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Analysis and Control of Multi-timescale Modular Directed Heterogeneous Networks

Anes Lazri Elena Panteley Antonio Loría

Abstract—We study the collective behavior of heterogeneous nonlinear systems, interconnected over generic directed graphs and in the scenario that, due to the nature of their interconnections, the agents self-organize in modules. These are sub-networks composed of agents that are densely connected with a strong coupling, while the modules themselves are sparsely interconnected. As we show, beyond certain coupling thresholds, the systems within each module synchronize rapidly with a weighted-average dynamical system that evolves more slowly than the individual systems. Then, the average dynamical systems corresponding to each and all the modules synchronize among themselves. Furthermore, we establish global asymptotic stability for the overall network under the conditions that the average dynamics admit the origin as a globally asymptotically stable equilibrium and each system be semi-passive. Finally, we explore stabilization techniques that consist in controlling the average dynamics to make the origin globally asymptotically stable.

I. INTRODUCTION

Diverse factors determine the collective behavior of networked systems, including: the network’s topology, the nature of the interconnections, the nodes’ individual dynamics, and the coupling strength. Each of these aspects imposes different challenges that can hardly be addressed simultaneously. For instance, earlier works addressing synchronization of heterogeneous nonlinear systems focus on undirected [1]–[3] or directed-and-balanced-graph networks [4], in which case the Laplacian matrix possess properties that ease the analysis. Works allowing for the latter are more recent—see *e.g.*, [5], [6] and several references therein.

In regards to the network’s topology and the coupling strength, we single out the so-called *modular networks*. These consist in networks of modules of densely interconnected nodes, while the interconnections amongst these modules are comparatively weaker [5], [7]. Modular networks appear naturally in very large-scale systems in varied domains, including neuroscience applications [8], social networks [9], or power networks [10], [11], to mention a few.

When the coupling gain is sufficiently high, the synchronized behavior of the network can be approximated by a reduced-order system. This reduction in complexity allows for a simplified analysis of the network dynamics. Indeed, when the interconnection within the modules are sufficiently high,

it is possible to obtain a reduced model that represents the overall emerging dynamics [9], [10].

Grouping multiple agents in one module has been employed in diverse contexts, whether for undirected networks [10], [12], and [13], or directed ones [14], [15]. On the other hand, empirical observations have consistently revealed that, subject to certain topological conditions, agents within a module tend to converge toward local consensus among themselves rapidly. However, achieving global consensus among modules transpires at a relatively slower pace when compared to intra-modular convergence. This suggests to adopt a multi-time-scale approach for the analysis of modular networks.

In [16] singular-perturbations theory is used to model a heterogeneous network with high coupling gain. In [17] the authors study modular undirected networks with linear dynamics and emphasize the presence of three-time scales in the synchronization of interconnected agents in modular networks. Then, under some assumptions, the network dynamics can be approximated using a two-parameter singular-perturbation form. The mean-field dynamics evolve on the slowest time scale, the intra-modular dynamics on the fastest, and the inter-modular dynamics are faster than mean-field but slower than the intra-modular dynamics.

In this paper, which is the outgrowth of [18], we analyze modular networks using a triple-time-scale model and give mild sufficient conditions for global asymptotic stability (GAS) of the origin. Relative to [14] we consider nonlinear heterogeneous systems; relative to [1]–[4] we consider generic directed-graph networks. For the purpose of modeling, our results follow the framework explained in [6] and for analysis, we use singular-perturbations theory. Our main assumptions are that the individual systems are semi-passive [19] and that the average dynamics admit a GAS equilibrium, but if the latter does not hold, we describe a control method to stabilize the network. The method is conceptually reminiscent of pinning control [3], but such a strategy does not necessarily consider the multiple time-scale behavior of modular networks.

II. MODEL AND PROBLEM FORMULATION

Let us consider N nonlinear dynamical systems

$$\dot{x}_i = f_i(x_i) + \nu_i, \quad i \leq N, \quad x_i \in \mathbb{R}^n, \quad (1)$$

where the functions f_i are continuously differentiable and ν_i are inputs defined as

$$\nu_i := - \sum_{j=1}^N \sigma_{ij} a_{ij} (x_i - x_j), \quad (2)$$

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where a_{ij} represents the (in)existence of a link between two agents and σ_{ij} represents the coupling strength. More precisely, $a_{ij} > 0$ and $\sigma_{ij} > 0$ if there is an interconnection arc from the j th node to the i th one and $a_{ij} = \sigma_{ij} = 0$ otherwise. In the case that all the interconnection strengths σ_{ij} are equal, say $\sigma_{ij} =: \sigma$, the networked system, in compact form, reads¹

$$\dot{x} = \mathcal{F}(x) - \sigma[\mathcal{L} \otimes I_n]x, \quad (3)$$

where $x := [x_1^\top \cdots x_N^\top]^\top$, $\mathcal{F}(x) := [f_1(x_1)^\top \cdots f_N(x_N)^\top]^\top$, and \mathcal{L} is a Laplacian matrix, whose elements are defined as

$$\ell_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^N a_{i\ell}, & i = j, \quad i, j \leq N. \end{cases}$$

We consider the case in which certain links have an interconnection strength $\sigma_{ij} := \sigma^I$ and for others $\sigma_{ij} := \sigma^E$, where $\sigma^I \gg \sigma^E \geq \sigma^*$, where $\sigma^* > 0$ is a certain threshold. In addition, it is assumed that there are many more arcs with strength σ^I than with strength σ^E . By virtue of this, the networked systems are naturally organized into densely interconnected modules that are sparsely interconnected among themselves.

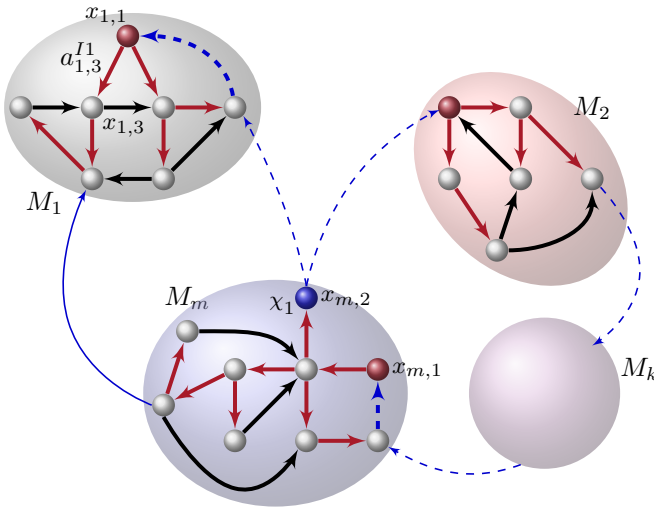


Fig. 1. Schematic representation of a directed modular network composed of m modules M_k , each constituting a sub-network. Within each module the interconnections are “strong” and dense (represented by thick arrows) while among modules the interconnections are of lesser strength and more sparse (represented by thinner arrows). Each module contains a directed spanning tree—represented by red arrows within the modules and with a root node also in red. The overall network also contains a directed spanning tree—represented by dashed blue and solid red arrows and with an overall root node represented in blue. This node is first in the overall network so it is originally labeled x_1 —see Eq. 1, but it does not necessarily correspond to the root of a module’s spanning tree. Indeed, it corresponds to the 2nd node in the m th module, so its state is $x_{m,1}$ in Eq. (4).

We are primarily interested in investigating sufficient conditions to guarantee global asymptotic stability of the origin for modular networks, but also in devising control design

¹In this paper we focus on the role of the coupling gain σ , so for simplicity we only consider state coupling. For networks under output coupling, e.g., with respect to the outputs $y_i = Hx_i$ the second term in (3) would read $\sigma[\mathcal{L} \otimes HH^\top]x$, but such systems are beyond the scope of this paper.

strategies to stabilize the origin. To those ends, we start by introducing a more suitable model that consists in a (non-unique) modular network decomposition. Such model is defined by relabeling the network’s nodes and the systems’ states and introducing an adequate notation.

We consider that under the assumptions made above on the coupling strength, m modules are formed, labeled M_k with $k \leq m$, and each containing N_k systems densely connected with high coupling strength. That is, each of such modules forms a sub-network of N_k nonlinear systems, whose models (1) are rewritten using a notation that makes explicit the module dependency. Let

$$\dot{x}_{k,i} = f_{k,i}(x_{k,i}) + u_{k,i}, \quad k \leq m, \quad i \leq N_k, \quad (4)$$

where $x_{k,i} \in \mathbb{R}^n$ denotes the state of the i th system within the k th module, and $f_{k,i}$ and $u_{k,i}$ define its dynamics and input respectively. Then, to distinguish the interconnection links depending on the coupling strength, the inputs are split in two parts as follows:

$$u_{k,i} = u_{k,i}^I + u_{k,i}^E, \quad (5a)$$

$$u_{k,i}^I = -\sigma^I \sum_{j=1}^{N_k} a_{ij}^{Ik} (x_{k,i} - x_{k,j}), \quad (5b)$$

$$u_{k,i}^E = -\sigma^E \sum_{\substack{\ell=1 \\ \ell \neq k}}^m \sum_{j=1}^{N_\ell} a_{ij}^{E\ell} (x_{k,i} - x_{\ell,j}). \quad (5c)$$

In the previous expressions, a_{ij}^{Ik} and $a_{ij}^{E\ell}$ are positive- or zero-valued, depending on the existence or absence of a link between two nodes within one module or between two modules respectively—see Figure 1 for an illustration.

Then, in compact form, we define the vectors of rearranged nodes’ states and dynamics, as follows:

$$x := \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{1,N_1} \\ \vdots \\ x_{m,1} \\ \vdots \\ x_{m,N_m} \end{bmatrix} \in \mathbb{R}^{Nn}, \quad F(x) := \begin{bmatrix} f_{1,1}(x_{1,1}) \\ \vdots \\ f_{1,N_1}(x_{1,N_1}) \\ \vdots \\ f_{m,1}(x_{m,1}) \\ \vdots \\ f_{m,N_m}(x_{m,N_m}) \end{bmatrix} \quad (6)$$

These vectors contain all the states x_i and respective dynamics f_i in (1), but not necessarily in the original order $[x_1^\top \cdots x_N^\top]^\top$. Note that $x = Tx$ where T is a permutation matrix (hence invertible). With this notation, the closed-loop networked system (4)–(5), takes the form,

$$\dot{x} = F(x) - \sigma^I[L^I \otimes I_n]x - \sigma^E[L^E \otimes I_n]x \quad (7)$$

where L^I and L^E are the Laplacian matrices

$$L^I := \text{blockdiag}[L_1^I \ L_2^I \ \cdots \ L_m^I], \quad (8a)$$

$$L_k^I := [\ell_{i,j}^{Ik}] \quad (8b)$$

$$\ell_{i,j}^{Ik} := \begin{cases} -a_{ij}^{Ik}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^{N_k} a_{i\ell}^{Ik}, & i = j \end{cases}, \quad (8c)$$

where, for each fixed $k \leq m$, $a_{ij}^{Ik} > 0$ if there is an interconnection arc from the j th node to the i th one within the k th module and $a_{ij}^{Ik} = 0$ otherwise; $L^E \in \mathbb{R}^{N \times N}$, which corresponds to the intra-modular Laplacian, is defined as $L^E = L - L^I$, where $L = T\mathcal{L}T^{-1}$. Both matrices L^I and L^E are Laplacians. Moreover, under the following assumption, L_k^I has exactly one eigenvalue with null real part.

Assumption 1 (topology): Each module M_k individually forms a strongly-connected sub-network; its topology is captured by the Laplacian L_k^I with elements defined in (8c). Furthermore, both the overall network and the network of modules, contain a spanning tree. In addition, the interconnection strengths satisfy $\sigma^E |L^E| < \sigma^I |L^I|$, where $|\cdot|$ denotes the induced L_2 norm. \square

Remark 1: There is little loss of generality in assuming that each module M_k is strongly connected because in the primary setting the modules are densely connected. The last part of Assumption 1 means that, for each agent, the intra-modular influence is higher than the inter-modular influence. In other words, either the interconnections within modules are denser than the interconnections among them or the interaction weights inside modules is higher than among them.

The rest of the paper is devoted to the stability analysis of, and control design for, Equation (7) under Assumption 1. To that end, we also pose the following hypothesis on the individual systems' dynamics.

Assumption 2 (regularity and passivity): For each pair (k, i) , the function $f_{k,i}$ in (4) is continuously differentiable and admits the origin as a unique equilibrium. In addition, all the units (1) are semi-passive [19] with respect to the input $u_{k,i}$ and the output $x_{k,i}$, with continuously differentiable and radially unbounded storage functions $V_{k,i} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$. That is, there exist positive definite and radially unbounded storage functions $V_{k,i}$, positive constants $\rho_{k,i}$, continuous functions $H_{k,i}$, and non-negative continuous functions $\psi_{k,i}$ such that

$$\dot{V}_{k,i}(x_{k,i}) \leq x_{k,i} u_{k,i} - H_{k,i}(x_{k,i})$$

and $H_{k,i}(x_{k,i}) \geq \psi_{k,i}(|x_{k,i}|)$ for all $|x_{k,i}| \geq \rho_{k,i}$. \square

In Assumption 2, the requirement that the origin is a common equilibrium point is imposed by the problem setting, that of establishing global asymptotic stability. Such assumption is not *a priori* needed in the context of analysis of synchronization—see *e.g.*, [2]–[4]. The hypothesis of semi-passivity, which is satisfied by many physical systems, such as neuronal networks [20], is useful to establish that the solutions are bounded, which is necessary for global asymptotic stability, but not for synchronization. Boundedness of solutions may also be established by imposing a stronger condition reminiscent of uniform global stability on the individual systems [21].

The following statement generalizes [6, Proposition 2] by establishing boundedness of the network's trajectories, for arbitrary directed networks of heterogeneous semi-passive

systems, containing a spanning tree. The proof is provided in [22].

Lemma 1: Consider a network of N interconnected dynamical systems as in (1)-(2), with f_i continuously differentiable, admitting the origin as the unique equilibrium, and such that each map $\nu_i \mapsto \chi_i$ is semi-passive. Assume, in addition, that the network's graph contains a directed spanning tree. Then, the trajectories $t \mapsto \chi_i(t)$, for all $i \leq N$, solutions to (1)-(2), are globally bounded. \square

The statement of Lemma 1 follows from the observation that by reordering the network's states, in similar fashion as done in [23] for weakly connected graphs, the Laplacian matrix L may be transformed into that of a connected network that consists in a spanning-tree of strongly-connected sub-graphs, so the transformed Laplacian matrix possesses a convenient lower-block-triangular form (see [22, Lemma 2]). Then, the statement follows using a cascades argument, from the fact that the trajectories of each strongly-connected sub-graph are bounded (see the proof of [6, Proposition 2]) and remain bounded under the effect of the interconnections (see [22, Lemma 3]).

Assumptions 1 and 2 are also instrumental to cast the analysis of (7) within the framework established in [6], which builds on the recognition that the networked systems' collective behavior is dichotomous. It consists in two distinct dynamical components that evolve in orthogonal spaces, that of an emerging average system with state χ_s and that of the synchronization errors e_i , defined as the difference between the dynamics of each individual system and the average dynamics, *i.e.*, $e_i := \chi_i - \chi_s$. More precisely, χ_s is a weighted average of χ_i s, defined via the left-eigenvector corresponding to the unique null eigenvalue of \mathcal{L} . In particular, if the synchronization manifold $\{i \leq N : e_i = 0\}$ is asymptotically stable the origin for the network system (7) is asymptotically stable if and only if so is $\{\chi_s = 0\}$.

In [24] it is recognized that global asymptotic stability of the origin for (3) is possible for sufficiently large values of σ . The analysis in this reference is based on the fact that the average dynamics $\dot{\chi}_s = f_s(\chi_s)$ evolves in scaled time t/σ , that is, much slower than the synchronization dynamics. For modular networks (7) the analysis starts with recognizing that the presence of two different coupling strengths entails *two* average dynamical systems, in addition to the synchronization dynamics of the individual systems. The precise modeling of these three dynamical systems is the subject of the next section.

III. THREE-TIME-SCALES MODELING

We derive an equivalent representation of the modular-network dynamics (7), under Assumptions 1 and 2, that is suitable for a singular-perturbations-based analysis (see Section IV). More precisely, we show that the following statement holds. bk

Proposition 1: Consider the system (7) under Assumptions 1 and 2. Then, there exists a globally invertible transformation $x \mapsto \mathcal{T}(x)$ such that $X := \mathcal{T}(x)$ may be split as $X := [x_e^\top \zeta^\top \xi^\top]^\top$, with $x_e \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{(m-1)n}$, and $\xi \in$

$\mathbb{R}^{n(N_k-1)m}$, and the dynamics of X has the form

$$\dot{x}_e = f_e(x_e, \eta, \xi) \quad (9a)$$

$$\varepsilon_E \dot{\eta} = -\bar{\Lambda}^E \eta + \varepsilon_E f_\eta(x_e, \eta, \xi) \quad (9b)$$

$$\varepsilon_I \dot{\xi} = -\frac{1}{\lambda_m(\bar{\Lambda}^I)} \bar{\Lambda}^I \xi + \varepsilon_I f_\xi(x_e, \eta, \xi), \quad (9c)$$

where f_e , f_η and f_ξ are continuously differentiable and admit the origin $\{X = 0\}$ as unique equilibrium, $\bar{\Lambda}_E \in \mathbb{R}^{(m-1)n \times (m-1)n}$ and $\bar{\Lambda}_I \in \mathbb{R}^{(N-m)n \times (N-m)n}$ are block-diagonal positive-definite matrices, $\varepsilon_E := 1/\sigma^E$ and $\varepsilon_I := \frac{1}{\sigma^I \lambda_m(\bar{\Lambda}^I)}$. Thus, the dynamical system (7) admits the equivalent representation (9) and the origin $\{X = 0\}$ for (9) is globally asymptotically stable if and only if so is the origin $\{x = 0\}$ for (7). \square

The importance of Proposition 1, whose proof is presented in Subsection C farther below, resides in the second statement, which follows directly from the first, but is added for clarity since it is used in Section IV to establish global asymptotic stability for (3). The proof of the existence of \mathcal{T} is constructive and follows from the contents of the following two subsections in which we exhibit the existence of the three dynamical systems in (9), evolving in three different time-scales. The fastest one corresponds to that in which the synchronization errors within each module evolve; as we show in Subsection III-A such errors are defined as a projection of the state ξ in (9c) and are defined relative to a module's average state. We refer to the latter average dynamics and that of ξ as the intra-modular dynamics. The second time scale, which is moderately fast, corresponds to that of the synchronization errors among the modules. Technically, these errors are a projection of the state η in (9b). Then, a third and slowest time scale corresponds to that in which the average of the modules' averages, represented by x_e , evolves. The dynamics evolving in the second and third time-scales constitute what we call inter-modular dynamics.

A. Intra-modular dynamics

Let k be arbitrarily fixed and let us focus our attention on the k th module. It consists in a connected (sub)network with Laplacian L_k^I , N_k nodes, and contains a spanning tree (Assumption 1). Therefore, L_k^I has a unique zero eigenvalue and $N_k - 1$ others with positive real part. The null eigenvalue has as an associated left eigenvector $v_{lk} := \frac{1}{N_k} \mathbf{1}_{N_k}$, with $\mathbf{1}_{N_k} := [1 \ 1 \ \dots \ 1]^\top$ and the other $N_k - 1$ eigenvalues admit associated linearly independent eigenvectors that are gathered in the matrix $Q_k \in \mathbb{R}^{N_k \times (N_k-1)}$. Therefore, L_k^I admits the Jordan decomposition

$$L_k^I = V_k \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_k^I \end{bmatrix} V_k^{-1} \quad (10)$$

where $\Lambda_k^I \in \mathbb{R}^{(N_k-1) \times (N_k-1)}$ is positive definite and $V_k \in \mathbb{R}^{N_k \times N_k}$ is the invertible transformation matrix $V_k := [\mathbf{1}_{N_k} \ Q_k]$. Therefore,

$$V_k^{-1} =: \begin{bmatrix} v_{lk}^\top \\ Q_k^\dagger \end{bmatrix}. \quad (11)$$

Then, akin to [24], we introduce a set of new state variables, $\zeta_k \in \mathbb{R}^n$ and $\xi_k \in \mathbb{R}^{n(N_k-1)}$, defined as

$$\begin{bmatrix} \zeta_k \\ \xi_k \end{bmatrix} =: \begin{bmatrix} [v_{lk}^\top \otimes I_n] \\ [Q_k^\dagger \otimes I_n] \end{bmatrix} \bar{x}_k,$$

where $\bar{x}_k := [x_{k,1}^\top \ x_{k,2}^\top \ \dots \ x_{k,N_k}^\top]^\top$.

The state variable ζ_k may be regarded as the weighted average of the k th module's nodes' states while, ξ_k is a projection of the synchronization errors, e_k , relative to that average. More precisely, we define $e_k \in \mathbb{R}^{nN_k}$ as

$$e_k := \bar{x}_k - [\mathbf{1}_{N_k} \otimes I_n] \zeta_k \quad (12)$$

$$= [Q_k \otimes I_n] \xi_k. \quad (13)$$

Note that all the systems within the module synchronize (that is, $x_{k,i} \rightarrow x_{k,j}$) if and only if $e_k \rightarrow 0$ or, equivalently, if and only if $\xi_k \rightarrow 0$. If, in addition, for the average system $\dot{\zeta}_k = [v_{lk}^\top \otimes I_n] \bar{x}_k$ we have $\zeta_k \rightarrow 0$, we conclude that $x_{k,i} \rightarrow 0$. The same reasoning applies to the stronger property of global asymptotic stability. Therefore, for our problem of interest, it is crucial to study the dynamic evolution of ζ_k and ξ_k . To that end, we gather these variables into the vectors $\zeta \in \mathbb{R}^{nm}$ and $\xi \in \mathbb{R}^{n(N_k-1)m}$, defined as $\zeta := [\zeta_1 \ \dots \ \zeta_m]$ and $\xi := [\xi_1^\top \ \dots \ \xi_m^\top]^\top$. The latter satisfy

$$\begin{bmatrix} \zeta \\ \xi \end{bmatrix} =: \begin{bmatrix} P^\dagger \otimes I_n \\ Q^\dagger \otimes I_n \end{bmatrix} x, \quad (14)$$

where x is defined in (6), $P \in \mathbb{R}^{N \times m}$ is defined as

$$P = \text{blockdiag}_{k \leq m} \{ \mathbf{1}_{N_k} \} \quad (15)$$

—cf. [25], $Q = \text{blockdiag}_{k \leq m} \{ Q_k \}$, $P = \text{blockdiag}_{k \leq m} \{ v_{lk} \}$ and $Q^\dagger = \text{blockdiag}_{k \leq m} \{ Q_k^\dagger \}$. Note that, since $\mathbf{1}_{N_k}$ and the columns of Q_k form the orthogonal transformation V_k in (10), we have $\mathbf{1}_{N_k}^\top Q_k = 0$, so $P^\dagger Q = 0$ and $Q^\dagger P = 0$, i.e., these are zero-matrices of dimensions $m \times (N_k - 1)m$ and $(N_k - 1)m \times m$, respectively. In turn, after (14) it follows that

$$x = \bar{P} \zeta + \bar{Q} \xi, \quad (16)$$

where we defined $\bar{P} := [P \otimes I_n]$ and $\bar{Q} := [Q \otimes I_n]$. Now, differentiating on both sides of (14) and using (7) and (16), we obtain

$$\begin{aligned} \dot{\zeta} &= \bar{P}^\dagger F(\bar{P} \zeta + \bar{Q} \xi) - \sigma^I \bar{P}^\dagger \bar{L}^I [\bar{P} \zeta + \bar{Q} \xi] \\ &\quad - \sigma^E \bar{P}^\dagger \bar{L}^E [\bar{P} \zeta + \bar{Q} \xi], \end{aligned} \quad (17)$$

$$\begin{aligned} \dot{\xi} &= \bar{Q}^\dagger F(\bar{P} \zeta + \bar{Q} \xi) - \sigma^I \bar{Q}^\dagger \bar{L}^I [\bar{P} \zeta + \bar{Q} \xi] \\ &\quad - \sigma^E \bar{Q}^\dagger \bar{L}^E [\bar{P} \zeta + \bar{Q} \xi], \end{aligned} \quad (18)$$

where $\bar{L}^I := [L^I \otimes I_n]$, and $\bar{L}^E := [L^E \otimes I_n]$. However, note that $L_k^I \mathbf{1}_{N_k} = 0$, so $L^I P = P^\dagger L^I = 0$, while $\bar{Q}^\dagger \bar{L}^I \bar{Q} \xi = \bar{\Lambda}^I \xi$. Thus, (17) and (18) become

$$\dot{\zeta} = F_\zeta(\zeta, \xi) \quad (19a)$$

$$\dot{\xi} = -\sigma^I \bar{\Lambda}^I \xi + F_\xi(\zeta, \xi), \quad (19b)$$

where $\bar{\Lambda}^I := [\Lambda^I \otimes I_n]$ and $\Lambda^I \in \mathbb{R}^{(N-m) \times (N-m)}$ is defined as $\Lambda^I := \text{blockdiag}_{k \leq m} \{ \Lambda_k^I \}$, and

$$F_\zeta(\zeta, \xi) = \bar{P}^\dagger F(\bar{P} \zeta + \bar{Q} \xi) - \sigma^E \bar{P}^\dagger \bar{L}^E [\bar{P} \zeta + \bar{Q} \xi], \quad (20)$$

$$F_\xi(\zeta, \xi) = \bar{Q}^\dagger F(\bar{P}\zeta + \bar{Q}\xi) - \sigma^E \bar{Q}^\dagger \bar{L}^E [\bar{P}\zeta + \bar{Q}\xi]. \quad (21)$$

Equation (19a) corresponds to the dynamics of a network of modules, *i.e.*, a network of sub-networks. On the synchronization manifold $\{\xi = 0\}$ each module may be assimilated to a single node with dynamics² $\dot{\zeta}_k = F_{\zeta_k}(\zeta, 0)$, but note that the dynamics of each module depend on that of other modules as well, so Eq. (19a) constitutes a reduced network of m nodes—*cf.* [26], [27], [28]. This is the inter-modular dynamics, which we study in detail next.

B. Inter-modular dynamics

To assess the behavior of the reduced network (19a) we start by applying on ζ a coordinate transformation similar to that defined above and performed on x . To that end, we observe that under Assumption 1 the network (19a) is also connected and contains a directed spanning tree. Furthermore, its associated Laplacian, which appears in the definition of $F_\zeta(\zeta, \xi)$ above, is given by

$$\tilde{L}^E := \begin{bmatrix} \bar{a}_{11}^E & -\bar{a}_{12}^E & \cdots & -\bar{a}_{1m}^E \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -\bar{a}_{m1}^E & -\bar{a}_{m2}^E & \cdots & \bar{a}_{mm}^E \end{bmatrix}, \quad (22)$$

where, for any two modules indexed i and j , $\bar{a}_{ij}^E := v_{\ell_i}^\top A_{ij} \mathbf{1}_{n_j}$ and A_{ij} is the $N_i \times N_j$ block of L^E gathering the edges from the i th to the j th module.

Thus, since there exists a directed spanning tree in the modules' interconnection graph, \tilde{L}^E admits the Jordan decomposition

$$\tilde{L}^E = W \begin{bmatrix} 0 & 0 \\ 0 & \Lambda^E \end{bmatrix} W^{-1}, \quad (23)$$

where $W \in \mathbb{R}^{m \times m}$ is nonsingular, and $\Lambda^E \in \mathbb{R}^{(m-1) \times (m-1)}$ is the diagonal matrix defined by the eigenvalues of \tilde{L}^E with positive real part, that is, $\Lambda^E := \text{diag}_{i \in \{2,3,\dots,m\}} \{\lambda_i(\tilde{L}^E)\}$. As a matter of fact,

$$W = [\mathbf{1}_m \ W_1], \quad (24)$$

where W_1 is full-column-rank, so it is left-invertible and W has full rank. Therefore,

$$W^{-1} =: \begin{bmatrix} w_\ell^\top \\ W_1^\dagger \end{bmatrix}. \quad (25)$$

Next, we introduce the new state variables

$$\begin{bmatrix} x_e \\ \eta \end{bmatrix} = \begin{bmatrix} w_\ell^\top \otimes I_n \\ W_1^\dagger \otimes I_n \end{bmatrix} \zeta, \quad (26)$$

where $x_e \in \mathbb{R}^n$ corresponds to the “weighted average of averages ζ_k ”, while η is a projection of all the synchronization errors among modules, $\zeta_k - x_e$. That is, the vector of synchronization errors corresponds to

$$e_\eta := \zeta - [\mathbf{1}_m \otimes I_n] x_e \quad (27a)$$

²In [2] synchronization is defined as the property of individual systems adopting a common trajectory $s(t)$ which, in the present context, is solution to $\dot{\zeta}_k = F_{\zeta_k}(\zeta, 0)$, but $s(t)$ is left undefined and is assumed to *exist*.

$$= [W_1 \otimes I_n] \eta. \quad (27b)$$

All the states of the reduced network (of modules) reach consensus with the x_e -system if and only if $\eta = 0$. Thus, the collective behavior of the reduced-network (19a) may be fully assessed by studying the equivalent dynamical system that results from differentiating on both sides of (26). That is,

$$\dot{x}_e = f_e(x_e, \eta, \xi) \quad (28a)$$

$$\dot{\eta} = -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi), \quad (28b)$$

where

$$f_e(x_e, \eta, \xi) = [w_\ell^\top \otimes I_n] F_\zeta(\zeta, \xi), \quad (29)$$

$$f_\eta(x_e, \eta, \xi) = \bar{W}_1^\dagger \bar{P}^\dagger F_\zeta(\bar{P}\zeta + \bar{Q}\xi) - \sigma^E \bar{W}_1^\dagger \bar{P}^\dagger \bar{L}^E \bar{Q}\xi \quad (30)$$

$$\zeta = [\mathbf{1}_m \otimes I_n] x_e + [W_1 \otimes I_n] \eta, \quad (31)$$

and $\bar{\Lambda}^E := [\Lambda^E \otimes I_n]$. Equation (28a) follows by direct differentiation of $x_e = [w_\ell^\top \otimes I_n] \zeta$, using (26) and (19a). The expression (31) follows from (27). Equation (28b), together with (30), is obtained as follows. After (26), (19a), and (20), we have

$$\dot{\eta} = \bar{W}_1^\dagger \bar{P}^\dagger F(\bar{P}\zeta + \bar{Q}\xi) - \sigma^E \bar{W}_1^\dagger [\bar{P}^\dagger \bar{L}^E \bar{P}\zeta + \bar{P}^\dagger \bar{L}^E \bar{Q}\xi].$$

Now, by definition—see (22)— $\bar{P}^\dagger \bar{L}^E \bar{P} = [\tilde{L}^E \otimes I_n]$. Then, after (31) and the fact that $\tilde{L}^E \mathbf{1}_m = 0$, we have $\bar{W}_1^\dagger \bar{P}^\dagger \tilde{L}^E \bar{P}\zeta = \bar{W}_1^\dagger [\tilde{L}^E \otimes I_n] \bar{W}_1 \eta$. Then, after (23) and (24), we obtain $\bar{W}_1^\dagger [\tilde{L}^E \otimes I_n] \bar{W}_1 = \bar{\Lambda}^E$.

C. Proof of Proposition 1

From the previous derivations we draw existence of two positive-definite matrices $\bar{\Lambda}^I$ and $\bar{\Lambda}^E$ as required—see above (20) and below (31), as well as below Eq. (23). Therefore, ε_E and ε_I in the Proposition are positive. Furthermore, it is clear that Eqs. (28) are equivalent to Eqs. (9a) and (9b) while Eq. (9c), with

$$f_\xi(x_e, \eta, \xi) = F_\xi \left([W \otimes I_n] \begin{bmatrix} x_e \\ \eta \end{bmatrix}, \xi \right), \quad (32)$$

is equivalent to Eq. (19b). In addition, after (20), (21), (29), (30), (32), and Assumption 2, it also follows that f_e , f_η , and f_ξ are all continuously differentiable and admit the origin as unique equilibrium point. It is left to show that after the previous developments we have $\mathcal{T}(x) := \mathcal{T}x$, where $\mathcal{T} \in \mathbb{R}^{mN_k n}$ is a constant invertible matrix.

To that end, we observe that, on one hand, after (14), we have $\xi = [\bar{Q}^\dagger \otimes I_n] x$ and $\zeta = [P^\dagger \otimes I_n] x$, so from (26) it follows that $x_e = [w_\ell^\top \otimes I_n] [P^\dagger \otimes I_n] x$ and $\eta = [W_1^\dagger \otimes I_n] [P^\dagger \otimes I_n] x$. That is,

$$\begin{bmatrix} x_e \\ \eta \\ \xi \end{bmatrix} = \begin{bmatrix} [w_\ell^\top \otimes I_n] [P^\dagger \otimes I_n] \\ [W_1^\dagger \otimes I_n] [P^\dagger \otimes I_n] \\ Q^\dagger \otimes I_n \end{bmatrix} x =: \mathcal{T}x. \quad (33)$$

On the other hand, after (27) we have $\zeta = [\mathbf{1}_m \otimes I_n] x_e + [W_1 \otimes I_n] \eta$, so using the latter in (16), we obtain that

$$x = \underbrace{\begin{bmatrix} [P \mathbf{1}_m \otimes I_n] & [P W_1 \otimes I_n] & [Q \otimes I_n] \end{bmatrix}}_{=: \mathcal{T}^{-1}} \begin{bmatrix} x_e \\ \eta \\ \xi \end{bmatrix} \quad (34)$$

Next, to verify that the product of the matrices in (33) and (34) satisfy $\mathcal{T}\mathcal{T}^{-1} = I_{N_k m n}$, we observe that

$$\mathcal{T}\mathcal{T}^{-1} = \begin{bmatrix} w_\ell^\top P^\dagger P \mathbf{1}_m & w_\ell^\top P^\dagger P W_1 & w_\ell^\top P^\dagger Q \\ W_1^\dagger P^\dagger P \mathbf{1}_m & W_1^\dagger P^\dagger P W_1 & W_1^\dagger P^\dagger Q \\ Q^\dagger P \mathbf{1}_m & Q^\dagger P W_1 & Q^\dagger Q \end{bmatrix} \otimes I_n$$

and the 3×3 block-matrix on the right-hand side of the expression above corresponds to $I_{N_k m}$. The latter follows from the fact that $W W^{-1} = W^{-1} W = P^\dagger P = I_m$ and $Q^\dagger Q = I_{(N_k - 1)m}$, so, after Eqs. (24) and (25), we have $w_\ell^\top P^\dagger P \mathbf{1}_m = 1$, $W_1^\dagger P^\dagger P W_1 = I_{m-1}$, $w_\ell^\top P^\dagger P W_1 = 0$ and $W_1^\dagger P^\dagger P \mathbf{1}_m = 0$. All the other off-diagonal blocks also equal to zero-matrices of appropriate dimensions since $Q^\dagger P = 0$ and $P^\dagger Q = 0$.

IV. NETWORK STABILITY ANALYSIS

In Section II we remarked that the networked system under study, given by (3), may be rewritten in the form (7) in which the different coupling strengths are made explicit. This is done by a simple permutation of the nodes' states. In turn, Proposition 1 establishes a less straightforward equivalent representation of (7) and, therefore, of the network (3), in a form that exhibits three time scales and is suitable for analysis using singular perturbations theory. This is given by Eqs. (9).

In this section we present our main statement on stability for the network (3). For this, we rely on the analysis of (9), via the following statement that establishes asymptotic stability of the origin for all initial conditions laying in arbitrary compacts of the state. This is a corollary of, but not a statement equivalent to, [29, Theorem 11.3], that we include for ease of reference. The proof is straightforward, so it is omitted.

Lemma 2 (corollary of Theorem 11.3 in [29]): Consider the nonlinear autonomous singularly-perturbed system,

$$\dot{x} = f(x, z) \quad (35a)$$

$$\varepsilon \dot{z} = Az + \varepsilon g(x, z), \quad (35b)$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$ and $A \in \mathbb{R}^{n_z \times n_z}$ Hurwitz. Assume that the equilibrium $(x, z) = (0, 0)$ is an isolated equilibrium point and, for any $R > 0$, f and g are Lipschitz for all $(x, z) \in B_R$, with $B_R := \{(x, z) \in \mathbb{R}^{n_x \times n_z} : |x|^2 + |z|^2 < R^2\}$, with a Lipschitz constant $L(R)$. In addition, assume that for each $R > 0$ there exist positive definite decrescent functions V and $W : B_R \rightarrow \mathbb{R}_{\geq 0}$, positive constants $\alpha_1, \alpha_2, \beta$, as well as positive-definite functions $\phi_1 : B_R \rightarrow \mathbb{R}$ and $\phi_2 : B_R \rightarrow \mathbb{R}$, given by $\phi_1(x) = |x|$, $\phi_2(z) = |z|$, such that, for all $(x, z) \in B_R$,

$$\frac{\partial V}{\partial x} f(x, 0) \leq -\alpha_1 \phi_1(x)^2, \quad (36)$$

$$\frac{\partial W}{\partial z} Az \leq -\alpha_2 \phi_2(z)^2, \quad (37)$$

$$\frac{\partial V}{\partial x} [f(x, z) - f(x, 0)] \leq \beta \phi_1(x) \phi_2(z). \quad (38)$$

Let $L_1(R) > 0$ and $L_2(R) > 0$ be Lipschitz constants satisfying, for all $x \in B_R$ and $z \in B_R$,

$$|g(x, z) - g(0, z)| \leq L_1(R)|x|, \quad (39a)$$

$$|g(0, z)| \leq L_2(R)|z|. \quad (39b)$$

Then, for all $\varepsilon < \varepsilon^* := \frac{\alpha_1 \alpha_2}{\alpha_1 L_2(R) + \beta L_1(R)} > 0$, the origin of (35) is asymptotically stable and attractive to all trajectories that are contained in B_R . \square

We are ready to present our main statement.

Theorem 1: (GAS of the networked system) Consider the networked system (3) in its first equivalent representation (7) and let Assumptions 1 and 2 hold. Consider, in addition, the second equivalent singular-perturbation form (9), established in Proposition 1, and assume that for any $R > 0$, there exists a positive-definite decrescent, once continuously differentiable, function $V_e : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and positive constants q_1 and c_1 , such that

$$\frac{\partial V_e}{\partial x_e} f_e(x_e, 0, 0) \leq -q_1 |x_e|^2 \quad (40)$$

$$\left| \frac{\partial V_e}{\partial x_e} \right| \leq c_1 |x_e|, \quad (41)$$

for all $x \in B_R$. Then, there exist $\sigma^{E^*} > 0$ and $\sigma^{I^*} > 0$ such that, for all $\sigma^I > \sigma^{I^*}$ and $\sigma^E > \sigma^{E^*}$, the origin for (7), and equivalently for (3), is globally asymptotically stable. \square

Proof: By Lemma 1 the solutions of the closed-loop system (1)-(2), given by Eq. (3), or equivalently those of (7), are globally bounded. That is, for any $r > 0$ there exist $R > 0$ such that

$$|x_0| \leq r \implies |x(t)| < R, \quad \forall t \geq 0. \quad (42)$$

In turn, in view of (14) and (26), $x_e(t)$, $\eta(t)$, $\xi(t)$ are also globally bounded. For simplicity, subject to a possible redefinition of R , we shall say that $|x_e(t)| < R$, $|\eta(t)| < R$, and $|\xi(t)| < R$ for all $t \geq 0$.

Then, the rest of the proof consists in applying Lemma 2 twice consecutively. One first time to show global asymptotic stability of the origin for the inter-modular dynamics on the synchronization manifold,

$$\dot{x}_e = f_e(x_e, \eta, 0) \quad (43a)$$

$$\dot{\eta} = -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, 0), \quad (43b)$$

where f_e and f_η are defined in (29) and (30) respectively, and a second time for the intra-modular dynamics

$$\dot{y} = f_y(y, \xi), \quad (44a)$$

$$\dot{\xi} = -\sigma^I \bar{\Lambda}^I \xi + g_\xi(y, \xi), \quad (44b)$$

where $y = [x_e^\top \ \eta^\top]^\top$,

$$g_\xi(y, \xi) = F_\xi([W^\dagger \otimes I_n]y, \xi),$$

$$f_y(y, \xi) := \begin{bmatrix} f_e(x_e, \eta, \xi) \\ -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi) \end{bmatrix}.$$

1) Stability of the inter-modular dynamics: We analyze the system (43). To that end, we apply Lemma 2 with $x := x_e$, $z := \eta$, $\varepsilon := 1/\sigma^E$, $f(x, z) := f_e(x_e, \eta, 0)$, $g(x, z) := f_\eta(x_e, \eta, 0)$, and $A := -\bar{\Lambda}^E$. Next, consider (42). To verify (36) we pose $V(x) := V_e(x_e)$ so, after (40), (36) holds for all $x_e \in B_R$, with $\phi_1(x) := \sqrt{q_1}|x_e|$. Next, under Assumption 1, $-\bar{\Lambda}^E$ is Hurwitz, so there exists $P_\eta = P_\eta^\top > 0$, such that $\bar{\Lambda}^{E\top} P_\eta + P_\eta \bar{\Lambda}^E = 2I$. Therefore, (37) holds with $W(z) :=$

$\eta^\top P_\eta \eta$ and $\phi_2(z) := |\eta|$. Finally, condition (38) holds under Assumption 2 and after (41). Indeed, after (29) and (31)

$$f_e(x_e, \eta, 0) = [v_e^\top \otimes I_n] F_\xi \left(\bar{P} [1_m \otimes I_n] x_e + \bar{P} [W_1 \otimes I_n] \eta, 0 \right),$$

so invoking the Mean-value Theorem on each component of $F_\xi(\cdot)$ above, we see that there exists $c(R)$ such that the left-hand side of (38) satisfies

$$\frac{\partial V_e}{\partial x_e} [f_e(x_e, \eta, 0) - f_e(x_e, 0, 0)] \leq c_1 c |x_e| |\eta| \quad (45)$$

for all $(x_e, \eta) \in B_R$. Therefore, (38) holds with $\beta := c_1 c$, $\alpha_1 = \sqrt{q_1}$ and $\alpha_2 = 1$. We conclude that for sufficiently large σ^{E^*} and any $\sigma^E \geq \sigma^{E^*}$, the origin $(x_e, \eta) = (0, 0)$ is asymptotically stable and all solutions that satisfy $x_e, \eta(t) \in B_R$ converge to zero. Furthermore, since $\sigma^{E^*} = \frac{1}{\epsilon}$, $\sigma^{E^*} := \frac{\alpha_1 L_{\eta_2}(R) + \beta L_{\eta_1}(R)}{\alpha_1}$, given the continuous differentiability of f_η in B_R and the existence of $L_{\eta_1}(R)$ and $L_{\eta_2}(R)$ satisfying (39) for $g(x, z) = f_\eta(x_e, \eta, 0)$ with $f_\eta(0, 0, 0) = 0$. In other words, the origin is asymptotically stable for the inter-modular dynamics (the reduced-order network), and the origin is attractive on B_R , provided that the individual systems in each module M_k synchronize with their respective averages. This is established next.

2) Stability of the intra-modular dynamics: We turn now our attention to the overall system (28)-(44b), rewritten in the equivalent form (9), so we can apply Lemma 2; this time, with $x := [x_e^\top \ \eta^\top]^\top$, $z := \xi$, $g(x, z) := f_\xi(x_e, \eta, \xi)$ in (32), and

$$f(x, z) := \begin{bmatrix} f_e(x_e, \eta, \xi) \\ -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi) \end{bmatrix}.$$

Let $V(x) := V_e(x_e) + \frac{1}{2} \eta^\top P_\eta \eta$, where P_η is defined in the previous paragraph and V_e is defined in the statement of the Theorem. Then, the left-hand side of Inequality (36) reads

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, 0) &= \frac{\partial V_e}{\partial x_e} f_e(x_e, 0, 0) \\ &+ \frac{\partial V_e}{\partial x_e} [f_e(x_e, \eta, 0) - f_e(x_e, 0, 0)] \\ &- \frac{\sigma^E}{2} \eta^\top [\bar{\Lambda}^{E\top} P_\eta + P_\eta \bar{\Lambda}^E] \eta + \eta^\top P_\eta f_\eta(0, \eta, 0) \\ &+ \eta^\top P_\eta [f_\eta(x_e, \eta, 0) - f_\eta(0, \eta, 0)]. \end{aligned} \quad (46)$$

The first term on the right-hand side of (46) satisfies (40) by assumption; the second term satisfies (45); the third is bounded by $-\sigma^E |\eta|^2$. Then, under Assumption 2, $f_\eta(0, \cdot, 0)$ is smooth and $f_\eta(0, 0, 0) = 0$, so by the Mean-value theorem (applied component-wise) it follows that for any $R > 0$ and all $\eta \in B_R$, $|f_\eta(0, \eta, 0)| \leq c_3(R) |\eta|$. After similar arguments and using (30) and (31), we conclude that the last term on the right-hand side of (46) satisfies

$$\eta^\top P_\eta [f_\eta(x_e, \eta, 0) - f_\eta(0, \eta, 0)] \leq c_4 |P_\eta| |x_e| |\eta| \quad (47)$$

for all $(x_e, \eta) \in B_R$. Thus, putting all these bounds together, we obtain

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, 0) &\leq -q_1 |x_e|^2 + c_1 c |x_e| |\eta| - \sigma^E |\eta|^2 \\ &+ c_3 |P_\eta| |\eta|^2 + c_4 |P_\eta| |\eta| |x_e|, \end{aligned}$$

so there exists $\beta < 1$, such that (36) holds with $\phi_1(x) := \sqrt{\beta \min\{q_1, \sigma^E\}} |x_e^\top \eta^\top|$ and $\alpha_1 = \sqrt{\beta \min\{q_1, \sigma^E\}}$.

The second inequality, (37), follows trivially with $W(z) := \frac{1}{2} \xi^\top P_\xi \xi$ where $P_\xi = P_\xi^\top$ solves the Lyapunov equation $\bar{\Lambda}^{I\top} P_\xi + P_\xi \bar{\Lambda}^I = 2I$, which holds since $-\bar{\Lambda}^I$ is Hurwitz. More precisely, since $\sigma^I = \frac{1}{\lambda_m(\bar{\Lambda}^I) \epsilon_I}$ it follows that $\alpha_2 = \frac{1}{\lambda_m(\bar{\Lambda}^I)}$.

Finally, to see that Inequality (38) holds, we first observe that the left-hand side of (38) equals to

$$\begin{aligned} \frac{\partial V_e}{\partial x_e} [f_e(x_e, \eta, \xi) - f_e(x_e, \eta, 0)] \\ + \eta^\top P_\eta [f_\eta(x_e, \eta, \xi) - f_\eta(x_e, \eta, 0)], \end{aligned}$$

both of which, again by virtue of the differentiability of f_e and f_η , satisfy upper bounds that are linear in $|\xi|$ for all $x \in B_R$ and all $\xi \in B_R$. Thus, after (41), we have

$$\frac{\partial V}{\partial x} [f(x, z) - f(x, 0)] \leq c [c_1 |x_e| + |P_\eta| |\eta|] |\xi|,$$

so, since $\max\{|\eta|, |x_e|\} \leq |x|$, (38) follows with $\phi_1(x) := \alpha_1 |x|$, $\phi_2(z) := |\xi|$, and $\beta := c \max\{c_1, |P_\eta|\}$.

Finally, given the continuous differentiability of f_ξ in B_R , there exist $L_{\xi_1}(R)$ and $L_{\xi_2}(R)$ such that $g(x, z) = f_\xi(x_e, \eta, \xi)$ with $f_\xi(0, 0, 0) = 0$ satisfies (39). Thus, we set $\sigma^{I^*} := \frac{\alpha_1 L_{\xi_2}(R) + \beta L_{\xi_1}(R)}{\alpha_1 \alpha_2}$.

The statement of Theorem 1 follows. \blacksquare

V. APPLICATION TO NETWORK STABILIZATION

Theorem 1 guarantees global asymptotic stability of the origin, provided that Inequality (40) holds. In other words, if the origin for the average dynamics (9a) on the synchronization manifold $\{(\eta, \xi) = (0, 0)\}$,

$$\dot{x}_e = f_e(x_e, 0, 0), \quad (48)$$

is globally asymptotically stable. In this section we explore two control methods to stabilize the origin for (7) in the case that the average dynamics (48) is not GAS. The standing assumption is that each control $u_{k,i}$ in (5a) may be endowed with an additional input $v_{k,i}$. That is, we redefine

$$u_{k,i} = u_{k,i}^I + u_{k,i}^E + v_{k,i}, \quad (49)$$

so, in compact form, the network equation (7) becomes

$$\dot{x} = F(x) - \sigma^I [L^I \otimes I_n] x - \sigma^E [L^E \otimes I_n] x + v(x, x_c), \quad (50)$$

where the new input

$$v := [v_{1,1}^\top \cdots v_{1,N_1}^\top] \cdots [v_{m,1}^\top \cdots v_{m,N_m}^\top]^\top$$

depends on the network's state x and a distributed dynamics controller's state x_c to be defined.

We describe below two approaches that rely on modifying both the network's topology (and dimension) and the dynamics of (48) to render the origin GAS. We do this by adding control nodes to the network owing to two different strategies. The first consists in adding nodes to selected modules and the second consists in adding one or several whole control modules. The new nodes may be regarded as dynamic controllers that are added strategically using the coupling control inputs $v_{k,i}$.

These control approaches are explained in further detail below and illustrated in Figure 2.

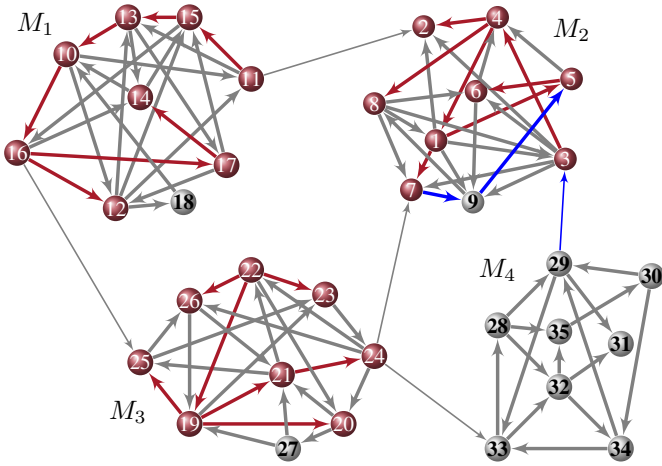


Fig. 2. Illustration of network control methods. Given a network of three modules M_1 , M_2 and M_3 containing eight nodes each, one control node is added to each module, and a whole control module, M_4 , is added to the network. Original nodes are in red, while added control nodes are in gray. Each module contains a directed spanning tree (emphasized in red) and, also, there is a spanning tree connecting M_1 to modules M_2 and M_3 . The two arcs colored in blue denote interconnections added by control design to ensure that the overall network is connected through a spanning tree.

A. Stabilization via control nodes added to modules

Let $k \leq M$ be arbitrarily fixed and let N'_k be a number of added nodes to the k th module. Then, the dynamics of each node *originally present* in the module, becomes

$$\dot{x}_{k,i} = f_{k,i}(x_{k,i}) + u_{k,i}^I + u_{k,i}^E + v_{k,i} \quad (51)$$

$$v_{k,i} = -\sigma^I \sum_{j=1}^{N'_k} a'_{k,i,j} (x_{k,i} - x_{k,N_k+j}), \quad (52)$$

where

$$a'_{k,i,j} := \begin{cases} 1 & \text{if there is an edge from } (k,i) \text{ to } (k,j) \\ 0 & \text{otherwise;} \end{cases} \quad (53)$$

that is, $v_{k,i} \neq 0$ only for existing nodes interconnected to new nodes. Furthermore, by convention, the states of the N'_k added control nodes are labeled $x_{k,N_k+j} \in \mathbb{R}^n$ and their individual dynamics are given by

$$\dot{x}_{k,N_k+j} = f_{k,N_k+j}(x_{k,N_k+j}) - \sigma^I \sum_{i=1}^{N_k} a'_{k,j,i} (x_{k,N_k+j} - x_{k,i}). \quad (54)$$

Eq. (54) represents the dynamics of the control nodes, *i.e.*, x_c in (50) is defined by all the applicable states x_{k,N_k+j} . As a result, the network (50) is transformed as follows. In compact form, the state vector, augmented by the states of the new nodes, corresponds to

$$\tilde{x} := [x_{1,1}^\top \cdots x_{1,N_1+1}^\top \cdots x_{1,N_1+N'_1}^\top \cdots x_{m,1}^\top \cdots x_{m,N_m+N'_m}^\top].$$

Then, proceeding as in Section II, akin to Eq. (7), we obtain the dynamics equation for the augmented network,

$$\dot{\tilde{x}} = \tilde{F}(\tilde{x}) - \sigma^I [\mathcal{L}^I \otimes I_n] \tilde{x} - \sigma^E [\mathcal{L}^E \otimes I_n] \tilde{x}, \quad (55)$$

where \tilde{F} corresponds to the function F in (6) augmented by the corresponding functions f_{k,N_k+j} in (54); while \mathcal{L}^I ,

$\mathcal{L}^E \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$, with $\mathcal{N} := N + \sum_{k=1}^M N'_k$ are, respectively, the augmented internal and external Laplacians, as per the definition introduced in Section II. Thus, the augmented system (55) has the same structure as (7). In particular, the stability of the origin $\{\tilde{x} = 0\}$ depends on the dynamics of the added nodes (54) rendering the origin globally asymptotically stable for the augmented average system, with $\{\xi = 0\}$ and $\{\eta = 0\}$. That is, for the system

$$\dot{x}_e = \sum_{k=1}^m \sum_{i=1}^{N_k+N'_k} w_{\ell k} v_{\ell k i} f_{k,i}(x_e), \quad (56)$$

where $w_{\ell}, v_{\ell k}$ are defined in the same way as in Section III for the new matrices $\mathcal{L}^I, \mathcal{L}^E$. We conclude this reasoning with the following statement, which stems directly from Theorem 1.

Corollary 1: Consider the networked system (50) under Assumptions 1 and 2, with modified individual dynamics as in (51)-(52) and the dynamic control extensions (54). Assume, in addition, that the origin for (56) is globally asymptotically stable and there exists a continuously differentiable Lyapunov function $V_e : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying (40)-(41), with $f_e(x_e, 0, 0)$ corresponding to the right-hand side of (56). Then, there exist $\sigma^{E*} > 0$ and $\sigma^{I*} > 0$ such that, for all $\sigma^I > \sigma^{I*}$ and $\sigma^E > \sigma^{E*}$, the origin for (55) system is GAS. \square

Illustrative example: Consider a network of $N = 24$ Lorenz oscillators with state $x_{k,i} := [x_{k,i} \ y_{k,i} \ z_{k,i}]^\top$ and dynamics

$$\dot{x}_{k,i} = H_{k,i}(x_{k,i})x_{k,i}, \quad H_{k,i} := \begin{bmatrix} -\sigma_{k,i} & \sigma_{k,i} & 0 \\ \rho_{k,i} & -1 & -x_{k,i} \\ 0 & x_{k,i} & -\beta_{k,i} \end{bmatrix}$$

where $\sigma_{k,i}, \beta_{k,i}, \rho_{k,i}$ are positive constants. Let these systems be interconnected over a strongly connected directed network that can be compartmentalized into $m = 3$ modules containing each $N_k = 8$ nodes. Let each module $k \in \{1, 2, 3\}$ constitute a strongly-connected sub-network with Laplacian

$$L_k^I = \begin{bmatrix} 4 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 5 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 3 \end{bmatrix}. \quad (57)$$

On the other hand, the elements of the external Laplacian $L^E \in \mathbb{R}^{24 \times 24}$ are set to $[L^E]_{1,1} = 2$, $[L^E]_{i,i} = 1$ for $i \in \{9, 17\}$, $[L^E]_{i,j} = -1$ for all $(i,j) \in \{(1,9), (1,17), (9,17), (17,1)\}$, and $[L^E]_{i,j} = 0$ otherwise. L^E contains mostly zero entries since the modules are sparsely connected. As matter of fact, the network may be represented as a strongly-connected network of modules. Then, the averaged dynamics x_e for this network yields

$$\dot{x}_e = \sum_{k=1}^m \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i} H_{k,i}(x_{e1}) x_e. \quad (58)$$

For certain values of $\sigma_{k,i}, \beta_{k,i}, \rho_{k,i}$, set differently for all oscillators, the network reach consensus. That is, all oscillators

converge to a stable equilibrium point, but different from the origin.

To enforce the global stabilization of the origin, to each module we add one node (*i.e.*, $N'_k = 1$) with dynamics $\dot{x}_{k,9} = [-\alpha x_{k,9} \ -\alpha y_{k,i} \ 0]^\top$ and we interconnect it to the first and third nodes. That is, in (51) we set

$$v_{k,i} = -\sigma^I (x_{k,i} - x_{k,9}) \quad \forall i \in \{1, 3\} \quad (59)$$

and $v_{k,i} = 0$ for all $i \in \{2, 4, 5, 6, 7, 8\}$. Then, we obtain the modified averaged dynamics, corresponding to the augmented network,

$$\begin{aligned} \dot{x}_e = & \sum_{k=1}^m \sum_{i=1}^{N_k} w_{lk} v_{lki} H_{k,i}(x_{e1}) x_e \\ & + \sum_{k=1}^m \sum_{i=N_k+1}^{N_k+N'_k} w_{lk} v_{lki} \begin{bmatrix} -\alpha \\ -\alpha \\ 0 \end{bmatrix} x_{e1} \end{aligned} \quad (60)$$

which corresponds to Eq. (56).

Now, let

$$\begin{aligned} \sigma^* &:= \sum_{k=1}^m \sum_{i=1}^{N_k} w_{lk} v_{lki} \sigma_{k,i}, & \rho^* &:= \sum_{k=1}^m \sum_{i=1}^{N_k} w_{lk} v_{lki} \rho_{k,i}, \\ \omega_1^* &:= \sum_{k=1}^m \sum_{i=1}^{N_k} w_{lk} v_{lki}, & \omega_2^* &:= \sum_{k=1}^m \sum_{i=N_k+1}^{N_k+N'_k} w_{lk} v_{lki}, \end{aligned}$$

Then, using these definitions, (60) becomes

$$\dot{x}_e = \begin{bmatrix} -\sigma^* - \alpha\omega_2^* & \sigma^* & 0 \\ \rho^* & -\alpha\omega_2^* - \omega_1^* & -\omega_1^* x_{e1} \\ 0 & \omega_1^* x_{e1} & -\beta^* \end{bmatrix} x_e. \quad (61)$$

It is left to set α so that the origin for (61) is GAS. To that end, we remark that σ^* , ρ^* , ω_1^* and ω_2^* are all positive since, for this example, so are w_{lk} and v_{lki} . A straightforward computation, using the Lyapunov function $\mathcal{V}(x_e) = \frac{1}{2} \|x_e\|^2$, shows that $\dot{\mathcal{V}}(x_e) \leq -q \|x_e\|^2$, with $q > 0$, for any $\alpha > \max\left\{\frac{(\rho^* + \sigma^*)^2 - 2\sigma^*}{2\omega_2^*}, \frac{(\rho^* + \sigma^*)^2 - 2\omega_1^*}{2\omega_2^*}\right\}$.

In Figure 3, we show the result of a simulation of the augmented network as described above, with $\alpha = 30$. It is appreciated that the origin is asymptotically stable.

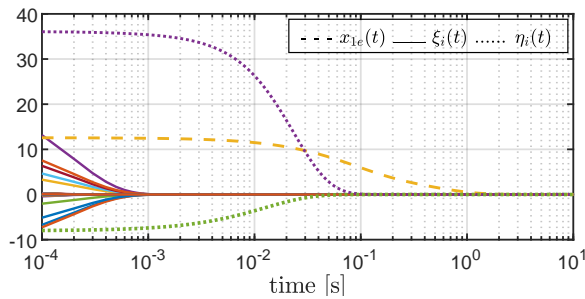


Fig. 3. Trajectories of the closed-loop system in logarithmic time scale. The solid lines represent the synchronization errors of the individual nodes relative to the modules' averages. The dotted lines represent the synchronization errors of each module relative to the modules' average. The dashed line depicts the overall average system's trajectories. In this simulation we used $\sigma^I = 5000$, $\sigma^E = 300$

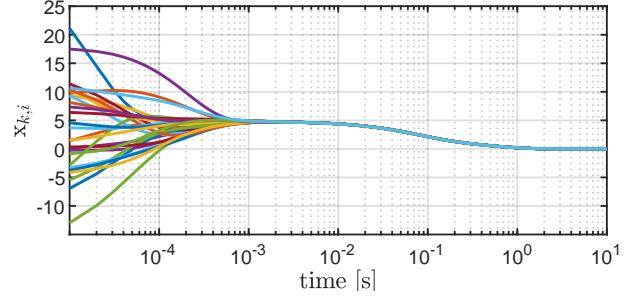


Fig. 4. Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 5000$, $\sigma^E = 12470 > \sigma^I \frac{|L^I|}{|L^E|}$. Since the difference between σ^I and σ^E is relatively small, we observe only a two-time-scale behavior. First, the trajectories synchronize, and then, they converge to zero.

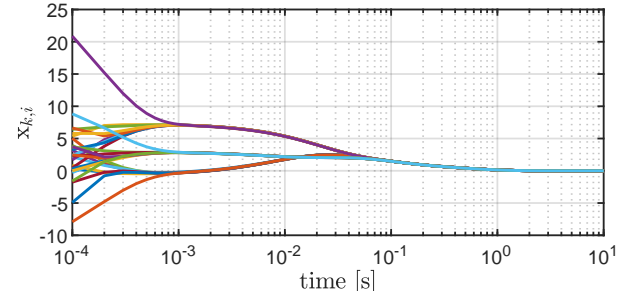


Fig. 5. Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 5000$, $\sigma^E = 300$. A three-time-scales behavior arises since the difference between σ^I and σ^E is relatively large. First, the trajectories synchronize within each module, then the modules' average trajectories synchronize, and finally, all the trajectories converge to zero.

Figure 4 shows the trajectories of the systems in the network when Assumption 1 is not satisfied. In this case, the system's three time-scale behavior is reduced to a two time-scale behavior. In fact, we can see in the figure that the network nodes find general agreement whether they are of the same module or different modules. This is in contrast to Fig 5, in which we see that, initially, nodes of the same module find local agreement, leading to the formation of three clusters. In a second phase, the three trajectories merge, before stabilizing at the origin.

B. Stabilization via added control modules

We present now the second control approach to stabilize the origin for the network (7). It consists in adding new whole modules with the aim, as in the previous section, to render the origin GAS for the resulting average system corresponding to the augmented network, on the synchronization manifold.

Let m' denote the number of new modules. For each $k' \leq m'$, the dynamics of the i th node within the k' th module is given by

$$\begin{aligned} \dot{x}_{m+k',i} = & f_{m+k',i}(x_{m+k',i}) \\ & - \sigma^I \sum_{l=1}^{N_{m+k'}} b_{m+k',i,l}(x_{m+k',i} - x_{m+k',l}) \end{aligned}$$

$$-\sigma^E \sum_{k=1}^m \sum_{j=1}^{N_k} a''_{(m+k',i),(k,j)} (x_{m+k',i} - x_{k,j}) \quad (62)$$

where the coefficients $b_{m+k',i,l}$ represent internal interconnections within the $(m+k')$ th module, *i.e.*,

$$b_{m+k',i,l} := \begin{cases} 1 & \text{if there is an edge from } (m+k',i) \text{ to} \\ & (m+k',l) \\ 0 & \text{if otherwise.} \end{cases}$$

The coefficients $a''_{(m+k',i),(k,j)}$ represent external interconnections from nodes within the new control modules to nodes in the original network. That is,

$$a''_{(m+k',i),(k,j)} := \begin{cases} 1 & \text{if there is an edge from the node } (k,j) \\ & \text{to the node } (m+k',i) \\ 0 & \text{if otherwise.} \end{cases}$$

Correspondingly, the coupling

$$v_{k,i} = -\sigma^E \sum_{k'=1}^{m'} \sum_{j=1}^{N_{m+k'}} a''_{k',k',i,j} (x_{k,i} - x_{m+k',j}) \quad (63)$$

is added to the existing nodes in the original network. Note that only for those existing nodes that are interconnected to nodes within new modules $v_{k,i} \neq 0$.

The state of the resulting augmented network is now $\tilde{x} := [x_{1,1}^\top \cdots x_{1,N_1}^\top \cdots x_{m+1,1}^\top \cdots x_{m+m',N_{m+m'}}^\top]^\top$. That is, the augmented network in closed loop yields

$$\dot{\tilde{x}} = \tilde{F}(\tilde{x}) - \sigma^I [\mathcal{M}^I \otimes I_n] \tilde{x} - \sigma^E [\mathcal{M}^E \otimes I_n] \tilde{x}, \quad (64)$$

where \tilde{F} corresponds to the function F in (6) augmented by the corresponding functions $f_{m+k',i}$ in (62), \mathcal{M}^I and \mathcal{M}^E are, respectively, the new internal and external Laplacians, akin to L^I and L^E as per the definition of the latter below (7). Hence, since the closed-loop system (64) has the same structure as (7), the arguments in Section III apply. Notably, one can compute eigenvectors $w_{\ell k}$ and $v_{\ell k i}$ as defined in (25) and (11) applying the appropriate transformations (as per in Section III) to \mathcal{M}^I and \mathcal{M}^E . Then, the new average dynamics, on the synchronization manifold, is

$$\dot{x}_e = \sum_{k=1}^{m+m'} \sum_{i=1}^{N_k} w_{\ell k} v_{\ell k i} f_{k,i}(x_e). \quad (65)$$

From this reasoning we draw the following statement, which follows directly from Theorem 1.

Corollary 2: Consider the networked system (50) with modified dynamics as in (51), additional inputs (63) and the dynamic control extensions (62). Let Assumptions 1 and 2 hold for the resulting augmented closed-loop system (64) and assume, in addition, that the origin for the average system (65) is globally asymptotically stable and there exists a continuously differentiable Lyapunov function $V_e : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ satisfying (40)-(41). Then, there exists $\sigma^{E*} > 0$ and $\sigma^{I*} > 0$ such that, for all $\sigma^I > \sigma^{I*}$ and $\sigma^E > \sigma^{E*}$, the origin of the closed-loop system (64) is GAS. \square

Illustrative example: To enforce global stabilization with the second strategy, we add a control module to the original network. This 4th module contains $N_{4^\top} = 8$ nodes with dynamics $\dot{x}_{4,i} = [-\alpha x_{4,i} \quad -\alpha y_{4,i} \quad 0]^\top$, furthermore, we consider that within this module, nodes are connected with respect to the Laplacian (57). A directed link is added from the first node of the control module to the first node of the other modules $k \in \{1, 2, 3\}$ and a directed link from the first node of the third module to the first node of the control module. That is, in (51) we set

$$v_{k,1} = -\sigma^E (x_{k,1} - x_{9,1}) \quad \forall k \in \{1, 2, 3\}, \quad (66)$$

and $v_{k,i} = 0$ for all $i \in \{2, 3, 4, 5, 6, 7, 8\}$. Finally, in (62), only $a''_{(4,1),(3,1)} = 1$ and all other coefficients are set to zero. The value of the control parameter α may be computed as for the previous example. The systems' trajectories are shown in Figure 6 below.

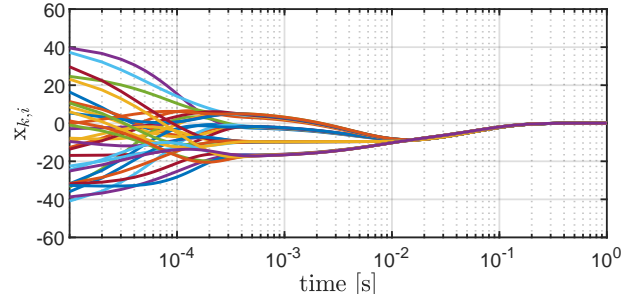


Fig. 6. Trajectories of the closed-loop system in logarithmic time scale, with $\sigma^I = 9000$, $\sigma^E = 900$. As in Figure 5 a three-time-scales behavior is appreciated, in view of the large discrepancy between σ^I and σ^E .

Thus, it may be appreciated from Figures 5 and 6 that, in spite of their simplicity, both control methods are efficacious to globally stabilize the origin for heterogeneous modular networks. In both figures the three-time scales behavior is well apparent. Indeed, in a first stage, the fastest trajectories $\xi(t)$ converge while $\eta(t)$ and $x_e(t)$ remain “constant” (on a logarithmic scale—see Figure 3). Then, as $\xi(t)$ approaches the origin, $\eta(t)$ converges as well, and $x_e(t)$ remains “constant”. Finally, $x_e(t)$ vanishes too, after $\eta(t)$ is close to zero.

VI. CONCLUSION

In large-scale networks having weighted interconnections, modules may be formed. Specifically if among groups of nodes the interconnections are considerably stronger and more dense than between such groups. In view of the weight discrepancy, such networks exhibit three time scales and may be analyzed via singular-perturbations theory. We showed that if the individual systems are semi-passive and the slowest underlying dynamics are globally asymptotically stable (stabilizable), so is the whole network.

The appearance of modular networks leads to the study of a whole range of new control problems for networked systems that deserve attention. In particular, the study of orbital stability for directed modular networks is pertinent in the context of synchronization of oscillators. An important

extension yet to establish concerns networks under output coupling and output synchronization.

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