On the 1d stochastic Schrödinger product

Aurélien Deya

ON THE 1D STOCHASTIC SCHRÖDINGER PRODUCT

AURÉLIEN DEYA

Abstract. We exhibit various restrictions about the wellposedness of the Schrödinger product

$$\mathcal{L} : z \longmapsto -\imath \int_0^t e^{\imath\Delta s}\big(z_s \cdot \Psi_s\big) ds$$

where Ψ refers to the so-called linear solution of the stochastic Schrödinger problem. We focus more specifically on the case where Ψ satisfies

$$(\imath\partial_t - \Delta)\Psi = \dot{B}, \quad \Psi_0 = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T}, \tag{0.1}$$

where $\dot{B}$ is a white noise in space with fractional time covariance of index $H > \frac{1}{2}$.

As an important consequence of our analysis, we obtain that if H is close to $\frac{1}{2}$ (that is $\dot{B}$ is close to a space-time white noise), then it is essentially impossible to treat the stochastic NLS problem

$$(\imath\partial_t - \Delta)u = \lambda\, u^p \overline{u}^q + \dot{B}, \quad u_0 = 0, \qquad \lambda \in \{-1, 1\},\ p, q \geqslant 1,\ t \in \mathbb{R},\ x \in \mathbb{T},$$

using only a first-order expansion of the solution ("$u = \Psi + z$").

1. Presentation of the problem

In this paper, we propose to point out some limitations in the analysis of the general class of non-linear stochastic Schrödinger models

$$(\imath\partial_t - \Delta)u = \lambda\, u^p \overline{u}^q + \dot{B}, \quad u(0, .) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{T}, \tag{1.1}$$

for fixed integers $p, q \geqslant 1$, $\lambda \in \{-1, 1\}$, and where $\dot{B}$ is a stochastic noise.

The study of noise influence on NLS models is a recurring topic in the SPDE literature. The most widely covered situation - by far - is that of a white noise in time with suitably colored spatial covariance. Provided such a noise is regular enough, the solution of (1.1) is expected to take values in a space of functions (almost surely); the powerful Itô integration tools then become available, which even opens the possibility to treat multiplicative perturbations (see e.g. [3, 7, 15, 16] for additive-noise models, and [1, 2, 5, 13] for a multiplicative noise).

In contrast with this "functional" case, we are here interested in rougher "distributional" situations - for which literature turns out to be much more scarce. Namely, in the continuation of [11], we will focus on examples for which the equation can only be handled in a space of *negative-order distributions*, and for which *renormalization procedures* are also required.

In order to implement these ideas, let us consider, throughout the analysis, the case of a white-in-space fractional-in-time noise. Thus, for some (fixed) index $H \in (\frac{1}{2}, 1)$, $\dot{B}$ is here defined as the centered Gaussian noise with covariance given by the formula

$$\mathbb{E}\Big[\dot{B}(s,x)\overline{\dot{B}(t,y)}\Big] = |t-s|^{2H-2}\delta_{\{x=y\}} \tag{1.2}$$

or otherwise stated: for all test-functions φ, ψ on $\mathbb{R} \times \mathbb{T}$,

$$\mathbb{E}\Big[\langle \dot{B}, \varphi\rangle\overline{\langle \dot{B}, \psi\rangle}\Big] = \int_{\mathbb{T}} dx \int_{\mathbb{R}^2} \frac{dsdt}{|t-s|^{2-2H}}\, \varphi(s,x)\overline{\psi(t,x)}. \tag{1.3}$$

Such a noise can be equivalently described through the expansion

$$\dot{B}(t,x) = \sum_{k\in\mathbb{Z}} \dot{\beta}_t^{(k)} e^{-\imath kx}, \tag{1.4}$$

where the $\dot{\beta}^{(k)}$'s are independent (real-valued) fractional noises on $\mathbb{R}$, with common Hurst index $H > \frac{1}{2}$.

Date: October 30, 2023.

Key words and phrases. Stochastic Schrödinger equation, fractional noise, renormalization.

Now, it is a classical fact that the expected regularity of the solution u in (1.1) is prescribed by that of the associated linear problem

$$(\imath\partial_t - \Delta)\Psi = \dot{B}, \quad \Psi(0,.) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{T}. \tag{1.5}$$

We will thus rely on the following preliminary lemma to identify the distributional cases of interest.

Lemma 1.1. *In the above setting, the following assertions hold true.*
(i) If $H > \frac{3}{4}$, then for every $T > 0$, one has $\Psi \in L^2([0,T] \times \mathbb{T})$ almost surely.
(ii) If $H \leqslant \frac{3}{4}$, then for every $T > 0$, one has $\mathbb{E}\Big[\big\|\Psi\big\|^2_{L^2([0,T]\times\mathbb{T})}\Big] = \infty$.

The statement of item *(ii)* corresponds to a slightly extended version of [11, Proposition 2.1, item (ii)] (take $H_0 = H$ and $H_1 = \frac{1}{2}$ therein), and it can be proved with the same arguments. As for item *(i)*, we have included a sketch of its proof in Section 6.1, for the sake of clarity.

Based on the result of Lemma 1.1, we henceforth focus on the case where $H \in (\frac{1}{2}, \frac{3}{4}]$, that is the situation where the solution Ψ of (1.5), and accordingly the solution u of (1.1), cannot be defined as functions.

Before we describe the so-called *first-order strategy* at the core of our investigations, let us introduce a few notations and spaces that will be used throughout the paper.

Notation 1.2. *From now on and for the rest of the paper:*

- *We use the classical convention $\mathbf{1}_{[0,a]} := -\mathbf{1}_{[a,0]}$ if $a < 0$.*
- *We fix a smooth symmetric function $\chi : \mathbb{R} \to [0,1]$ such that $\chi \equiv 1$ on $[-1,1]$ and $supp(\chi) \subset [-\frac{3}{2}, \frac{3}{2}]$.*
- *For every function $f : \mathbb{R} \to \mathbb{R}$ and every $\lambda \in \mathbb{R}$, we denote the Fourier transform (in time) as*

$$\mathcal{F}(f)(\lambda) := \int_{\mathbb{R}} dx\, e^{-\imath\lambda x} f(x).$$

- *We denote by $\mathcal{I}_\chi$ the local integration operator, that is for all $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$,*

$$\mathcal{I}_\chi f(t) := -\imath\chi(t) \int_0^t ds\, \chi(s) f(s). \tag{1.6}$$

We also denote by $\mathcal{I}_\chi(.,.)$ the Fourier kernel associated with this operator, and defined through the identity

$$\mathcal{F}\big(\mathcal{I}_\chi f\big)(\lambda) = \int_{\mathbb{R}} d\lambda_1\, \mathcal{I}_\chi(\lambda, \lambda_1) \mathcal{F}(f)(\lambda_1). \tag{1.7}$$

- *For every function $\varphi : \mathbb{T} \to \mathbb{R}$ and every $k \in \mathbb{Z}$, we set*

$$\varphi_k := \int_{\mathbb{T}} dx\, e^{-\imath kx} \varphi(x) = \frac{1}{2\pi} \int_0^{2\pi} dx\, e^{-\imath kx} \varphi(x). \tag{1.8}$$

- *We consider the general scale of anisotropic Besov spaces $\{X^b_c,\ b, c \in [0,1)\}$ defined through the norm*

$$\big\|z\big\|^2_{X^b_c} := \int_{\mathbb{R}} d\lambda \sum_{k\in\mathbb{Z}} \big\{\langle\lambda\rangle^{2b} + \langle k\rangle^{2c}\big\} \big|\mathcal{F}(z_k)(\lambda)\big|^2, \tag{1.9}$$

where the notation $\langle.\rangle$ refers to $\langle\lambda\rangle := (1 + |\lambda|^2)^{\frac{1}{2}}$. Also, we set $X^b := X^b_0$ and $X_c := X^0_c$.

- *We denote by $(\Omega, \mathfrak{F}, \mathbb{P})$ the complete probability space which accommodates the fractional noise $\dot{B}$ under consideration (or equivalently the one-parameter fractional noises $(\dot{\beta}^{(k)})$ in (1.4)).*
- *For any Banach space E, we define*

$$L^{\infty,-}(\Omega; E) := \bigcap_{p \geqslant 1} L^p(\Omega; E).$$

• *Last but not least, we denote by* $\dot{B}^{(n)}$ *the (spatial) regularization of* $\dot{B}$ *derived from* (1.4) *through the formula*

$$\dot{B}^{(n)}(t,x) = \sum_{k:\,\langle k\rangle \leqslant 2^n} \dot{\beta}_t^{(k)} e^{-\imath kx}. \tag{1.10}$$

1.1. **The first-order strategy and related questions.**

Recall that we concentrate here on situations where the solution u of (1.1) is not expected to be a function, so that there is no a priori interpretation of the product term $u^p\overline{u}^q$.

To overcome this problem, a natural idea - which has been extensively implemented in parabolic frameworks, as well as in [11, 12] - is to consider some first-order expansion of the solution. We propose to explain the details behind this approach at the level of the approximated equation first.

For $\dot{B}^{(n)}$ defined as in (1.10), let $u^{(n)}$ be the (well-defined) solution of the approximated equation

$$(\imath\partial_t - \Delta)u^{(n)} = \lambda\,(u^{(n)})^p(\overline{u^{(n)}})^q + \dot{B}^{(n)}, \quad u^{(n)}(0,.) = 0, \quad t\in\mathbb{R},\ x\in\mathbb{T}. \tag{1.11}$$

With the notation in (1.6), and setting

$$\varrho^{(n)} := \mathcal{I}_\chi\big(e^{\imath t\Delta}\cdot \dot{B}^{(n)}\big), \quad v^{(n)}(t) := e^{\imath t\Delta}u^{(n)}(t), \tag{1.12}$$

it is easy to check that we can (locally) recast (1.11) under the mild formulation

$$v^{(n)}(t) = \varrho^{(n)}(t) + \mathcal{I}_\chi\mathcal{M}^{(p,q)}\big(v^{(n)},\dots,v^{(n)}\big)(t), \tag{1.13}$$

where the product operator $\mathcal{M}^{(p,q)}$ is defined in Fourier coordinates by the formula

$$\begin{aligned}&\mathcal{M}^{(p,q)}(v^{(1)},\dots,v^{(p)},w^{(1)},\dots,w^{(q)})_k(t)\\ &= \sum_m \sum_{\substack{k_1,\dots,k_p\\ k_1+\dots+k_p=m}} \sum_{\substack{\ell_1,\dots,\ell_p\\ \ell_1+\dots+\ell_p=m-k}} e^{\imath t\Omega_{k,\underline{k},\underline{\ell}}} v_{k_1}^{(1)}(t)\cdots v_{k_p}^{(p)}(t)\overline{w_{\ell_1}^{(1)}}(t)\cdots\overline{w_{\ell_q}^{(q)}}(t),\end{aligned} \tag{1.14}$$

with

$$\Omega_{k,\underline{k},\underline{\ell}} := -k^2 - (k_1^2+\dots+k_p^2) + (\ell_1^2+\dots+\ell_q^2). \tag{1.15}$$

In light of (1.13), a natural first-order transformation of the problem simply consists in the consideration of the difference process

$$z^{(n)} := v^{(n)} - \varrho^{(n)},$$

which now satisfies the equation

$$z^{(n)}(t) = \mathcal{I}_\chi\mathcal{M}^{(p,q)}\big(z^{(n)}+\varrho^{(n)},\dots,z^{(n)}+\varrho^{(n)}\big)(t), \tag{1.16}$$

where the right-hand side can be readily expanded by multilinearity.

The whole point of this change of perspective can be roughly summed up as follows: taking the action of the operator $\mathcal{I}_\chi\mathcal{M}^{(p,q)}$ into account, we *hope* the difference-process $z^{(n)}$ to be sufficiently regular (and at least more regular than $\varrho^{(n)}$ or $v^{(n)}$) so that the product terms in (1.16) can now make sense as $n\to\infty$, with a suitable control in the scale X_c^b (see (1.9)).

As a first step of this analysis, we are naturally led to examine the "deterministic" component arising from the multilinear expansion of (1.16), that is the operation

$$z \longmapsto \mathcal{I}_\chi\mathcal{M}^{(p,q)}\big(z,\dots,z\big).$$

The wellposedness and stability of such a product in X_c^b is classically guaranteed by the condition $b>\frac{1}{2}$ (see e.g. [14]). For this reason, we shall henceforth assume that $b>\frac{1}{2}$ and thus concentrate on the restricted scale

$$\big\{X_c^b,\ b\in(\frac{1}{2},1),\ c\in[0,1)\big\}.$$

Then, when it comes to the analysis of the "stochastic" components in (1.16) (i.e., the terms involving at least one $\varrho^{(n)}$ in the expansion of the right-hand side), the following important observation immediately comes to mind.

Observation. In order to guarantee the convergence of equation (1.16), one should *at least* be able to address the following two problems:

(**P1**) Does the sequence of - explicit - stochastic processes

$$\mathcal{I}_\chi \mathcal{M}^{(p,q)}\big(\Xi^{(n)}, \dots, \Xi^{(n)}\big) \tag{1.17}$$

converge in $L^{\infty,-}(\Omega; X_c^b)$ as $n \to \infty$ (for b, c to be determined) ?

(**P2**) Does the stochastic Schrödinger product operation

$$\mathcal{L}^{(n)} : z \longmapsto \mathcal{I}_\chi \mathcal{M}^{(p,q)}\big(z, \Xi^{(n)}, \dots, \Xi^{(n)}\big) \tag{1.18}$$

converge as a *random operator* from X_c^b to X_c^b as $n \to \infty$ (for b, c to be determined) ?

The two above problems (**P1**) and (**P2**) will be our guidelines in the subsequent study. However, for a proper examination of these questions in the rough stochastic framework, the above formulations both need to be refined, which is the purpose of the two next sections.

1.1.1. *About renormalization.* Owing to the pathwise irregularity of $\dot{B}$ (especially as H gets close to $\frac{1}{2}$), the convergence of the process in (1.17) can only be achieved through a renormalization trick, as developed in Section 2.1 below. Consequently, equation (1.16) can only be handled in a renormalized sense.

Nevertheless, in order not to deviate from the existing SPDE literature (especially the known results in the heat or wave settings), we shall impose this renormalization procedure to be *explicit*. In other words, the transformation should only give rise to explicit renormalizing constants at the level of the approximated equation.

In order to achieve this objective, our strategy will obey the following two rules:

(**C1**) We *do allow* the use of (natural) renormalization procedures for the explicit process

$$\mathcal{I}_\chi \mathcal{M}^{(p,q)}\big(\Xi^{(n)}, \dots, \Xi^{(n)}\big).$$

(**C2**) We *do not allow* any renormalization (and so any deformation) for the general product operation

$$\mathcal{L}^{(n)} : z \longmapsto \mathcal{I}_\chi \mathcal{M}^{(p,q)}\big(z, \Xi^{(n)}, \dots, \Xi^{(n)}\big).$$

Remark 1.3. Observe that the condition (**C2**) immediately rules out any "a priori deformation of the product", such as a Skorohod-type interpretation of the problem (see e.g. [6]), or the renormalization trick implemented in [12] for the cubic model, namely:

$$(\imath\partial_t - \Delta)u = \left(|u|^2 - \int_{\mathbb{T}} |u|^2\right)u + \dot{B}, \quad u(0,.) = \Phi, \quad t \in \mathbb{R},\ x \in \mathbb{T}.$$

It is clear indeed that the correction term $\int_{\mathbb{T}} |u|^2$ derived from the latter transformation is not explicit, i.e. it is not explicitly defined in terms of $\dot{B}$.

With these considerations in mind, and following the condition (**C1**), we can refine the formulation of the problem (**P1**) as follows:

(**P1′**) Can the sequence of stochastic processes

$$\mathcal{I}_\chi \mathcal{M}^{(p,q)}\big(\Xi^{(n)}, \dots, \Xi^{(n)}\big)$$

be suitably renormalized so as to converge in $L^{\infty,-}(\Omega; X_c^b)$ (for b, c to be determined) ?

1.1.2. *Stochastic Schrödinger product as a random operator.*

Let us now go back to the formulation of the problem (**P2**), related to the control of the (Schrödinger) product operation $\mathcal{L}^{(n)}$ in (1.18). Following the above condition (**C2**), we intend to tackle this product operation directly, that is without any renormalization trick.

Recall that by (1.14), one has

$$\mathcal{L}^{(n)}(z)_k = \mathcal{I}_\chi\Big(t \mapsto \sum_m \sum_{\substack{k_1,\dots,k_p \\ k_1+\dots+k_p=m}} \sum_{\substack{\ell_1,\dots,\ell_p \\ \ell_1+\dots+\ell_p=m-k}} e^{\imath t\Omega_{k,\underline{k},\underline{\ell}}} z_{k_1}(t) \mathring{\mathcal{I}}^{(n)}_{k_2}(t)\cdots\mathring{\mathcal{I}}^{(n)}_{k_p}(t)\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_1}}(t)\cdots\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_q}}(t)\Big), \quad (1.19)$$

with $\Omega_{k,\underline{k},\underline{\ell}}$ given by (1.15).

Based on this expression, one can morally expect the desired regularization effect (in space) to stem from the integration (in time) of the exponential factor $e^{\imath t\Omega_{k,\underline{k},\underline{\ell}}}$ in (1.19). We are thus naturally led to decompose the expression of $\mathcal{L}^{(n)}$ as a sum of a "degenerate" (Schrödinger) product operator

$$\mathcal{L}^{\circ,(n)}(z)_k := \sum_m \sum_{\substack{k_1,\dots,k_p \\ k_1+\dots+k_p=m}} \sum_{\substack{\ell_1,\dots,\ell_p \\ \ell_1+\dots+\ell_p=m-k}} \mathbf{1}_{\{\Omega_{k,\underline{k},\underline{\ell}}=0\}} \mathcal{I}_\chi\Big(t \mapsto z_{k_1}(t) \mathring{\mathcal{I}}^{(n)}_{k_2}(t)\cdots\mathring{\mathcal{I}}^{(n)}_{k_p}(t)\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_1}}(t)\cdots\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_q}}(t)\Big), \quad (1.20)$$

and a "non-degenerate" (Schrödinger) product

$$\mathcal{L}^{\sharp,(n)}(z)_k := \mathcal{I}_\chi\Big(t \mapsto \sum_m \sum_{\substack{k_1,\dots,k_p \\ k_1+\dots+k_p=m}} \sum_{\substack{\ell_1,\dots,\ell_p \\ \ell_1+\dots+\ell_p=m-k}} \mathbf{1}_{\{\Omega_{k,\underline{k},\underline{\ell}}\neq 0\}} e^{\imath t\Omega_{k,\underline{k},\underline{\ell}}} z_{k_1}(t) \mathring{\mathcal{I}}^{(n)}_{k_2}(t)\cdots\mathring{\mathcal{I}}^{(n)}_{k_p}(t)\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_1}}(t)\cdots\overline{\mathring{\mathcal{I}}^{(n)}_{\ell_q}}(t)\Big). \quad (1.21)$$

Let us now particularize the formulation of the problem (**P2**) to each of these two components.

As far as $\mathcal{L}^{\circ,(n)}$ is concerned, observe that the degeneracy condition $\mathbf{1}_{\{\Omega_{k,\underline{k},\underline{\ell}}=0\}}$ actually offers drastic simplifications in the expression of the product, by introducing a strong correlation between the indexes $k, \underline{k}, \underline{\ell}$ (see for instance (1.26) in the case $p = q = 1$). Thanks to these simplifications, one can legitimately hope for a direct analysis of the operator and a direct treatment of the following question:

(**P2′**) Does the operator norm $\big\|\mathcal{L}^{\circ,(n)}\big\|_{X^b_c\to X^b_c}$ converge (in $L^{\mathscr{O},-}(\Omega)$) as $n\to\infty$ (for suitable b, c) ?

Unfortunately, due to the much higher sophistication of the non-degenerate component $\mathcal{L}^{\sharp,(n)}$ (see Remark 1.6 for a brief comparison between $\mathcal{L}^{\sharp,(n)}$ and $\mathcal{L}^{\circ,(n)}$ when $p = q = 1$), the evaluation of the random - implicitly defined - norm $\big\|\mathcal{L}^{\sharp,(n)}\big\|_{X^b_c\to X^b_c}$ turns out to be a much more difficult task. In fact, capturing the value of an operator norm in $L(X^b_c, X^b_c)$ (or in any other distributions scale) is known to be a rarely attainable objective in general, and this observation is all the more true in our random setting. For this reason, we shall instead focus on the analysis of a more tractable estimate of $\big\|\mathcal{L}^{\sharp,(n)}\big\|_{X^b_c\to X^b_c}$.

In order to introduce the latter quantity, observe that $\mathcal{L}^{\sharp,(n)}$ can be written in Fourier coordinates as

$$\mathcal{F}\big(\mathcal{L}^{\sharp,(n)}(z)_k\big)(\lambda) = \sum_{k_1}\int d\lambda_1 \big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\mathcal{F}(z_{k_1})(\lambda_1), \quad (1.22)$$

for some specific kernel $\mathcal{K}^{(n)}_\chi$ (see the expression (2.6) below for a description of $\mathcal{K}^{(n)}_\chi$ in the case $p = q = 1$). Now, based on this kernel formulation, it is easily checked (see Section 6.2.1 for details) that for every $z \in X^b_c$,

$$\big\|\mathcal{L}^{\sharp,(n)}(z)\big\|_{X^b_c} \leqslant \|z\|_{X^b_c}\cdot \mathcal{P}^{(n)}_{b,c}, \quad (1.23)$$

where $\mathcal{P}^{(n)}_{b,c}$ is given by

$$\mathcal{P}^{(n)}_{b,c} := \sum_{k_1,k_1'} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1'\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \Big| \sum_k \int d\lambda \, \{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\} \big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1'}(\lambda,\lambda_1')}\Big|^2. \tag{1.24}$$

With bound (1.23) in mind, the - explicit - estimate $\mathcal{P}^{(n)}_{b,c}$ is thus the quantity that will serve us as a landmark in the analysis of $\mathcal{L}^{\sharp,(n)}$, and more exactly as an approximation of the operator norm $\big\|\mathcal{L}^{\sharp,(n)}\big\|_{X^b_c\to X_c}$. Along this idea, let us particularize the previous control issue (**P2**) to $\mathcal{L}^{\sharp,(n)}$ through the following simplified version of the problem.

(**P2″**) Does the quantity $\mathcal{P}^{(n)}_{b,c}$ converge in $L^{\infty,-}(\Omega)$ (or even in $L^1(\Omega)$) as $n\to\infty$ (for suitable b,c) ?

Remark 1.4. In accordance with the developments in [11], we consider the approximation of $\big\|\mathcal{L}^{\sharp,(n)}\big\|_{X^b_c\to X_c}$ by $\mathcal{P}^{(n)}_{b,c}$ as a full part of the first-order strategy here described. Note that similar kernel-based estimates of operator norms are also extensively used in the so-called *theory of random tensors* developed by Deng, Nahmod and Yue (see e.g. item (5) of Proposition 5.1 or the proof of Proposition 6.1 in [10]).

We have included a short discussion about the sharpness of this approximation in Sections 6.2.1 and 6.2.2 below. In particular, it is therein shown that in the Young integration setting, the consideration of the corresponding quantity $\mathcal{P}^{(n)}$ allows us to recover the well-known threshold value $H=\frac{1}{2}$ for the Hurst index.

1.2. **Objective of the study.**

In the case of the simplest possible nonlinearity in (1.1) (i.e., $p=q=1$), the above-described first-order strategy is precisely the method that was implemented - with success - in [11], for a fractional noise of index $H>\frac{5}{8}$. In particular, the three questions (**P1′**), (**P2′**) and (**P2″**) all received *positive* answers in the latter situation.

Our aim in the present study is to show that this first-order approach is however not sufficient to cover the whole range $\frac{1}{2}<H<1$, thus advocating for more sophisticated developments in rougher situations than those treated in [11].

This conclusion will be derived from a close examination of the challenging convergence issues (**P1′**), (**P2′**) and (**P2″**). Just as in [11], we will restrict to the most elementary nonlinearity in (1.1), i.e. $p=q=1$, keeping in mind that even stronger restrictions should apply for larger p,q.

Notation 1.5. *We set from now on* $\mathcal{M}:=\mathcal{M}^{(1,1)}$ *(see* (1.14)*), and so*

$$\mathcal{M}(v,w)_k(t) := \sum_{k_1} e^{\imath t\Omega_{k,k_1}} v_{k+k_1}\overline{w_{k_1}}, \quad \textit{with} \quad \Omega_{k,k_1} := |k+k_1|^2-|k_1|^2-|k|^2 = 2kk_1. \tag{1.25}$$

In particular, $\mathbf{1}_{\{\Omega_{k,k_1}=0\}}=\mathbf{1}_{\{k=0\}\cup\{k_1=k\}}$*, and so*

$$\mathcal{L}^{\circ,(n)}(z)_k = \sum_{k_1}\mathbf{1}_{\{k=0\}\cup\{k_1=k\}}\mathcal{I}_\chi\Big(t\mapsto z_{k_1}(t)\overline{\mathcal{O}^{(n)}_{k_1-k}}(t)\Big), \tag{1.26}$$

while

$$\mathcal{L}^{\sharp,(n)}(z)_k = \mathbf{1}_{\{k\neq 0\}}\mathcal{I}_\chi\bigg(t\mapsto \sum_{k_1\neq k} e^{\imath t\Omega_{k,k_1-k}} z_{k_1}(t)\overline{\mathcal{O}^{(n)}_{k_1-k}}(t)\bigg). \tag{1.27}$$

Remark 1.6. By writing $\mathcal{L}^{\circ,(n)}(z)_k=\sum_{k_1}\mathcal{L}^{\circ,(n)}(z)_{kk_1}$ and similarly $\mathcal{L}^{\sharp,(n)}(z)_k=\sum_{k_1}\mathcal{L}^{\sharp,(n)}(z)_{kk_1}$, one immediately sees from (1.26)-(1.27) that $(\mathcal{L}^{\circ,(n)}(z)_{kk_1})$ morally reduces to two (time-dependent) vectors, while $(\mathcal{L}^{\circ,(n)}(z)_{kk_1})$ consists of an almost full (time-dependent) matrix. This of course gives us an idea of how much more sophisticated the non-degenerate operator $\mathcal{L}^{\sharp,(n)}$ with respect to $\mathcal{L}^{\circ,(n)}$.

2. Main results

With the above presentation of the problem in mind, we are in a position to state our main results related to the three central questions (**P1′**), (**P2′**) and (**P2″**). Recall that we have fixed $p = q = 1$ for the rest of the paper.

2.1. Problem (P1'): control of the renormalized Schrödinger product tree.

Let us start with the examination of the tree process $\Psi^{(n)} := \mathcal{I}_\chi \mathcal{M}\big(\Psi^{(n)}, \Psi^{(n)}\big)$, which we shall refer to as the *(Schrödinger) product tree* in the sequel.

As we evoked it earlier, this quantity needs to be renormalized before we can study its convergence. To this end, we will successively rely on three classical rescaling steps:

(**R**) A first partial space averaging

$$\mathcal{M}(\Psi^{(n)}, \Psi^{(n)}) \quad \longrightarrow \quad \mathcal{M}(\Psi^{(n)}, \Psi^{(n)}) - \Psi^{(n)} \int \overline{\Psi^{(n)}},$$

in the spirit of Bourgain's renormalization method for the cubic NLS model (see e.g. [4, 8]).

(**R′**) Then a more standard space averaging:

$$\mathcal{M}(\Psi^{(n)}, \Psi^{(n)}) - \Psi^{(n)} \int \overline{\Psi^{(n)}} \longrightarrow \widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)}) := \Big[\mathcal{M}(\Psi^{(n)}, \Psi^{(n)}) - \Psi^{(n)} \int \overline{\Psi^{(n)}}\Big] - \int \Big[\mathcal{M}(\Psi^{(n)}, \Psi^{(n)}) - \Psi^{(n)} \int \overline{\Psi^{(n)}}\Big]. \quad (2.1)$$

(**R″**) Finally, a stochastic Wick renormalization trick:

$$\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)}) \quad \longrightarrow \quad \widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)}) - \mathbb{E}\big[\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)})\big].$$

Remark 2.1. Observe that the rescaling terms in Steps (**R′**) and (**R″**) only involve reduced quantities, i.e. quantities depending on at most two of the three parameters (t, x, ω), which can indeed be expected from any reasonable renormalization trick.

On the other hand, the rescaling term in (**R**) still appeals to the "fully-dependent" quantity $\Psi^{(n)}$. However, the latter can easily be turned into a linear correction drift at the level of the approximated equation, making it acceptable in the procedure (see [11, Section 1.2] for details).

As a result of the three steps above, we derive the following renormalized version of the product tree:

$$\widetilde{\Psi\Psi}^{(n)}_k(t) := \mathcal{I}_\chi\Big(\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)})_k(.) - \mathbb{E}\big[\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)})_k(.)\big]\Big)(t), \quad (2.2)$$

where the renormalized product $\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)})$ is defined by (2.1). Based on the expression (1.25) of $\mathcal{M}$, this renormalized product can also be written in Fourier coordinates as

$$\widetilde{\mathcal{M}}(\Psi^{(n)}, \Psi^{(n)})_k(t) := \mathbf{1}_{\{k\neq 0\}} \sum_{k_1 \neq 0} e^{it\Omega_{k,k_1}} \Psi^{(n)}_{k+k_1} \overline{\Psi^{(n)}_{k_1}}. \quad (2.3)$$

In other words, we are here dealing with the "non-degenerate" part of $\Psi\Psi^{(n)}$, i.e. the summands for which the quantity Ω_{k,k_1} does not vanish.

Our main result about this (renormalized) product tree can now be stated as follows.

Proposition 2.2. *Assume that* $H \in (\frac{1}{2}, \frac{3}{4}]$. *Then the following picture holds true.*

(*i*) *For every* $0 \leqslant c < \frac{1}{2}$, *one has*

$$\sup_{n\geqslant 1} \mathbb{E}\Big[\Big\|\widetilde{\Psi\Psi}^{(n)}\Big\|^2_{X_c}\Big] < \infty.$$

(ii) For every $\frac{1}{2} < b < 2H - \frac{1}{2}$*, one has*

$$\sup_{n\geqslant 1} \mathbb{E}\Big[\Big\| \widetilde{\Psi}^{(n)} \Big\|_{X^b}^2 \Big] < \infty.$$

(iii) If $b = 2H - \frac{1}{2}$*, then*

$$\mathbb{E}\Big[\Big\| \widetilde{\Psi}^{(n)} \Big\|_{X^b}^2 \Big] \overset{n\to\infty}{\longrightarrow} \infty.$$

The above properties thus provide us with the exact time regularity of $\widetilde{\Psi}$ (namely $(2H-\frac{1}{2})-$), which, given the central role of the process in the dynamics of (1.1), can be seen as an important result of independent interest.

In view of our present objective, the optimal condition $\frac{1}{2} < b < 2H - \frac{1}{2}$ will be our first restriction in the application of the first-order strategy described in Section 1.1.

2.2. Problem (P2'): about the Schrödinger product operator $\mathcal{L}^{\circ,(n)}$.

Let us now turn to the issues related to the Schrödinger product operator $\mathcal{L}^{(n)}$, starting from its degenerate - and relatively simple - component $\mathcal{L}^{\circ,(n)}$ (see (1.26)).

Our main result about $\mathcal{L}^{\circ,(n)}$ will actually be derived from a careful examination of the action of the operator on suitable (Gaussian) processes. The property can be summed as follows.

Proposition 2.3. *Assume that* $H \in (\frac{1}{2}, \frac{3}{4})$ *and fix* $b \in (\frac{1}{2}, 1)$.

If $0 \leqslant c < \frac{3}{2} - 2H$*, then there exists a sequence* $(Z^{(n)})_{n\geqslant 1}$ *of random functions such that*

$$\sup_{n\geqslant 1} \mathbb{E}\Big[\big\| Z^{(n)} \big\|_{X_c^b}^q \Big] < \infty \quad \text{for every } q \geqslant 2, \quad \text{and} \quad \mathbb{E}\Big[\big\| \mathcal{L}^{\circ,(n)} Z^{(n)} \big\|_{X_c^b}^2 \Big] \overset{n\to\infty}{\longrightarrow} \infty. \tag{2.4}$$

In particular, if $0 \leqslant c < \frac{3}{2} - 2H$*, then for every* $p > 2$*, one has*

$$\mathbb{E}\Big[\big\| \mathcal{L}^{\circ,(n)} \big\|_{X_c^b \to X_c^b}^p \Big] \overset{n\to\infty}{\longrightarrow} \infty.$$

Proposition 2.4 therefore offers a partial answer to the guideline question (**P2'**). In particular here, the result gives birth to our second restriction on the spaces X_c^b involved in the first-order analysis: namely, one must have $c \geqslant \frac{3}{2} - 2H$.

2.3. Problem (P2"): about the Schrödinger product operator $\mathcal{L}^{\sharp,(n)}$.

As explained in the above Section 1.1.2, we only focus here on the convergence issue for the approximation $\mathcal{P}_{b,c}^{(n)}$ of $\big\| \mathcal{L}^{\sharp,(n)} \big\|_{X_c^b \to X_c^b}$ given by the expression

$$\mathcal{P}_{b,c}^{(n)} := \sum_{k_1,k_1'} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c} + \langle \lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1'\rangle^{2c} + \langle \lambda_1'\rangle^{2b}} \Big| \sum_k \int d\lambda \, \{\langle k\rangle^{2c} + \langle \lambda\rangle^{2b}\} \big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) \overline{\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1'}(\lambda,\lambda_1')} \Big|^2, \tag{2.5}$$

where the Fourier kernel $\mathcal{K}_\chi^{(n)}$ is defined through the identity

$$\mathcal{F}\big(\mathcal{L}^{\sharp,(n)}(z)_k\big)(\lambda) = \sum_{k_1} \int d\lambda_1 \big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) \mathcal{F}(z_{k_1})(\lambda_1).$$

In the setting $p = q = 1$ under consideration, it is readily checked that $\mathcal{K}_\chi^{(n)}$ is in fact given by

$$\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) := \mathbf{1}_{\{k\neq 0\}} \mathbf{1}_{\{k_1\neq k\}} \int_{\mathbb{R}} d\lambda_2 \, \overline{\mathcal{F}\big(\Psi_{k_1-k}^{(n)}\big)(\lambda_2)} \mathcal{I}_\chi(\lambda, \Omega_{k,k_1-k} + \lambda_1 - \lambda_2), \tag{2.6}$$

where the integration kernel $\mathcal{I}_\chi(.,.)$ has been introduced in (1.7).

Our main result about $\mathcal{P}_{b,c}^{(n)}$, partially responding to the problem (**P2″**), reads as follows.

Proposition 2.4. *Let $H \in (\frac{1}{2}, \frac{3}{4})$, $b \in (\frac{1}{2}, 1)$ and $c \in (0,1)$.*

If $c \geqslant b - \frac{1}{4}$, then one has

$$\mathbb{E}\Big[\big|\mathcal{P}_{b,c}^{(n)}\big|\Big] \overset{n\to\infty}{\longrightarrow} \infty.$$

Proposition 2.4 therefore points out our third (and last) restriction on the class of spaces X_c^b suitable for a first-order strategy: one must impose that $c < b - \frac{1}{4}$.

2.4. **Conclusion.**

By gathering the constraints exhibited in Propositions 2.2, 2.3 and 2.4, we can conclude our investigations about the limits of the first-order strategy for the stochastic Schrödinger problem (1.1).

Corollary 2.5. *If $\frac{1}{2} < H \leqslant \frac{9}{16}$, then for any pair $(b,c) \in (\frac{1}{2}, 1) \times [0,1)$, one has either*

$$\Big\| \widetilde{\varphi\varphi}^{(n)} \Big\|_{X^b} \overset{n\to\infty}{\longrightarrow} \infty, \quad \big\|\mathcal{L}^{\circ,(n)}\big\|_{X_c^b \to X_c^b} \overset{n\to\infty}{\longrightarrow} \infty, \quad \text{or} \quad \mathcal{P}_{b,c}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty \quad \text{in } L^{\infty,-}(\Omega). \tag{2.7}$$

In particular, for $\frac{1}{2} < H \leqslant \frac{9}{16}$, the stochastic Schrödinger problem (1.1) cannot be treated with the first-order strategy described in Section 1.1.

Proof. For none of the two explosions in (2.7) to happen, it is necessary that $b < 2H - \frac{1}{2}$ (Proposition 2.2), $c \geqslant \frac{3}{2} - 2H$ (Proposition 2.3) and $c < b - \frac{1}{4}$ (Proposition 2.4), which can be summed up as

$$\frac{3}{2} - 2H \leqslant c < b - \frac{1}{4} < 2H - \frac{3}{4},$$

and hence one must have $H > \frac{9}{16}$. □

These results thus call for the development of more sophisticated methods, such as paracontrolled or random-tensor-type strategies, in order to cover the whole range $H > \frac{1}{2}$ for $\dot{B}$. To be more specific, we do not expect any possible improvement regarding the constraints on the Schrödinger product tree (Proposition 2.2), and we only advocate for a more sophisticated treatment of the product operation $z \mapsto \mathcal{I}_\chi \mathcal{M}^{(p,q)}(z, \varphi^{(n)})$. These further developments could for instance be derived from a suitable "ansatz" formulation of the problem (see [10, Section 5.2]), which we plan to investigate in a future study.

Finally, it is worth noting that there is no hope to reach the case of a space-time white noise $\dot{B}$ through the present renormalization method, as it can be seen from the following explosion result.

Proposition 2.6. *Assume that $H = \frac{1}{2}$, that is $\dot{B}$ is a space-time white noise on $\mathbb{R} \times \mathbb{T}$. Then, with the notation of Section 2.1 and for every $b \geqslant \frac{1}{2}$, it holds that*

$$\mathbb{E}\Big[\Big\| \widetilde{\varphi\varphi}^{(n)} \Big\|_{X^b}^2\Big] \overset{n\to\infty}{\longrightarrow} \infty.$$

The latter property can be established with similar arguments as those in the proof of Proposition 2.2, item (*iii*). For the sake of conciseness, we prefer to leave the details as an exercise to the reader.

The rest of the paper is organized as follows. In Section 3, we focus on the analysis of the Schrödinger product tree and on the proof of Proposition 2.2. Sections 4 and 5 are then devoted to the study of the (stochastic) Schrödinger product operator $\mathcal{L}^{(n)}$: Section 4 contains the proof of Proposition 2.3, while Section 5 contains the proof of Proposition 2.4. Finally, the Appendix section contains a partial proof of Lemma 1.1 (Section 6.1), as well as a discussion about the relevance of $\mathcal{P}_{b,c}^{(n)}$ as an approximation of $\big\|\mathcal{L}^{(n)}\big\|_{X_c^b \to X_c}$ (Section 6.2).

3. Convergence of the Schrödinger product tree

We start with the study of the convergence issue for the renormalized Schrödinger product tree

$$\widetilde{\Psi}^{(n)}_k(t) := \mathcal{I}_\chi\Big(\widetilde{\mathcal{M}}(\Psi^{(n)},\Psi^{(n)})_k(.) - \mathbb{E}\big[\widetilde{\mathcal{M}}(\Psi^{(n)},\Psi^{(n)})_k(.)\big]\Big)(t), \tag{3.1}$$

introduced in Section 2.1. We recall in particular that the (rescaled) linear solution $\Psi^{(n)}$ has been introduced in (1.12), while the renormalized product operator $\widetilde{\mathcal{M}}$ is defined in Fourier coordinates by

$$\widetilde{\mathcal{M}}(v,w)_k(t) := \mathbf{1}_{\{k\neq 0\}} \sum_{k_1\neq 0} e^{\imath t\Omega_{k,k_1}} v_{k+k_1}\overline{w_{k_1}}, \quad \text{with } \Omega_{k,k_1} := |k+k_1|^2 - |k_1|^2 - |k|^2 = 2kk_1. \tag{3.2}$$

As a preliminary step, let us point out some useful expressions and estimates for the covariance of the process $\Psi^{(n)}$, at the core of expression (3.1).

3.1. Covariance of the linear solution.

One has by definition

$$\mathcal{F}(\Psi^{(n)}_k)(\lambda) = \mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\mathcal{F}\big(\mathcal{I}_\chi\big(e^{-\imath.k^2}\dot\beta^{(k)}\big)\big)(\lambda)$$

and we can use the integration kernel $\mathcal{I}_\chi(.,.)$ (see (1.7)) to express this quantity as

$$\begin{aligned}\mathcal{F}(\Psi^{(n)}_k)(\lambda) &= \mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1)\mathcal{F}\big(e^{-\imath.k^2}\dot\beta^{(k)}\big)(\lambda_1)\\ &= \mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1)\int dt\, e^{-\imath(\lambda_1+k^2)t}\dot\beta^{(k)}_t.\end{aligned}$$

Based on this expression, and since

$$\mathbb{E}\Big[\dot\beta^{(k)}_t\dot\beta^{(k')}_{t'}\Big] = \mathbf{1}_{\{k=k'\}}|t-t'|^{2H-2},$$

we get that

$$\begin{aligned}&\mathbb{E}\Big[\mathcal{F}(\Psi^{(n)}_k)(\lambda)\overline{\mathcal{F}(\Psi^{(n)}_{k'})(\lambda')}\Big]\\ &= \mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k'\rangle\leqslant 2^n\}}\int d\lambda_1 d\lambda_1'\,\mathcal{I}_\chi(\lambda,\lambda_1)\overline{\mathcal{I}_\chi(\lambda',\lambda_1')}\int dtdt'\, e^{-\imath(\lambda_1+|k|^2)t}e^{\imath(\lambda_1'+|k'|^2)t'}\mathbb{E}\Big[\dot\beta^{(k)}_t\dot\beta^{(k')}_{t'}\Big]\\ &= \mathbf{1}_{\{k=k'\}}\mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int d\lambda_1 d\lambda_1'\,\mathcal{I}_\chi(\lambda,\lambda_1)\overline{\mathcal{I}_\chi(\lambda',\lambda_1')}\int dtdt'\, e^{-\imath(\lambda_1+|k|^2)t}e^{\imath(\lambda_1'+|k|^2)t'}|t-t'|^{2H-2}\\ &= c\,\mathbf{1}_{\{k=k'\}}\mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int d\lambda_1 d\lambda_1'\,\mathcal{I}_\chi(\lambda,\lambda_1)\overline{\mathcal{I}_\chi(\lambda',\lambda_1')}\int dtdt'\, e^{-\imath(\lambda_1+|k|^2+\xi)t}e^{\imath(\lambda_1'+|k|^2+\xi)t'}\\ &= c\,\mathbf{1}_{\{k=k'\}}\mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\mathcal{I}_\chi(\lambda,-\xi-k^2)\overline{\mathcal{I}_\chi(\lambda',-\xi-k^2)}.\end{aligned} \tag{3.3}$$

For a more detailed expression of this quantity, let us introduce the following notation.

Notation 3.1. *For all $s,t,\lambda\in\mathbb{R}$, we set*

$$\Xi_t(\lambda) := \int_{\mathbb{R}} dr\, e^{\imath\lambda r}\chi(r)\mathbf{1}_{[0,t]}(r) = \int_0^t dr\, e^{\imath\lambda r}\chi(r),$$

$$\Xi^t(\lambda) := \int_{\mathbb{R}} dr\, e^{\imath\lambda r}\chi(r)\mathbf{1}_{[0,r]}(t) = \mathbf{1}_{\{t\geqslant 0\}}\int_t^\infty dr\, e^{\imath\lambda r}\chi(r) - \mathbf{1}_{\{t<0\}}\int_{-\infty}^t dr\, e^{\imath\lambda r}\chi(r). \tag{3.4}$$

With this notation, recall that

$$\mathcal{I}_\chi(\lambda,\lambda') = \int_{\mathbb{R}} dt\, e^{-\imath\lambda t}\chi(t)\int_{\mathbb{R}} ds\, e^{\imath\lambda' s}\chi(s)\mathbf{1}_{[0,t]}(s) = \int_{\mathbb{R}} dt\, e^{-\imath\lambda t}\chi(t)\,\Xi_t(\lambda'), \tag{3.5}$$

and so we can rephrase (3.3) as

$$\begin{aligned}&\mathbb{E}\Big[\mathcal{F}(\mathring{\wp}_{k}^{(n)})(\lambda)\overline{\mathcal{F}(\mathring{\wp}_{k'}^{(n)})(\lambda')}\Big]\\&= c\,\mathbf{1}_{\{k=k'\}}\mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int dtdt'\,\chi(t)\chi(t')e^{-\imath\lambda t}e^{\imath\lambda' t'}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\Xi_t(-\xi-k^2)\overline{\Xi_{t'}(-\xi-k^2)}.\end{aligned}\tag{3.6}$$

Besides, based on (3.5), it is readily checked that

$$\big|\mathcal{I}(\lambda,\lambda_1)\big|\lesssim\frac{1}{\langle\lambda\rangle\langle\lambda-\lambda_1\rangle},\tag{3.7}$$

and therefore the expression in (3.3) leads us to the following uniform estimate (with respect to n): for all $k,k'\in Z$ and $\lambda,\lambda'\in\mathbb{R}$,

$$\begin{aligned}\Big|\mathbb{E}\Big[\mathcal{F}(\mathring{\wp}_{k}^{(n)})(\lambda)\overline{\mathcal{F}(\mathring{\wp}_{k'}^{(n)})(\lambda')}\Big]\Big|&\lesssim\mathbf{1}_{\{k=k'\}}\frac{1}{\langle\lambda\rangle\langle\lambda'\rangle}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\frac{1}{\langle\xi+\lambda+k^2\rangle}\frac{1}{\langle\xi+\lambda'+k^2\rangle}\\&\lesssim\mathbf{1}_{\{k=k'\}}\frac{1}{\langle\lambda\rangle\langle\lambda'\rangle}\frac{1}{\langle\lambda-\lambda'\rangle^{1-\varepsilon}}\bigg[\frac{1}{\langle\lambda+k^2\rangle^{2H-1}}+\frac{1}{\langle\lambda'+k^2\rangle^{2H-1}}\bigg],\end{aligned}\tag{3.8}$$

where the second inequality follows from [11, Lemma 3.8].

With similar arguments, we obtain that

$$\mathbb{E}\Big[\mathcal{F}(\mathring{\wp}_{k}^{(n)})(\lambda)\mathcal{F}(\mathring{\wp}_{k'}^{(n)})(\lambda')\Big]=c\,\mathbf{1}_{\{k=k'\}}\mathbf{1}_{\{\langle k\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\mathcal{I}_\chi(\lambda,-\xi-k^2)\mathcal{I}_\chi(\lambda',\xi-k^2),\tag{3.9}$$

and then, for all $k,k'\in Z$ and $\lambda,\lambda'\in\mathbb{R}$,

$$\begin{aligned}\Big|\mathbb{E}\Big[\mathcal{F}(\mathring{\wp}_{k}^{(n)})(\lambda)\mathcal{F}(\mathring{\wp}_{k'}^{(n)})(\lambda')\Big]\Big|&\lesssim\mathbf{1}_{\{k=k'\}}\frac{1}{\langle\lambda\rangle\langle\lambda'\rangle}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\frac{1}{\langle\xi+\lambda+k^2\rangle}\frac{1}{\langle\xi-\lambda'-k^2\rangle}\\&\lesssim\mathbf{1}_{\{k=k'\}}\frac{1}{\langle\lambda\rangle\langle\lambda'\rangle}\frac{1}{\langle\lambda+\lambda'+2k^2\rangle^{1-\varepsilon}}\bigg[\frac{1}{\langle\lambda+k^2\rangle^{2H-1}}+\frac{1}{\langle\lambda'+k^2\rangle^{2H-1}}\bigg].\end{aligned}\tag{3.10}$$

We are now in a position to tackle the proof of our main asymptotic result.

3.2. **Proof of Proposition 2.2.**

First, using the integration kernel $\mathcal{I}_\chi(.,.)$ (see (1.7)) and the Fourier expression of $\widetilde{\mathcal{M}}$ (see (3.2)), we can write the Fourier transform of the process under consideration as

$$\begin{aligned}&\mathcal{F}\big(\mathcal{I}_\chi\widetilde{\mathcal{M}}(\mathring{\wp}^{(n)},\mathring{\wp}^{(n)})_k\big)(\lambda)=\int_{\mathbb{R}}d\lambda'\,\mathcal{I}_\chi(\lambda,\lambda')\mathcal{F}\big(\widetilde{\mathcal{M}}(\mathring{\wp}^{(n)},\mathring{\wp}^{(n)})_k\big)(\lambda')\\&=\mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int_{\mathbb{R}}d\lambda'\,\mathcal{I}_\chi(\lambda,\lambda')\int dt\,e^{-\imath\lambda' t}e^{\imath t\Omega_{k,k_1}}\mathring{\wp}_{k+k_1}^{(n)}(t)\overline{\mathring{\wp}_{k_1}^{(n)}(t)}\\&=\mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1\,\overline{\mathcal{F}(\mathring{\wp}_{k_1}^{(n)})(\lambda_1)}\int d\lambda_2\,\mathcal{F}(\mathring{\wp}_{k+k_1}^{(n)})(\lambda_2)\int_{\mathbb{R}}d\lambda'\,\mathcal{I}_\chi(\lambda,\lambda')\int dt\,e^{-\imath\lambda' t}e^{\imath t\Omega_{k,k_1}}e^{\imath t\lambda_2}e^{-\imath t\lambda_1},\end{aligned}$$

and thus

$$\mathcal{F}\big(\mathcal{I}_\chi\big(\widetilde{\mathcal{M}}(\mathring{\wp}^{(n)},\mathring{\wp}^{(n)})_k\big)\big)(\lambda)=\mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2\,\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{F}(\mathring{\wp}_{k_1}^{(n)})(\lambda_1)}\mathcal{F}(\mathring{\wp}_{k+k_1}^{(n)})(\lambda_2).$$

Based on this expression, and by applying Wick's formula, we can compute

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{F}\big(\mathcal{I}_\chi\big(\widetilde{\mathcal{M}}(\Upsilon^{(n)},\Upsilon^{(n)})_k\big)(\lambda)\big|^2\Big] \\
&= \mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2 \sum_{k_1'\neq 0}\int d\lambda_1' d\lambda_2'\ \mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1'}+\lambda_2'-\lambda_1')} \\
&\qquad\qquad \mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big] \\
&= \Big|\mathbb{E}\big[\mathcal{F}\big(\mathcal{I}_\chi\big(\widetilde{\mathcal{M}}(\Upsilon^{(n)},\Upsilon^{(n)})_k\big)(\lambda)\big]\Big|^2 \\
&\quad + \mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2 \sum_{k_1'\neq 0}\int d\lambda_1' d\lambda_2'\ \mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1'}+\lambda_2'-\lambda_1')} \\
&\qquad\qquad \mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big] \\
&\quad + \mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2 \sum_{k_1'\neq 0}\int d\lambda_1' d\lambda_2'\ \mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1'}+\lambda_2'-\lambda_1')} \\
&\qquad\qquad \mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big]. \qquad (3.11)
\end{aligned}$$

As far as the last term of this expansion is concerned, observe that according to (3.9), one has

$$\begin{aligned}
&\mathbf{1}_{\{k\neq 0\}}\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big] \\
&= \mathbf{1}_{\{k\neq 0\}}\mathbf{1}_{\{k_1=k+k_1'\}}\mathbf{1}_{\{k+k_1=k_1'\}}\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big] \\
&= \mathbf{1}_{\{k\neq 0\}}\mathbf{1}_{\{k_1=k+k_1'\}}\mathbf{1}_{\{2k+k_1'=k_1'\}}\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big] \ = 0.
\end{aligned}$$

For the same reason,

$$\begin{aligned}
&\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1'})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1'})(\lambda_2')}\Big] \\
&= \mathbf{1}_{\{k_1=k_1'\}}\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2')}\Big].
\end{aligned}$$

Thus, going back to (3.11) and recalling the definition (3.1) of $\widetilde{\Upsilon}^{(n)}$, we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{F}\big(\widetilde{\Upsilon}^{(n)}_k\big)(\lambda)\big|^2\Big] = \mathbb{E}\Big[\big|\mathcal{F}\big(\mathcal{I}_\chi\big(\widetilde{\mathcal{M}}(\Upsilon^{(n)},\Upsilon^{(n)})_k\big)(\lambda)\big|^2\Big] - \Big|\mathbb{E}\big[\mathcal{F}\big(\mathcal{I}_\chi\big(\widetilde{\mathcal{M}}(\Upsilon^{(n)},\Upsilon^{(n)})_k\big)(\lambda)\big]\Big|^2 \\
&= \mathbf{1}_{\{k\neq 0\}}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2 \int d\lambda_1' d\lambda_2'\ \mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2'-\lambda_1')} \\
&\qquad\qquad \mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2')}\Big]. \qquad (3.12)
\end{aligned}$$

Using the estimates (3.7) and (3.8), we get that for all $0 \leqslant c < \frac{1}{2}$ and $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\mathbb{E}\Big[\big\|\widetilde{\mathbb{Y}}^{(n)}\big\|_{X_c}^2\Big] \\
&\lesssim \sum_{k\neq 0}\sum_{k_1\neq 0}\langle k\rangle^{2c}\int\frac{d\lambda}{\langle\lambda\rangle^2}\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\frac{d\lambda_2}{\langle\lambda_2\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2+\lambda_1\rangle}\int\frac{d\lambda_1'}{\langle\lambda_1'\rangle}\frac{d\lambda_2'}{\langle\lambda_2'\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2'+\lambda_1'\rangle} \\
&\lesssim \sum_{k\neq 0}\sum_{k_1\neq 0}\langle k\rangle^{2c}\int\frac{d\lambda}{\langle\lambda\rangle^2}\bigg|\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\int\frac{d\lambda_2}{\langle\lambda_2\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2+\lambda_1\rangle}\bigg|^2 \\
&\lesssim \sum_{k\neq 0}\sum_{k_1\neq 0}\langle k\rangle^{2c}\int\frac{d\lambda}{\langle\lambda\rangle^2}\bigg|\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}+\lambda_1\rangle^{1-\frac{\varepsilon}{2}}}\bigg|^2 \\
&\lesssim \sum_{k\neq 0}\sum_{k_1\neq 0}\langle k\rangle^{2c}\int\frac{d\lambda}{\langle\lambda\rangle^2}\frac{1}{\langle\lambda-\Omega_{k,k_1}\rangle^{2-2\varepsilon}} \lesssim \sum_{k\neq 0}\langle k\rangle^{2c}\sum_{k_1\neq 0}\frac{1}{\langle\Omega_{k,k_1}\rangle^{2-2\varepsilon}} \lesssim \sum_k\frac{1}{\langle k\rangle^{2-2c-2\varepsilon}}\sum_{k_1}\frac{1}{\langle k_1\rangle^{2-2\varepsilon}} < \infty,
\end{aligned}$$

which already proves item (i) of the statement.

Let us now focus on the norm in X^b, which, thanks to (3.12), can be expanded as

$$\begin{aligned}
\mathbb{E}\Big[\big\|\widetilde{\mathbb{Y}}^{(n)}\big\|_{X^b}^2\Big] = \sum_{k\neq 0}\int d\lambda\,\langle\lambda\rangle^{2b}\sum_{k_1\neq 0}\int d\lambda_1 d\lambda_2\int d\lambda_1' d\lambda_2'\;\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2'-\lambda_1')} \\
\mathbb{E}\Big[\overline{\mathcal{F}(\mathbb{Y}_{k_1}^{(n)})(\lambda_1)}\mathcal{F}(\mathbb{Y}_{k_1}^{(n)})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\mathbb{Y}_{k+k_1}^{(n)})(\lambda_2)\overline{\mathcal{F}(\mathbb{Y}_{k+k_1}^{(n)})(\lambda_2')}\Big].
\end{aligned}$$

3.2.1. *Proof of item (ii).* Assume here that $\frac{1}{2} < b < 2H - \frac{1}{2}$.

Using the estimates (3.7) and (3.8), we get that

$$\begin{aligned}
&\mathbb{E}\Big[\big\|\widetilde{\mathbb{Y}}^{(n)}\big\|_{X^b}^2\Big] \\
&\lesssim \sum_{k\neq 0}\int\frac{d\lambda}{\langle\lambda\rangle^{2-2b}}\sum_{k_1\neq 0}\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\frac{d\lambda_2}{\langle\lambda_2\rangle}\int\frac{d\lambda_1'}{\langle\lambda_1'\rangle}\frac{d\lambda_2'}{\langle\lambda_2'\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2+\lambda_1\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2'+\lambda_1'\rangle} \\
&\qquad\Big[\frac{1}{\langle\lambda_1+k_1^2\rangle^{2H-1}}+\frac{1}{\langle\lambda_1'+k_1^2\rangle^{2H-1}}\Big]\Big[\frac{1}{\langle\lambda_2+(k+k_1)^2\rangle^{2H-1}}+\frac{1}{\langle\lambda_2'+(k+k_1)^2\rangle^{2H-1}}\Big] \\
&\lesssim \mathbb{A}_b+\mathbb{B}_b,
\end{aligned}$$

with

$$\begin{aligned}
\mathbb{A}_b := \sum_{k\neq 0}\sum_{k_1\neq 0}\int\frac{d\lambda}{\langle\lambda\rangle^{2-2b}}\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\frac{d\lambda_2}{\langle\lambda_2\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2+\lambda_1\rangle}\frac{1}{\langle\lambda_1+k_1^2\rangle^{2H-1}}\frac{1}{\langle\lambda_2+(k+k_1)^2\rangle^{2H-1}} \\
\int\frac{d\lambda_1'}{\langle\lambda_1'\rangle}\frac{d\lambda_2'}{\langle\lambda_2'\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2'+\lambda_1'\rangle}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{B}_b := \sum_{k\neq 0}\sum_{k_1\neq 0}\int\frac{d\lambda}{\langle\lambda\rangle^{2-2b}}\int\frac{d\lambda_1}{\langle\lambda_1\rangle}\frac{d\lambda_2}{\langle\lambda_2\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2+\lambda_1\rangle}\frac{1}{\langle\lambda_1+k_1^2\rangle^{2H-1}} \\
\int\frac{d\lambda_1'}{\langle\lambda_1'\rangle}\frac{d\lambda_2'}{\langle\lambda_2'\rangle}\frac{1}{\langle\lambda-\Omega_{k,k_1}-\lambda_2'+\lambda_1'\rangle}\frac{1}{\langle\lambda_2'+(k+k_1)^2\rangle^{2H-1}}.
\end{aligned}$$

Since $\mathbb{A}_b$ and $\mathbb{B}_b$ no longer depend on n, we only need to prove that these two quantities are finite.

For $\mathbb{A}_b$, note first that

$$\int \frac{d\lambda_1'}{\langle \lambda_1' \rangle} \frac{d\lambda_2'}{\langle \lambda_2' \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} - \lambda_2' + \lambda_1' \rangle} \lesssim \int \frac{d\lambda_1'}{\langle \lambda_1' \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} + \lambda_1' \rangle^{1-\frac{\varepsilon}{2}}} \lesssim \frac{1}{\langle \lambda - \Omega_{k,k_1} \rangle^{1-\varepsilon}}.$$

On the other hand,

$$\begin{aligned} &\int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{d\lambda_2}{\langle \lambda_2 \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} - \lambda_2 + \lambda_1 \rangle} \frac{1}{\langle \lambda_1 + k_1^2 \rangle^{2H-1}} \frac{1}{\langle \lambda_2 + (k+k_1)^2 \rangle^{2H-1}} \\ &\lesssim \int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{1}{\langle \lambda_1 + k_1^2 \rangle^{2H-1}} \int \frac{d\lambda_2}{\langle \lambda_2 \rangle} \frac{1}{\langle \lambda_2 + (k+k_1)^2 \rangle^{2H-1}} \lesssim \frac{1}{\langle k_1 \rangle^{4H-2-\varepsilon}} \frac{1}{\langle k+k_1 \rangle^{4H-2-\varepsilon}}. \end{aligned}$$

Thus, going back to the definition of $\mathbb{A}_b$, we obtain that for $\varepsilon > 0$ small enough,

$$\begin{aligned} \mathbb{A}_b &\lesssim \sum_{k_1 \neq 0} \frac{1}{\langle k_1 \rangle^{4H-2-\varepsilon}} \sum_{k \neq 0} \frac{1}{\langle k+k_1 \rangle^{4H-2-\varepsilon}} \int \frac{d\lambda}{\langle \lambda \rangle^{2-2b}} \frac{1}{\langle \lambda - \Omega_{k,k_1} \rangle^{1-\varepsilon}} \\ &\lesssim \sum_{k_1 \neq 0} \frac{1}{\langle k_1 \rangle^{4H-2b-2\varepsilon}} \sum_{k \neq 0} \frac{1}{\langle k+k_1 \rangle^{4H-2-\varepsilon}} \frac{1}{\langle k \rangle^{2-2b-\varepsilon}}. \end{aligned} \tag{3.13}$$

Since $b < 2H - \frac{1}{2}$, one has $4H - 2b > 1$, and so we can deduce that

$$\mathbb{A}_b \lesssim \sum_{k_1 \neq 0} \frac{1}{\langle k_1 \rangle^{8H-4b-1-4\varepsilon}} < \infty,$$

for $\varepsilon > 0$ small enough.

As far as $\mathbb{B}_b$ is concerned, one has

$$\begin{aligned} &\int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{d\lambda_2}{\langle \lambda_2 \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} - \lambda_2 + \lambda_1 \rangle} \frac{1}{\langle \lambda_1 + k_1^2 \rangle^{2H-1}} \\ &\lesssim \int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{1}{\langle \lambda_1 + k_1^2 \rangle^{2H-1}} \frac{1}{\langle \lambda - \Omega_{k,k_1} + \lambda_1 \rangle^{1-\varepsilon}} \\ &\lesssim \bigg(\int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{1}{\langle \lambda_1 + k_1^2 \rangle^{4H-2}} \bigg)^{\frac{1}{2}} \bigg(\int \frac{d\lambda_1}{\langle \lambda_1 \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} + \lambda_1 \rangle^{2-2\varepsilon}} \bigg)^{\frac{1}{2}} \lesssim \frac{1}{\langle k_1 \rangle^{4H-2-\varepsilon}} \frac{1}{\langle \lambda - \Omega_{k,k_1} \rangle^{\frac{1}{2}}} \end{aligned}$$

and in the same way

$$\int \frac{d\lambda_1'}{\langle \lambda_1' \rangle} \frac{d\lambda_2'}{\langle \lambda_2' \rangle} \frac{1}{\langle \lambda - \Omega_{k,k_1} - \lambda_2' + \lambda_1' \rangle} \frac{1}{\langle \lambda_2' + (k+k_1)^2 \rangle^{2H-1}} \lesssim \frac{1}{\langle k+k_1 \rangle^{4H-2-\varepsilon}} \frac{1}{\langle \lambda - \Omega_{k,k_1} \rangle^{\frac{1}{2}}},$$

which gives

$$\mathbb{B}_b \lesssim \sum_{k_1 \neq 0} \frac{1}{\langle k_1 \rangle^{4H-2-\varepsilon}} \sum_{k \neq 0} \frac{1}{\langle k+k_1 \rangle^{4H-2-\varepsilon}} \int \frac{d\lambda}{\langle \lambda \rangle^{2-2b}} \frac{1}{\langle \lambda - \Omega_{k,k_1} \rangle}.$$

Thus, we are in the same position as in (3.13), and we can use the same arguments to assert that $\mathbb{B}_b < \infty$.

This achieves to prove that for every b such that $\frac{1}{2} < b < 2H - \frac{1}{2}$, one has

$$\sup_{n \geqslant 1} \mathbb{E}\Big[\Big\| \widetilde{\Psi}^{(n)} \Big\|_{X^b}^2 \Big] < \infty,$$

as desired.

3.2.2. *Proof of item (iii).* Assume now that $b = 2H - \frac{1}{2}$.

Using the expression in (3.6), we can write

$$\begin{aligned}
&\mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2')}\Big]\\
&= \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int dtdt'\,\chi(t)\chi(t')e^{\imath\lambda_1 t}e^{-\imath\lambda_1' t'}\overline{\Xi_t(-\xi-k_1^2)}\Xi_{t'}(-\xi-k_1^2)\\
&\quad \mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\eta}{|\eta|^{2H-1}}\int dsds'\,\chi(s)\chi(s')e^{-\imath\lambda_2 s}e^{\imath\lambda_2' s'}\Xi_s(-\eta-(k+k_1)^2)\overline{\Xi_{s'}(-\eta-(k+k_1)^2)}. \quad (3.14)
\end{aligned}$$

Then observe that by (3.5),

$$\begin{aligned}
&\int d\lambda_1 d\lambda_2\,\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)e^{\imath\lambda_1 t}e^{-\imath\lambda_2 s}\\
&= \int_{\mathbb{R}} dv\,e^{-\imath\lambda v}\chi(v)\int dr\,\mathbf{1}_{[0,v]}(r)\chi(r)\int d\lambda_1 d\lambda_2\,e^{\imath r(\Omega_{k,k_1}+\lambda_2-\lambda_1)}e^{\imath\lambda_1 t}e^{-\imath\lambda_2 s}\\
&= \int_{\mathbb{R}} dv\,e^{-\imath\lambda v}\chi(v)\int dr\,\mathbf{1}_{[0,v]}(r)\chi(r)e^{\imath r\Omega_{k,k_1}}\Big(\int d\lambda_1\,e^{-\imath r\lambda_1}e^{\imath\lambda_1 t}\Big)\Big(\int d\lambda_2\,e^{\imath r\lambda_2}e^{-\imath\lambda_2 s}\Big)\\
&= \delta_{\{s=t\}}\chi(t)e^{\imath t\Omega_{k,k_1}}\int_{\mathbb{R}} dv\,e^{-\imath\lambda v}\chi(v)\mathbf{1}_{[0,v]}(t), \quad (3.15)
\end{aligned}$$

and in a similar way

$$\int d\lambda_1' d\lambda_2'\,\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2'-\lambda_1')}e^{-\imath\lambda_1' t'}e^{\imath\lambda_2' s'} = \delta_{\{s'=t'\}}\chi(t')e^{-\imath t'\Omega_{k,k_1}}\int_{\mathbb{R}} dw\,e^{\imath\lambda w}\chi(w)\mathbf{1}_{[0,w]}(t'). \quad (3.16)$$

Combining (3.14), (3.15) and (3.16), we obtain that

$$\begin{aligned}
&\int d\lambda_1 d\lambda_2\int d\lambda_1' d\lambda_2'\,\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2-\lambda_1)\overline{\mathcal{I}_\chi(\lambda,\Omega_{k,k_1}+\lambda_2'-\lambda_1')}\\
&\qquad\qquad \mathbb{E}\Big[\overline{\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1)}\mathcal{F}(\Upsilon^{(n)}_{k_1})(\lambda_1')\Big]\mathbb{E}\Big[\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2)\overline{\mathcal{F}(\Upsilon^{(n)}_{k+k_1})(\lambda_2')}\Big]\\
&= \int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int_{\mathbb{R}}\frac{d\eta}{|\eta|^{2H-1}}\\
&\quad\Big(\int_{\mathbb{R}} dv\,e^{-\imath\lambda v}\chi(v)\int dt\,\mathbf{1}_{[0,v]}(t)e^{\imath t\Omega_{k,k_1}}\chi(t)^3\overline{\Xi_t(-\xi-k_1^2)}\Xi_t(-\eta-(k+k_1)^2)\Big)\\
&\qquad\Big(\int_{\mathbb{R}} dw\,e^{\imath\lambda w}\chi(w)\int dt'\,\mathbf{1}_{[0,w]}(t')e^{-\imath t'\Omega_{k,k_1}}\chi(t')^3\Xi_{t'}(-\xi-k_1^2)\overline{\Xi_{t'}(-\eta-(k+k_1)^2)}\Big)
\end{aligned}$$

and so, going back to (3.12), we get the expression

$$\mathbb{E}\Big[\big\|\widetilde{\Upsilon}^{(n)}\big\|^2_{X^b}\Big] = \sum_{k\neq 0}\sum_{k_1\neq 0}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,\langle\lambda\rangle^{2b}\big|Q^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2,$$

where we have set

$$Q^L_{\beta,\beta'}(\lambda) := \int dt\,e^{-\imath\lambda t}\chi(t)\int_0^t ds\,e^{\imath sL}\chi(s)^3\overline{\Xi_s(-\beta)}\Xi_s(-\beta').$$

In particular,

$$\begin{aligned}
\mathbb{E}\Big[\big\|\widetilde{\Upsilon}^{(n)}\big\|^2_{X^b}\Big] &\geqslant \sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|Q^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2\\
&\geqslant \sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|Q^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda\Omega_{k,k_1})\big|^2.
\end{aligned}$$

For all $L \geqslant 1$, we can use an integration-by-parts argument to decompose $Q^L_{\beta,\beta'}(\lambda L)$ as

$$Q^L_{\beta,\beta'}(\lambda L) = \int dt\, e^{-\imath\lambda L t}\chi(t)\int_0^t ds\, e^{\imath sL}\chi(s)^3\overline{\Xi_s(-\beta)}\Xi_s(-\beta') = M^L_{\beta,\beta'}(\lambda) + R^L_{\beta,\beta'}(\lambda),$$

where

$$M^L_{\beta,\beta'}(\lambda) := \frac{1}{\imath\lambda L}\int dt\, e^{-\imath Lt(\lambda-1)}\chi(t)^4\,\overline{\Xi_t(-\beta)}\Xi_t(-\beta') \tag{3.17}$$

and

$$R^L_{\beta,\beta'}(\lambda) := \frac{1}{\imath\lambda L}\int dt\, e^{-\imath\lambda Lt}\chi'(t)\int_0^t ds\, e^{\imath sL}\chi(s)^3\overline{\Xi_s(-\beta)}\Xi_s(-\beta'). \tag{3.18}$$

Using this decomposition, we can write

$$\begin{aligned}
\mathbb{E}\Big[\big\|\widetilde{\Upsilon}^{(n)}\big\|^2_{X^b}\Big] &\geqslant \sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2\\
&\quad+\sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|R^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2\\
&\quad-2\sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|\big|R^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|\\
&\geqslant \sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2\\
&\quad-2\sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|\big|R^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|.
\end{aligned}$$

By applying Lemma 3.2 below, we get that for all $\Omega \geqslant 1$, $k,\ell \geqslant 1$ and $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega}_{\xi+k,\eta+\ell}(\lambda)\big|\big|R^{\Omega}_{\xi+k,\eta+\ell}(\lambda)\big|\\
&\lesssim\frac{1}{\Omega^{4-\varepsilon}}\Bigg[\int_{|\lambda|\geqslant 2}\frac{d\lambda}{|\lambda|^{4-2b}}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\frac{1}{\langle\xi+k\rangle}\frac{1}{\langle\eta+\ell\rangle}\\
&\quad+\int_{|\lambda|\leqslant 2}\frac{d\lambda}{|\lambda|^{2-2b}|\lambda-1|^{1-\varepsilon}}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\\
&\qquad\Big(\frac{1}{\langle\xi+k\rangle\langle\eta+\ell\rangle}+\frac{1}{\langle\xi+(k+\Omega)\rangle\langle\eta+\ell\rangle}+\frac{1}{\langle\xi+(k+\Omega)\rangle\langle\eta+(\ell-\Omega-\xi-k)\rangle}\\
&\qquad\quad+\frac{1}{\langle\eta+(\ell-\Omega)\rangle\langle\xi+k\rangle}\Big)+\frac{1}{\langle\eta+(\ell-\Omega)\rangle\langle\xi+(k+\Omega-\eta-\ell)\rangle}\Big)\Bigg]\lesssim\frac{1}{\Omega^{4-\varepsilon}},
\end{aligned}$$

where we have used the fact that for all $0\leqslant\alpha<\beta\leqslant 1$ such that $\alpha+\beta>1$, one has

$$\sup_{a\in\mathbb{R}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^\alpha}\frac{1}{\langle\xi+a\rangle^\beta}<\infty.$$

Therefore, for $\varepsilon>0$ small enough,

$$\begin{aligned}
&\sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi d\eta}{|\xi|^{2H-1}|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|\big|R^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|\\
&\lesssim\sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\frac{1}{\Omega_{k,k_1}^{3-2b-\varepsilon}}\lesssim\Big(\sum_{k\geqslant 1}\frac{1}{k^{3-2b-\varepsilon}}\Big)^2<\infty,
\end{aligned}$$

due to $b<1$.

We are thus left with the analysis of

$$\mathfrak{M}^{(n)} := \sum_{k\geqslant 1}\sum_{k_1\geqslant 1}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k+k_1\rangle\leqslant 2^n\}}\Omega_{k,k_1}^{1+2b}\int\frac{d\xi}{|\xi|^{2H-1}}\int\frac{d\eta}{|\eta|^{2H-1}}\int d\lambda\,|\lambda|^{2b}\big|M^{\Omega_{k,k_1}}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2. \tag{3.19}$$

To this end, write for all $\Omega \geqslant 1$,

$$\begin{aligned}
&\int \frac{d\xi}{|\xi|^{2H-1}} \int \frac{d\eta}{|\eta|^{2H-1}} \int d\lambda\, |\lambda|^{2b} \big| M^{\Omega}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2 \\
&= \frac{1}{|\Omega|^2} \int \frac{d\xi}{|\xi|^{2H-1}} \int \frac{d\eta}{|\eta|^{2H-1}} \int \frac{d\lambda}{|\lambda|^{2-2b}} \left| \int dt\, e^{-\imath\Omega(\lambda-1)t} \chi(t)^4\, \overline{\Xi_t(-\xi-k_1^2)} \Xi_t(-\eta-(k+k_1)^2) \right|^2 \\
&= \frac{1}{|\Omega|^2} \int dt dt'\, \chi(t)^4 \chi(t')^4 \bigg(\int \frac{d\lambda}{|\lambda|^{2-2b}} e^{-\imath\lambda\Omega(t-t')} \bigg) e^{\imath\Omega(t-t')} \mathcal{J}_{k_1}(t,t') \overline{\mathcal{J}_{k+k_1}(t,t')},
\end{aligned}$$

where we have set

$$\mathcal{J}_\ell(t,t') := \int \frac{d\xi}{|\xi|^{2H-1}} \overline{\Xi_t(-\xi-\ell^2)} \Xi_{t'}(-\xi-\ell^2).$$

Then, by elementary transformations,

$$\begin{aligned}
&\frac{1}{|\Omega|^2} \int dt dt'\, \chi(t)^4 \chi(t')^4 \bigg(\int \frac{d\lambda}{|\lambda|^{2-2b}} e^{-\imath\lambda\Omega(t-t')} \bigg) e^{\imath\Omega(t-t')} \mathcal{J}_{k_1}(t,t') \overline{\mathcal{J}_{k+k_1}(t,t')} \\
&= \frac{1}{|\Omega|^{1+2b}} \int \frac{dt dt'}{|t-t'|^{2b-1}}\, \chi(t)^4 \chi(t')^4\, e^{\imath\Omega(t-t')} \mathcal{J}_{k_1}(t,t') \overline{\mathcal{J}_{k+k_1}(t,t')} \\
&= \frac{1}{|\Omega|^{1+2b}} \int \frac{dt ds}{|s|^{2b-1}}\, \chi(t)^4 \chi(t-s)^4\, e^{\imath\Omega s} \mathcal{J}_{k_1}(t,t-s) \overline{\mathcal{J}_{k+k_1}(t,t-s)} \\
&= \frac{1}{|\Omega|^3} \int \frac{dt ds}{|s|^{2b-1}}\, \chi(t)^4 \chi\Big(t-\frac{s}{\Omega}\Big)^4\, e^{\imath s} \mathcal{J}_{k_1}\Big(t,t-\frac{s}{\Omega}\Big) \overline{\mathcal{J}_{k+k_1}\Big(t,t-\frac{s}{\Omega}\Big)}. \qquad (3.20)
\end{aligned}$$

Let us now expand the two quantities $\mathcal{J}_{k_1}\big(t,t-\frac{s}{\Omega}\big)$ and $\overline{\mathcal{J}_{k+k_1}\big(t,t-\frac{s}{\Omega}\big)}$. On the one hand,

$$\begin{aligned}
\mathcal{J}_{k_1}\Big(t,t-\frac{s}{\Omega}\Big) &= \int \frac{d\xi}{|\xi|^{2H-1}} \overline{\Xi_t(-\xi-k_1^2)} \Xi_{t-\frac{s}{\Omega}}(-\xi-k_1^2) \\
&= \int \frac{d\xi}{|\xi|^{2H-1}} \bigg(\int_0^t dr\, \chi(r) e^{\imath(\xi+k_1^2)r} \bigg) \bigg(\int_0^{t-\frac{s}{\Omega}} dr'\, \chi(r') e^{-\imath(\xi+k_1^2)r'} \bigg) \\
&= \int \frac{d\xi}{|\xi|^{2H-1}} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \chi(r') \int dr\, \mathbf{1}_{[0,t]}(r) \chi(r) e^{\imath(\xi+k_1^2)(r-r')} \\
&= \int \frac{d\xi}{|\xi|^{2H-1}} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \chi(r') \int dr\, \mathbf{1}_{[0,t]}(r'+r) \chi(r'+r) e^{\imath(\xi+k_1^2)r} \\
&= \frac{1}{k_1^{4H-4}} \int \frac{d\xi'}{|\xi'|^{2H-1}} \int dr\, e^{\imath r k_1^2(\xi'+1)} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \mathbf{1}_{[0,t]}(r'+r) \chi(r') \chi(r'+r) \\
&= \frac{1}{k_1^{4H-2}} \int \frac{d\xi'}{|\xi'|^{2H-1}} \int dr\, e^{\imath r(\xi'+1)} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \mathbf{1}_{[0,t]}\Big(r'+\frac{r}{k_1^2}\Big) \chi(r') \chi\Big(r'+\frac{r}{k_1^2}\Big) \\
&= \frac{1}{k_1^{4H-2}} \int \frac{dr}{|r|^{2-2H}}\, e^{\imath r} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \mathbf{1}_{[0,t]}\Big(r'+\frac{r}{k_1^2}\Big) \chi(r') \chi\Big(r'+\frac{r}{k_1^2}\Big),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\overline{\mathcal{J}_{k+k_1}\Big(t,t-\frac{s}{\Omega}\Big)} \\
&= \frac{1}{(k+k_1)^{4H-2}} \int \frac{du}{|u|^{2-2H}}\, e^{-\imath u} \int du'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(u') \mathbf{1}_{[0,t]}\Big(u'+\frac{u}{(k+k_1)^2}\Big) \chi(u') \chi\Big(u'+\frac{u}{(k+k_1)^2}\Big).
\end{aligned}$$

By injecting these two expansions into (3.20), we deduce the expression

$$\begin{aligned}
&\int \frac{d\xi}{|\xi|^{2H-1}} \int \frac{d\eta}{|\eta|^{2H-1}} \int d\lambda\, |\lambda|^{2b} \big| M^{\Omega}_{\xi+k_1^2,\eta+(k+k_1)^2}(\lambda)\big|^2 \\
&= \frac{1}{|\Omega|^3} \frac{1}{k_1^{4H-2}} \frac{1}{(k+k_1)^{4H-2}} \int dt\, \chi(t)^4 \int \frac{ds}{|s|^{2b-1}}\, e^{\imath s} \chi\Big(t - \frac{s}{\Omega}\Big)^4 \\
&\qquad \Big(\int \frac{dr}{|r|^{2-2H}}\, e^{\imath r} \int dr'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \mathbf{1}_{[0,t]}\Big(r' + \frac{r}{k_1^2}\Big) \chi(r') \chi\Big(r' + \frac{r}{k_1^2}\Big)\Big) \\
&\qquad\quad \Big(\int \frac{du}{|u|^{2-2H}}\, e^{-\imath u} \int du'\, \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(u') \mathbf{1}_{[0,t]}\Big(u' + \frac{u}{(k+k_1)^2}\Big) \chi(u') \chi\Big(u' + \frac{u}{(k+k_1)^2}\Big)\Big) \\
&= \frac{1}{|\Omega|^3} \frac{1}{k_1^{4H-2}} \frac{1}{(k+k_1)^{4H-2}} Q_{b,H}(\Omega, k, k_1), \qquad (3.21)
\end{aligned}$$

where we have set

$$\begin{aligned}
Q_{b,H}(\Omega,k,k_1) := &\int dt \int dr' \int du'\, \chi(t)^4 \chi(r') \chi(u') \\
&\Big(\int \frac{ds}{|s|^{2b-1}}\, e^{\imath s} \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(r') \mathbf{1}_{[0,t-\frac{s}{\Omega}]}(u') \chi\Big(t - \frac{s}{\Omega}\Big)^4 \Big) \Big(\int \frac{dr}{|r|^{2-2H}}\, e^{\imath r} \mathbf{1}_{[0,t]}\Big(r' + \frac{r}{k_1^2}\Big) \chi\Big(r' + \frac{r}{k_1^2}\Big)\Big) \\
&\qquad \Big(\int \frac{du}{|u|^{2-2H}}\, e^{-\imath u} \mathbf{1}_{[0,t]}\Big(u' + \frac{u}{(k+k_1)^2}\Big) \chi\Big(u' + \frac{u}{(k+k_1)^2}\Big)\Big).
\end{aligned}$$

Using the subsequent Lemma 3.3, we can easily check that

$$Q_{b,H}(\Omega_{k,k_1}, k, k_1) \to q_{b,H} \quad \text{as } k, k_1 \to \infty,$$

where

$$q_{b,H} := \Big(\int \frac{ds}{|s|^{2b-1}}\, e^{\imath s} \Big) \Big| \int \frac{dr}{|r|^{2-2H}}\, e^{\imath r} \Big|^2 \Big(\int dt\, \chi(t)^8 \int_0^t dr'\, \chi(r')^2 \int_0^t du'\, \chi(u')^2 \Big).$$

Since $q_{b,H} > 0$, there exists $K \geqslant 1$ large enough such that for all $k_1 \geqslant k \geqslant K$, one has

$$Q_{b,H}(\Omega_{k,k_1}, k, k_1) \geqslant \frac{q_{b,H}}{2}.$$

Therefore, going back to (3.19) and (3.21), we get that

$$\mathfrak{M}^{(n)} \gtrsim \sum_{k \geqslant K} \frac{1}{|k|^{2-2b}} \sum_{k_1 \geqslant k} \mathbf{1}_{\{\langle k_1 \rangle \leqslant 2^n\}} \mathbf{1}_{\{\langle k+k_1 \rangle \leqslant 2^n\}} \frac{1}{|k_1|^{4H-2b}} \frac{1}{|k+k_1|^{4H-2}}.$$

and since K does not depend on n, we deduce that

$$\begin{aligned}
\liminf_{n\to\infty} \mathfrak{M}^{(n)} &\gtrsim \sum_{k \geqslant K} \frac{1}{|k|^{2-2b}} \sum_{k_1 \geqslant k} \frac{1}{|k_1|^{4H-2b}} \frac{1}{|k+k_1|^{4H-2}} \\
&\gtrsim \sum_{k \geqslant K} \frac{1}{|k|^{8H-4b-1}} \Big(\frac{1}{k} \sum_{k_1 \geqslant k} \frac{1}{|\frac{k_1}{k}|^{4H-2b}} \frac{1}{|1+\frac{k_1}{k}|^{4H-2}} \Big) \\
&\gtrsim \sum_{k \geqslant K} \frac{1}{|k|^{8H-4b-1}} \int_1^\infty \frac{dx}{|x|^{4H-2b} |1+x|^{4H-2}} \gtrsim \sum_{k \geqslant K} \frac{1}{|k|^{8H-4b-1}} \int_1^\infty \frac{dx}{|x|^{8H-2b-2}}.
\end{aligned}$$

At this point, recall that $b = 2H - \frac{1}{2}$, and so the latter bound reduces in fact to

$$\sum_{k \geqslant K} \frac{1}{|k|^{8H-4b-1}} \int_1^\infty \frac{dx}{|x|^{8H-2b-2}} = \Big(\sum_{k \geqslant K} \frac{1}{|k|} \Big) \Big(\int_1^\infty \frac{dx}{|x|^{4H-1}} \Big) = \infty,$$

which leads us to the desired conclusion

$$\lim_{n\to\infty} \mathfrak{M}^{(n)} = \infty.$$

3.3. **Auxiliary lemmas.**

The few technical results below have been used in the proof of Proposition 2.2.

Lemma 3.2. *For all $L \geqslant 1$ and $\beta, \beta' \in \mathbb{R}$, let $M^L_{\beta,\beta'}$ and $R^L_{\beta,\beta'}$ be the functions on $\mathbb{R}$ defined by* (3.17) *and* (3.18)*, respectively. Then the following estimates hold true:*
(i) For all $|\lambda| \geqslant 2$, one has

$$\big|M^L_{\beta,\beta'}(\lambda)\big| \lesssim \frac{1}{L^2|\lambda|^2} \quad \textit{and} \quad \big|R^L_{\beta,\beta'}(\lambda)\big| \lesssim \frac{1}{L^2|\lambda|^2}\frac{1}{\langle\beta\rangle\langle\beta'\rangle}.$$

(ii) For all $|\lambda| \leqslant 2$, one has

$$\big|M^L_{\beta,\beta'}(\lambda)\big| \lesssim \frac{1}{L^{2-\varepsilon}}\frac{1}{|\lambda||\lambda-1|^{1-\varepsilon}}$$

and

$$\begin{aligned}&\big|R^L_{\beta,\beta'}(\lambda)\big|\\ &\lesssim \frac{1}{L^2|\lambda|}\bigg[\frac{1}{\langle\beta\rangle\langle\beta'\rangle} + \frac{1}{\langle L+\beta\rangle\langle\beta'\rangle} + \frac{1}{\langle L+\beta\rangle\langle L+\beta-\beta'\rangle} + \frac{1}{\langle L-\beta'\rangle\langle\beta\rangle} + \frac{1}{\langle L-\beta'\rangle\langle L+\beta-\beta'\rangle}\bigg].\end{aligned} \tag{3.22}$$

Proof. *(i)* If $|\lambda| \geqslant 2$, then by an elementary integration-by-parts argument, we get that

$$\begin{aligned}\big|M^L_{\beta,\beta'}(\lambda)\big| \lesssim \frac{1}{L^2|\lambda|^2}\int dt\,\Big[&\big|\partial_t(\chi(.)^4)(t)\big|\big|\Xi_t(-\beta)\big|\big|\Xi_t(-\beta')\big|\\ &+ \big|\chi(t)\big|^4\big|\partial_t(\Xi_.(-\beta))(t)\big|\big|\Xi_t(-\beta')\big| + \big|\chi(t)\big|^4\big|\Xi_t(-\beta)\big|\big|\partial_t(\Xi_.(-\beta'))(t)\big|\Big] \lesssim \frac{1}{L^2|\lambda|^2},\end{aligned}$$

where we have used the uniform bound

$$|\Xi_t(-\beta)| + |\partial_t\big(\Xi_t(-\beta)\big)| \lesssim 1. \tag{3.23}$$

In the same way,

$$\begin{aligned}|R^L_{\beta,\beta'}(\lambda)| \lesssim \frac{1}{L^2|\lambda|^2}\bigg[&\int dt\,|\chi''(t)|\int_0^t ds\,|\chi(s)|^3|\Xi_s(-\beta)||\Xi_s(-\beta')|\\ &+ \int dt\,|\chi'(t)||\chi(t)|^3|\Xi_t(-\beta)||\Xi_t(-\beta')|\bigg] \lesssim \frac{1}{L^2|\lambda|^2}\frac{1}{\langle\beta\rangle\langle\beta'\rangle},\end{aligned}$$

where we have used the fact that

$$|\Xi_t(-\beta)| \lesssim \frac{1}{\langle\beta\rangle}. \tag{3.24}$$

(ii) For $M^L_{\beta,\beta'}(\lambda)$, we can use again (3.23) and an integration-by-parts argument to write, for $|\lambda| \leqslant 2$,

$$\big|M^L_{\beta,\beta'}(\lambda)\big| \lesssim \frac{1}{L|\lambda|}\bigg|\int dt\,e^{-\imath Lt(\lambda-1)}\chi(t)^4\,\overline{\Xi_t(-\beta)}\Xi_t(-\beta')\bigg|^{1-\varepsilon} \lesssim \frac{1}{L^{2-\varepsilon}|\lambda||\lambda-1|^{1-\varepsilon}}.$$

As for $R^L_{\beta,\beta'}(\lambda)$, observe first that

$$\begin{aligned}&\int_0^t ds\,e^{\imath sL}\chi(s)^3\overline{\Xi_s(-\beta)}\Xi_s(-\beta') = \frac{1}{\imath L}e^{\imath tL}\chi(t)^3\overline{\Xi_t(-\beta)}\Xi_t(-\beta')\\ &- \frac{1}{\imath L}\int_0^t ds\,e^{\imath sL}\partial_s(\chi^3)(s)\overline{\Xi_s(-\beta)}\Xi_s(-\beta') - \frac{1}{\imath L}\int_0^t ds\,e^{\imath s(L+\beta)}\chi(s)^4\Xi_s(-\beta') - \frac{1}{\imath L}\int_0^t ds\,e^{\imath s(L-\beta')}\chi(s)^4\overline{\Xi_s(-\beta)},\end{aligned}$$

and so, by (3.24), we get that

$$\begin{aligned}&|R^L_{\beta,\beta'}(\lambda)| = \frac{1}{L|\lambda|}\bigg|\int dt\,e^{-\imath\lambda Lt}\chi'(t)\int_0^t ds\,e^{\imath sL}\chi(s)^3\overline{\Xi_s(-\beta)}\Xi_s(-\beta')\bigg|\\ &\lesssim \frac{1}{L^2|\lambda|}\bigg[\frac{1}{\langle\beta\rangle\langle\beta'\rangle} + \int dt\,|\chi'(t)|\bigg|\int_0^t ds\,e^{\imath s(L+\beta)}\chi(s)^4\Xi_s(-\beta')\bigg| + \int dt\,|\chi'(t)|\bigg|\int_0^t ds\,e^{\imath s(L-\beta')}\chi(s)^4\overline{\Xi_s(-\beta)}\bigg|\bigg].\end{aligned}$$

Then, using the same elementary arguments as for the derivation of (3.7), we obtain that

$$\Big| \int_0^t ds\, e^{\imath s(L+\beta)} \chi(s)^4 \Xi_s(-\beta') \Big| \lesssim \frac{1}{\langle L+\beta \rangle} \Big[\frac{1}{\langle \beta' \rangle} + \frac{1}{\langle L+\beta-\beta' \rangle} \Big],$$

as well as

$$\Big| \int_0^t ds\, e^{\imath s(L-\beta')} \chi(s)^4 \overline{\Xi_s(-\beta)} \Big| \lesssim \frac{1}{\langle L-\beta' \rangle} \Big[\frac{1}{\langle \beta \rangle} + \frac{1}{\langle L+\beta-\beta' \rangle} \Big],$$

which yields the desired bound (3.22).

□

Lemma 3.3. *Fix $\alpha \in (0,1)$.*

(i) It holds that

$$\sup_{\substack{L \geqslant 1 \\ t,r',u' \in [-2,2]}} \Big| \int_{\mathbb{R}} \frac{ds}{|s|^\alpha} e^{\imath s} \mathbf{1}_{[0,t-\frac{s}{L}]}(r') \mathbf{1}_{[0,t-\frac{s}{L}]}(u') \chi\Big(t-\frac{s}{L}\Big) \Big| \lesssim 1 \tag{3.25}$$

and

$$\sup_{\substack{L \geqslant 1 \\ r' \in [-2,2]}} \Big| \int_{\mathbb{R}} \frac{dr}{|r|^\alpha}\, e^{\imath r} \mathbf{1}_{[0,t]}\Big(r'+\frac{r}{L}\Big) \chi\Big(r'+\frac{r}{L}\Big) \Big| \lesssim 1. \tag{3.26}$$

(ii) For all $L \geqslant 1$ and $t,r',u' \in [-2,2]$, one has

$$\Big| \int_{\mathbb{R}} \frac{ds}{|s|^\alpha} e^{\imath s} \Big\{ \mathbf{1}_{[0,t-\frac{s}{L}]}(r') \mathbf{1}_{[0,t-\frac{s}{L}]}(u') \chi\Big(t-\frac{s}{L}\Big) - \mathbf{1}_{[0,t]}(r') \mathbf{1}_{[0,t]}(u') \chi(t) \Big\} \Big| \lesssim \frac{1}{L^\alpha} \Big\{ \frac{1}{|t-r'|^\alpha} + \frac{1}{|t-u'|^\alpha} \Big\} \tag{3.27}$$

and

$$\Big| \int_{\mathbb{R}} \frac{dr}{|r|^\alpha} e^{\imath r} \Big\{ \mathbf{1}_{[0,t]}\Big(r'+\frac{r}{L}\Big) \chi\Big(r'+\frac{r}{L}\Big) - \mathbf{1}_{[0,t]}(r') \chi(r') \Big\} \Big| \lesssim \frac{1}{L^\alpha} \Big\{ \frac{1}{|t|^\alpha} + \frac{1}{|t-r'|^\alpha} + \frac{1}{|r'|^\alpha} \Big\}. \tag{3.28}$$

Proof.
(i) Let $L \geqslant 1$ and $t,r',u' \in [-2,2]$. One has, for almost every $s \in \mathbb{R}$,

$$\mathbf{1}_{[0,t-\frac{s}{L}]}(r') = \mathbf{1}_{\{r' \geqslant 0\}} \mathbf{1}_{\{s \leqslant L(t-r')\}} - \mathbf{1}_{\{r'<0\}} \mathbf{1}_{\{s \geqslant L(t-r')\}},$$

and so we can write

$$\begin{aligned}
&\mathbf{1}_{[0,t-\frac{s}{L}]}(r') \mathbf{1}_{[0,t-\frac{s}{L}]}(u') \\
&= \Big(\mathbf{1}_{\{r' \geqslant 0\}} \mathbf{1}_{\{s \leqslant L(t-r')\}} - \mathbf{1}_{\{r'<0\}} \mathbf{1}_{\{s \geqslant L(t-r')\}} \Big) \Big(\mathbf{1}_{\{u' \geqslant 0\}} \mathbf{1}_{\{s \leqslant L(t-u')\}} - \mathbf{1}_{\{u'<0\}} \mathbf{1}_{\{s \geqslant L(t-u')\}} \Big) \\
&= \mathbf{1}_{\{r',u' \geqslant 0\}} \mathbf{1}_{\{s \leqslant L[(t-r') \wedge (t-u')]\}} + \mathbf{1}_{\{r',u'<0\}} \mathbf{1}_{\{s \geqslant L[(t-r') \vee (t-u')]\}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{ds}{|s|^\alpha} e^{\imath s} \mathbf{1}_{[0,t-\frac{s}{L}]}(r') \mathbf{1}_{[0,t-\frac{s}{L}]}(u') \chi\Big(t-\frac{s}{L}\Big) \\
&= \mathbf{1}_{\{r',u' \geqslant 0\}} \int_{-\infty}^{L[(t-r') \wedge (t-u')]} \frac{ds}{|s|^\alpha} e^{\imath s} \chi\Big(t-\frac{s}{L}\Big) + \mathbf{1}_{\{r',u'<0\}} \int_{L[(t-r') \vee (t-u')]}^{\infty} \frac{ds}{|s|^\alpha} e^{\imath s} \chi\Big(t-\frac{s}{L}\Big),
\end{aligned}$$

and we can apply the estimate (3.31) in Lemma 3.4 below to deduce (3.25).

We can then use the same arguments to prove (3.26), by noting that

$$\mathbf{1}_{[0,t]}\Big(r'+\frac{r}{L}\Big) = \mathbf{1}_{\{t \geqslant 0\}} \mathbf{1}_{\{-r'L \leqslant r \leqslant -r'L+Lt\}} - \mathbf{1}_{\{t<0\}} \mathbf{1}_{\{-r'L+Lt \leqslant r \leqslant -r'L\}}.$$

(ii) Regarding (3.27), we naturally start with the bound

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\frac{ds}{|s|^{\alpha}}e^{\imath s}\Big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(r')\mathbf{1}_{[0,t-\frac{s}{L}]}(u')\chi\Big(t-\frac{s}{L}\Big)-\mathbf{1}_{[0,t]}(r')\mathbf{1}_{[0,t]}(u')\chi\big(t\big)\Big\}\Big|\\
&\leqslant\Big|\int_{\mathbb{R}}\frac{ds}{|s|^{\alpha}}e^{\imath s}\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(r')-\mathbf{1}_{[0,t]}(r')\big\}\mathbf{1}_{[0,t-\frac{s}{L}]}(u')\chi\Big(t-\frac{s}{L}\Big)\Big|\\
&\qquad+\mathbf{1}_{[0,t]}(r')\Big|\int_{\mathbb{R}}\frac{ds}{|s|^{\alpha}}e^{\imath s}\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(u')-\mathbf{1}_{[0,t]}(u')\big\}\chi\Big(t-\frac{s}{L}\Big)\Big|\\
&\qquad\qquad+\mathbf{1}_{[0,t]}(r')\mathbf{1}_{[0,t]}(u')\Big|\int_{\mathbb{R}}\frac{ds}{|s|^{\alpha}}e^{\imath s}\Big\{\chi\Big(t-\frac{s}{L}\Big)-\chi\big(t\big)\Big\}\Big| \qquad (3.29)
\end{aligned}$$

For the first integral in (3.29), observe for instance that for almost every $s\geqslant 0$,

$$\begin{aligned}
\mathbf{1}_{[0,t-\frac{s}{L}]}(r')-\mathbf{1}_{[0,t]}(r')&=\mathbf{1}_{\{r'\geqslant 0\}}\big\{\mathbf{1}_{\{t\geqslant r'+\frac{s}{L}\}}-\mathbf{1}_{\{t\geqslant r'\}}\big\}-\mathbf{1}_{\{r'<0\}}\big\{\mathbf{1}_{\{t\leqslant r'+\frac{s}{L}\}}-\mathbf{1}_{\{t\leqslant r'\}}\big\}\\
&=-\big\{\mathbf{1}_{\{r'\geqslant 0\}}+\mathbf{1}_{\{r'<0\}}\big\}\mathbf{1}_{\{r'\leqslant t\leqslant r'+\frac{s}{L}\}}=-\mathbf{1}_{\{r'\leqslant t\}}\mathbf{1}_{\{s\geqslant L(t-r')\}}, \qquad (3.30)
\end{aligned}$$

which, combined with the previous decomposition

$$\mathbf{1}_{[0,t-\frac{s}{L}]}(u')=\mathbf{1}_{\{u'\geqslant 0\}}\mathbf{1}_{\{s\leqslant L(t-u')\}}-\mathbf{1}_{\{u'<0\}}\mathbf{1}_{\{s\geqslant L(t-u')\}},$$

yields

$$\begin{aligned}
&\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(r')-\mathbf{1}_{[0,t]}(r')\big\}\mathbf{1}_{[0,t-\frac{s}{L}]}(u')\\
&=-\big\{\mathbf{1}_{\{0\leqslant u'\leqslant r'\leqslant t\}}\mathbf{1}_{\{L(t-r')\leqslant s\leqslant L(t-u')\}}-\mathbf{1}_{\{r'\leqslant t\}}\mathbf{1}_{\{u'<0\}}\mathbf{1}_{\{s\geqslant L(t-u')\}}\mathbf{1}_{\{s\geqslant L(t-r')\}}\big\}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\int_0^{\infty}\frac{ds}{s^{\alpha}}e^{\imath s}\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(r')-\mathbf{1}_{[0,t]}(r')\big\}\mathbf{1}_{[0,t-\frac{s}{L}]}(u')\chi\Big(t-\frac{s}{L}\Big)\\
&=-\mathbf{1}_{\{0\leqslant u'\leqslant r'\leqslant t\}}\int_{L|t-r'|}^{L|t-u'|}\frac{ds}{s^{\alpha}}e^{\imath s}\chi\Big(t-\frac{s}{L}\Big)+\mathbf{1}_{\{r'\leqslant t\}}\mathbf{1}_{\{u'<0\}}\int_{\max(L|t-r'|,L(t-u'))}^{\infty}\frac{ds}{s^{\alpha}}e^{\imath s}\chi\Big(t-\frac{s}{L}\Big),
\end{aligned}$$

and we can use the estimate (3.32) in Lemma 3.4 to assert that

$$\Big|\int_0^{\infty}\frac{ds}{s^{\alpha}}e^{\imath s}\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(r')-\mathbf{1}_{[0,t]}(r')\big\}\mathbf{1}_{[0,t-\frac{s}{L}]}(u')\chi\Big(t-\frac{s}{L}\Big)\Big|\lesssim\frac{1}{L^{\alpha}}\max\Big(\frac{1}{|t-r'|^{\alpha}},\frac{1}{|t-u'|^{\alpha}}\Big),$$

which corresponds to the desired bound.

The bound for the second integral in (3.29) immediately follows from the same arguments, that is from the combination of (3.30) with (3.32): we get here that

$$\Big|\int_{\mathbb{R}}\frac{ds}{|s|^{\alpha}}e^{\imath s}\big\{\mathbf{1}_{[0,t-\frac{s}{L}]}(u')-\mathbf{1}_{[0,t]}(u')\big\}\chi\Big(t-\frac{s}{L}\Big)\Big|\lesssim\frac{1}{L^{\alpha}|t-u'|^{\alpha}}.$$

As for the third integral in (3.29), we can write

$$\begin{aligned}
\Big|\int_0^{\infty}\frac{ds}{s^{\alpha}}e^{\imath s}\Big\{\chi\big(t-\frac{s}{L}\big)-\chi(t)\Big\}\Big|&=\Big|\frac{\alpha}{i}\int_0^{\infty}\frac{ds}{s^{\alpha+1}}e^{\imath s}\Big\{\chi\Big(t-\frac{s}{L}\Big)-\chi(t)\Big\}+\frac{1}{\imath L}\int_0^{\infty}\frac{ds}{s^{\alpha}}e^{\imath s}\chi'\Big(t-\frac{s}{L}\Big)\Big|\\
&\lesssim\frac{1}{L^{\alpha}}\int_0^{\infty}\frac{ds}{s^{\alpha+1}}|\chi(t-s)-\chi(t)|+\frac{1}{L}\int_0^{4L}\frac{ds}{s^{\alpha}}\lesssim\frac{1}{L^{\alpha}},
\end{aligned}$$

which completes the proof of (3.27).

The proof of (3.28) stems from a similar strategy: observe for instance the decomposition, valid for almost all $t,r>0$,

$$\begin{aligned}
&\mathbf{1}_{[0,t]}\Big(r'+\frac{r}{L}\Big)-\mathbf{1}_{[0,t]}(r')=\mathbf{1}_{\{-\frac{r}{L}\leqslant r'\leqslant t-\frac{r}{L}\}}-\mathbf{1}_{\{0\leqslant r'\leqslant t\}}\\
&=\mathbf{1}_{\{0\leqslant t\leqslant\frac{r}{L}\}}\big\{\mathbf{1}_{\{-\frac{r}{L}\leqslant r'\leqslant t-\frac{r}{L}\}}-\mathbf{1}_{\{0\leqslant r'\leqslant t\}}\big\}+\mathbf{1}_{\{t>\frac{r}{L}\}}\big\{\mathbf{1}_{\{-\frac{r}{L}\leqslant r'\leqslant 0\}}-\mathbf{1}_{\{t-\frac{r}{L}\leqslant r'\leqslant t\}}\big\}\\
&=\mathbf{1}_{\{r\geqslant L|t|\}}\mathbf{1}_{\{r\geqslant -Lr'\}}\mathbf{1}_{\{r\leqslant L(t-r')\}}-\mathbf{1}_{\{r\geqslant L|t|\}}\mathbf{1}_{\{0\leqslant r'\leqslant t\}}+\mathbf{1}_{\{r\geqslant L|r'|\}}\mathbf{1}_{\{r<Lt\}}\mathbf{1}_{\{r'\leqslant 0\}}-\mathbf{1}_{\{r\geqslant L|t-r'|\}}\mathbf{1}_{\{r<Lt\}}\mathbf{1}_{\{r'\leqslant t\}},
\end{aligned}$$

which paves the way toward the application of (3.32), just as above. □

Lemma 3.4. *Fix $\alpha \in (0,1)$. For all test-function $\varphi : \mathbb{R} \to \mathbb{R}$ with support in $[-K,K]$ ($K \geqslant 1$) and all $\lambda \in \{-1,1\}$, it holds that*

$$\sup_{\substack{A_1,A_2 \in [-\infty,\infty] \\ L \geqslant 1}} \Big| \int_{A_1}^{A_2} \frac{dr}{|r|^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big| \lesssim \|\varphi\|_\infty + \|\varphi'\|_\infty K^{1-\alpha}, \tag{3.31}$$

and for all $0 < B_1 \leqslant B_2 \leqslant \infty$, $L \geqslant 1$,

$$\Big| \int_{B_1}^{B_2} \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big| \lesssim \frac{1}{B_1^\alpha} \|\varphi\|_\infty + \frac{K^{1-\alpha}}{L^\alpha} \|\varphi'\|_\infty. \tag{3.32}$$

Proof. For (3.32), let us write, with an elementary integrations by parts,

$$\begin{aligned} &\int_{B_1}^{B_2} \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \\ &= \frac{1}{\imath\lambda} \frac{1}{B_2^\alpha} e^{\imath\lambda B_2} \varphi\Big(\frac{B_2}{L}\Big) - \frac{1}{\imath\lambda} \frac{1}{B_1^\alpha} e^{\imath\lambda B_1} \varphi\Big(\frac{B_1}{L}\Big) + \frac{\alpha}{\imath\lambda} \int_{B_1}^{B_2} \frac{dr}{r^{\alpha+1}} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) - \frac{1}{\imath\lambda L} \int_{B_1}^{B_2} \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi'\Big(\frac{r}{L}\Big), \end{aligned}$$

which, since $\operatorname{supp} \varphi' \subset [-K,K]$, gives

$$\Big| \int_{B_1}^{B_2} \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big| \lesssim \frac{1}{B_1^\alpha} \|\varphi\|_\infty + \|\varphi\|_\infty \int_{B_1}^{\infty} \frac{dr}{r^{\alpha+1}} + \frac{1}{L} \int_0^{KL} \frac{dr}{r^\alpha} \Big|\varphi'\Big(\frac{r}{L}\Big)\Big| \lesssim \frac{1}{B_1^\alpha} \|\varphi\|_\infty + \frac{K^{1-\alpha}}{L^\alpha} \|\varphi'\|_\infty.$$

As for (3.31), it suffices to observe that for every $A \in [0,\infty]$, one has

$$\begin{aligned} \Big| \int_0^A \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big| &\leqslant \mathbf{1}_{\{0 \leqslant A \leqslant 1\}} \|\varphi\|_\infty \int_0^1 \frac{dr}{r^\alpha} + \mathbf{1}_{\{A \geqslant 1\}} \Big[\|\varphi\|_\infty \int_0^1 \frac{dr}{r^\alpha} + \Big| \int_1^A \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big| \Big] \\ &\lesssim \|\varphi\|_\infty + \mathbf{1}_{\{A \geqslant 1\}} \Big| \int_1^A \frac{dr}{r^\alpha} e^{\imath\lambda r} \varphi\Big(\frac{r}{L}\Big) \Big|. \end{aligned}$$

We can then use (3.32) to (uniformly) bound the latter integral and derive (3.31).

□

4. Proof of Proposition 2.3

4.1. **Proof of Proposition 2.3.**

Fix $b \in (\frac{1}{2}, 1)$ and $c \in [0, \frac{3}{2} - 2H)$. For every $n \geqslant 1$, define $Z^{(n)}$ through the formula

$$\mathcal{F}\big(Z_k^{(n)}\big)(\lambda) := \mathbf{1}_{\{k \neq 0\}} \frac{1}{\langle k \rangle^{3-4H}} \frac{1}{\langle \lambda \rangle^2} \mathcal{F}\big(\wr_k^{(n)}\big)(\lambda).$$

For every $q \geqslant 1$, one has

$$\begin{aligned} &\mathbb{E}\Big[\big\| Z^{(n)} \big\|_{X_c^b}^{2q} \Big] \lesssim \mathbb{E}\Big[\Big(\sum_k \langle k \rangle^{2c} \int d\lambda \, \langle \lambda \rangle^{2b} \big| \mathcal{F}\big(Z_k^{(n)}\big)(\lambda) \big|^2 \Big)^q \Big] \\ &\lesssim \sum_{k_1,\dots,k_q} \frac{1}{\langle k_1 \rangle^{6-8H-2c}} \cdots \frac{1}{\langle k_q \rangle^{6-8H-2c}} \int \frac{d\lambda_1}{\langle \lambda_1 \rangle^{4-2b}} \cdots \frac{d\lambda_q}{\langle \lambda_q \rangle^{4-2b}} \, \mathbb{E}\Big[\big| \mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_1) \big|^2 \cdots \big| \mathcal{F}\big(\wr_{k_q}^{(n)}\big)(\lambda_q) \big|^2 \Big] \\ &\lesssim \Big(\sum_k \frac{1}{\langle k \rangle^{6-8H-2c}} \int \frac{d\lambda}{\langle \lambda \rangle^{4-2b}} \, \mathbb{E}\Big[\big| \mathcal{F}\big(\wr_k^{(n)}\big)(\lambda) \big|^{2q} \Big]^{\frac{1}{q}} \Big)^q \lesssim \Big(\sum_k \frac{1}{\langle k \rangle^{6-8H-2c}} \int \frac{d\lambda}{\langle \lambda \rangle^{4-2b}} \, \mathbb{E}\Big[\big| \mathcal{F}\big(\wr_k^{(n)}\big)(\lambda) \big|^2 \Big] \Big)^q, \end{aligned}$$

where we have used the fact that $\mathcal{F}\big(\wr_k^{(n)}\big)(\lambda)$ is a Gaussian variable to derive the last inequality.

Now we can use the basic covariance estimate (3.8) to assert that

$$\sum_k \frac{1}{\langle k\rangle^{6-8H-2c}} \int \frac{d\lambda}{\langle\lambda\rangle^{4-2b}} \mathbb{E}\Big[\big|\mathcal{F}\big(\wr_k^{(n)}\big)(\lambda)\big|^2\Big] \lesssim \sum_k \frac{1}{\langle k\rangle^{6-8H-2c}} \int \frac{d\lambda}{\langle\lambda\rangle^{6-2b}} \frac{1}{\langle \lambda + k^2\rangle^{2H-1}}$$
$$\lesssim \sum_k \frac{1}{\langle k\rangle^{4-4H-2c}} \ < \ \infty,$$

due to $c < \frac{3}{2} - 2H$. We have thus checked the first part of (2.4): namely, for every $q \geqslant 2$,

$$\sup_{n\geqslant 0} \mathbb{E}\Big[\big\|Z^{(n)}\big\|_{X_c^b}^q\Big] < \infty.$$

We now intend to show that

$$\mathbb{E}\Big[\big\|\mathcal{L}^{\circ,(n)} Z^{(n)}\big\|_{X^0}^2\Big] \overset{n\to\infty}{\longrightarrow} \infty, \tag{4.1}$$

which, since $b, c \geqslant 0$, will immediately prove the second part of (2.4).

Let us start the analysis by writing, along (1.22),

$$\mathcal{F}\big(\mathcal{L}^{\circ,(n)}(Z^{(n)})_k\big)(\lambda)$$
$$= \sum_{k_1\neq 0} \mathbf{1}_{\{k=0\}\cup\{k_1=k\}} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, \overline{\mathcal{F}\big(\wr_{k_1-k}^{(n)}\big)(\lambda_2-\lambda_1)}\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_2),$$

and so

$$\mathbb{E}\Big[\big\|\mathcal{L}^{\circ,(n)} Z^{(n)}\big\|_{X^0}^2\Big] = \sum_k \int d\lambda\, \mathbb{E}\Big[\big|\mathcal{F}\big(\mathcal{L}^{\circ,(n)}(Z^{(n)})_k\big)(\lambda)\big|^2\Big]$$
$$\geqslant \int d\lambda\, \mathbb{E}\Big[\big|\mathcal{F}\big(\mathcal{L}^{\circ,(n)}(Z^{(n)})_0\big)(\lambda)\big|^2\Big]$$
$$\geqslant \int d\lambda\, \mathbb{E}\Big[\Big|\sum_{k_1\neq 0} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, \overline{\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_2-\lambda_1)}\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_2)\Big|^2\Big]$$
$$\geqslant \int d\lambda \Big|\sum_{k_1\neq 0} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, \mathbb{E}\Big[\overline{\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_2-\lambda_1)}\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda_2)\Big]\Big|^2 \ =: \ \mathbb{I}^{(n)},$$

where we have used Jensen's inequality to derive the last inequality.

It remains us to prove that

$$\mathbb{I}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty. \tag{4.2}$$

To this end, we can use the notation of the subsequent Lemma 4.1 to write

$$\mathbb{I}^{(n)} = \int d\lambda \Big|\sum_{k_1\neq 0} \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \Big[\frac{1}{|k_1|^{4H-2}} M(\lambda_2-\lambda_1,\lambda_2) + R_{k_1}(\lambda_2-\lambda_1,\lambda_2)\Big]\Big|^2$$
$$= \int d\lambda \Big|\Big(\sum_{k_1} \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \frac{1}{|k_1|^{4H-2}}\Big)\Big(\int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, M(\lambda_2-\lambda_1,\lambda_2)\Big)$$
$$+ \sum_{k_1\neq 0} \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, R_{k_1}(\lambda_2-\lambda_1,\lambda_2)\Big|^2$$
$$\geqslant |\mathbb{S}^{(n)}|^2 \int d\lambda \Big|\int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, M(\lambda_2-\lambda_1,\lambda_2)\Big|^2$$
$$+ \int d\lambda \Big|\sum_{k_1\neq 0} \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, R_{k_1}(\lambda_2-\lambda_1,\lambda_2)\Big|^2$$
$$- 2\,|\mathbb{S}^{(n)}| \cdot \Big[\int d\lambda \Big|\int d\lambda_1\, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2}\, M(\lambda_2-\lambda_1,\lambda_2)\Big|$$
$$\Big|\sum_{k_1\neq 0} \mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \int d\lambda_1'\, \mathcal{I}_\chi(\lambda,\lambda_1') \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2}\, R_{k_1}(\lambda_2'-\lambda_1',\lambda_2')\Big|\Big],$$

where we have set

$$\mathbb{S}^{(n)} := \sum_{k_1} \mathbf{1}_{\{\langle k_1\rangle \leqslant 2^n\}} \frac{1}{\langle k_1\rangle^{3-4H}} \frac{1}{|k_1|^{4H-2}}.$$

Note that the previous inequality entails in particular

$$\begin{aligned}\mathbb{I}^{(n)} \geqslant |\mathbb{S}^{(n)}|^2 \int d\lambda \bigg| \int d\lambda_1 \, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, M(\lambda_2-\lambda_1,\lambda_2)\bigg|^2 \\ - 2\,|\mathbb{S}^{(n)}| \cdot \int d\lambda \bigg[\int d\lambda_1 \, \big|\mathcal{I}_\chi(\lambda,\lambda_1)\big| \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, \big|M(\lambda_2-\lambda_1,\lambda_2)\big|\bigg] \\ \bigg[\sum_{k_1\neq 0} \int d\lambda_1' \, \big|\mathcal{I}_\chi(\lambda,\lambda_1')\big| \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2} \, \big|R_{k_1}(\lambda_2'-\lambda_1',\lambda_2')\big|\bigg]. \end{aligned} \tag{4.3}$$

By combining the definition (4.6) and the estimate (4.7) below, it is easy to check that the integral

$$\int d\lambda \bigg[\int d\lambda_1 \, \big|\mathcal{I}_\chi(\lambda,\lambda_1)\big| \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, \big|M(\lambda_2-\lambda_1,\lambda_2)\big|\bigg]\bigg[\sum_{k_1\neq 0} \int d\lambda_1' \, \big|\mathcal{I}_\chi(\lambda,\lambda_1')\big| \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2} \, \big|R_{k_1}(\lambda_2'-\lambda_1',\lambda_2')\big|\bigg]$$

is finite. Indeed, on the one hand, it holds that

$$\begin{aligned}\int d\lambda_1 \, \big|\mathcal{I}_\chi(\lambda,\lambda_1)\big| \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, \big|M(\lambda_2-\lambda_1,\lambda_2)\big| &\lesssim \frac{1}{\langle\lambda\rangle} \int \frac{d\lambda_1}{\langle\lambda-\lambda_1\rangle} \int \frac{d\lambda_2}{\langle\lambda_2\rangle^3} \, \frac{1}{\langle\lambda_2-\lambda_1\rangle} \\ &\lesssim \frac{1}{\langle\lambda\rangle} \int \frac{d\lambda_1}{\langle\lambda-\lambda_1\rangle\langle\lambda_1\rangle} \lesssim \frac{1}{\langle\lambda\rangle^{2-\varepsilon}},\end{aligned}$$

and on the other hand

$$\begin{aligned}&\sum_{k_1\neq 0} \int d\lambda_1' \, \big|\mathcal{I}_\chi(\lambda,\lambda_1')\big| \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2} \, \big|R_{k_1}(\lambda_2'-\lambda_1',\lambda_2')\big| \\ &\lesssim \sum_{k_1\neq 0} \int d\lambda_1' \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2} \, \big|R_{k_1}(\lambda_2'-\lambda_1',\lambda_2')\big| \lesssim \sum_{k_1\neq 0} \int d\lambda_1' \int \frac{d\lambda_2'}{\langle\lambda_2'\rangle^2} \, \big|R_{k_1}(\lambda_1',\lambda_2')\big| \lesssim \sum_{k_1\neq 0} \frac{1}{|k_1|^{4H-2\varepsilon}} \lesssim 1,\end{aligned}$$

where we have used (4.7).

We can thus rephrase (4.3) as

$$\mathbb{I}^{(n)} \geqslant c_0\,|\mathbb{S}^{(n)}|^2 - c_1\,|\mathbb{S}^{(n)}| = |\mathbb{S}^{(n)}|^2\bigg[c_0 - \frac{c_1}{|\mathbb{S}^{(n)}|}\bigg], \tag{4.4}$$

where $c_1 \geqslant 0$ is a finite constant and

$$c_0 := \int d\lambda \bigg| \int d\lambda_1 \, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, M(\lambda_2-\lambda_1,\lambda_2)\bigg|^2.$$

Observe that $\mathbb{S}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty$, and therefore, based on (4.4), it only remains us to guarantee that $c_0 > 0$. To this end, write

$$\begin{aligned}c_0 &= \int d\lambda \bigg| \int d\lambda_1 \, \mathcal{I}_\chi(\lambda,\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \, M(\lambda_2-\lambda_1,\lambda_2)\bigg|^2 \\ &= c\int d\lambda \bigg| \int dt\, e^{-\imath\lambda t}\chi(t) \int d\lambda_1 \, \Xi_t(\lambda_1) \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \int ds\, \chi(s)^2 \Xi^s(\lambda_2-\lambda_1)\overline{\Xi^s(\lambda_2)}\bigg|^2 \\ &= c\int dt\, \chi(t)^2 \bigg| \int ds\, \chi(s)^2 \int \frac{d\lambda_2}{\langle\lambda_2\rangle^2} \bigg(\int d\lambda_1 \, \Xi_t(\lambda_1)\Xi^s(\lambda_2-\lambda_1)\bigg)\overline{\Xi^s(\lambda_2)}\bigg|^2.\end{aligned}$$

Then

$$\begin{aligned}\int d\lambda_1 \, \Xi_t(\lambda_1)\Xi^s(\lambda_2-\lambda_1) &= \int dr\, \chi(r)\mathbf{1}_{[0,t]}(r) \int dv\, \chi(v)\mathbf{1}_{[0,v]}(s) \int d\lambda_1 \, e^{\imath r\lambda_1} e^{\imath v(\lambda_2-\lambda_1)} \\ &= \int dv\, \chi(v)^2 \mathbf{1}_{[0,t]}(v)\mathbf{1}_{[0,v]}(s) e^{\imath v\lambda_2},\end{aligned}$$

which yields

$$c_0 = c\int dt\,\chi(t)^2\bigg|\int ds\,\chi(s)^2\int dr\,\chi(r)\mathbf{1}_{[0,r]}(s)\int dv\,\chi(v)^2\mathbf{1}_{[0,t]}(v)\mathbf{1}_{[0,v]}(s)\int\frac{d\lambda_2}{\langle\lambda_2\rangle^2}e^{-\imath\lambda_2(r-v)}\bigg|^2$$
$$= c\int dt\,\chi(t)^2\bigg|\int ds\,\chi(s)^2\int dr\,\chi(r)\mathbf{1}_{[0,r]}(s)\int dv\,\chi(v)^2\mathbf{1}_{[0,t]}(v)\mathbf{1}_{[0,v]}(s)e^{-|r-v|}\bigg|^2,$$

where we have used the classical identity

$$\int\frac{d\lambda}{\langle\lambda\rangle^2}e^{-\imath\lambda t} = c\,e^{-|t|}.$$

Finally, since $\chi\geqslant 0$ and $\chi\equiv 1$ on $[-1,1]$, we obtain that

$$c_0 \geqslant c\int_0^1 dt\,\chi(t)^2\bigg|\int_0^t ds\,\chi(s)^2\int_s^t dr\,\chi(r)\mathbf{1}_{[0,r]}(s)\int_s^t dv\,\chi(v)^2\mathbf{1}_{[0,t]}(v)\mathbf{1}_{[0,v]}(s)e^{-|r-v|}\bigg|^2$$
$$\geqslant c\int_0^1 dt\bigg|\int_0^t ds\int_s^t dr\int_s^t dv\,e^{-|r-v|}\bigg|^2 > 0,$$

as desired.

4.2. **Auxiliary lemma.**

The following decomposition result has been used in the proof of Proposition 2.3.

Lemma 4.1. *In the setting of Proposition 2.3, one has for every* $(k_1,k_1')\neq(0,0)$,

$$\mathbb{E}\Big[\overline{\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda)}\mathcal{F}\big(\wr_{k_1'}^{(n)}\big)(\lambda')\Big] = \mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\bigg[\frac{1}{|k_1|^{4H-2}}M(\lambda,\lambda')+R_{k_1}(\lambda,\lambda')\bigg], \tag{4.5}$$

where, using the notation introduced in (3.4)*, we have set*

$$M(\lambda,\lambda') := c\int_{\mathbb{R}} ds\,\chi(s)^2\,\Xi^s(\lambda)\overline{\Xi^s(\lambda')} \tag{4.6}$$

for some constant $c>0$*, and the function* R *is such that for all* $k_1\neq 0$ *and* $\varepsilon>0$,

$$\int d\lambda_1\int\frac{d\lambda_2}{\langle\lambda_2\rangle^2}\big|R_{k_1}(\lambda_1,\lambda_2)\big| \lesssim \frac{1}{|k_1|^{4H-2\varepsilon}}. \tag{4.7}$$

Proof. Recall that by (3.3), one has

$$\mathbb{E}\Big[\overline{\mathcal{F}\big(\wr_{k_1}^{(n)}\big)(\lambda)}\mathcal{F}\big(\wr_{k_1'}^{(n)}\big)(\lambda')\Big] = c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\overline{\mathcal{I}_\chi(\lambda,-\xi-k_1^2)}\mathcal{I}_\chi(\lambda',-\xi-k_1^2),$$

and thus

$$\begin{aligned}
&\mathbb{E}\Big[\overline{\mathcal{F}\big(\mathring{\Psi}^{(n)}_{k_1}\big)(\lambda)}\mathcal{F}\big(\mathring{\Psi}^{(n)}_{k_1'}\big)(\lambda')\Big] = c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}} \\
&\quad \int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\int_{\mathbb{R}} ds\, e^{\imath(\xi+k_1^2)s}\chi(s)\mathbf{1}_{[0,t]}(s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\int_{\mathbb{R}} ds'\, e^{-\imath(\xi+k_1^2)s'}\chi(s')\mathbf{1}_{[0,t']}(s') \\
&= c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int_{\mathbb{R}} ds\int_{\mathbb{R}} ds'\, e^{-\imath(\xi+k_1^2)(s'-s)} \\
&\qquad\qquad \Big[\chi(s)\chi(s')\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s')\Big] \\
&= c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\int_{\mathbb{R}} dr\, e^{-\imath(\xi+k_1^2)r} \\
&\qquad\qquad \Big[\int_{\mathbb{R}} ds\ \chi(s)\chi(s+r)\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s+r)\Big] \\
&= c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi|^{2H-1}}\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi+k_1^2) \\
&= c\,\mathbf{1}_{\{k_1=k_1'\}}\mathbf{1}_{\{\langle k_1\rangle\leqslant 2^n\}}\int_{\mathbb{R}}\frac{d\xi}{|\xi-k_1^2|^{2H-1}}\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi), \qquad (4.8)
\end{aligned}$$

where we have set

$$A_{\lambda,\lambda'}(r) := \int_{\mathbb{R}} ds\ \chi(s)\chi(s+r)\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s+r).$$

With this notation, let us define

$$M(\lambda,\lambda') := c\int_{\mathbb{R}} d\xi\,\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi) = c\,A_{\lambda,\lambda'}(0)$$

and

$$R_{k_1}(\lambda,\lambda') := c\int_{\mathbb{R}} d\xi\left[\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right]\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi),$$

which, going back to (4.8), yields the decomposition (4.5).

In order to check (4.7), let us write

$$\begin{aligned}
\big|R_{k_1}(\lambda,\lambda')\big| &\lesssim \int_{-\infty}^{\frac{1}{2}k_1^2} d\xi\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| \\
&\qquad + \int_{\frac{1}{2}k_1^2}^{\infty} d\xi\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big|. \qquad (4.9)
\end{aligned}$$

On the one hand, for every $\xi\in(-\infty,\frac{1}{2}k_1^2)$, one has $|\xi-k_1^2|\geqslant\frac{1}{2}k_1^2$, and so

$$\begin{aligned}
\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right| &\lesssim \left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|^{\varepsilon}\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|^{1-\varepsilon} \\
&\lesssim \frac{1}{|k_1^2|^{(2H-1)\varepsilon}}\frac{1}{|k_1^2|^{2H(1-\varepsilon)}}|\xi|^{1-\varepsilon} \lesssim \frac{|\xi|^{1-\varepsilon}}{|k_1|^{4H-2\varepsilon}},
\end{aligned}$$

which entails

$$\int_{-\infty}^{\frac{1}{2}k_1^2} d\xi\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| \lesssim \frac{1}{|k_1|^{4H-2\varepsilon}}\int_{\mathbb{R}} d\xi\,|\xi|^{1-\varepsilon}\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big|. \qquad (4.10)$$

On the other hand,

$$\int_{\frac{1}{2}k_1^2}^{\infty} d\xi\left|\frac{1}{|\xi-k_1^2|^{2H-1}}-\frac{1}{|k_1^2|^{2H-1}}\right|\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| = |k_1|^{4-4H}\int_{\frac{1}{2}}^{\infty} d\xi\left|\frac{1}{|\xi-1|^{2H-1}}-1\right|\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(k_1^2\xi)\big|. \qquad (4.11)$$

By injecting (4.10) and (4.11) into (4.9), we obtain that

$$\big|R_{k_1}(\lambda,\lambda')\big| \lesssim \frac{1}{|k_1|^{4H-2\varepsilon}} \int_{\mathbb{R}} d\xi\, |\xi|^{1-\varepsilon} \big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| + |k_1|^{4-4H} \int_{\frac{1}{2}}^{\infty} d\xi \left| \frac{1}{|\xi-1|^{2H-1}} - 1 \right| \big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(k_1^2\xi)\big|. \tag{4.12}$$

In order to estimate $\mathcal{F}\big(A_{\lambda,\lambda'}\big)$ in the above integrals, observe that

$$\begin{aligned} A_{\lambda,\lambda'}(r) &= \int_{\mathbb{R}} ds\ \chi(s)\chi(s+r) \int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(s) \int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s+r) \\ &= \int_{\mathbb{R}} ds \left[\chi(r-s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(r-s)\right]\left[\chi(-s)\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(-s)\right] \\ &= \big(f_{\lambda'} * g_{\lambda}\big)(r), \end{aligned}$$

with

$$f_{\lambda'}(s) := \chi(s)\int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s) \quad \text{and} \quad g_{\lambda}(s) := \chi(-s)\int_{\mathbb{R}} dt\, e^{\imath\lambda t}\chi(t)\mathbf{1}_{[0,t]}(-s).$$

As a result,

$$\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| \lesssim \big|\mathcal{F}\big(f_{\lambda'}\big)(\xi)\big|\big|\mathcal{F}\big(g_{\lambda}\big)(\xi)\big|.$$

Then it is readily checked that

$$\begin{aligned} \big|\mathcal{F}\big(f_{\lambda'}\big)(\xi)\big| &= \left| \int_{\mathbb{R}} ds\, e^{-\imath\xi s}\chi(s) \int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t')\mathbf{1}_{[0,t']}(s) \right| \\ &= \left| \int_{\mathbb{R}} dt'\, e^{-\imath\lambda' t'}\chi(t') \int_0^{t'} ds\, e^{-\imath\xi s}\chi(s) \right| \ \lesssim\ \frac{1}{\langle\lambda'\rangle}\frac{1}{\langle\xi+\lambda'\rangle}. \end{aligned}$$

In the same way,

$$\big|\mathcal{F}\big(g_{\lambda}\big)(\xi)\big| \lesssim \frac{1}{\langle\lambda\rangle}\frac{1}{\langle\xi+\lambda\rangle},$$

which yields

$$\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| \lesssim \frac{1}{\langle\lambda\rangle}\frac{1}{\langle\lambda'\rangle}\frac{1}{\langle\xi+\lambda\rangle}\frac{1}{\langle\xi+\lambda'\rangle}.$$

Therefore

$$\int_{\mathbb{R}} d\xi\, |\xi|^{1-\varepsilon}\big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(\xi)\big| \lesssim \frac{1}{\langle\lambda\rangle\langle\lambda'\rangle}\int_{\mathbb{R}} d\xi\, \frac{|\xi|^{1-\varepsilon}}{\langle\xi+\lambda'\rangle\langle\xi+\lambda\rangle},$$

while

$$\int_{\frac{1}{2}}^{\infty} d\xi \left| \frac{1}{|\xi-1|^{2H-1}} - 1 \right| \big|\mathcal{F}\big(A_{\lambda,\lambda'}\big)(k_1^2\xi)\big| \lesssim \frac{1}{\langle\lambda\rangle}\frac{1}{\langle\lambda'\rangle}\int_{\frac{1}{2}}^{\infty} d\xi \left| \frac{1}{|\xi-1|^{2H-1}} - 1 \right| \frac{1}{\langle\lambda+k_1^2\xi\rangle}\frac{1}{\langle\lambda'+k_1^2\xi\rangle}.$$

Going back to (4.12), we obtain the estimate

$$\begin{aligned} &\big|R_{k_1}(\lambda,\lambda')\big| \\ &\lesssim \frac{1}{\langle\lambda\rangle}\frac{1}{\langle\lambda'\rangle}\left[\frac{1}{|k_1|^{4H-2\varepsilon}}\int_{\mathbb{R}} d\xi\, \frac{|\xi|^{1-\varepsilon}}{\langle\xi+\lambda'\rangle\langle\xi+\lambda\rangle} + |k_1|^{4-4H}\int_{\frac{1}{2}}^{\infty} d\xi \left| \frac{1}{|\xi-1|^{2H-1}} - 1 \right| \frac{1}{\langle\lambda+k_1^2\xi\rangle}\frac{1}{\langle\lambda'+k_1^2\xi\rangle}\right]. \end{aligned}$$

Once endowed with this bound, the derivation of (4.7) easily follows. Namely, for $k_1 \neq 0$,

$$\begin{aligned}
\int d\lambda_1 \int \frac{d\lambda_2}{\langle \lambda_2\rangle^2} \big| R_{k_1}(\lambda_1,\lambda_2)\big| &\lesssim \frac{1}{|k_1|^{4H-2\varepsilon}} \int \frac{d\lambda_1}{\langle \lambda_1\rangle} \int \frac{d\lambda_2}{\langle \lambda_2\rangle^3} \int_{\mathbb{R}} d\xi\, \frac{|\xi|^{1-\varepsilon}}{\langle \xi+\lambda_1\rangle \langle \xi+\lambda_2\rangle} \\
&\quad + \frac{1}{|k_1|^{4H-4}} \int \frac{d\lambda_1}{\langle \lambda_1\rangle} \int \frac{d\lambda_2}{\langle \lambda_2\rangle^3} \int_{\frac{1}{2}}^{\infty} d\xi \bigg| \frac{1}{|\xi-1|^{2H-1}} - 1 \bigg| \frac{1}{\langle \lambda_1 + k_1^2\xi\rangle} \frac{1}{\langle \lambda_2 + k_1^2\xi\rangle} \\
&\lesssim \frac{1}{|k_1|^{4H-2\varepsilon}} \int_{\mathbb{R}} d\xi\, |\xi|^{1-\varepsilon} \int \frac{d\lambda_1}{\langle \lambda_1\rangle} \frac{1}{\langle \lambda_1+\xi\rangle} \int \frac{d\lambda_2}{\langle \lambda_2\rangle^3} \frac{1}{\langle \lambda_2+\xi\rangle} \\
&\quad + \frac{1}{|k_1|^{4H-4}} \int_{\frac{1}{2}}^{\infty} d\xi \bigg| \frac{1}{|\xi-1|^{2H-1}} - 1 \bigg| \int \frac{d\lambda_1}{\langle \lambda_1\rangle} \frac{1}{\langle \lambda_1 + k_1^2\xi\rangle} \int \frac{d\lambda_2}{\langle \lambda_2\rangle^3} \frac{1}{\langle \lambda_2 + k_1^2\xi\rangle} \\
&\lesssim \frac{1}{|k_1|^{4H-2\varepsilon}} \int_{\mathbb{R}} d\xi\, \frac{|\xi|^{1-\varepsilon}}{\langle \xi\rangle^{2-\frac{\varepsilon}{2}}} + \frac{1}{|k_1|^{4H-4}} \int_{\frac{1}{2}}^{\infty} d\xi \bigg| \frac{1}{|\xi-1|^{2H-1}} - 1 \bigg| \frac{1}{\langle k_1^2\xi\rangle^{2-\varepsilon}} \\
&\lesssim \frac{1}{|k_1|^{4H-2\varepsilon}} \bigg[\int_{\mathbb{R}} d\xi\, \frac{|\xi|^{1-\varepsilon}}{\langle \xi\rangle^{2-\frac{\varepsilon}{2}}} + \int_{\frac{1}{2}}^{\infty} \frac{d\xi}{|\xi|^{2-\varepsilon}} \bigg| \frac{1}{|\xi-1|^{2H-1}} - 1 \bigg| \bigg] \lesssim \frac{1}{|k_1|^{4H-2\varepsilon}}.
\end{aligned}$$

□

5. Proof of Proposition 2.4

5.1. **Notation.**

As a preliminary observation, note that due to (3.3), the covariance of the kernel $\mathcal{K}^{(n)}$ satisfies

$$\begin{aligned}
&\mathbb{E}\Big[\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) \overline{\big(\mathcal{K}_\chi^{(n)}\big)_{k'k_1'}(\lambda',\lambda_1')} \Big] \\
&= \mathbf{1}_{\{k,k'\neq 0\}} \mathbf{1}_{\{k_1\neq k\}} \mathbf{1}_{\{k_1'\neq k'\}} \\
&\qquad \int d\lambda_2 d\lambda_2'\, \mathcal{I}_\chi(\lambda, \Omega_{k,k_1-k} + \lambda_1 - \lambda_2) \overline{\mathcal{I}_\chi(\lambda', \Omega_{k',k_1'-k'} + \lambda_1' - \lambda_2')} \mathbb{E}\Big[\overline{\mathcal{F}(\mathfrak{I}_{\bullet}^{(n)}_{k_1-k})(\lambda_2)} \mathcal{F}(\mathfrak{I}_{\bullet}^{(n)}_{k_1'-k'})(\lambda_2') \Big] \\
&= \mathbf{1}_{\{k_1-k=k_1'-k'\}} \mathbb{E}\Big[\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) \overline{\big(\mathcal{K}_\chi^{(n)}\big)_{k'k_1'}(\lambda',\lambda_1')} \Big]. \qquad (5.1)
\end{aligned}$$

For future reference, let us also rephrase the above expression (when $k = k', k_1 = k_1'$) using (3.3), which gives

$$\begin{aligned}
&\mathbb{E}\Big[\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda,\lambda_1) \overline{\big(\mathcal{K}_\chi^{(n)}\big)_{kk_1}(\lambda',\lambda_1')} \Big] \\
&= \mathbf{1}_{\{k\neq 0\}} \mathbf{1}_{\{k_1\neq k\}} \\
&\qquad \int d\lambda_2 d\lambda_2'\, \mathcal{I}_\chi(\lambda, \Omega_{k,k_1-k} + \lambda_1 - \lambda_2) \overline{\mathcal{I}_\chi(\lambda', \Omega_{k,k_1-k} + \lambda_1' - \lambda_2')} \mathbb{E}\Big[\overline{\mathcal{F}(\mathfrak{I}_{\bullet}^{(n)}_{k_1-k})(\lambda_2)} \mathcal{F}(\mathfrak{I}_{\bullet}^{(n)}_{k_1-k})(\lambda_2') \Big] \\
&= \mathbf{1}_{\{k\neq 0\}} \mathbf{1}_{\{k_1\neq k\}} \mathbf{1}_{\{\langle k_1-k\rangle \leqslant 2^n\}} \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}}\, A^\xi_{(k,\lambda)}(k_1,\lambda_1) \overline{A^\xi_{(k,\lambda)}(k_1,\lambda_1')}, \qquad (5.2)
\end{aligned}$$

where we have set

$$A^\xi_{(k,\lambda)}(k_1,\lambda_1) := \int d\lambda_2\, \mathcal{I}_\chi(\lambda, \Omega_{k,k_1-k} + \lambda_1 - \lambda_2) \overline{\mathcal{I}_\chi(\lambda_2, -\xi - (k_1-k)^2)}.$$

5.2. **Proof of Proposition 2.4.**

We fix $b, c \in (0,1)$ such that $b - c \leqslant \frac{1}{4}$ and consider the related quantity $\mathcal{P}^{(n)}_{b,c}$ in (2.5). Using Jensen's inequality, we immediately obtain that

$$
\begin{aligned}
&\mathbb{E}\big[\mathcal{P}^{(n)}_{b,c}\big]\\
&= \sum_{k_1,k_1'\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1'\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \mathbb{E}\bigg[\bigg|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} d\lambda\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\,\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1'}(\lambda,\lambda_1')}\bigg|^2\bigg]\\
&\geqslant \sum_{k_1,k_1'\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1'\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \bigg|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} d\lambda\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\,\mathbb{E}\Big[\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1'}(\lambda,\lambda_1')}\Big]\bigg|^2\\
&\qquad\qquad =: \mathfrak{P}^{(n)}_{b,c}.
\end{aligned}
$$

Then, thanks to (5.1), we can assert that

$$
\begin{aligned}
\mathfrak{P}^{(n)}_{b,c} &= \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \bigg|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} d\lambda\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\,\mathbb{E}\Big[\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1')}\Big]\bigg|^2\\
&= \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle k_1\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \sum_{k,k'\in\mathbb{Z}} \int_{\mathbb{R}^2} d\lambda d\lambda'\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\{\langle k'\rangle^{2c}+\langle\lambda'\rangle^{2b}\}\\
&\qquad\qquad \mathbb{E}\Big[\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{kk_1}(\lambda,\lambda_1')}\Big]\mathbb{E}\Big[\big(\mathcal{K}^{(n)}_\chi\big)_{k'k_1}(\lambda',\lambda_1)\overline{\big(\mathcal{K}^{(n)}_\chi\big)_{k'k_1}(\lambda',\lambda_1')}\Big],
\end{aligned}
$$

which, combined with (5.2), yields that

$$
\begin{aligned}
\mathfrak{P}^{(n)}_{b,c} &= \sum_{k,k'\neq 0} \int_{\mathbb{R}^2} d\lambda d\lambda'\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\{\langle k'\rangle^{2c}+\langle\lambda'\rangle^{2b}\} \sum_{k_1\notin\{k,k'\}} \mathbf{1}_{\{\langle k_1-k\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k_1-k'\rangle\leqslant 2^n\}}\\
&\qquad \int_{\mathbb{R}} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \int_{\mathbb{R}} \frac{d\lambda_1'}{\langle k_1\rangle^{2c}+\langle\lambda_1'\rangle^{2b}} \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}}\, A^{\xi}_{(k,\lambda)}(k_1,\lambda_1)\overline{A^{\xi}_{(k,\lambda)}(k_1,\lambda_1')}\\
&\qquad\qquad \int_{\mathbb{R}} \frac{d\xi'}{|\xi'|^{2H-1}}\, \overline{A^{\xi'}_{(k',\lambda')}(k_1,\lambda_1)}A^{\xi'}_{(k',\lambda')}(k_1,\lambda_1')\\
&= \sum_{k,k'\neq 0} \int_{\mathbb{R}^2} d\lambda d\lambda'\,\{\langle k\rangle^{2c}+\langle\lambda\rangle^{2b}\}\{\langle k'\rangle^{2c}+\langle\lambda'\rangle^{2b}\} \sum_{k_1\notin\{k,k'\}} \mathbf{1}_{\{\langle k_1-k\rangle\leqslant 2^n\}}\mathbf{1}_{\{\langle k_1-k'\rangle\leqslant 2^n\}}\\
&\qquad \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} \frac{d\xi'}{|\xi'|^{2H-1}} \bigg|\int_{\mathbb{R}} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} A^{\xi}_{(k,\lambda)}(k_1,\lambda_1)\overline{A^{\xi'}_{(k',\lambda')}(k_1,\lambda_1)}\bigg|^2.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathfrak{P}^{(n)}_{b,c} &\geqslant \sum_{k\neq 0}\langle k\rangle^{4c} \sum_{k_1\neq k} \mathbf{1}_{\{\langle k_1-k\rangle\leqslant 2^n\}}\\
&\qquad\qquad \int_{\mathbb{R}^2} d\lambda d\lambda' \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} \frac{d\xi'}{|\xi'|^{2H-1}} \bigg|\int_{\mathbb{R}} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} A^{\xi}_{(k,\lambda)}(k_1,\lambda_1)\overline{A^{\xi'}_{(k,\lambda')}(k_1,\lambda_1)}\bigg|^2\\
&\geqslant \sum_{k\neq 0}\langle k\rangle^{4c} \sum_{k_1\neq k} \mathbf{1}_{\{\langle k_1-k\rangle\leqslant 2^n\}} \int_{\mathbb{R}} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \int_{\mathbb{R}} \frac{d\lambda_1'}{\langle k_1\rangle^{2c}+\langle\lambda_1'\rangle^{2b}}\\
&\qquad \bigg[\int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, A^{\xi}_{(k,\lambda)}(k_1,\lambda_1)\overline{A^{\xi}_{(k,\lambda)}(k_1,\lambda_1')}\bigg]\bigg[\int_{\mathbb{R}} \frac{d\xi'}{|\xi'|^{2H-1}} \int_{\mathbb{R}} d\lambda'\,\overline{A^{\xi'}_{(k,\lambda')}(k_1,\lambda_1)}A^{\xi'}_{(k,\lambda')}(k_1,\lambda_1')\bigg]\\
&\geqslant \sum_{k\neq 0}\langle k\rangle^{4c} \sum_{k_1\neq k} \mathbf{1}_{\{\langle k_1-k\rangle\leqslant 2^n\}} \int_{\mathbb{R}} \frac{d\lambda_1}{\langle k_1\rangle^{2c}+\langle\lambda_1\rangle^{2b}} \int_{\mathbb{R}} \frac{d\lambda_1'}{\langle k_1\rangle^{2c}+\langle\lambda_1'\rangle^{2b}}\\
&\qquad\qquad \bigg|\int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, A^{\xi}_{(k,\lambda)}(k_1,\lambda_1)\overline{A^{\xi}_{(k,\lambda)}(k_1,\lambda_1')}\bigg|^2.
\end{aligned}
$$

In turn, this entails that

$$\begin{aligned}\mathfrak{P}_{b,c}^{(n)} &\geqslant \sum_{2\leqslant k\leqslant 2^n} \langle k\rangle^{4c} \int_{-2k-\frac{1}{2}}^{-2k+\frac{1}{2}} \frac{d\lambda_1}{\langle k-1\rangle^{2c}+\langle \lambda_1\rangle^{2b}} \int_{-2k-\frac{1}{2}}^{-2k+\frac{1}{2}} \frac{d\lambda_1'}{\langle k-1\rangle^{2c}+\langle \lambda_1'\rangle^{2b}} \\ &\qquad\qquad \Big| \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, A^{\xi}_{(k,\lambda)}(k-1,\lambda_1)\overline{A^{\xi}_{(k,\lambda)}(k-1,\lambda_1')}\Big|^2 \\ &\gtrsim \sum_{2\leqslant k\leqslant 2^n} \frac{\langle k\rangle^{4c}}{\langle k\rangle^{4c}+\langle k\rangle^{4b}} \int_{-2k-\frac{1}{2}}^{-2k+\frac{1}{2}} d\lambda_1 \int_{-2k-\frac{1}{2}}^{-2k+\frac{1}{2}} d\lambda_1' \Big| \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, A^{\xi}_{(k,\lambda)}(k-1,\lambda_1)\overline{A^{\xi}_{(k,\lambda)}(k-1,\lambda_1')}\Big|^2 \\ &\gtrsim \sum_{2\leqslant k\leqslant 2^n} \frac{\langle k\rangle^{4c}}{\langle k\rangle^{4c}+\langle k\rangle^{4b}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1' \Big| \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, A^{\xi}_{(k,\lambda)}(k-1,\lambda_1-2k)\overline{A^{\xi}_{(k,\lambda)}(k-1,\lambda_1'-2k)}\Big|^2. \end{aligned} \tag{5.3}$$

At this point, observe that due to $\Omega_{k,-1}=2k$, one has in fact

$$\begin{aligned} A^{\xi}_{(k,\lambda)}(k-1,\lambda_1-2k) &= \int d\lambda_2\, \mathcal{I}_\chi(\lambda,\Omega_{k,-1}+\lambda_1-2k-\lambda_2)\overline{\mathcal{I}_\chi(\lambda_2,-\xi-1)} \\ &= \int d\lambda_2\, \mathcal{I}_\chi(\lambda,\lambda_1-\lambda_2)\overline{\mathcal{I}_\chi(\lambda_2,-\xi-1)} =: B(\xi,\lambda,\lambda_1), \end{aligned}$$

and therefore, going back to (5.3), we deduce that

$$\mathfrak{P}_{b,c}^{(n)} \gtrsim \Big(\sum_{2\leqslant k\leqslant 2^n} \frac{\langle k\rangle^{4c}}{\langle k\rangle^{4c}+\langle k\rangle^{4b}}\Big)\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1' \Big| \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, B(\xi,\lambda,\lambda_1)\overline{B(\xi,\lambda,\lambda_1')}\Big|^2\Big).$$

Recall now that $b-c\leqslant \frac{1}{4}$. Thus, in order to ensure that $\mathfrak{P}_{b,c}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty$, we only need to guarantee that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda_1' \Big| \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, B(\xi,\lambda,\lambda_1)\overline{B(\xi,\lambda,\lambda_1')}\Big|^2 > 0. \tag{5.4}$$

To this end, observe for instance that for $(\lambda_1,\lambda_1')=(0,0)$,

$$\int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, B(\xi,\lambda,0)\overline{B(\xi,\lambda,0)} = \int_{\mathbb{R}} \frac{d\xi}{|\xi|^{2H-1}} \int_{\mathbb{R}} d\lambda\, \big|B(\xi,\lambda,0)\big|^2,$$

and then, for $(\xi,\lambda)=(-1,0)$,

$$\begin{aligned} B(-1,0,0) &= \int d\lambda_2\, \mathcal{I}_\chi(0,-\lambda_2)\overline{\mathcal{I}_\chi(\lambda_2,0)} \\ &= \int d\lambda_2 \Big(\int_{\mathbb{R}} dt\, \chi(t) \int_0^t ds\, e^{\imath\lambda_2 s}\chi(s)\Big)\Big(\int_{\mathbb{R}} dt'\, e^{\imath\lambda_2 t'}\chi(t')\int_0^{t'} ds'\, \chi(s')\Big) \\ &= \int_{\mathbb{R}} dt\, \chi(t)\int_0^t ds\, \chi(s)\chi(-s)\int_0^{-s} ds'\, \chi(s') = \int_{\mathbb{R}} dt\, \chi(t)\int_0^t ds\, \chi(s)^2 \int_0^s ds'\, \chi(s') > 0, \end{aligned}$$

where we have used the fact that χ is symmetric and $\chi\equiv 1$ on $[-1,1]$.

For obvious continuity reasons, we deduce that (5.4) is satisfied, and accordingly one has

$$\mathfrak{P}_{b,c}^{(n)} \overset{n\to\infty}{\longrightarrow} \infty,$$

as desired.

6. Appendix

6.1. **Proof of Lemma 1.1, item** (i)**.**

We know that the linear solution Ψ is explicitly given by the convolution formula

$$\Psi(t) = \int_0^t e^{-\imath\Delta(t-s)}\dot{B}(s),$$

and hence, using the notation introduced in (1.8),

$$\begin{aligned}\mathbb{E}\Big[\big\|\Psi\big\|^2_{L^2([0,T]\times\mathbb{T})}\Big] &= \sum_k \int_0^T \mathbb{E}\Big[\big|\Psi_k(t)\big|^2\Big]\,dt \\ &= \sum_k \int_0^T \mathbb{E}\bigg[\bigg|\int_0^t e^{-\imath k^2 s}\dot{\beta}^{(k)}_{t-s}\,ds\bigg|^2\bigg]\,dt = \sum_k \int_0^T\int_0^t ds\int_0^t ds' e^{-\imath k^2(s-s')}\mathbb{E}\big[\dot{\beta}^{(k)}_{t-s}\dot{\beta}^{(k)}_{t-s'}\big].\end{aligned}$$

Recall that the fractional covariance is given by $\mathbb{E}\big[\dot{\beta}^{(k)}_s\dot{\beta}^{(k)}_{s'}\big] = |s-s'|^{2H-2}$, which gives

$$\begin{aligned}\mathbb{E}\Big[\big\|\Psi\big\|^2_{L^2([0,T]\times\mathbb{T})}\Big] &= \sum_k \int_0^T\int_0^t ds\int_0^t ds'\,\frac{e^{-\imath k^2(s-s')}}{|s-s'|^{2-2H}} = \sum_k \int_0^T\int_0^t ds\int_{-s}^{t-s} dr\,\frac{e^{\imath k^2 r}}{|r|^{2-2H}} \\ &= \sum_k \int_0^T\int_0^t ds\int_{-s}^{0} dr\,\frac{e^{\imath k^2 r}}{|r|^{2-2H}} + \sum_k \int_0^T\int_0^t ds\int_{0}^{t-s} dr\,\frac{e^{\imath k^2 r}}{|r|^{2-2H}} \\ &= \sum_k \int_0^T\int_0^t ds\int_{0}^{s} dr\,\frac{e^{-\imath k^2 r}}{|r|^{2-2H}} + \sum_k \int_0^T\int_0^t ds\int_{0}^{s} dr\,\frac{e^{\imath k^2 r}}{|r|^{2-2H}} \\ &= 2\sum_k \int_0^T\int_0^t ds\int_0^s dr\,\frac{\cos(k^2 r)}{r^{2-2H}} \\ &= 2\int_0^T\int_0^t ds\int_0^s \frac{dr}{r^{2-2H}} + 2\sum_{k\neq 0}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^t ds\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}} \\ &\lesssim 1 + 2\sum_{k\neq 0}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^t ds\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}. \end{aligned} \tag{6.1}$$

Now observe that

$$\begin{aligned}&\bigg|\sum_{k\neq 0}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^t ds\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg| \\ &\lesssim \sum_{k\neq 0:|k|\leqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^t ds\int_0^1\frac{dr}{r^{2-2H}} + \sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^t ds\bigg|\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg| \\ &\lesssim \sum_{k\neq 0}\frac{1}{|k|^{4H-2}} + \sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^T ds\bigg|\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg|. \end{aligned} \tag{6.2}$$

Then

$$\begin{aligned}&\sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^T ds\bigg|\int_0^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg| \\ &\lesssim \sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_0^{\frac{1}{k^2}} ds\int_0^1\frac{dr}{r^{2-2H}} + \sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_{\frac{1}{k^2}}^T ds\bigg[\int_0^1\frac{dr}{r^{2-2H}} + \bigg|\int_1^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg|\bigg] \\ &\lesssim \sum_{k\neq 0}\frac{1}{|k|^{4H-2}} + \sum_{k\neq 0:|k|\geqslant\frac{1}{\sqrt{T}}}\frac{1}{|k|^{4H-2}}\int_0^T\int_{\frac{1}{k^2}}^T ds\bigg|\int_1^{k^2 s} dr\,\frac{\cos(r)}{r^{2-2H}}\bigg| \end{aligned} \tag{6.3}$$

and for every $s > \frac{1}{k^2}$, we can use an integration-by-parts argument to derive that

$$\left| \int_1^{k^2 s} dr\, \frac{\cos(r)}{r^{2-2H}} \right| \lesssim 1 + \int_1^{k^2 s} dr\, \frac{|\sin(r)|}{r^{3-2H}} \lesssim 1 + \int_1^{\infty} \frac{dr}{r^{3-2H}} \lesssim 1. \tag{6.4}$$

Injecting (6.2), (6.3) and (6.4) into (6.1), we can conclude that

$$\mathbb{E}\Big[\big\| \Psi \big\|^2_{L^2([0,T]\times\mathbb{T})} \Big] \lesssim 1 + \sum_{k\neq 0} \frac{1}{|k|^{4H-2}} < \infty,$$

due to the assumption $H > \frac{3}{4}$.

6.2. **Estimate of the product-operator norm.** We gather here a few elements to justify our consideration of the quantity $\mathcal{P}^{(n)}_{b,c}$ in (2.5) as an estimate of the operator norm $\big\| \mathcal{L}^{\sharp,(n)} \big\|_{X^b_c \to X^b_c}$.

6.2.1. *General operator estimates in Besov spaces.* Given a general kernel $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, define the operator $\mathcal{L}_K : L^2(\mathbb{R}^d) \to \mathcal{L}^2(\mathbb{R}^d)$ by

$$\mathcal{F}(\mathcal{L}_K f)(\lambda) = \int_{\mathbb{R}^d} d\lambda_1\, K(\lambda,\lambda_1)(\mathcal{F}f)(\lambda_1), \quad \lambda \in \mathbb{R}^d. \tag{6.5}$$

For each $\underline{b} = (b_1,\dots,b_d)$, consider the anisotropic Besov space $\mathcal{H}^{\underline{b}}$ on $\mathbb{R}^d$ related to the Fourier multiplier

$$\langle \lambda \rangle_{\underline{b}} := \Big(\sum_{i=1}^d |\lambda_i|^{2b_i} \Big)^{\frac{1}{2}}.$$

If K has no specific a priori structure, then we are essentially confined to very general transformations toward the operator norm of $\mathcal{L}_K$ (from $\mathcal{H}^{\underline{b}}$ to $\mathcal{H}^{\underline{b}'}$): namely,

$$\begin{aligned}
\big\| \mathcal{L}_K f \big\|^2_{\mathcal{H}^{\underline{b}'}} &= \int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'} \left| \int d\lambda_1\, K(\lambda,\lambda_1)(\mathcal{F}f)(\lambda_1) \right|^2 \\
&= \int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'} \left| \int d\lambda_1\, \frac{1}{\langle \lambda_1 \rangle_{\underline{b}}} K(\lambda,\lambda_1) \big(\langle \lambda_1 \rangle_{\underline{b}} (\mathcal{F}f)(\lambda_1) \big) \right|^2 \\
&= \int d\lambda_1 d\lambda_1' \left[\frac{1}{\langle \lambda_1 \rangle_{\underline{b}}} \frac{1}{\langle \lambda_1' \rangle_{\underline{b}}} \int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'}\, K(\lambda,\lambda_1) \overline{K(\lambda,\lambda_1')} \right] \Big[\big(\langle \lambda_1 \rangle_{\underline{b}} (\mathcal{F}f)(\lambda_1) \big) \big(\langle \lambda_1' \rangle_{\underline{b}} \overline{(\mathcal{F}f)(\lambda_1')} \big) \Big] \\
&\leqslant \left(\int \frac{d\lambda_1}{\langle \lambda_1 \rangle^2_{\underline{b}}} \frac{d\lambda_1'}{\langle \lambda_1' \rangle^2_{\underline{b}}} \left| \int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'}\, K(\lambda,\lambda_1) \overline{K(\lambda,\lambda_1')} \right|^2 \right)^{\frac{1}{2}} \big\| f \big\|^2_{\mathcal{H}^{\underline{b}}},
\end{aligned} \tag{6.6}$$

which provides us with the general bound

$$\big\| \mathcal{L}_K \big\|_{\mathcal{H}^{\underline{b}} \to \mathcal{H}^{\underline{b}'}} \leqslant \mathcal{P}^{\frac{1}{4}}_{\underline{b},\underline{b}'}, \quad \text{where } \mathcal{P}_{\underline{b},\underline{b}'} := \int \frac{d\lambda_1}{\langle \lambda_1 \rangle^2_{\underline{b}}} \frac{d\lambda_1'}{\langle \lambda_1' \rangle^2_{\underline{b}}} \left| \int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'}\, K(\lambda,\lambda_1) \overline{K(\lambda,\lambda_1')} \right|^2. \tag{6.7}$$

Remark 6.1. The operator norm $\big\| \mathcal{L}_K \big\|_{\mathcal{H}^{\underline{b}} \to \mathcal{H}^{\underline{b}'}}$ can be more basically - but less sharply - bounded by the Hilbert-Schmidt norm of $\mathcal{L}_K$, that is the quantity

$$\mathcal{Q}^{\frac{1}{2}}_{\underline{b},\underline{b}'} := \left(\int d\lambda\, \langle \lambda \rangle^2_{\underline{b}'} \int \frac{d\lambda_1}{\langle \lambda_1 \rangle^2_{\underline{b}}} |K(\lambda,\lambda_1)|^2 \right)^{\frac{1}{2}},$$

as it can be seen from a straightforward application of Cauchy-Schwarz inequality in (6.6).

A classical example where the three quantities $\big\| \mathcal{L}_K \big\|_{\mathcal{H}^{\underline{b}} \to \mathcal{H}^{\underline{b}'}}$, $\mathcal{P}^{\frac{1}{4}}_{\underline{b},\underline{b}'}$ and $\mathcal{Q}^{\frac{1}{2}}_{\underline{b},\underline{b}'}$ can be compared is given by the elementary kernel $K(t,s) := \mathbf{1}_{\{0\leqslant s\leqslant t\leqslant 1\}}$ in $L^2(\mathbb{R})$, or more exactly by the integration operator

$$\mathcal{L}f(t) := \mathbf{1}_{[0,1]}(t) \int_0^t f(s)\, ds.$$

In this case, it can be checked that

$$\big\|\mathcal{L}\big\|_{L^2\to L^2} = \frac{2}{\pi} \approx 0.637, \quad \mathcal{P}_{0,0}^{\frac{1}{4}} = \Big(\frac{1}{6}\Big)^{\frac{1}{4}} \approx 0.639, \quad \text{and} \quad \mathcal{Q}_{0,0}^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.707.$$

6.2.2. *Example: Young integration as random operator.* As an additional justification of our focus on $\mathcal{P}^{(n)}$, let us consider the (well-known) case of the fractional integration operator.

Thus, consider a fractional noise $\dot{B}$ on $\mathbb{R}$ with index $H \in (0,1)$, as well as a smooth approximation $\dot{B}^{(n)}$ given for instance by

$$\dot{B}^{(n)}(t) := \int_{\{|\xi|\leqslant 2^n\}} e^{-\imath t\xi}\, \widehat{W}(d\xi), \tag{6.8}$$

where $\widehat{W}$ stands for the Fourier transform of a Wiener process W on $\mathbb{R}$.

Then define the (local) fractional integration operator in a standard way: for any regular $z : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$,

$$\big(\mathcal{L}^{(n)} z\big)(t) := \chi(t) \int_0^t ds\, \chi(s) z(s) \dot{B}^{(n)}(s), \tag{6.9}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ only stands for a smooth localizing cut-off function.

In Fourier mode, the operator $\mathcal{L}^{(n)}$ can easily be written along the pattern of (6.5), that is as

$$\mathcal{F}\big(\mathcal{L}^{(n)} z\big)(\lambda) = \int d\lambda_1\, K^{(n)}(\lambda,\lambda_1)\, (\mathcal{F}z)(\lambda_1),$$

for some unique kernel $K^{(n)}$. Setting

$$\big\|z\big\|_{X^b}^2 := \int d\lambda\, \langle\lambda\rangle^{2b} \big|\mathcal{F}(z)\big|^2$$

and applying (6.7), we deduce that

$$\big\|\mathcal{L}^{(n)}\big\|_{X^b\to X^b} \lesssim \big(\mathcal{P}_b^{(n)}\big)^{\frac{1}{4}}, \quad \text{with} \quad \mathcal{P}_b^{(n)} := \int \frac{d\lambda_1}{\langle\lambda_1\rangle^{2b}} \frac{d\lambda_1'}{\langle\lambda_1'\rangle^{2b}} \bigg| \int d\lambda\, \langle\lambda\rangle^{2b}\, K^{(n)}(\lambda,\lambda_1) \overline{K^{(n)}(\lambda,\lambda_1')} \bigg|^2. \tag{6.10}$$

Focusing exclusively on this explicit quantity $\mathcal{P}_b^{(n)}$ allows us to recover the classical "Young" dichotomy of fractional integration theory.

Proposition 6.2. *In the above setting, the following picture holds true:*
(i) If $H > \frac{1}{2}$, then for every $b \in (\frac{1}{2}, H)$, one has

$$\sup_n \mathbb{E}\Big[\mathcal{P}_b^{(n)}\Big] < \infty.$$

(ii) If $H = \frac{1}{2}$, then for every $b \in \mathbb{R}$, one has

$$\mathbb{E}\Big[\mathcal{P}_b^{(n)}\Big] \overset{n\to\infty}{\longrightarrow} \infty.$$

For the sake of conciseness, we prefer to leave the proof of this assertion as an exercise to the reader.

References

[1] V. Barbu, M. Röckner and D. Zhang: Stochastic nonlinear Schrödinger equations. *Nonlinear Analysis: Theory, Methods and Applications* **136** (2016), 168-194.

[2] A. de Bouard and A. Debussche: A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.* **205** (1999), 161-181.

[3] A. de Bouard and A. Debussche: The stochastic nonlinear Schrödinger equation in H^1. *Stoch. Anal. Appl.* **21** (2003), no. 1, 97-126.

[4] J. Bourgain: Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Commun. Math. Phys.* **176** (1996), 421-445.

[5] Z. Brzézniak and A. Millet: On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Analysis* **41** (2013), no. 2, 269-315.

[6] X. Chen, A. Deya, C. Ouyang and S. Tindel: Moment estimates for some renormalized parabolic Anderson models. *Ann. Prob.* **49** (2021), no. 5, 2599-2636.

[7] K. Cheung and R. Mosincat: Stochastic nonlinear Schrödinger equations on tori. *Stoch PDE: Anal. Comp.* **7** (2019), 169-208.

[8] J. Colliander and T. Oh: Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. *Duke Math. J.* **161** (2012), no. 3, 367–414.

[9] Y. Deng, A. Nahmod and H. Yue: Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two. *Arxiv preprint.*

[10] Y. Deng, A. Nahmod and H. Yue: Random tensors, propagation of randomness, and nonlinear dispersive equations. *Invent. math.* **228** (2022), 539-686.

[11] A. Deya, R. Fukuizumi and L. Thomann: Renormalization of a 1d quadratic Schrödinger model with additive noise. *Arxiv preprint.*

[12] J. Forlano, T. Oh and Y. Wang: Stochastic nonlinear Schrödinger equation with almost space-time white noise. *Journal of the Australian Mathematical Society* **10** (2020), no. 1, 44-67.

[13] F. Hornung: The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. *J. Evol. Equ.* **18** (2018), 1085–1114.

[14] C. Kenig, G. Ponce and L. Vega. Quadratic forms for the 1-D semilinear Schrödinger equation. *Trans. Amer. Math. Soc.* 348 (1996), no. 8, 3323-3353.

[15] T. Oh and M. Okamoto: On the stochastic nonlinear Schrödinger equations at critical regularities. *Stoch PDE: Anal. Comp.* **8** (2020), 869-894.

[16] T. Oh, O. Pocovnicu and Y. Wang: On the stochastic nonlinear Schrödinger equations with non-smooth additive noise. *Kyoto J. Math.* **60** (2020), no. 4, 1227-1243.