with orientation consensus:** Let  $p^* := [p_x^* \ p_y^*]^\top$  be a given reference constant set-point on the plane. Let  $\delta_i := [\delta_{xi} \ \delta_{yi}]^\top$  be a given offset relative to  $p^*$ , so that, defining  $\bar{p}_i := p_i - \delta_i - p^*$ , we have that  $\bar{p}_i = 0$  for all  $i \leq N$  means that all the vehicles are positioned in a geometric formation with center at  $p^*$ . Then, the formation-stabilization-with-common-orientation control problem consists in designing decentralized control laws such that

$$\lim_{t \rightarrow \infty} |\bar{p}_i(t)| = 0, \quad \lim_{t \rightarrow \infty} \theta_i(t) = \theta_c, \quad (2a)$$

$$\lim_{t \rightarrow \infty} v_i(t) = 0, \quad \lim_{t \rightarrow \infty} \omega_i(t) = 0, \quad \forall i \in \{1, \dots, N\}, \quad (2b)$$

where  $\theta_c$  is a non-given constant that depends on the system's dynamics, initial conditions, and the network's topology. This is a problem of partial consensus control—cf. [1], [7]–[9]; from a practical viewpoint, it is pertinent, *e.g.*, as the first part of a more complex maneuver [24] which may consist in making the robots gather and advance in formation, for instance in a scouting mission—see Section III-C.

### III. CONTROL APPROACH AND MAIN RESULTS

#### A. Rationale

Formation control with consensual orientation, as stated above, may be solved by designing a formation-stabilization and a consensus controller independently. However, this presents two difficulties: the first stems from the fact that nonholonomic systems are known not to be stabilizable via smooth time-invariant feedback; the second is that of analyzing consensus over generic directed graph-networks for highly nonlinear systems, such as nonholonomic integrators. To circumvent the first difficulty we use the approach of so-called  $\delta$ -persistently-exciting ( $\delta$ -PE) controllers [12], which are designed to inject an external time-varying perturbation that is persistently exciting [17], as long as a certain function of the state is away from the origin. To achieve the consensus goal we use the control law

$$\omega_i = -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\theta_i - \theta_j), \quad k_w > 0, \quad (3)$$

where  $a_{ij} = 1$  if an interconnection from the node  $j$  to the node  $i$  exists and  $a_{ij} = 0$  if otherwise. Note that since

the interconnections are directed, in general,  $a_{ij} \neq a_{ji}$ . Disregarding the stabilization task, orientation consensus is achieved provided that the network's graph contains a directed spanning tree [25]. That is, (3) guarantees that  $\theta_i \rightarrow \theta_j \rightarrow \theta_c$  for all  $i, j \leq N$ , where  $\theta_c$  is constant.

Now, in relation to the stabilization objective we start by introducing the error coordinates

$$z_i := \begin{bmatrix} \phi(\theta_i)^\perp \\ \phi(\theta_i)^\top \end{bmatrix} \bar{p}_i, \quad \phi(\theta_i) = [\cos(\theta_i) \ \sin(\theta_i)]^\top \\ \phi(\theta_i)^\perp = [-\sin(\theta_i) \ \cos(\theta_i)]. \quad (4)$$

Note that  $\phi(\theta_i)^\perp \phi(\theta_i) \equiv 0$ , so, after (4) and (1), we have

$$\dot{z}_i = \begin{bmatrix} \dot{z}_{1i} \\ \dot{z}_{2i} \end{bmatrix} = \begin{bmatrix} -z_{2i}\omega_i \\ z_{1i}\omega_i + v_i \end{bmatrix} \quad (5)$$

and since the matrix in (4) is globally invertible, we have that  $z_i = 0$  if and only if  $\bar{p}_i = 0$ , so the formation-stabilization objective boils down to ensuring that  $z_i \rightarrow 0$  holds for (5). To that end, we remark that Equations (5) may be assimilated to those of a stable (oscillatory) system with input  $v_i$ . Hence, for any  $\omega_i \neq 0$ , the control law

$$v_i = -k_{2i} z_{2i} + (k_{1i} - 1)\omega_i z_{1i}, \quad k_{1i}, k_{2i} > 0, \quad (6)$$

makes  $z_i \rightarrow 0$ . To see this more clearly note that the closed-loop system takes the form

$$\begin{bmatrix} \dot{z}_{2i} \\ \dot{z}_{1i} \end{bmatrix} = \begin{bmatrix} -k_{2i} & k_{1i}\omega_i \\ -\omega_i & 0 \end{bmatrix} \begin{bmatrix} z_{2i} \\ z_{1i} \end{bmatrix}, \quad (7)$$

which has a structure familiar in adaptive control—cf. [12]. Hence, it may be showed that if  $k_{2i} > 0$ ,  $z_{2i} \rightarrow 0$  (see the proof of our main result below). Moreover, if  $\omega_i$  depended only on  $t$ , it would be sufficient to make  $\omega_i$  persistently exciting (PE), *i.e.*, to ensure that there exist  $\mu$  and  $T > 0$  such that

$$\int_t^{t+T} |\omega_i(s)| ds \geq \mu \quad \forall t \geq 0,$$

to ensure that  $z_{1i} \rightarrow 0$ . Making  $\omega_i$  be PE, however, means that  $\theta_i$  cannot converge to a constant value, which is in clear conflict with the orientation-consensus objective. In summary, persistency of excitation is needed to attain the formation control objectives, while it is required that  $\omega_i$  converge to zero to achieve orientation consensus. To overcome this conundrum, we endow the angular-motion consensus control law (3) with an oscillatory perturbation that persists as long as  $|z_{1i}| \neq 0$ . That is, with a  $\delta$ -persistently exciting function—cf. [12]. Hence, we redesign the consensus control law (3) to take the form

$$\omega_i = -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\theta_i - \theta_j) + f_i(t) \beta_i(z_i), \quad k_w > 0. \quad (8)$$

In the previous expression,  $f_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is designed to be twice continuously differentiable, bounded with bounded derivatives, and  $\dot{f}_i$  is persistently exciting, that is, there exist  $T, \mu > 0$  such that

$$\int_t^{t+T} |\dot{f}_i(s)| ds \geq \mu \quad \forall t \geq 0. \quad (9)$$

In addition, the function  $\beta_i : \mathbb{R}^2 \rightarrow \mathbb{R}$  is chosen to be twice continuously differentiable and to satisfy  $\beta_i(z_i) = 0$

if and only if  $z_i = 0$ . Loosely speaking, the purpose of the term  $f_i(t)\beta_i(z_i)$  is to prevent  $\omega_i$  from vanishing to zero (achieving consensus) as long as  $z_i \neq 0$  (the stabilization task has not been achieved). Indeed, provided that  $z_i(t)$  is bounded,  $f_i(t)\beta_i(z_i)$  acts as a bounded disturbance that vanishes only if both objectives, consensus and stabilization are attained simultaneously. This rationale is formalized in following statement given next.

### B. Formation stabilization

*Proposition 1 (Formation stabilization):* Consider a network of autonomous vehicles modeled as in (1), communicating over a directed-graph containing a directed spanning tree, in closed loop with the controller (6)-(8), where for each  $i \leq N$ ,  $f_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is twice continuously differentiable, bounded with bounded derivatives, and satisfies (9), whereas  $\beta_i : \mathbb{R}^2 \rightarrow \mathbb{R}$  is twice continuously differentiable and satisfies  $\beta_i(z_i) = 0$  if and only if  $z_i = 0$ . Then, the formation-stabilization-with-common-orientation goals in (2) are attained.  $\square$

*Proof:* Replacing the control laws (6)-(8) in the kinematics equations (5)-(1c) we obtain the closed-loop system

$$\dot{z}_{1i} = -\omega_i z_{2i} \quad (10a)$$

$$\dot{z}_{2i} = k_{1i}\omega_i z_{1i} - k_{2i}z_{2i} \quad (10b)$$

$$\dot{\theta}_i = -k_w \sum_{j \in \mathcal{N}_i} a_{ij}(\theta_i - \theta_j) + f_i(t)\beta_i(z_i). \quad (10c)$$

To establish the statement of the proposition we start by showing that  $z_i \in \mathcal{L}_\infty$ , where  $z_i := [z_{1i} \ z_{2i}]^\top$ . To that end, we use the positive-definite, decrescent Lyapunov function

$$V_z(z) = \frac{1}{2} \sum_{i=1}^N [k_{1i}z_{1i}^2 + z_{2i}^2]. \quad (11)$$

Its total derivative along the trajectories of (10a)-(10b) yields

$$\dot{V}_z(z(t)) = -k_{2i} \sum_{i=1}^N z_{2i}(t)^2 \leq 0. \quad (12)$$

Integrating on both sides of  $\dot{V}_z(z(t)) \leq 0$ , from  $t_o \geq 0$  to  $t$ , we obtain  $[k_{1i}z_{1i}(t)^2 + z_{2i}(t)^2] \leq [k_{1i}z_{1i}(t_o)^2 + z_{2i}(t_o)^2]$  which implies that

$$|z_i(t)|^2 \leq \frac{\max\{k_{1i}, 1\}}{\min\{k_{1i}, 1\}} |z_i(t_o)|^2. \quad (13)$$

That is,  $z_i$  is uniformly globally bounded.

Strictly speaking, the latter holds only on the interval of existence of  $t \mapsto \omega(t)$ . Say, for a certain  $\infty > t_f > t_o$ , any  $t_o \geq 0$  and for all  $t \in [t_o, t_o + t_f)$ . In the sequel we show that  $t_f = +\infty$ , that  $\omega := [\omega_1 \ \dots \ \omega_N]^\top$  is also uniformly bounded and, therefore, the arguments above hold for all  $t \geq t_o \geq 0$ .

To that end, we start by rewriting (8) in multi-variable compact form,

$$\omega = -L\theta + \alpha(t, z), \quad (14)$$

where  $\alpha(t, z) := [f_1(t)\beta_1(z_1) \ \dots \ f_N(t)\beta_N(z_N)]^\top$ ,  $\theta := [\theta_1 \ \dots \ \theta_N]^\top$ , and

$$L := [\ell_{ij}] \in \mathbb{R}^{N \times N}, \quad \ell_{ij} := \begin{cases} -a_{ij} & \forall i \neq j \\ \sum_{k \in \mathcal{N}_i} a_{ik}, & \forall i = j \end{cases} \quad (15)$$

is the Laplacian matrix. Then, following [26], we introduce the synchronization errors

$$e_\theta = [I_N - \mathbf{1}_N v_\ell^\top] \theta, \quad (16)$$

which correspond to the vector  $e_\theta := [e_{\theta 1} \ \dots \ e_{\theta N}]^\top$ , where  $e_{\theta i} := \theta_i - v_\ell^\top \theta$ ,  $v_\ell$  is the left eigenvector associated to the unique zero eigenvalue of  $L$ , and  $v_\ell^\top \theta$  is a weighted average of  $\theta$ . That is, consensus is reached if  $\theta_i \rightarrow v_\ell^\top \theta$  for all  $i \leq N$  or, equivalently,  $e_\theta \rightarrow 0$ .

*Remark 1:* The errors  $e_\theta$  satisfy

$$\dot{e}_\theta = -L e_\theta + [I_N - \mathbf{1}_N v_\ell^\top] \alpha(t, z). \quad (17)$$

To see this, we note that from (16),

$$\dot{e}_\theta = -L[I_N - \mathbf{1}_N v_\ell^\top] \theta + [I_N - \mathbf{1}_N v_\ell^\top] \alpha(t, z),$$

so (17) follows observing that  $L\mathbf{1}_N = 0$ .  $\bullet$

Next, let us assume that the graph contains a directed spanning tree then the statement of [26, Lemma 2] generates  $P = P^\top \in \mathbb{R}^{N \times N}$  on solving, for any matrix  $Q_L \in \mathbb{R}^{N \times N}$ ,  $Q_L = Q_L^\top > 0$  and for  $\sigma > 0$ , the equation

$$PL + L^\top P = Q_L - \sigma[P\mathbf{1}_N v_\ell^\top + v_\ell \mathbf{1}_N^\top P], \quad (18)$$

where  $v_\ell$  is the left eigenvector associated to the single zero eigenvalue of  $L$ . Then, consider the Lyapunov function  $V_{e_\theta} = e_\theta^\top P e_\theta$ . Its total derivative along the trajectories of (17) yields

$$\dot{V}_{e_\theta} = -e_\theta^\top [PL + L^\top P] e_\theta + 2e_\theta^\top P [I_N - \mathbf{1}_N v_\ell^\top] \alpha(t, z),$$

so, using (18), we obtain

$$\begin{aligned} \dot{V}_{e_\theta} &= -e_\theta^\top [Q_L - \sigma[P\mathbf{1}_N v_\ell^\top + v_\ell \mathbf{1}_N^\top P]] e_\theta \\ &\quad + 2e_\theta^\top P [I_N - \mathbf{1}_N v_\ell^\top] \alpha(t, z). \end{aligned}$$

However, since  $v_\ell^\top \mathbf{1}_N = 1$ , the second term on the right-hand side of the equation above equals to zero. Indeed, after (16),  $P\mathbf{1}_N v_\ell^\top e_\theta = P\mathbf{1}_N v_\ell^\top [I_N - \mathbf{1}_N v_\ell^\top] \theta = P[\mathbf{1}_N v_\ell^\top - \mathbf{1}_N v_\ell^\top] \theta = 0_N$ . Therefore,

$$\dot{V}_{e_\theta} = -e_\theta^\top Q_L e_\theta + 2e_\theta^\top P [I_N - \mathbf{1}_N v_\ell^\top] \alpha(t, z).$$

Now, let  $\Delta := 2\|P[I_N - \mathbf{1}_N v_\ell^\top]\|$  where  $\|\cdot\|$  denotes here the induced matrix norm. For any  $t_f > 0$ , let

$$\|\alpha\|_{t_f} := \sup_{t \in [t_o, t_o + t_f)} |\alpha(t, z(t))|.$$

Then, defining  $q_m := \lambda_{\min}(Q_L)$ , we have

$$\dot{V}_{e_\theta}(e_\theta(t)) \leq -q_m |e_\theta(t)|^2 + \Delta |e_\theta(t)| \|\alpha\|_{t_f} \quad (19)$$

for all  $t \in [t_o, t_o + t_f)$ . We show by contradiction that  $t_f = +\infty$ , so all of the above holds for all  $t \geq t_o$ .

Assume that  $|\omega(t)| \rightarrow \infty$  as  $t \rightarrow t_f < \infty$ . Then, since  $\theta(t) = \int_{t_o}^t \omega(t) dt$  we have  $|\theta(t)| \rightarrow \infty$  as  $t \rightarrow t_f$ . On the other hand,  $V_{e_\theta}(e_\theta(t)) := \theta(t)^\top [I_N - \mathbf{1}_N v_\ell^\top]^\top [I_N - \mathbf{1}_N v_\ell^\top] \theta(t)$ , so  $V_{e_\theta}(e_\theta(t)) \rightarrow \infty$  as  $t \rightarrow t_f$ . Next, defining  $\nu(s) := V_{e_\theta}(e_\theta(s))$ , after (19), we have

$$\dot{\nu}(t) \leq \frac{\Delta^2}{2q_m} \|\alpha\|_{t_f}^2. \quad (20)$$

Integrating on both sides of (20), from  $t_o$  to  $t_f$ , we obtain

$$\lim_{t \rightarrow t_f} \nu(t) \leq \frac{\Delta^2}{2q_m} \|\alpha\|_{t_f}^2 [t_f - t_o] + \nu(t_o).$$

By assumption  $t_f < \infty$ , so the right-hand side of the inequality above is finite, whereas

$$\lim_{t \rightarrow t_f} \nu(t) = \lim_{t \rightarrow t_f} V_{e_\theta}(e_\theta(t)) = +\infty,$$

which is a contradiction. We conclude that  $t_f = +\infty$  so (12) holds along trajectories for all  $t \geq t_o$  and  $z_i(t)$  is uniformly globally bounded. Also, in view of the latter, the continuity of  $\alpha(t, \cdot)$  and the uniform boundedness of  $\alpha(\cdot, z)$ ,  $t \mapsto \alpha(t, z(t))$  is uniformly bounded, that is,  $\|\alpha\|_{t_f}^2 < \infty$ . From this and integrating on both sides of (19) along the trajectories, from  $t_o$  to  $\infty$ , we obtain that  $e_\theta$  is uniformly globally bounded.

We show next that  $\omega$  is also uniformly globally bounded. For this, we note that the Laplacian matrix  $L$  admits the decomposition  $L = UMU^{-1}$  where  $U$  is a matrix whose columns are the right eigenvectors of  $L$ , *i.e.*,  $U = [\mathbf{1}_N U_1]$ ,  $U^{-1} = [v_\ell U_1^\dagger]^\top$ , and  $M$  contains the Jordan blocks corresponding to the eigenvalues of  $L$ . The first eigenvalue of  $M$  being zero and simple, we also have  $L = U_1 \bar{M} U_1^\dagger$ , where  $\bar{M}$  contains all the Jordan blocks of  $L$  corresponding to the non-zero eigenvalues. It follows that  $L\theta = U_1 \bar{M} U_1^\dagger \theta$ , so  $|L\theta| \leq \lambda_{\max}(L) |U_1 U_1^\dagger \theta| = \lambda_{\max}(L) |[I_N - \mathbf{1}_N v_\ell^\top] \theta| = \lambda_{\max}(L) |e_\theta|$ . We conclude that  $|L\theta(t)|$  is uniformly globally bounded. From this, and the boundedness of  $f_i$  and  $\alpha(t, z(t))$ , it follows that  $\omega$  in (14) is also uniformly globally bounded, and so is  $\dot{\theta} = \omega$ . Consequently,  $\dot{z}_i$  is uniformly globally bounded as it is a function of  $z_i$  and  $\omega_i$ —see Eq. (7).

In addition, for further development, we remark that  $\dot{\omega}$  is also globally uniformly bounded, since so are all the terms on the right hand side of

$$\dot{\omega}_i = -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\omega_i - \omega_j) + f_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i + \dot{f}_i(t) \beta_i(z_i). \quad (21)$$

Next, we establish the convergence of  $z$  and  $L\theta$ . This part of the proof relies on a recursive application of Barbälät's Lemma.

First, we integrate on both sides of (12) from  $t_o$  to  $\infty$ . Since  $z$  is globally uniformly bounded we obtain that  $z_2 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ . Since, moreover,  $\dot{z}_2 \in \mathcal{L}_\infty$  we obtain, by [17, Lemma 3.2.5] that  $z_2 \rightarrow 0$  asymptotically. To establish the convergence of  $z_1$  we proceed by establishing, first, the convergence of  $\omega_i z_{1i}$  for any  $i \leq N$ . This follows from the fact that both terms on the right hand side of  $k_{1i} \omega_i z_{1i} = k_{2i} z_{2i} - \dot{z}_{2i}$ —cf. (10b), converge to zero. That  $z_{2i} \rightarrow 0$  was established above, while the convergence of  $\dot{z}_{2i}$  follows from Barbälät's Lemma. Indeed, on one hand,  $\ddot{z}_{2i}$  is uniformly globally bounded since so are all terms on the right hand side of

$$\begin{aligned} \ddot{z}_{2i} &= k_{1i} \dot{\omega}_i z_{1i} + k_{1i} \omega_i \dot{z}_{1i} - k_{2i} \dot{z}_{2i} \\ &= k_{1i} \left[ -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\omega_i - \omega_j) + f_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i \right] z_{1i} \\ &\quad + k_{1i} \dot{f}_i(t) \beta_i(z_i) z_{1i} - k_{1i} \omega_i^2 z_{2i} - k_{1i} k_{2i} \omega_i z_{1i} + k_{2i}^2 z_{2i}. \end{aligned} \quad (22)$$

On the other hand,

$$\lim_{t \rightarrow \infty} \int_{t_o}^t \dot{z}_{2i}(\tau) d\tau = \lim_{t \rightarrow \infty} z_{2i}(t) - z_{2i}(t_o) = -z_{2i}(t_o).$$

Now, since  $|\omega_i z_{1i}| \rightarrow 0$  and both  $\omega_i$  and  $z_{1i}$  are uniformly globally bounded, either  $z_{1i} \rightarrow 0$  or  $\omega_i \rightarrow 0$ . In the first case, the proof is completed. Hence, let us assume that  $\omega_i \rightarrow 0$  and turn our attention to the term  $k_{1i} \dot{f}_i(t) \beta_i(z_i) z_{1i}$  on the right-hand side of (22). By assumption  $\beta_i(z_i) = 0$  if and only if  $z_i = 0$  and  $\dot{f}_i$  is persistently exciting. Therefore, since  $t \mapsto \beta_i(z_i(t))$  is uniformly globally bounded we see that  $z_{1i} \rightarrow 0$  if and only if so does

$$k_{1i} \dot{f}_i(t) \beta_i(z_i) z_{1i} = \ddot{z}_{2i} + k_{1i} \omega_i^2 z_{2i} + k_{1i} k_{2i} \omega_i z_{1i} - k_{2i}^2 z_{2i} - k_{1i} \left[ -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\omega_i - \omega_j) + f_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i \right] z_{1i}.$$

We proceed to establish the convergence of all the terms on the right-hand side of the previous expression, individually. As a matter of fact, this has already been done for the second, third, and fourth terms, since both  $z_{2i}$  and  $\omega_i z_{1i}$  converge to zero. Furthermore, the convergence of the last term on the right-hand side follows from that of  $\dot{z}_i$  and the uniform boundedness of  $f_i(\cdot)$  and  $\beta_i(z_i(\cdot))$ . In addition, by assumption  $\omega_i \rightarrow 0$ , so it is only left to show that  $\ddot{z}_{2i} \rightarrow 0$ . To that end, we use Barbälät's Lemma once more. First, we observe that, for any  $t_o \geq 0$

$$\lim_{t \rightarrow \infty} \int_{t_o}^t \ddot{z}_{2i} = \lim_{t \rightarrow \infty} \dot{z}_{2i}(t) - \dot{z}_{2i}(t_o) = -\dot{z}_{2i}(t_o). \quad (23)$$

On the other hand, we observe that  $t \mapsto \ddot{z}_{2i}(t)$  is uniformly continuous since  $z_{2i}^{(3)}$  is uniformly globally bounded. Indeed, so are all the terms bounded on the right-hand side of

$$\begin{aligned} z_{2i}^{(3)} &= k_{1i} \left[ -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\dot{\omega}_i - \dot{\omega}_j) + \dot{f}_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i \right. \\ &\quad + f_i(t) \dot{z}_i^\top \frac{\partial^2 \beta_i(z_i)}{\partial^2 z_i} \dot{z}_i + f_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \ddot{z}_i \\ &\quad \left. + \ddot{f}_i(t) \beta_i(z_i) + \dot{f}_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i \right] z_{1i} \\ &\quad + k_{1i} \left[ -k_w \sum_{j \in \mathcal{N}_i} a_{ij} (\omega_i - \omega_j) + f_i(t) \frac{\partial \beta_i(z_i)}{\partial z_i} \dot{z}_i \right. \\ &\quad \left. + \dot{f}_i(t) \beta_i(z_i) \right] \dot{z}_{1i} - 2k_{1i} \omega_i \dot{\omega}_i z_{2i} - k_{1i} \omega_i^2 \dot{z}_{2i} \\ &\quad - k_{1i} k_{2i} \dot{\omega}_i z_{1i} - k_{1i} k_{2i} \omega_i \dot{z}_{1i} + k_{2i}^2 \dot{z}_{2i}. \end{aligned}$$

This completes the proof.  $\blacksquare$

### C. Application to formation scouting

After Proposition 1, the controller (6)-(8) guarantees that a group of vehicles meet at a specified rendezvous set-point and acquire a consensual orientation. This task may be considered as the starting of a gather-and-scout maneuver in which it is also required to advance in formation along a path composed of straight lines and occasional turns. Besides the practical motivations of such a mission, it is worth noticing that a

formation cannot be rigidly maintained over arbitrary paths, but it can if they mainly consist in straight lines [27], [28].

This task may be simply achieved by introducing *individual* reference models for each vehicle, given by  $\dot{p}_i^* := \phi(\theta_i)v^*$  with  $p_i^*(0) = p_j^*(0)$  for all  $i, j \leq N$ , and constant speed  $v^*$ . Then, we redefine the control law (6) as

$$v_i = -k_{2i}z_{2i} + (k_{1i} - 1)\omega_i z_{1i} + v^*, \quad k_{1i}, k_{2i} > 0. \quad (24)$$

For the vehicles to agree on a consensual orientation, we add to the network an orientation leader with dynamics  $\dot{\theta}^* = \omega^*$ , where  $\omega^*$  is set to zero (to move on a straight line) or to a constant (to take a turn). More precisely, for  $\nu$  turns in a path, let  $M := 2\nu + 1$  and, for each  $k \in \{1, 2, \dots, M + 1\}$ , let  $\{t_k\}$  be an increasing finite sequence of times, such that  $\omega^*(t) = 0$  for all  $t \in [t_k, t_{k+1})$  with  $k$  odd, and  $\omega^*(t) \neq 0$  for all  $t \in [t_k, t_{k+1})$ , with  $k$  even. In addition, let  $(t_{j+1} - t_j) \gg (t_{j+2} - t_{j+1})$  for any odd number  $j \in \{1, 2, \dots, M - 2\}$ , and  $t_{M+1} = +\infty$ , so  $\omega^*(t) = 0$  for all  $t \geq t_M$ .

Since an orientation-leader node is added, we also modify the orientation-consensus controller so that, on the consensus manifold  $\{\theta_i = \theta^*\}$ , the individual reference models generate a consensual trajectory, solution of  $\dot{p}^* := \phi(\theta^*)v^*$ , that is,  $p_i^*(t) = p_j^*(t)$  for all  $t \geq 0$  and  $i, j \leq N$ . As a result, they advance in the same direction while keeping the formation. The orientation-consensus controller, which is given in the following statement, is decentralized and modifies the generic directed graph into one that typically appears in leader-follower configurations—cf. [21].

*Proposition 2 (formation scouting):* Let  $v^*$  be a given constant velocity and, for  $\omega^*$  as defined above, let  $\dot{\theta}^* = \omega^*$  and  $\dot{p}^* := \phi(\theta_i)v^*$ . In addition, let

$$\omega_i = -k_w \left[ \sum_{j \in \mathcal{N}_i} a_{ij}(\theta_i - \theta_j) + b_i(\theta_i - \theta^*) \right] + f_i(t)\beta_i(z_i), \quad (25)$$

where  $\beta_i$  and  $f_i$  are as in Proposition 1,  $b_i > 0$  for at least one node from which any other node can be reached while  $b_i \geq 0$  for any other node, and  $a_{ij}$  are such that the Laplacian matrix  $L$  defined in (15) contains a unique zero eigenvalue and all others have strictly positive real parts (the graph contains a spanning tree). Then, for the closed-loop system,  $\lim_{t \rightarrow \infty} |\bar{p}_i(t)| = 0$  and  $\lim_{t \rightarrow \infty} \theta_i(t) = \lim_{t \rightarrow \infty} \theta^*(t)$ .  $\square$

*Proof:* The closed-loop equations correspond to Eqs. (10a)-(10b) and, in terms of the synchronization errors,  $e_\theta = \theta - \mathbf{1}_N \theta^*$ ,  $\dot{e}_\theta = -k_w[L + B]e_\theta + \alpha(t, z) - \mathbf{1}_N \omega^*$ , where  $B = \text{diag}[b_i]$ . Under the above-stated assumptions,  $[L + B]$  is a non-singular  $M$ -matrix, so for any  $Q = Q^\top > 0$ , there exists a matrix  $P = P^\top > 0$  that solves the Lyapunov equation  $-P[L + B] - [L + B]^\top P = -Q$  (for details, see Theorem 4.25 in [29]). By construction,  $\omega^*(t) = 0$  for all  $t \geq t_M$ , so the proof follows as in that of Proposition 1.  $\blacksquare$

#### IV. SIMULATION RESULTS

We simulated two experiments involving a group of eight vehicles interconnected as in the directed graph showed in Fig. 1, which contains a directed spanning tree. The first experiment consists in making all robots meeting around the origin, so  $p^* = (0, 0)^\top$ , at the vertices of an octagon and

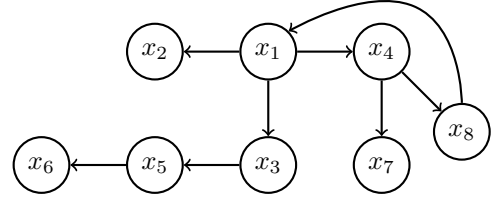


Fig. 1. Connection digraph for the eight communicating agents

acquiring a common non pre-imposed orientation  $\theta_c$ , so we use the controller (6)-(8). The initial positions on the plane are depicted in Fig. 2 with a ‘o’ while the initial orientations are set to  $\theta(0) := [\pi/2 \ 0 \ \pi/8 \ -\pi/8 \ \pi/3 \ \pi/5 \ -\pi/7 \ \pi/6]$ . The control gains were set to the same values for all robots, in a way to produce a little oscillatory transient; we used  $k_{1i} = 3.5$ ,  $k_{2i} = 30$ , and  $k_w = 1$ . In addition, we used  $f_i(t) = \cos(15t) - \sin(35t)$  and  $\beta_i(z_i) = 22z_{1i}$ .

The paths followed by the robots on the plane are also showed in Fig. 2 below, and their final consensual orientation ( $\theta_c \approx 35$  [deg]) is illustrated with pointing arrows.

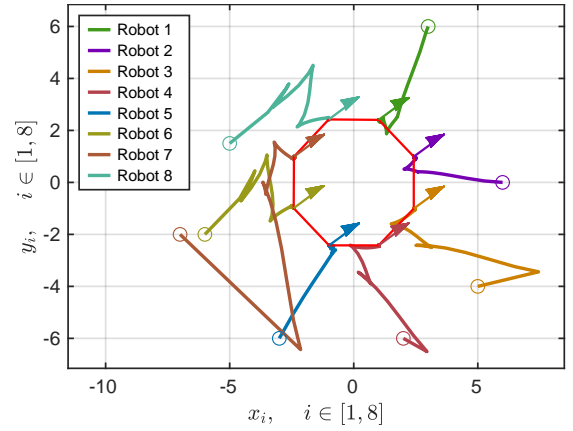


Fig. 2. Vehicles on the phase plane converging to a rendezvous point

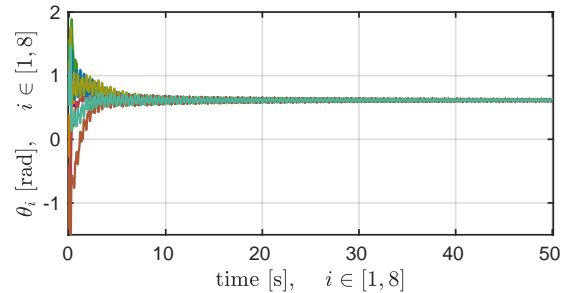


Fig. 3. Systems' orientations achieving consensus

In Fig. 3 we show the variation of orientations. The consensus errors in Cartesian positions rapidly become very small in norm while the orientations take a longer to settle due to the persistent, but vanishing perturbation  $f_i(t)\beta_i(z_i)$  in (8), needed to achieve set-point stabilization.

In a second simulated experiment, the robots gather in an octagonal formation and advance in formation along the consensual direction, only that in this case a virtual orientation reference  $\dot{\theta}^* = \omega^*$  is accessed only by node 1 so  $b_1 = 1$  and

$b_k = 0, k \in \{2, \dots, 8\}$ . The squad is required to advance at a constant speed  $v^* = 0.05$  [m/s] along straight paths and turning at an angular velocity of  $\omega^* = 0.1$  [rad/s] for pre-planned short periods of time. For this, we used the controller (24)-(25). The paths followed by the vehicles are showed in Fig. 4.

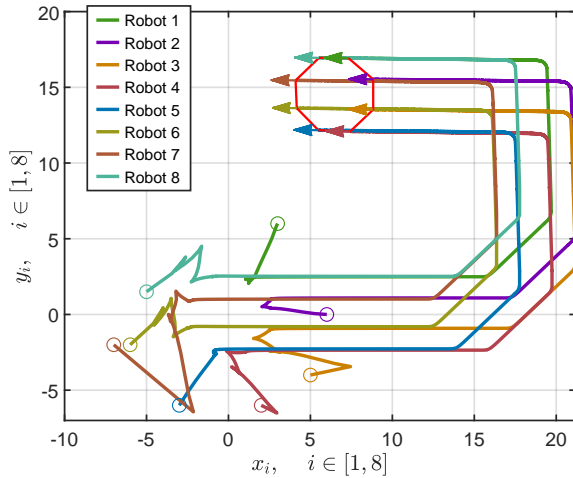


Fig. 4. Systems' paths on the phase plane converging to a formation and moving with constant speed

## V. CONCLUSIONS

The decentralized controller for formation-stabilization and orientation consensus in Section III-B addresses, individually for each robot, the problem of formation stabilization and, in a decentralized multi-agent fashion, orientation-consensus control. With such controller, moreover, a simple task of formation-scouting may be achieved. Our contributions, however, are limited as they do not cover the case of full-consensus formation control, *i.e.*, including position-consensus, nor forward velocity consensus. Specifically for the latter, the use of second-order models appears imperative. Designing a fully distributed controller to address these problems over a directed graph containing a spanning-tree for nonholonomic systems is to the best of our knowledge an open problem. The Lyapunov-based analysis that we provide may serve as basis to extend our results to these scenarii and, in turn analyze robustness with respect to input disturbances.

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