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# Analog of Linear Landau Damping in a coupled Vlasov-Euler system for thick sprays

C. Buet\*, B. Després<sup>†</sup> and V. Fournet<sup>‡</sup>

## Abstract

We study a linearized thick sprays model. We recast the system as a linear Schrödinger equation and we prove that the corresponding operator only has absolutely continuous spectrum. This proves that the system displays an analog of the Landau damping and one observes a decay of the acoustic energy. We construct the dispersion relation of the problem and we compute the physical root with the smallest imaginary part. We construct a numerical scheme, and the numerical results confirm the theoretical findings.

## 1 Introduction

Collisionless motion of an electrostatic plasma is well-described by the Vlasov-Poisson system

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0 \\ \mathbf{E} = \nabla \Delta^{-1} (\int_{\mathbf{R}^3} f dv) . \end{cases} \quad (1.1)$$

It is known since the seminal work of Landau [25] that such model can exhibit a decay of the electric field for large times. This property of plasma, known today as *Landau damping*, has received a lot of interest since then and there exists an extensive literature on the linear Landau damping and more recently on the nonlinear Landau damping. See for example [11, 28, 33] and the reference therein. See also [3, 20, 21] for similar effect in other systems.

In this work, we present new results of linear damping properties of solutions of a multiphase model describing suspensions of particles in a underlying gas. Such complex flow is usually referred to as a *spray* [14]. A typical system which describes this kind of spray writes

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (\mathbf{\Gamma} f) = 0 \\ \partial_t (\alpha \varrho) + \nabla_x \cdot (\alpha \varrho \mathbf{u}) = 0 \\ \partial_t (\alpha \varrho \mathbf{u}) + \nabla_x \cdot (\alpha \varrho \mathbf{u} \otimes \mathbf{u}) + \alpha \nabla_x p(\varrho) = D_\star \int_{\mathbf{R}^3} (\mathbf{v} - \mathbf{u}) f dv . \end{cases} \quad (1.2)$$

In (1.2), the first equation is a Vlasov-type equation and it describes the evolution of the particles through a distribution function  $f = f(t, \mathbf{x}, \mathbf{v}) \geq 0$  in the phase

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space  $\mathbf{T}^3 \times \mathbf{R}^3$  for times  $t > 0$ . The second and third equations of (1.2) are the barotropic compressible Euler equations and they describe the evolution of the density  $\varrho = \varrho(t, \mathbf{x}) \geq 0$  and the velocity  $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) \in \mathbf{R}^3$  of the fluid, for times  $t > 0$  and  $\mathbf{x} \in \mathbf{T}^3$ . The system (1.2) describes the so-called *thick* sprays regime, in which the total volume occupied by the particles is not negligible compared to the one of the fluid. This has the consequence that the fluid and the particles are coupled through the following quantities. The first quantity is the volume fraction of the fluid  $\alpha = \alpha(t, \mathbf{x})$ . It is linked to the  $f$  by the formula

$$\alpha(t, \mathbf{x}) = 1 - \frac{4}{3}\pi r_p^3 \int_{\mathbf{R}^3} f(t, \mathbf{x}, \mathbf{v}) dv.$$

Here, we assume that  $\alpha(t, \mathbf{x}) \in [0, 1]$ . This is in strong contrast with the *thin* sprays regime where this quantity is assumed to be very closed to 1 and is therefore absent from the equations [2]. The radius of the particles  $r_p > 0$  is a constant in this work. The force field acting on the particles  $\mathbf{\Gamma} := \mathbf{\Gamma}(t, \mathbf{x}, \mathbf{v})$  is given by

$$\mathbf{\Gamma}(t, \mathbf{x}, \mathbf{v}) = -\nabla_x p(\varrho) - \frac{D_\star}{\frac{4}{3}\pi r_p^3}(\mathbf{v} - \mathbf{u}).$$

It is composed of two terms. The term  $-\nabla_x p(\varrho)$  is a specific feature of thick sprays (it is absent in the thin sprays regime). The term  $D_\star(\mathbf{v} - \mathbf{u})$  is a drag force exerted on the particles by the fluid. The retroaction of the drag force on the fluid appears in the third equation of (1.2) through the term  $D_\star \int_{\mathbf{R}^3} (\mathbf{v} - \mathbf{u}) f dv$  and is sometimes referred to as the *Brinkman force*. The drag coefficient  $D_\star \geq 0$  is usually given in semi-empirical manners. This model can be used to describe various physical phenomena at different length scales, such as aerosols for medical use [6, 7], the combustion in engines [1], aerosols in the atmosphere [27], and also in astrophysics for the modelling of gas giants and exoplanets [18]. See also [4, 5, 15–17, 29, 34].

In parallel with the system (1.1) where collisions are neglected, we neglect the drag force in (1.2). Thus,  $D_\star = 0$  and the model problem that we will use in most of this work writes

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f - \nabla_x p(\varrho) \cdot \nabla_v f = 0 \\ \partial_t(\alpha \varrho) + \nabla_x \cdot (\alpha \varrho \mathbf{u}) = 0 \\ \partial_t(\alpha \varrho \mathbf{u}) + \nabla_x \cdot (\alpha \varrho \mathbf{u} \otimes \mathbf{u}) + \alpha \nabla_x p(\varrho) = 0. \end{cases} \quad (1.3)$$

A description of our main results is the following.

- The first result investigates damping properties of small perturbations of homogenous profiles at the linear level. It is described in the Theorem 1 below. We linearize the system (1.3) around a reference solution and we analyze the decay of the perturbation in  $L^2$  norm. We adapt ideas from plasma physics [9, 12, 13] and exploit the scattering structure of the linearized system, rewriting it as a linear Schrödinger equation of the form

$$\frac{d}{dt} \mathbf{U} = iH\mathbf{U}.$$

The analysis is based on a careful analysis of the spectrum of the unbounded operator  $H$ . The main difficulty is that the system at stakes is more complex than the Vlasov-Poisson system, making the computations more tedious.

- The second result is the computation of the rate of decay of the perturbation of the linearized equations for analytic initial data. Mimicking computations done by Landau [25] for the Vlasov-Poisson system, we obtain the dispersion relation of the linearized system. This function is written as the sum of the dispersion relation of the Vlasov-Poisson equation and an additional term

which is specific to the model (1.3). The computation of the dispersion relation is done in the Proposition 4 below. Then, we formally derive an expansion for the perturbation of the density.

- Finally we show numerical simulation illustrating the damping investigated in this paper. We present schemes for the linearized and the nonlinear equations. The schemes are derived from classical methods. It shows that the damping predicted by the linear equations is observed in the linear and the nonlinear simulations. We also present preliminary formal and numerical results for the case with nonzero friction  $D_\star > 0$ .

**Overview of the paper** This work is organized as follows. In section 2.2, we prove Theorem 1. In section 2.3, we show formal computation leading to the dispersion relation. In section 3, we show numerical results illustrating Theorem 1, and we show that the computation done in section 2.3 are verified numerically. Finally in section 4, we discuss remaining open questions about this work.

## 2 Linear Damping

In this section, our aim is to analyze the damping properties of the system (1.3) in the linear regime. We follow a classical approach where we linearize the equations around gaussian equilibrium. In contrast with plasma physics, the physical validity of such equilibrium is questionable for thick sprays, and should be justified. Another possibility is to use a more general class of equilibrium as in [8]. In this work we limit ourselves to gaussian equilibrium since it simplifies the equations and already brings valuable insight on the mathematical structure of the system.

### 2.1 The linear thick sprays model

We consider the following homogeneous solution of (1.3)

$$\begin{cases} \varrho(t, \mathbf{x}) = \varrho_0 > 0, \\ \mathbf{u}(t, \mathbf{x}) = 0, \\ f(t, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{v}) := \frac{n_0}{\sqrt{2\pi}} e^{-|\mathbf{v}|^2/2}, \end{cases}$$

where the density  $n_0 = \int_{\mathbf{R}^3} f^0 dv$  is related to the volume fraction  $\alpha_0$  by

$$1 - \frac{4}{3} \pi r_p^3 n_0 = \alpha_0 \in (0, 1). \quad (2.1)$$

Following [8], we perform the linearization

$$\begin{cases} \varrho(t, \mathbf{x}) = \varrho_0 + \varepsilon \varrho_1(t, \mathbf{x}) + O(\varepsilon^2) \\ \mathbf{u}(t, \mathbf{x}) = \varepsilon \mathbf{u}_1(t, \mathbf{x}) + O(\varepsilon^2) \\ f(t, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{v}) + \varepsilon \sqrt{f^0(\mathbf{v})} f_1(t, \mathbf{x}, \mathbf{v}) + O(\varepsilon^2). \end{cases}$$

Dropping the quadratic terms and the subscripts, one obtains the following linear thick sprays equations with  $\tau(t, \mathbf{x}) = -\varrho_1(t, \mathbf{x})/\varrho_0^2$  and  $c_0 = \sqrt{p'(\varrho_0)}$

$$\begin{cases} \alpha_0 \varrho_0 \partial_t \tau = \alpha_0 \nabla_x \cdot \mathbf{u} + \frac{4}{3} \pi r_p^3 \nabla_x \cdot \int_{\mathbf{R}^3} \mathbf{v} \sqrt{f^0} f dv \\ \alpha_0 \varrho_0 \partial_t \mathbf{u} = \alpha_0 \varrho_0^2 c_0^2 \nabla_x \tau \\ \partial_t f + \mathbf{v} \cdot \nabla_x f - \varrho_0^2 c_0^2 \sqrt{f^0(v)} \mathbf{v} \cdot \nabla_x \tau = 0. \end{cases} \quad (2.2)$$

## 2.2 Spectral theory for linear thick sprays

The main result of this section is the following.

**Theorem 1.** *Let  $f^0 = f^0(v)$  satisfies (2.1), and initial condition  $(\tau_{\text{ini}}, \mathbf{u}_{\text{ini}}, f_{\text{ini}}) \in L^2(\mathbf{T}^3) \times (L^2(\mathbf{T}^3))^3 \times L^2(\mathbf{T}^3 \times \mathbf{R}^3)$  such that*

$$\iint_{\mathbf{T}^3 \times \mathbf{R}^3} f_{\text{ini}}(\mathbf{x}, \mathbf{v}) \, d\mathbf{v} d\mathbf{x} = 0, \quad \int_{\mathbf{T}^3} \tau_{\text{ini}}(\mathbf{x}) \, d\mathbf{x} = 0, \quad \int_{\mathbf{T}^3} \mathbf{u}_{\text{ini}}(\mathbf{x}) \, d\mathbf{x} = 0.$$

*Then as  $t \rightarrow +\infty$  the solution  $(\tau(t, \cdot), u(t, \cdot), f(t, \cdot, \cdot))$  of the linearized thick sprays equations (2.2) with initial data  $(\tau_{\text{ini}}, u_{\text{ini}}, f_{\text{ini}})$  verifies*

$$\|\tau(t)\|_{L^2} + \|\mathbf{u}(t)\|_{L^2} \rightarrow 0,$$

*and  $f(t, \cdot, \cdot)$  converges weakly to 0 in  $L^2(\mathbf{T}^3 \times \mathbf{R}^3)$ .*

**Remark.** As the quadratic energy  $\int_{\mathbf{T}^3 \times \mathbf{R}^3} |f|^2 \, d\mathbf{x} d\mathbf{v} + \int_{\mathbf{R}^3} |\tau|^2 + |\mathbf{u}|^2 \, d\mathbf{x}$  is conserved, Theorem 1 implies that all the acoustic energy is transferred to the particles. It will be visible in Figure 4 where fast oscillations show up in the velocity variable. It is known in the plasma community as *filamentation*.

The proof of Theorem 1 is done using tools from spectral theory [24, 26]. For the sake of simplicity of the notations, we place ourselves in 1D, but the proof is valid in dimension  $d \in \mathbf{N}^*$ . After setting all constants to 1, the system (2.2) rewrites

$$\begin{cases} \partial_t \tau = \partial_x u + \partial_x \int_{\mathbf{R}} v \sqrt{f^0} f \, dv \\ \partial_t u = \partial_x \tau \\ \partial_t f + v \partial_x f - \sqrt{f^0(v)} v \partial_x \tau = 0. \end{cases} \quad (2.3)$$

We start from initial data  $(\tau_{\text{ini}}, u_{\text{ini}}, f_{\text{ini}})$  satisfying

$$\iint_{\mathbf{T} \times \mathbf{R}} f_{\text{ini}}(x, v) \, dv dx = 0, \quad \int_{\mathbf{T}} \tau_{\text{ini}}(x) \, dx = 0, \quad \int_{\mathbf{T}} u_{\text{ini}}(x) \, dx = 0.$$

One can show that solutions of (2.3) conserve the quadratic energy

$$\frac{d}{dt} \left( \iint_{\mathbf{T} \times \mathbf{R}} |f|^2 \, dv dx + \int_{\mathbf{T}} |\tau|^2 + |u|^2 \, dx \right) = 0. \quad (2.4)$$

**Functional setting** We define the following functional spaces

$$L_0^2(\mathbf{T}) = \left\{ u \in L^2(\mathbf{T}), \quad \int_{\mathbf{T}} u(x) \, dx = 0 \right\},$$

and

$$L_0^2(\mathbf{T} \times \mathbf{R}) = \left\{ f \in L^2(\mathbf{T} \times \mathbf{R}), \quad \iint_{\mathbf{T} \times \mathbf{R}} f(x, v) \, dx dv = 0 \right\},$$

and

$$\ell_0^2(\mathbf{Z}) = \{ u \in \ell^2(\mathbf{Z}), \quad u_0 = 0 \},$$

and the unbounded operator  $H : L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T} \times \mathbf{R}) \longrightarrow L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T} \times \mathbf{R})$ , by

$$iH = \begin{pmatrix} 0 & \partial_x & \partial_x \int_{\mathbf{R}} v \sqrt{f^0(v)} \cdot \, dv \\ \partial_x & 0 & 0 \\ v \sqrt{f^0(v)} \partial_x & 0 & -v \partial_x \end{pmatrix},$$

with domain

$$D[H] = \{(\tau, u, f) \in L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T}) \times L_0^2(\mathbf{T} \times \mathbf{R}), \text{ such that } \partial_x \tau \in L^2(\mathbf{T}), \partial_x u \in L^2(\mathbf{T}), v \partial_x f \in L^2(\mathbf{T} \times \mathbf{R})\}.$$

The system (2.3) rewrites as a linear Schrödinger equation

$$\frac{d}{dt} \mathbf{U} = iH\mathbf{U}, \quad \mathbf{U} = (\tau, u, f).$$

To study the operator  $H$ , we introduce the Fourier transform  $\mathcal{F} : L_0^2(\mathbf{T}) \rightarrow \ell_0^2(\mathbf{Z})$  defined by

$$\mathcal{F}(f)(k) = f_k := \int_{\mathbf{T}} e^{-ikx} f(x) dx, \quad \forall k \in \mathbf{Z}.$$

In this context, the Fourier inverse  $\mathcal{F}^{-1} : \ell_0^2(\mathbf{Z}) \rightarrow L_0^2(\mathbf{T})$  is well-defined and  $\mathcal{F}$  is a unitary operator. Denote  $\ell_k^2 L_v^2 = \mathcal{F}(L_0^2(\mathbf{T} \times \mathbf{R}))$ . We work in the Hilbert space

$$X = \ell_0^2(\mathbf{Z}) \times \ell_0^2(\mathbf{Z}) \times \ell_k^2 L_v^2,$$

equipped with the hermitian product

$$\langle (\tau, u, f), (\sigma, w, g) \rangle_X = \sum_{k \in \mathbf{Z}} \tau_k \bar{\sigma}_k + \sum_{k \in \mathbf{Z}} u_k \bar{w}_k + \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}} f_k(v) \overline{g_k(v)} dv.$$

We introduce  $\mathbf{k}$  the operator defined by  $(\mathbf{k}\tau)_k = k\tau_k$  for all  $k \in \mathbf{Z}$ . Let  $\mathcal{H} : X \rightarrow X$  the operator defined by

$$i\mathcal{H} = \mathbf{k} \begin{pmatrix} 0 & 1 & \int_{\mathbf{R}} v \sqrt{f^0(v)} \cdot dv \\ 1 & 0 & 0 \\ v \sqrt{f^0(v)} \cdot & 0 & -v \end{pmatrix},$$

with domain

$$D[\mathcal{H}] = \{(\tau, u, f) \in X, \text{ such that } \mathbf{k}\tau \in \ell^2(\mathbf{Z}), \mathbf{k}u \in \ell^2(\mathbf{Z}), \mathbf{k}vf \in \ell_k^2 L_v^2\}.$$

By construction,  $H$  and  $\mathcal{H}$  are unitarily equivalent. The system (2.3) rewrites in the Fourier space, as

$$\frac{d}{dt} \mathbf{U} = i\mathcal{H}\mathbf{U}.$$

**Proposition 2.** *The operators  $H$  and  $\mathcal{H}$  are self adjoint.*

*Proof.* Let us write  $\mathcal{H} = \mathcal{H}_0 + \mathcal{K}$ , with

$$\mathcal{H}_0 = \mathbf{k} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -v \end{pmatrix}, \quad \mathcal{K} = \mathbf{k} \begin{pmatrix} 0 & 0 & \int_{\mathbf{R}} v \sqrt{f^0(v)} \cdot dv \\ 0 & 0 & 0 \\ v \sqrt{f^0(v)} \cdot & 0 & 0 \end{pmatrix},$$

with domain

$$D[\mathcal{H}_0] = D[\mathcal{H}],$$

$$D[\mathcal{K}] = \left\{ (\tau, u, f) \in X, \mathbf{k} \int_{\mathbf{R}} v \sqrt{f^0(v)} f dv \in \ell^2(\mathbf{Z}), \mathbf{k}v \sqrt{f^0(v)} \tau \in \ell_k^2 L_v^2 \right\}.$$

Let us show that  $(\mathcal{H}_0, D[\mathcal{H}_0])$  is self-adjoint. A simple computation show that  $\mathcal{H}_0$  is symmetric,  $\langle \mathcal{H}_0 U, V \rangle_X = \langle U, \mathcal{H}_0 V \rangle_X$  for all  $U, V \in D[\mathcal{H}_0]$ . It remains to show that  $D[\mathcal{H}_0^*] \subset D[\mathcal{H}_0]$ . Let  $(\sigma, w, g) \in D[\mathcal{H}_0^*]$ . By definition of the adjoint, for every  $(\tau, u, f) \in D[\mathcal{H}_0]$ , there exists  $(s, \omega, h) \in X$  such that

$$\langle \mathcal{H}(\tau, u, f), (\sigma, w, g) \rangle = \langle (\tau, u, f), (s, \omega, h) \rangle.$$

In particular, for

$$(\tau, u, f) = \mathbf{k} \begin{pmatrix} w \mathbf{1}_{[-n, n]} \\ \sigma \mathbf{1}_{[-n, n]} \\ -v g \mathbf{1}_{[-n, n] \times [-m, m]} \end{pmatrix},$$

one has

$$\begin{aligned} \langle \mathcal{H}_0(\tau, u, f), (\sigma, w, g) \rangle &= \sum_{k=-n}^n k^2 |w_k|^2 + \sum_{k=-n}^n k^2 |\sigma_k|^2 + \sum_{k=-n}^n \int_{-m}^m k^2 v^2 |g_k|^2 dv \\ &= \langle (\tau, u, f), (s, \omega, h) \rangle_X \\ &\leq \|(\tau, u, f)\|_X \|s, \omega, h\|_X \\ &= \left[ \sum_{k=-n}^n k^2 |w_k|^2 + \sum_{k=-n}^n k^2 |\sigma_k|^2 + \sum_{k=-n}^n \int_{-m}^m k^2 v^2 |g_k(v)|^2 dv \right]^{1/2} \|s, \omega, h\|_X. \end{aligned}$$

So

$$\left[ \sum_{k=-n}^n k^2 |w_k|^2 + \sum_{k=-n}^n k^2 |\sigma_k|^2 + \sum_{k=-n}^n \int_{-m}^m k^2 v^2 |g_k(v)|^2 dv \right]^{1/2} \leq \|s, \omega, h\|_X,$$

and, taking the limit  $m, n \rightarrow \infty$ , it yields

$$\begin{aligned} \mathbf{k}w &\in \ell^2(\mathbf{Z}), \\ \mathbf{k}\sigma &\in \ell^2(\mathbf{Z}), \\ -\mathbf{k}vg &\in \ell_k^2 L_v^2. \end{aligned}$$

This show that  $\mathcal{H}_0$  is self adjoint. Moreover, the operator  $\mathcal{K}$  is clearly symmetric, and  $D[\mathcal{H}_0] \subset D[\mathcal{K}]$ . Furthermore

$$\begin{aligned} \|\mathcal{K}(\tau, u, f)\|_X^2 &= \sum_{k \in \mathbf{Z}} k^2 \int_{\mathbf{R}} v^2 f^0(v) |\tau_k|^2 dv + \sum_{k \in \mathbf{Z}} k^2 \left| \int_{\mathbf{R}} v \sqrt{f^0(v)} f_k(v) dv \right|^2 \\ &\leq \int_{\mathbf{R}} v^2 f^0(v) dv \sum_{k \in \mathbf{Z}} k^2 |\tau_k|^2 + n_0 \sum_{k \in \mathbf{Z}} k^2 |u_k|^2 + \int_{\mathbf{R}} f^0(v) dv \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}} v^2 k^2 |f_k(v)|^2 dv \\ &\leq n_0 \|\mathcal{H}_0(\tau, u, f)\|_X^2. \end{aligned}$$

So the operator  $\mathcal{K}$  is  $\mathcal{H}_0$ -bounded. Thanks to the hypothesis (2.1),  $n_0 < 1$  and the Kato-Rellich theorem applies. One concludes that  $\mathcal{H}$  is self adjoint on  $D[\mathcal{H}] = D[\mathcal{H}_0]$ . Since  $H$  is unitary equivalent to  $\mathcal{H}$ , the operator  $(H, D[H])$  is self adjoint.  $\square$

Since the operators  $H$  and  $\mathcal{H}$  are self-adjoint, their spectrums can be decomposed in terms of theory of measure [24, 26]. It yields the following decomposition

$$X = X^{\text{ac}} \oplus X^{\text{sc}} \oplus X^{\text{pp}}.$$

where  $X^{\text{ac}}$  (resp.  $X^{\text{sc}}$ , resp.  $X^{\text{pp}}$ ) corresponds to the absolutely continuous (resp. singular continuous, resp. pure point) part of the spectrum. The pure point subspace  $X^{\text{pp}}$  is spanned by the eigenvectors

$$X^{\text{pp}} = \text{Span}\{\varphi \in X, \mathcal{H}\varphi = \lambda\varphi \text{ for some } \lambda \in \mathbf{R}\}.$$

On the other hand, the subspace  $X^{\text{ac}}$  is characterized [22, 23, 30] by the existence of a dense subset  $A \subset X^{\text{ac}}$  such that

$$\varphi \in A \Rightarrow \left\| (\mathcal{H} - \lambda - i\varepsilon)^{-1} \varphi \right\|_X = O\left(\frac{1}{\sqrt{\varepsilon}}\right). \quad (2.5)$$

This characterisation is known as the Christensen criterion. Before moving on, we recall the so-called Plemelj formula,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x + i\varepsilon} = \text{P.V} \left( \frac{1}{x} \right) - i\pi\delta_0, \quad \text{in } \mathcal{D}'.$$

**Theorem 3.** *The operator  $H$  only has absolutely continuous spectrum.*

*Proof.* Since  $H$  is unitary equivalent to  $\mathcal{H}$ , if  $\mathcal{H}$  only has absolutely continuous spectrum, then the same statement holds for  $H$ . We verify the criterion (2.5) by means of a series of elementary calculations.

Let  $A$  be the set of triplet of function  $(\tau, u, f)$  such that  $\tau, u \in \ell_0^2(\mathbf{Z})$  with finite support and  $f \in \ell_k^2 L_v^2$  with finite support in  $k$ , and bounded in  $v$ . The set  $A$  is dense in  $X$ . Let us compute  $(\mathcal{H} - (\lambda + i\varepsilon))^{-1}(\tau, u, f)$  by solving for  $(\sigma, w, g) \in X$  the equation  $(\mathcal{H} - (\lambda + i\varepsilon))(\sigma, w, g) = (\tau, u, f)$  which writes, for all  $k \neq 0, v \in \mathbf{R}$

$$\begin{cases} kw_k + k \int_{\mathbf{R}} v \sqrt{f^0(v)} g_k(v) \, dv - (\lambda + i\varepsilon)\sigma_k = \tau_k \\ k\sigma_k - (\lambda + i\varepsilon)w_k = u_k \\ -vk g_k(v) + kv \sqrt{f^0(v)} \sigma_k - (\lambda + i\varepsilon)g_k(v) = f_k(v). \end{cases}$$

Taking the third equation, one has

$$g_k(v) = \frac{kv \sqrt{f^0(v)}}{\lambda + i\varepsilon + vk} \sigma_k - \frac{f_k(v)}{\lambda + i\varepsilon + vk}. \quad (2.6)$$

The second equation gives

$$w_k = \frac{1}{\lambda + i\varepsilon} (k\sigma_k - u_k). \quad (2.7)$$

Then, the first equation gives

$$\frac{k^2}{\lambda + i\varepsilon} \sigma_k - \frac{k}{\lambda + i\varepsilon} u_k + k \int_{\mathbf{R}} v \sqrt{f^0(v)} g_k(v) \, dv - (\lambda + i\varepsilon)\sigma_k = \tau_k.$$

Using (2.6) and (2.7)

$$\frac{k^2}{(\lambda + i\varepsilon)^2} \sigma_k + \frac{k^2}{\lambda + i\varepsilon} \sigma_k \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} \, dv - \sigma_k = \frac{k}{(\lambda + i\varepsilon)^2} u_k + \frac{1}{\lambda + i\varepsilon} \tau_k + \frac{k}{\lambda + i\varepsilon} \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + i\varepsilon + vk} \, dv.$$

Finally

$$\begin{aligned} \sigma_k &= \frac{-1}{1 - \frac{k^2}{(\lambda + i\varepsilon)^2} - \frac{k^2}{\lambda + i\varepsilon} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} \, dv} \left( \frac{\tau_k}{\lambda + i\varepsilon} + \frac{k}{(\lambda + i\varepsilon)^2} u_k + \frac{k}{\lambda + i\varepsilon} \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + i\varepsilon + vk} \, dv \right) \\ &= - \frac{(\lambda + i\varepsilon)^2}{(\lambda + i\varepsilon)^2 - k^2 - k^2(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} \, dv} \left( \frac{\tau_k}{\lambda + i\varepsilon} + \frac{k}{(\lambda + i\varepsilon)^2} u_k + \frac{k}{\lambda + i\varepsilon} \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + i\varepsilon + vk} \, dv \right) \\ &= - \frac{(\lambda + i\varepsilon)\tau_k + ku_k + k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + i\varepsilon + vk} \, dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} \, dv}. \end{aligned} \quad (2.8)$$

To satisfy the criterion (2.5), it is enough, from the formula (2.6)-(2.7), to bound  $\sigma$  and  $w$  in  $\ell^\infty$ . One obtains using Plemelj formula

$$\begin{aligned} \sigma_k &= - \frac{(\lambda + i\varepsilon)\tau_k + ku_k + k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + i\varepsilon + vk} \, dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} \, dv} \\ &\xrightarrow{\varepsilon \rightarrow 0} - \frac{\lambda\tau_k + ku_k + k\lambda \left[ \text{P.V} \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(v)}{\lambda + vk} \, dv + i\pi\lambda \sqrt{f^0(\lambda)} f_k(\lambda) \right]}{\lambda^2 - k^2 - k\lambda \left[ \text{P.V} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + vk} \, dv + i\pi\lambda^2 f^0(\lambda) \right]}. \end{aligned}$$



So  $\sigma$  is uniformly bounded in  $\varepsilon$  in  $\ell^\infty(\mathbf{Z})$ . In the same way, from (2.7)

$$\begin{aligned}
w_k &= \frac{1}{\lambda + i\varepsilon} \left( -\frac{k(\lambda + i\varepsilon)\tau_k + k^2 u_k + k^2(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - u_k \right) \\
&= -\frac{k\tau_k + k^2 \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - \frac{k^2 \frac{u_k}{\lambda + i\varepsilon}}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - \frac{u_k}{\lambda + i\varepsilon} \\
&= -\frac{k\tau_k + k^2 \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - \frac{u_k}{\lambda + i\varepsilon} \left[ 1 + \frac{k^2}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} \right] \\
&= -\frac{k\tau_k + k^2 \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - \frac{u_k}{\lambda + i\varepsilon} \left[ \frac{(\lambda + i\varepsilon)^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} \right] \\
&= -\frac{k\tau_k + k^2 \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} - u_k \left[ \frac{(\lambda + i\varepsilon) - k \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv} \right] \\
&= -\frac{k\tau_k + k^2 \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + i\varepsilon + vk} dv + u_k \left[ (\lambda + i\varepsilon) - k \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv \right]}{(\lambda + i\varepsilon)^2 - k^2 - k(\lambda + i\varepsilon) \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + i\varepsilon + vk} dv}.
\end{aligned}$$

Then one concludes using again Plemelj formula that

$$w_k \xrightarrow{\varepsilon \rightarrow 0} -\frac{k\tau_k + \text{P.V} \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(v)}{\lambda + vk} dv + i\pi\lambda\sqrt{f^0(\lambda)}f_k(\lambda) + u_k \left[ \lambda - k\text{P.V} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + vk} dv - i\pi k\lambda^2 f^0(\lambda) \right]}{\lambda^2 - k^2 - k\text{P.V} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{\lambda + vk} dv - i\pi k\lambda^2 f^0(\lambda)}.$$

In the end, one obtains that  $w$  is uniformly bounded in  $\varepsilon$  in  $\ell^\infty(\mathbf{Z})$ . We then turn to an estimate of  $g$  in  $\ell_k^2 L_v^2$ . The triangle inequality yields

$$\|g\|_{\ell_k^2 L_v^2} \leq \left\| \frac{kv\sqrt{f^0}\sigma}{\lambda + i\varepsilon + vk} \right\|_{\ell_k^2 L_v^2} + \left\| \frac{f}{\lambda + i\varepsilon + vk} \right\|_{\ell_k^2 L_v^2}.$$

One has

$$\begin{aligned}
\left\| \frac{kv\sqrt{f^0}\sigma}{\lambda + i\varepsilon + vk} \right\|_{\ell_k^2 L_v^2}^2 &= \sum_{k \in \mathbf{Z} \setminus \{0\}} \int_{v \in \mathbf{R}} \left| \frac{kv\sqrt{f^0}\sigma_k}{\lambda + i\varepsilon + vk} \right|^2 dv \\
&= \sum_{k \in \mathbf{Z} \setminus \{0\}} k^2 \sigma_k^2 \int_{v \in \mathbf{R}} \frac{v^2 f^0}{(\lambda + vk)^2 + \varepsilon^2} dv \\
&\leq \|v^2 f^0\|_{L_v^\infty} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{k^2 \sigma_k^2}{k\varepsilon} \\
&\leq \frac{\|v^2 f^0\|_{L_v^\infty} \|\mathbf{k}\sigma\|_{\ell^2}^2}{\varepsilon},
\end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{f}{\lambda + i\varepsilon + vk} \right\|_{\ell_k^2 L_v^2}^2 &= \sum_{k \in \mathbf{Z} \setminus \{0\}} \int_{v \in \mathbf{R}} \frac{|f_k(v)|^2}{(\lambda + vk)^2 + \varepsilon^2} dv \\
&\leq \sum_{k \in \mathbf{Z} \setminus \{0\}} \int_{v \in \mathbf{R}} \frac{\|f_k\|_{L_v^\infty}^2}{(\lambda + vk)^2 + \varepsilon^2} dv \\
&\leq \frac{C}{\varepsilon} \|\mathbf{k}^{-1} f \mathbf{1}_{k \neq 0}\|_{\ell_k^2 L_v^\infty}^2.
\end{aligned}$$

Finally, it yields for  $\varepsilon$  small enough,

$$\forall(\tau, u, f) \in A, \quad \|(\mathcal{H} - (\lambda + i\varepsilon))^{-1}(\tau, u, f)\|_X \leq \frac{C_{\lambda, \tau, u, f}}{\sqrt{\varepsilon}}.$$

With  $C_{\tau, u, f, \lambda}$  a constant that depends only on  $\tau, u, f, \lambda$ . One concludes that  $\mathcal{H}$  only has absolutely continuous spectrum.  $\square$

We can then conclude this section and prove Theorem 1.

*Proof of Theorem 1.* Since the operator  $H$  has absolutely continuous spectrum and does not have pure point and singular spectrum, it is a standard consequence of scattering theory [24, 26] that the solution  $(\tau(t), u(t), f(t)) = e^{-itH}(\tau_{\text{ini}}, u_{\text{ini}}, f_{\text{ini}})$  converges weakly to 0. Furthermore it is easy to see that

$$\forall k, \quad \tau_k(t) \rightarrow 0, \quad u_k(t) \rightarrow 0.$$

Using the conservation of energy, one has

$$\begin{aligned} \sum_{k \in \mathbf{Z}} (|\tau_k(t)|^2 + |u_k(t)|^2) &\leq \sum_{k \in \mathbf{Z}} \left( |\tau_k(t)|^2 + |u_k(t)|^2 + \int_{\mathbf{R}} |f_k(t, v)|^2 dv \right) \\ &= \sum_{k \in \mathbf{Z}} \left( |\tau_k(0)|^2 + |u_k(0)|^2 + \int_{\mathbf{R}} |f_k(0, v)|^2 dv \right). \end{aligned}$$

Let  $\varepsilon > 0$ , then there exists  $N > 0$  such that

$$\sum_{|k| > N} \left( |\tau_k(0)|^2 + |u_k(0)|^2 + \int_{\mathbf{R}} |f_k(0, v)|^2 dv \right) < \varepsilon.$$

Then one writes

$$\begin{aligned} \sum_{k \in \mathbf{Z}} (|\tau_k(t)|^2 + |u_k(t)|^2) &= \sum_{|k| \leq N} (|\tau_k(t)|^2 + |u_k(t)|^2) + \sum_{|k| > N} (|\tau_k(t)|^2 + |u_k(t)|^2) \\ &< \sum_{|k| \leq N} (|\tau_k(t)|^2 + |u_k(t)|^2) + \varepsilon. \end{aligned}$$

Taking  $t \rightarrow +\infty$ ,

$$\lim_{t \rightarrow +\infty} \sum_{k \in \mathbf{Z}} (|\tau_k(t)|^2 + |u_k(t)|^2) \leq \varepsilon, \quad \forall \varepsilon > 0.$$

One concludes by Plancherel identity.  $\square$

### 2.3 Dispersion relation

In this section, we perform a more quantitative analysis than in the previous section by assuming that the initial data are analytic functions. For simplicity of the notations we take  $n_0 = 1$ . The conclusion (2.12) is that the damping is a combination of a decreasing exponential part and an oscillatory part. This behaviour is qualitatively the same to the linear Landau damping in plasma physics for the Vlasov-Poisson system, even if the physical nature of the force that acts on the particles is different.

For a function  $\tau \in L^\infty(\mathbf{R}_+; L^2(\mathbf{T}))$ , we define the Laplace-Fourier transform as

$$\tilde{\tau}_k(\omega) := \int_0^{+\infty} \int_{\mathbf{T}} e^{i\omega t} e^{-ikx} \tau(t, x) dx dt, \quad \Im(\omega) > 0.$$

**Proposition 4.** Let  $(\tau, u, f)$  solutions of the linearized system (2.3), and  $\tilde{\tau}_k$  be the Fourier-Laplace transform of  $\tau$ . Then one has, for  $\Im(\omega) > 0$

$$\tilde{\tau}_k(\omega) = \frac{\mathcal{N}(\omega, k)}{\mathcal{D}(\omega, k)},$$

with

$$\mathcal{N}(\omega, k) = \frac{i}{\omega} \left( \tau_k(0) - \frac{k}{\omega} u_k(0) + \int_{\mathbf{R}} \frac{v \sqrt{f^0(v)} f_k(0, v)}{v - \frac{\omega}{k}} dv \right),$$

and

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} + \left[ 1 + \frac{\omega}{\sqrt{2\pi}k} \int_{\mathbf{R}} \frac{e^{-v^2/2}}{v - \frac{\omega}{k}} dv \right].$$

The function  $\mathcal{D}$  is the dispersion relation of the linear system.

**Remark.** It is interesting to compare  $\mathcal{D}$  to the dispersion relation of the linear Vlasov-Poisson system

$$\mathcal{D}_{\text{VP}}(\omega, k) = 1 + \frac{1}{k^2} \left[ 1 + \frac{\omega}{\sqrt{2\pi}k} \int_{\mathbf{R}} \frac{e^{-v^2/2}}{v - \frac{\omega}{k}} dv \right].$$

We remark that

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} + k^2(\mathcal{D}_{\text{VP}}(\omega, k) - 1) = 1 - k^2 - \frac{k^2}{\omega^2} + k^2 \mathcal{D}_{\text{VP}}(\omega, k).$$

*Proof.* We consider the system (2.3) written for Fourier modes  $k \neq 0$ .

$$\begin{cases} \partial_t \tau_k(t) = iku_k(t) + ik \int_{\mathbf{R}} \sqrt{f^0} f_k(t, v) v dv \\ \partial_t u_k(t) = ik\tau_k(t) \\ \partial_t f_k(t, v) + ikv f_k(t, v) - ikv \sqrt{f^0} \tau_k(t) = 0. \end{cases} \quad (2.9)$$

The solutions of (2.9) are bounded, so the following Fourier-Laplace transforms are well-defined

$$\begin{aligned} \tilde{\tau}_k(\omega) &= \int_0^{+\infty} e^{i\omega t} \tau_k(t) dt, \quad \Im(\omega) > 0, \\ \tilde{u}_k(\omega) &= \int_0^{+\infty} e^{i\omega t} u_k(t) dt, \quad \Im(\omega) > 0, \\ \tilde{f}_k(\omega, v) &= \int_0^{+\infty} e^{i\omega t} f_k(t, v) dt, \quad \Im(\omega) > 0, \quad v \in \mathbf{R}. \end{aligned}$$

Multiplying the Vlasov equation by  $e^{i\omega t}$  and integrating in  $t$ , (2.9) writes

$$\begin{cases} -i\omega \tilde{\tau}_k(\omega) = ik\tilde{u}_k(\omega) + ik \int_{\mathbf{R}} v \sqrt{f^0(v)} \tilde{f}_k(\omega, v) dv + \tau_k(0) \\ -i\omega \tilde{u}_k(\omega) = ik\tilde{\tau}_k(\omega) + u_k(0) \\ (-i\omega + ikv) \tilde{f}_k(\omega, v) - ikv \sqrt{f^0} \tilde{\tau}_k(\omega) = f_k(0, v). \end{cases}$$

Then one has

$$\begin{aligned}
-i\omega\tilde{\tau}_k(\omega) &= ik\tilde{u}_k(\omega) + ik \int_{\mathbf{R}} v\sqrt{f^0(v)}\tilde{f}_k(\omega, v) dv + \tau_k(0) \\
&= ik \left( -\frac{ik\tilde{\tau}_k(\omega) + u_k(0)}{i\omega} \right) + ik \int_{\mathbf{R}} v\sqrt{f^0(v)}\tilde{f}_k(\omega, v) dv + \tau_k(0) \\
&= -i\frac{k^2}{\omega}\tilde{\tau}_k(\omega) + ik \int_{\mathbf{R}} v\sqrt{f^0(v)}\tilde{f}_k(\omega, v) dv + \tau_k(0) - \frac{k}{\omega}u_k(0) \\
&= -i\frac{k^2}{\omega}\tilde{\tau}_k(\omega) + ik \int_{\mathbf{R}} v\sqrt{f^0(v)} \left( \frac{f_k(0, v) + ikv\sqrt{f^0(v)}\tilde{\tau}_k(\omega)}{-i\omega + ikv} \right) dv + \tau_k(0) - \frac{k}{\omega}u_k(0) \\
&= -i\frac{k^2}{\omega}\tilde{\tau}_k(\omega) + ik\tilde{\tau}_k(\omega) \int_{\mathbf{R}} \frac{ikv^2 f^0(v)}{-i\omega + ikv} dv + \tau_k(0) - \frac{k}{\omega}u_k(0) + ik \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(0, v)}{-i\omega + ikv} dv.
\end{aligned}$$

Then one has

$$\tilde{\tau}_k(\omega) = \left( \frac{k^2}{\omega^2} - \frac{k}{\omega} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{v - \frac{\omega}{k}} dv \right) \tilde{\tau}_k(\omega) + \frac{i}{\omega} \left( \tau_k(0) - \frac{k}{\omega}u_k(0) + \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}f_k(0, v)}{v - \frac{\omega}{k}} dv \right),$$

which rewrites

$$\tilde{\tau}_k(\omega) = \frac{\mathcal{N}(\omega, k)}{\mathcal{D}(\omega, k)} \tag{2.10}$$

with

$$\mathcal{N}(\omega, k) = \frac{i}{\omega} \left( \hat{\tau}(0, k) - \frac{k}{\omega}\hat{u}(0, k) + \int_{\mathbf{R}} \frac{v\sqrt{f^0(v)}\hat{f}_0(k, v)}{v - \frac{\omega}{k}} dv \right)$$

and

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} + \frac{k}{\omega} \int_{\mathbf{R}} \frac{v^2 f^0(v)}{v - \frac{\omega}{k}} dv.$$

Furthermore, since

$$\int_{\mathbf{R}} \frac{v^2 e^{-v^2/2}}{v - \frac{\omega}{k}} dv = \int_{\mathbf{R}} \frac{v(v - \frac{\omega}{k})e^{-v^2/2}}{v - \frac{\omega}{k}} dv + \frac{\omega}{k} \int_{\mathbf{R}} \frac{ve^{-v^2/2}}{v - \frac{\omega}{k}} dv = \frac{\omega}{k} \int_{\mathbf{R}} \frac{ve^{-v^2/2}}{v - \frac{\omega}{k}} dv$$

and

$$\begin{aligned}
\int_{\mathbf{R}} \frac{ve^{-v^2/2}}{v - \frac{\omega}{k}} dv &= \int_{\mathbf{R}} \frac{(v - \frac{\omega}{k})e^{-v^2/2}}{v - \frac{\omega}{k}} dv + \frac{\omega}{k} \int_{\mathbf{R}} \frac{e^{-v^2/2}}{v - \frac{\omega}{k}} dv \\
&= \sqrt{2\pi} + \frac{\omega}{k} \int_{\mathbf{R}} \frac{e^{-v^2/2}}{v - \frac{\omega}{k}} dv.
\end{aligned}$$

One obtains that  $\mathcal{D}$  writes

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} + \left[ 1 + \frac{\omega}{\sqrt{2\pi}k} \int_{\mathbf{R}} \frac{e^{-v^2/2}}{v - \frac{\omega}{k}} dv \right].$$

□

For all  $k \neq 0$ , the function  $\mathcal{D}(\cdot, k)$  is analytic on the half-space  $\Im(\omega) > 0$ . One can define an analytic continuation of  $\mathcal{D}$  on  $\mathbf{C}^*$ . Let  $Z$  be a function defined on  $\mathbf{C}$  by

$$Z(\zeta) = \sqrt{\pi}e^{-\zeta^2} \left( i - \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{t^2} dt \right).$$

Then  $Z$  is analytic on  $\mathbf{C}$  and  $\forall \Im(\zeta) > 0$ ,

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} \frac{e^{-v^2}}{v - \zeta} dv.$$

Therefore, one can do an analytic continuation of  $\zeta \mapsto \int_{\mathbf{R}} \frac{e^{-v^2}}{v - \zeta} dv$  on  $\mathbf{C}$ . Then, the function  $\mathcal{D}(\omega, k)$  can also be continued analytically on  $\mathbf{C}^*$  by the formula

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} + \left[ 1 + \frac{\sqrt{\pi} \omega}{\sqrt{2} k} e^{-\frac{\omega^2}{2k^2}} \left( i - \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{\sqrt{2}k}} e^{t^2} dt \right) \right].$$

The function  $\mathcal{N}$  can also be continued analytically in the same manner, and then  $\tilde{\tau}_k(\omega)$  is continued analytically on  $\mathbf{C}^*$ . At this point, we follow what is classically done in the plasma physics community [31, 33] for the Vlasov-Poisson system and we assume that  $\mathcal{D}$  only has finitely many zeros (which seems reasonable to prove) and that they are all simple (a proof of this fact seems much more difficult). So that the Laplace inversion theorem applies and one has

$$\tau_k(t) = \frac{1}{2i\pi} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-i\omega t} \tilde{\tau}_k(\omega) d\omega.$$

One can use the residue Theorem to compute this integral. One obtains the following expansion

$$\tau_k(t) = \sum_{\omega_j} \text{Res}(\tilde{\tau}_k, \omega_j) e^{-i\omega_j t}. \quad (2.11)$$

where the  $\omega_j$  are the roots of  $\mathcal{D}$ , which is a finite set, and  $\text{Res}(\tilde{\tau}_k, \omega_j)$  is the residue of  $\tilde{\tau}_k$  at  $\omega_j$ . We then obtain that the dominant term of this expansion decay exponentially. The expression of  $\mathcal{D}$  when the constant are not normalized to 1 (one must redo the computation of the section 2.3 with the system (2.2)) is

$$\mathcal{D}(\omega, k) = 1 - c_0^2 \frac{k^2}{\omega^2} + \frac{4\pi r_p^3 \varrho_0 c_0^2}{3\alpha_0} \left[ 1 + \frac{\sqrt{\pi} \omega}{\sqrt{2} k} e^{-\frac{\omega^2}{2k^2}} \left( i - \frac{2}{\sqrt{\pi}} \int_0^{\frac{\omega}{\sqrt{2}k}} e^{t^2} dt \right) \right].$$

The  $\omega_j$  are the solutions of  $\mathcal{D}(\omega, k) = 0$  for a given  $k$ . As all the terms in the sum (2.11) decay exponentially fast, only the term corresponding to the  $\omega_j$  with the largest imaginary part will remain for large time, and the others will quickly become negligible. Denote  $\omega_r = \Re(\omega_j)$ ,  $\omega_i = \Im(\omega_j)$ , and  $\text{Res}(\tilde{\tau}_k, \omega_j) = r e^{i\varphi}$ , one notices that there is always a root of the form  $\omega_r + i\omega_j$  associated to  $r e^{i\varphi}$  and one of the form  $-\omega_r + i\omega_j$  associated to  $r e^{-i\varphi}$ . Therefore, by considering only the two roots with the largest  $\omega_i$ , one has

$$\tau_k(t) \approx r e^{i\varphi} e^{-(\omega_r + i\omega_i)t} + r e^{-i\varphi} e^{(-\omega_r + i\omega_i)t} = 2r e^{\omega_i t} \cos(\omega_r t - \varphi). \quad (2.12)$$

To compute the two roots with the largest imaginary part of  $\mathcal{D}$ , we use a standard numerical method such as a Newton's scheme from the python library SCIPY. For example with  $\frac{4}{3}\pi r_p^3 = 0.3$ ,  $\varrho_0 = 1$  and  $k = 0.5$ , one obtains that the two roots with the largest imaginary part are

$$\omega \approx \pm 0.4969 - 0.1295i.$$

### 3 Numerical illustrations

In this section, we illustrate the results of Theorem 1 in dimension 1+1. We show results on the linear and nonlinear equations. We consider a uniform discretisation of the phase space  $\mathbf{T} \times \mathbf{R}$ , with steps  $\Delta x$ ,  $\Delta v$ . In all the numerical results, we chose  $r_p$  so that  $\frac{4}{3}\pi r_p^3 = 0.3$ .

### 3.1 Numerical methods

We describe here the numerical schemes used in the next section for the purpose of illustration. We present two schemes. The first one is for the nonlinear problem, which is the most interesting one in view of possible physical applications. The second one is for the linearized problem because it is a direct illustration of the theoretical results.

#### 3.1.1 Nonlinear equations

We start with the nonlinear equations

$$\begin{cases} \partial_t f + v \partial_x f - \partial_x p \partial_v f = 0 \\ \partial_t(\alpha \varrho) + \partial_x(\alpha \varrho u) = 0 \\ \partial_t(\alpha \varrho u) + \partial_x(\alpha \varrho u^2) + \alpha \partial_x p = 0 \\ p(\varrho) = \varrho^\gamma, \gamma = 1.4. \end{cases} \quad (3.1)$$

We use a semi-Lagrangian method for the Vlasov equation and a Lax-Wendroff scheme for the fluid part. Multiplying the Vlasov equation in (3.1) by  $v$  and integrating in  $v$ , then summing with the momentum equation of the fluid, one sees that the fluid part of the equation can be written in a conservative form

$$\begin{cases} \partial_t(\alpha \varrho) + \partial_x(\alpha \varrho u) = 0 \\ \partial_t \left( \alpha \varrho u + \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v \, dv \right) + \partial_x(\alpha \varrho u^2) + \partial_x p + \partial_x \left( \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} v^2 f \, dv \right) = 0 \\ \partial_t f + v \partial_x f - \partial_x p \partial_v f = 0. \end{cases}$$

It writes

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f) = 0$$

with

$$\mathbf{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} := \begin{pmatrix} \alpha \varrho \\ \alpha \varrho u + \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v \, dv \end{pmatrix}$$

and

$$\mathbf{F}(\mathbf{U}, f) = \begin{pmatrix} \alpha \varrho u \\ \alpha \varrho u^2 + p + \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v^2 \, dv \end{pmatrix} = \begin{pmatrix} U_2 - m_* \int_{\mathbf{R}} f v \, dv \\ \frac{(U_2 - \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v \, dv)^2}{U_1} + \frac{U_1^\gamma}{\alpha^\gamma} + \int_{\mathbf{R}} f v^2 \, dv \end{pmatrix}.$$

The jacobian matrix of  $\mathbf{F}$  with respect to  $\mathbf{U}$  is

$$\mathbf{A}(\mathbf{U}, f) := \nabla_{\mathbf{U}} \mathbf{F}(\mathbf{U}, f) = \begin{pmatrix} 0 & 1 \\ -\frac{(U_2 - \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v \, dv)^2}{U_1^2} + \frac{\gamma U_1^{\gamma-1}}{\alpha^\gamma} & \frac{2(U_2 - \frac{4}{3} \pi r_p^3 \int_{\mathbf{R}} f v \, dv)}{U_1} \end{pmatrix}. \quad (3.2)$$

Its eigenvalues are

$$\lambda_{\pm} = u \pm \sqrt{\frac{p'(\varrho)}{\alpha}}.$$

This allows us to use a classical hyperbolic numerical scheme to discretize the fluid part, and to immediately obtain a scheme preserving the mass and the total momentum.

The numerical method to discretize the system (3.1) is described as follow: Given  $(\mathbf{U}^n, f^n)$  at a given time  $t^n$ .

- Compute  $f^*$  by solving the free transport  $\partial_t f + v \partial_x f = 0$  with a semi-lagrangian scheme during a timestep  $\Delta t$  with initial condition  $f^n$ .

- Compute  $f^{n+1}$  by solving  $\partial_t f - \partial_x p \partial_v f = 0$  with a semi-lagrangian scheme during a timestep  $\Delta t$  with initial condition  $f^*$ .
- Compute  $\mathbf{U}^{n+1}$  by solving  $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f)$  with a Lax-Wendroff scheme during a timestep  $\Delta t$  with initial condition  $(\mathbf{U}^n, f^{n+1})$ .

The semi-lagrangian method [10, 32] is a classical method to discretize transport equations, so we do not describe it here, only the Lax-Wendroff scheme used here is somehow non standard and we describe it hereafter.

Given initial data  $(\mathbf{U}^n, f^{n+1})$ , the quantity  $\mathbf{U}^{n+1}$  is computed by the Lax-Wendroff scheme [19]:

$$\begin{aligned} \mathbf{U}^{n+1} &= \mathbf{U}^n - \frac{\Delta t}{2\Delta x} (\mathbf{F}(\mathbf{U}_{j+1}^n, f_{j+1}^{n+1}) - \mathbf{F}(\mathbf{U}_{j-1}^n, f_{j-1}^{n+1})) \\ &+ \frac{\Delta t^2}{2\Delta x^2} \left( \mathbf{A}_{j+1/2}^n (\mathbf{F}(\mathbf{U}_{j+1}^n, f_{j+1}^{n+1}) - \mathbf{F}(\mathbf{U}_j^n, f_j^{n+1})) - \mathbf{A}_{j-1/2}^n (\mathbf{F}(\mathbf{U}_j^n, f_j^{n+1}) - \mathbf{F}(\mathbf{U}_{j-1}^n, f_{j-1}^{n+1})) \right), \end{aligned}$$

where  $\mathbf{A}_{j+1/2}^n$  is the jacobian (3.2) evaluated at some average state:

$$\mathbf{A}_{j+1/2}^n = \mathbf{A} \left( \frac{\mathbf{U}_{j+1}^n + \mathbf{U}_j^n}{2}, \frac{f_{j+1}^{n+1} + f_j^{n+1}}{2} \right).$$

Following classical results, the scheme is stable under the CFL condition:

$$\lambda_{\pm} \frac{\Delta t}{\Delta x} \leq \frac{1}{2}.$$

Notice that  $\lambda$  is in  $1/\alpha$  so small value of  $\alpha$  could lead very small time step  $\Delta t$ .

### 3.1.2 Linear equations

We start with the linearized equations

$$\begin{cases} \alpha_0 \varrho_0 \partial_t \tau = \alpha_0 \partial_x u + \frac{4}{3} \pi r_p^3 \partial_x \int_{\mathbf{R}} v \sqrt{f^0} f \, dv \\ \alpha_0 \varrho_0 \partial_t u = \alpha_0 \varrho_0^2 c_0^2 \partial_x \tau \\ \partial_t f + v \partial_x f - \varrho_0^2 c_0^2 \sqrt{f^0(v)} v \partial_x \tau = 0. \end{cases} \quad (3.3)$$

One notices that the variable  $v$  only acts as a parameter. A discretization in  $v$  yields

$$\begin{cases} \alpha_0 \varrho_0 \partial_t \tau = \alpha_0 \partial_x u + \frac{4}{3} \pi r_p^3 \partial_x \sum_k v_k \sqrt{f^0(v_k)} f(t, x, v_k) \Delta v \\ \alpha_0 \varrho_0 \partial_t u = \alpha_0 \varrho_0^2 c_0^2 \partial_x \tau \\ \partial_t f(t, x, v_k) + v_k \partial_x f(t, x, v_k) - \varrho_0^2 c_0^2 \sqrt{f^0(v_k)} v_k \partial_x \tau = 0, \quad \forall k \in \llbracket 0, N \rrbracket \end{cases}$$

It can be recast as a linear system of conservations laws

$$\partial_t \mathbf{W} + A \partial_x \mathbf{W} = 0,$$

with

$$\mathbf{W} = \begin{pmatrix} \alpha_0 \varrho_0 \tau \\ \alpha_0 \varrho_0 u \\ f(\cdot, \cdot, v_0) \\ \vdots \\ f(\cdot, \cdot, v_N) \end{pmatrix}$$

and  $A$  a constant matrix. We use a Lax-Wendroff scheme under the form [19]

$$\mathbf{W}^{n+1} = \mathbf{W}^n - \frac{\Delta t}{2\Delta x} A (\mathbf{W}_{j+1}^n - \mathbf{W}_{j-1}^n) + \frac{\Delta t^2}{2\Delta x^2} A^2 (\mathbf{W}_{j+1}^n - 2\mathbf{W}_j^n + \mathbf{W}_{j-1}^n).$$

### 3.2 Numerical results

In this section we consider the nonlinear system (3.1) with initial data

$$\begin{cases} \varrho(t=0, x) = 1, \\ u(t=0, x) = 0, \\ f(t=0, x, v) = \frac{1}{\sqrt{2\pi}}(1 + \varepsilon \cos(kx)\sqrt{f^0(v)})e^{-v^2/2}, \end{cases}$$

and the linear system (3.3),

$$\begin{cases} \tau(t=0, x) = 0, \\ u(t=0, x) = 0, \\ f(t=0, x, v) = \frac{1}{\sqrt{2\pi}} \cos(kx)e^{-v^2/2}. \end{cases}$$

The coefficient are  $\varepsilon = 0.001$  and  $k = 0.5$ . The domain of computation  $[0, 4\pi] \times [-10, 10]$  with periodic boundary condition. Using (2.12), an approximation of  $\tau(t, x)$  writes

$$\tau(t, x) \approx 4re^{\omega_i t} \sin(kx) \cos(\omega_r t - \varphi).$$

As it is done in plasma physics [31,33], this allows in our case to write the following family of reference solutions

$$\tau_{\text{ref}}(t, x) = 4re^{-0.1295t} \sin(0.5x) \cos(0.4969t - \varphi),$$

where  $r \in \mathbf{R}$  is the amplitude and  $\varphi \in \mathbf{R}$  is the phase.

The numerical results obtained with the schemes described in section 3.1 are visible in Figure 1 for the nonlinear case and in Figure 2 for the linear case for  $r = -6.5$  and  $\varphi = 4$ . One notices that the numerical solutions computed by both schemes are of the form predicted by the linear theory. Moreover the expected rate of decay is observed with great accuracy, both in the linear and nonlinear case. This indicates that the linear damping effect survives the nonlinear effect of the nonlinear system. In Figure 3 and 4, one can see the strong oscillations in the  $v$ -variable. This is a typical effect seen in plasma physics and it is sometime called *filamentation*. It is also an illustration of the weak convergence of the perturbation of the distribution function to 0.



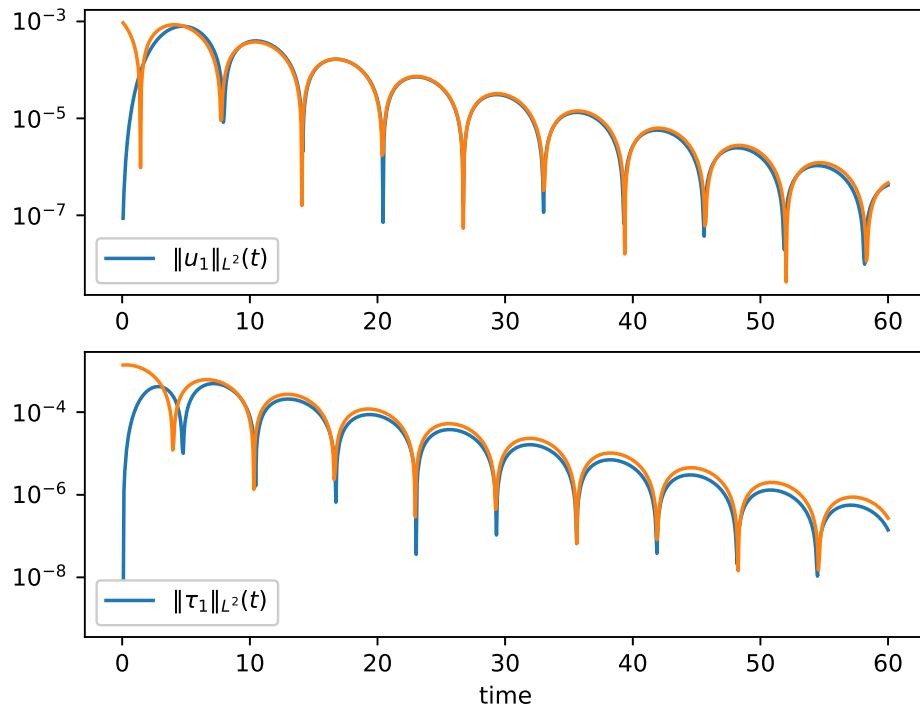


Figure 1: Decay of the acoustic energy computed with the nonlinear equations. In blue, the numerical solution and in orange the solution predicted by the linear theory.

## Damping

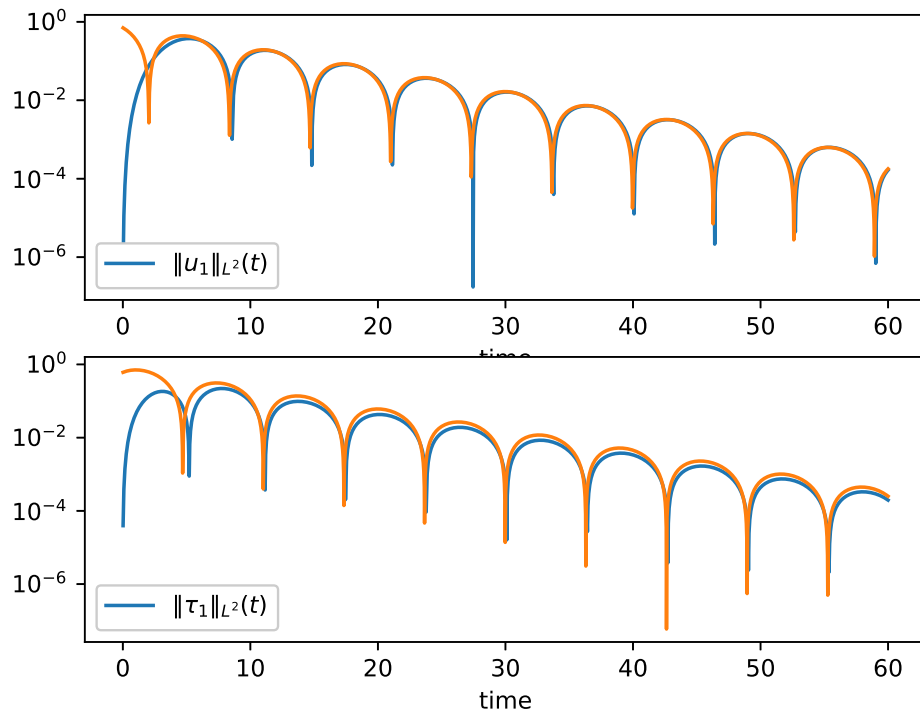


Figure 2: Decay of the acoustic energy computed with the linear equations. In blue, the numerical solution and in orange the solution predicted by the linear theory.

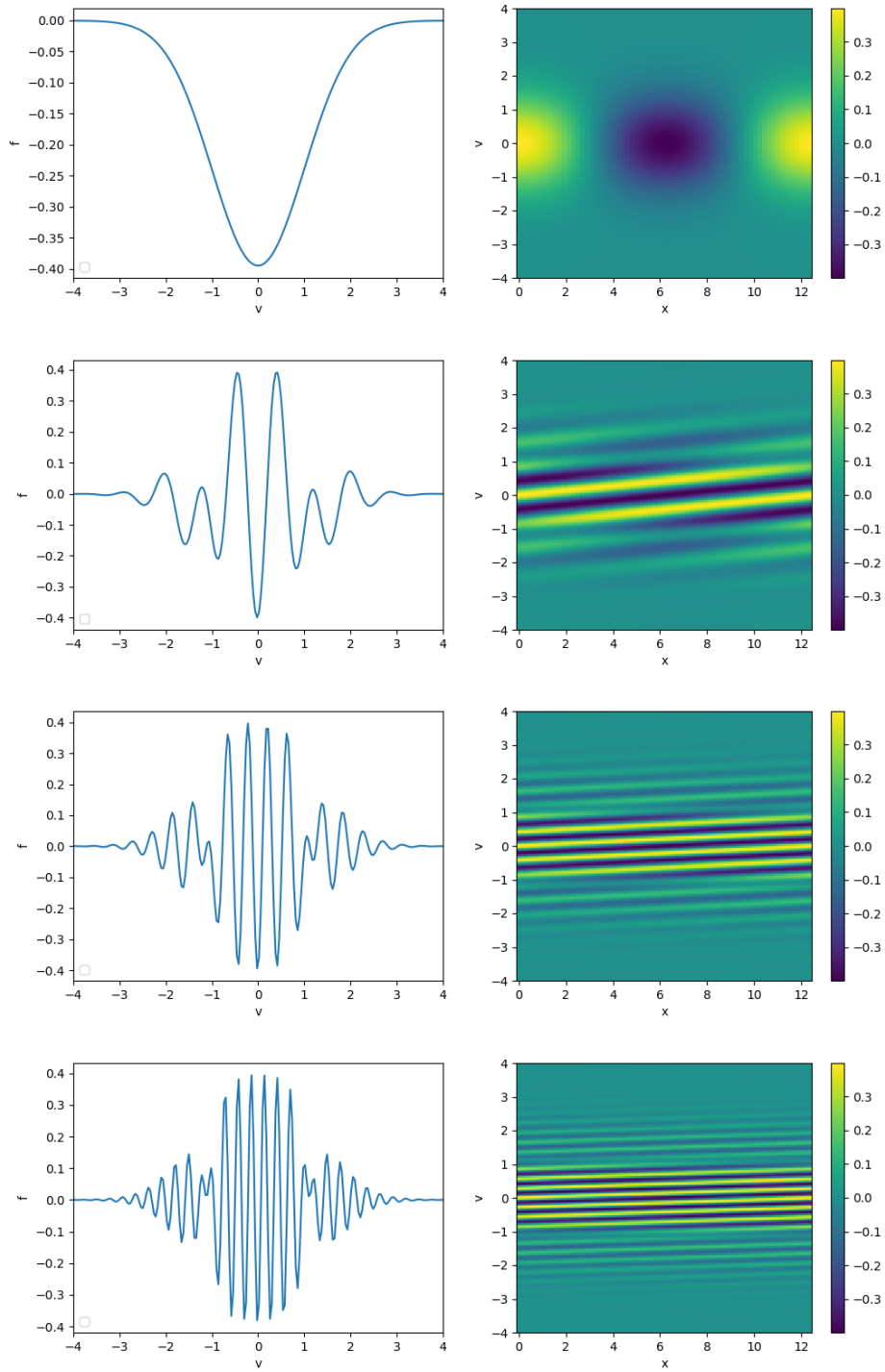


Figure 3: Filamentation or weak convergence of the distribution function in the linear equations for different times. In descending order,  $t = 0$ ,  $t = 15$ ,  $t = 30$ ,  $t = 45$ .

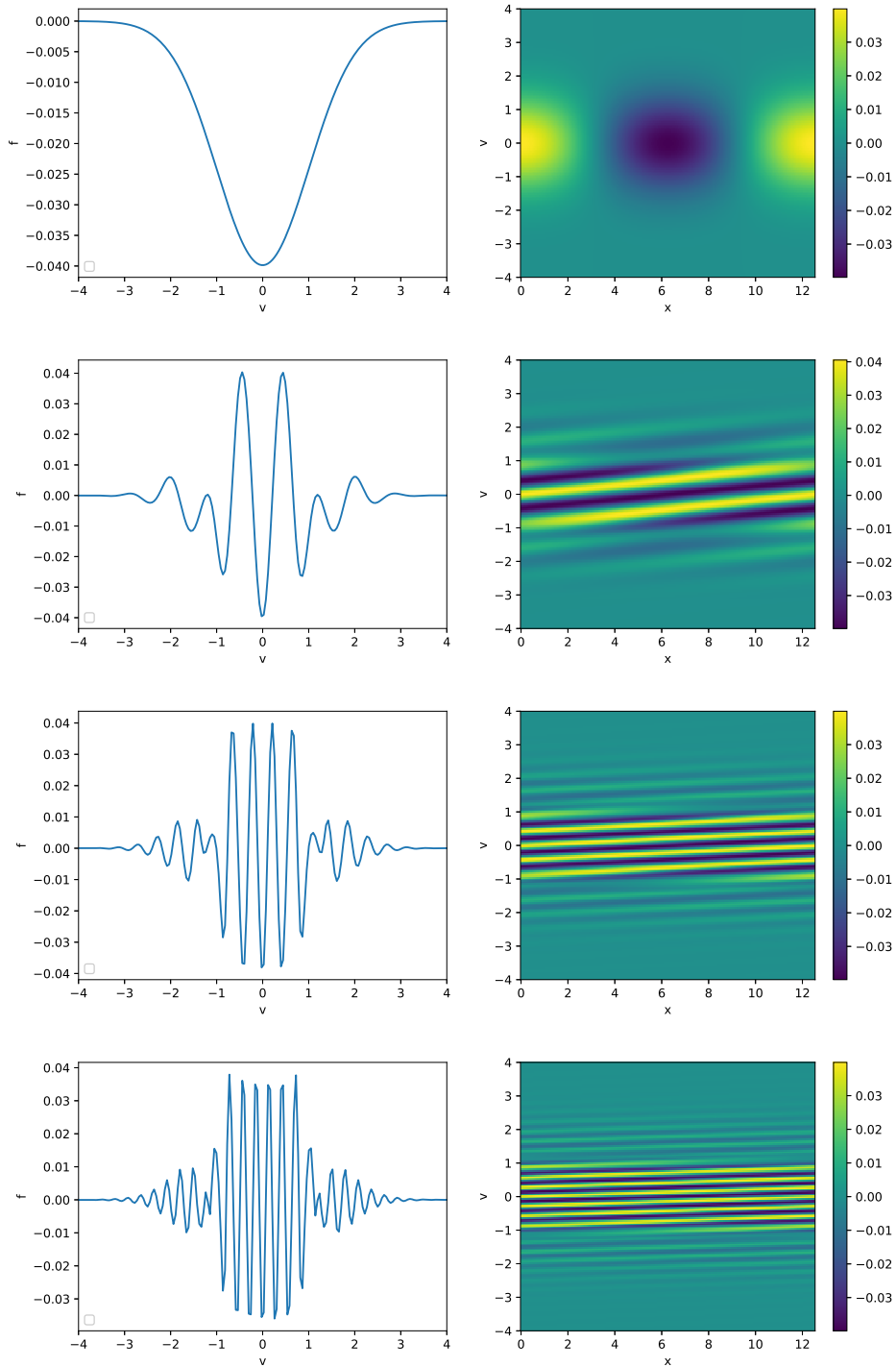


Figure 4: Filamentation or weak convergence of  $f - f^0$  in the nonlinear equations for different times. In descending order,  $t = 0$ ,  $t = 15$ ,  $t = 30$ ,  $t = 45$ .

## 4 The case with friction

The introduction of a nonzero friction  $D_* > 0$  is important in view of physical applications [8]. It introduces an explicit dissipation mechanism in the equations.

Since there is no such mechanism in the Vlasov-Poisson equations, the way to generalise the mathematical analysis is not obvious. In what follows, we present preliminary results on the problem for small friction.

Following [8], the linearization of the system (1.2) with friction  $d_\star = \frac{D_\star}{\frac{4}{3}\pi r_p^3} > 0$  yields

$$\begin{cases} \partial_t \tau = \partial_x u + \partial_x \int_{\mathbf{R}} \sqrt{f^0} e^{d_\star t} f v \, dv \\ \partial_t u = \partial_x \tau + d_\star \int_{\mathbf{R}} v \sqrt{f^0} e^{d_\star t} g \, dv - d_\star u \int_{\mathbf{R}} f^0 \, dv \\ \partial_t f + v \partial_x f - \sqrt{f^0} e^{d_\star t} v \partial_x \tau = d_\star \sqrt{f^0} e^{d_\star t} v u - d_\star f + d_\star \partial_v (v f). \end{cases}$$

Applying the Fourier-Laplace transform, one gets

$$\begin{cases} -i\omega \tilde{\tau}_k(\omega) = ik \tilde{u}_k(\omega) + ik \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega - id_\star) \, dv + \tau_k(0) \\ -i\omega \tilde{u}_k(\omega) = ik \tilde{\tau}_k(\omega) + d_\star \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega - id_\star) \, dv - d_\star \tilde{u}_k(\omega) \int_{\mathbf{R}} f^0 \, dv + u_k(0) \\ (-i\omega + ikv) \tilde{f}_k(\omega, v) = ikv \sqrt{f^0} \tilde{\tau}_k(\omega - id_\star) + d_\star \tilde{u}_k(\omega - id_\star) v \sqrt{f^0} + d_\star v \partial_v \tilde{f}_k(\omega, v) + f_k(0, v). \end{cases}$$

If one assumes that the coefficient  $d_\star$  is small, another linearization procedure based on a Taylor expansion with respect to  $d_\star$  yields

$$\tilde{\tau}_k(\omega - id_\star) = \tau_k(\omega) - id_\star \partial_\omega \tau_k(\omega) + O(d_\star^2),$$

with similar formula for  $\tilde{u}_k$  and  $\tilde{f}_k$ . Injecting those expansions and dropping the quadratic terms in  $d_\star$ , it yields

$$\begin{cases} -i\omega \tilde{\tau}_k(\omega) = ik \tilde{u}_k(\omega) + ik \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega) \, dv + d_\star k \int_{\mathbf{R}} v \sqrt{f^0} \partial_\omega \tilde{f}_k(\omega) \, dv + \tau_k(0) \\ -i\omega \tilde{u}_k(\omega) = ik \tilde{\tau}_k(\omega) + d_\star \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega) \, dv - d_\star \tilde{u}_k(\omega) \int_{\mathbf{R}} f^0 \, dv + u_k(0) \\ (-i\omega + ikv) \tilde{f}_k(\omega, v) = ikv \sqrt{f^0} \tilde{\tau}_k(\omega) - d_\star v \sqrt{f^0} \partial_\omega \tilde{\tau}_k(\omega) + d_\star \tilde{u}_k(\omega) v \sqrt{f^0} + d_\star v \partial_v \tilde{f}_k(\omega, v) + f_k(0, v). \end{cases}$$

One obtains

$$\begin{aligned} \tilde{f}_k(\omega, v) &= \frac{ikv \sqrt{f^0} \tilde{\tau}_k(\omega)}{ikv - i\omega} + \frac{f_k(0, v)}{ikv - i\omega} \\ &\quad + \frac{d_\star}{ikv - i\omega} \left[ \tilde{u}_k(\omega) v \sqrt{f^0} + v \partial_v \tilde{f}_k(\omega, v) - v \sqrt{f^0} \partial_\omega \tilde{\tau}_k(\omega) \right], \end{aligned}$$

and

$$\begin{aligned} \tilde{u}_k(\omega) &= \frac{-\frac{k}{\omega} \tilde{\tau}_k(\omega) + \frac{i}{\omega} u_k(0) - \frac{id_\star}{\omega} \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega) \, dv}{1 - \frac{id_\star}{\omega} \int_{\mathbf{R}} f^0 \, dv} \\ &= \frac{-k \tilde{\tau}_k(\omega) + i u_k(0) - id_\star \int_{\mathbf{R}} v \sqrt{f^0} \tilde{f}_k(\omega) \, dv}{\omega - id_\star \int_{\mathbf{R}} f^0 \, dv}. \end{aligned}$$

Then  $\tilde{f}_k(\omega, v)$  writes (dropping quadratic terms in  $d_\star$ )

$$\begin{aligned} \tilde{f}_k(\omega, v) &= \frac{ikv \sqrt{f^0} \tilde{\tau}_k(\omega)}{ikv - i\omega} + \frac{f_k(0, v)}{ikv - i\omega} \\ &\quad + \frac{d_\star}{ikv - i\omega} \frac{v \sqrt{f^0}}{\omega - id_\star \int_{\mathbf{R}} f^0 \, dv} \left[ -k \tilde{\tau}_k(\omega) + i u_k(0) \right] \\ &\quad + \frac{d_\star}{ikv - i\omega} \left[ v \partial_v \tilde{f}_k(\omega, v) - v \sqrt{f^0} \partial_\omega \tilde{\tau}_k(\omega) \right]. \end{aligned}$$

For  $\tilde{u}_k$ , one has

$$\tilde{u}_k(\omega) = \frac{-k\tilde{\tau}_k(\omega) + iu_k(0) + d_\star k \int_{\mathbf{R}} \frac{v^2 f^0}{ikv - i\omega} dv \tilde{\tau}_k(\omega) - id_\star \int_{\mathbf{R}} \frac{v\sqrt{f^0} f_k(0,v)}{ikv - i\omega} dv}{\omega - id_\star \int_{\mathbf{R}} f^0 dv}.$$

For  $\tilde{\tau}_k$ , one has

$$\begin{aligned} -i\omega\tilde{\tau}_k(\omega) &= \frac{-k\tilde{\tau}_k(\omega) + iu_k(0) + d_\star k \int_{\mathbf{R}} \frac{v^2 f^0}{ikv - i\omega} dv \tilde{\tau}_k(\omega) - id_\star \int_{\mathbf{R}} \frac{v\sqrt{f^0} f_k(0,v)}{ikv - i\omega} dv}{\omega - id_\star \int_{\mathbf{R}} f^0 dv} \\ &+ ik \int_{\mathbf{R}} \frac{v^2 f^0}{ikv - i\omega} dv \tilde{\tau}_k(\omega) + \int_{\mathbf{R}} \frac{v\sqrt{f^0} f_k(0,v)}{ikv - i\omega} dv \\ &+ \frac{d_\star ik}{\omega - id_\star \int_{\mathbf{R}} f^0 dv} \int_{\mathbf{R}} \frac{v^2 f^0}{ikv - i\omega} dv [-k\tilde{\tau}_k(\omega) + iu_k(0)] \\ &+ id_\star k \int_{\mathbf{R}} \frac{v^2 \sqrt{f^0} \partial_v \tilde{f}_k(\omega, v)}{ikv - i\omega} dv - id_\star k \int_{\mathbf{R}} \frac{v^2 f^0}{ikv - i\omega} dv \partial_\omega \tilde{\tau}_k(\omega) \\ &+ ikd_\star \frac{\partial}{\partial \omega} \int_{\mathbf{R}} \frac{v^2 f^0 \tilde{\tau}_k(\omega)}{ikv - i\omega} dv + d_\star k \frac{\partial}{\partial \omega} \int_{\mathbf{R}} \frac{f_k(0,v)}{ikv - i\omega} dv \\ &+ \tau_k(0). \end{aligned}$$

Rewriting this equation, one gets a differential equation of the form

$$d_\star A(\omega) \partial_\omega \tilde{\tau}_k + B(\omega, d_\star) \tilde{\tau}_k + C(\omega, d_\star) = 0,$$

where  $A, B, C$  are functions of  $\omega$ . If  $d_\star = 0$ , one recovers equation (2.10). One would need to study this equation to see how the friction interacts with the damping mechanism. This is left for further research.

We nevertheless show numerical illustration for small value of  $d_\star$ . The results have been obtained with the nonlinear equations. We observe that for sufficiently small value of  $d_\star$ , the behaviour of the numerical solution is similar to the previous results, see Figure 5. But, with larger value of  $d_\star$ , we observe initially a damping phenomenon, which is replaced by the purely oscillatory phenomenon, see Figure 6. For even larger value of  $d_\star$ , we observe an oscillatory phenomenon, see Figure 7. The interpretation of those results with respect to the Lyapunov functional constructed in [8] remains to be done.

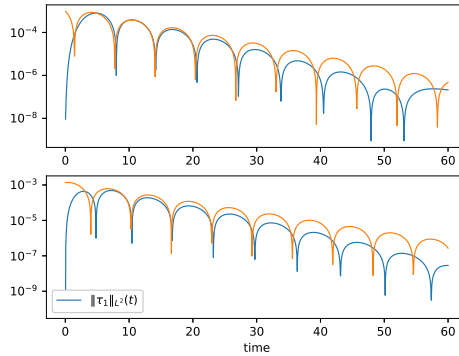


Figure 5: Decay of the acoustic energy with friction for the nonlinear equations for  $D_\star = 10^{-3}$ . In blue, the numerical solution and in orange, the solution in the case with no friction

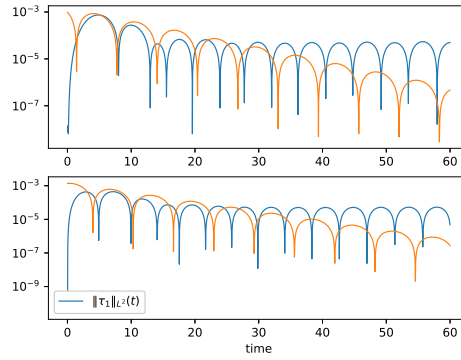


Figure 6: Decay of the acoustic energy with friction for the nonlinear equations for  $D_\star = 10^{-2}$ . In blue, the numerical solution and in orange, the solution in the case with no friction

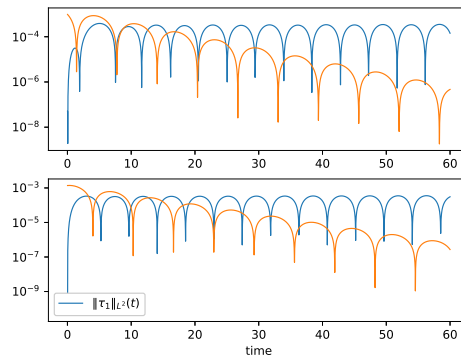


Figure 7: Decay of the acoustic energy with friction for the nonlinear equations for  $D_\star = 10^{-1}$ . In blue, the numerical solution and in orange, the solution in the case with no friction

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