

# Explicit analytical solutions for the full plane-stress field in sandwich beams under flexure governed by zigzag warping

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## Abstract

We provide analytical solutions for the full stress field of straight sandwich beams with identical skins subject to linear elastic flexure governed by zigzag warping, where all layers obey Timoshenko's kinematics. As a main novelty, we make use of a balance equation for the Cauchy continuum to recover of the through-the-thickness normal stress component,  $\sigma_y$ . The new estimates are accurate for a wide range of relative stiffness between skins and core and suitable boundary conditions, as it can be demonstrated through the comparison with detailed finite element simulations where the sandwich is modelled as a two-dimensional continuum. As a main practical result concerned with the study of delamination, we find that at a core-skin interface of a cantilever sandwich subjected to a uniformly distributed load, in a region close to the fully-clamped cross-section,  $\sigma_y$  is a tensile stress of magnitude larger than that of the shear stress. On this basis, we infer that the availability of good estimates for  $\sigma_y$ , along with those for the longitudinal and shear stresses, may be important for the accurate design of sandwich

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panels.

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## 1 Introduction

We focus on the model for sandwich beams developed since the pivotal contributions of Yu [31], Heller [12], and Krajinovic [15, 16], whose main assumption is that both core and skins obey Timoshenko kinematics. We call this model the Krajinovic model because, to the best of our knowledge, Krajinovic [15] was the first one to establish a way to provide general analytical solutions for the case here of interest, that is the accurate evaluation of the linear elastic flexure of straight sandwich beams with identical skins.

The Krajinovic model, here summarised in Sect. 2, has been shown to provide good analytical estimates not only in terms of displacements, but also in terms of longitudinal and shear stress fields [3, 2]. This holds for a wide range of *relative stiffness between core and skins*, where this parameter mainly depends on the elastic moduli and the thicknesses of the layers [28]. The range of engineering adequacy of Krajinovic model even becomes extremely large if the boundary conditions are applied in such a way as to avoid as much as possible stress concentrations. Otherwise, dealing with “extreme” boundary conditions, in general referring to forces applied on a single skin with constraints applied on the opposite skins, may easily require more complex higher-order theories involving *soft* cores, typically including the midlayer through-the-thickness deformability [8, 9, 30, 24, 19, 22, 17]. Of course, the more complex the theory the more difficult finding useful analytical solutions.

Analytical solutions not only are very important in the optimal design, but may also be one of the very few reliable tools under specific circumstances, where otherwise powerful numerical methods fail. This is for instance the case of the stress field in the sandwich region next to a fully-clamped cross-section, where the displacement-based finite element (FE) method cannot provide good results when modelling the sandwich as a two-dimensional (2D) continuum [11, 29, 2].

By focusing on the through-the-thickness normal stress component  $\sigma_y$ , this contribution complements our previous efforts [3, 2] where we provided analytical solutions for the longitudinal and shear stresses within Krajinovic theory. As presented in Sect. 3, we obtain  $\sigma_y$  by a double stress recovery. In fact, if one is strictly consistent with the zigzag warping assumed by the theory, Krajinovic model delivers very accurate predictions of the longitudinal stress field *only*. Obtaining accurate estimates for the shear stress  $\tau_{xy}$  requires one to resort to the application of a Jourawski-like approach [14] to the longitudinal stress  $\sigma_x$  ensuing from the zigzag kinematics, which is actually equivalent to integrate, with respect to the transverse axis  $y$ , the balance equation for the Cauchy continuum

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad (1)$$

along with averaging over the out-of-plane axis  $z$ . This *first* stress recovery has been adopted also in some numerical contributions [18, 20], where one has to take care of the required derivative of  $\sigma_x$  with respect to the beam axis  $x$ , which leads to higher continuity conditions for the functions adopted to approximate the primal fields.

In this investigation, we adopt a *second* stress recovery to determine the through-the-

thickness stress  $\sigma_y$ , which is just indeterminate on the basis of the kinematics assumed by Krajinovic theory, totally disregarding the through-the-thickness normal deformation. To this purpose, we consider the balance equation for the Cauchy continuum

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad (2)$$

where we plug the shear stress obtained from the *first* stress recovery. Hence, this *second* stress recovery for  $\sigma_y$  relies on a stress field, i.e.  $\tau_{xy}$ , that has already been obtained by a recovery procedure. While there is no problem in our analytical approach, numerical methods using the foregoing double stress recovery have to deal with high continuity conditions of the approximating functions, given that  $\sigma_y$  turns out to depend on  $\partial^2 \sigma_x / \partial x^2$ . This numerical issue has been formerly dealt with by Matsunaga [18] for laminated beams governed by a higher-order shear deformation theory obtained by power series expansions for the displacement components. In the case of laminated plates [4, 18, 7, 23] the through-the-thickness normal stress field is recovered after two shear stress fields are obtained by two parallel stress recovery procedures.

About the role played by  $\sigma_y$  in laminated beams, here we demonstrate, through the cantilever benchmark of Sect. 4, that there are important boundary value problems in which it may not be disregarded. In particular, we show that at the interfaces between core and skins  $\sigma_y$  may be a tensile stress of much larger magnitude than  $\tau_{xy}$ , thus surely impacting the analysis of delamination. Among the earlier studies on the sandwich theory here adopted, Heller [12] already focused on the interfacial stress, but the focus was restricted to the shear stress only. Very recently, the interfacial normal stress has been considered as well [26],

although in the context of a simpler structural model, where the skins are rigid in shear, thus behaving as Euler-Bernoulli beams. Here, we not only adopt the richer Krajinovic model, but we provide explicit analytical solutions for  $\sigma_y$  over the whole sandwich.

## 2 Summary of the Krajinovic model

We adopt the notation and select the independent structural functions as in [3, 2] to describe the Krajinovic model. The model relies on the zigzag warping depicted in Fig. 1, where

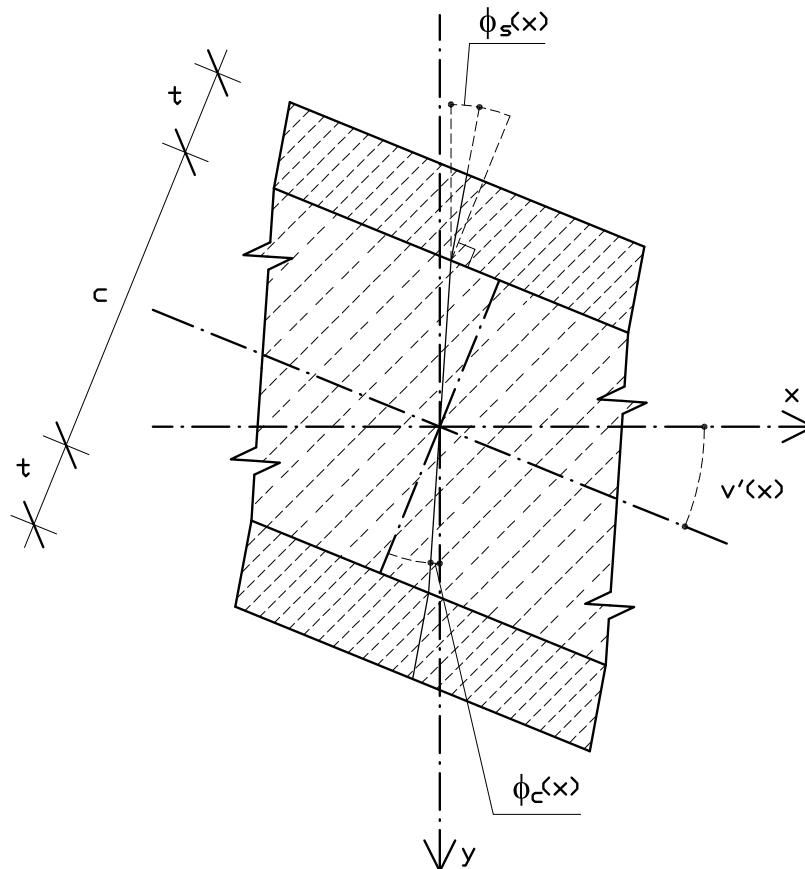


Figure 1: Zigzag kinematics for a sandwich beam with identical skins. Taken from [2].

$v(x)$  is the transverse displacement (directed as the  $y$  axis, such as  $z$  is the neutral axis),  $t$  and  $c$  are the thicknesses of skins and core, respectively,  $\phi_c(x)$  and  $\phi_s(x)$  are the rotations of

the core and skin cross-sections, respectively. Instead of  $\phi_c(x)$  and  $\phi_s(x)$ , for the independent structural variables, Krajinovic selected a mean rotation of the whole cross-section and a warping function associated with null bending moment. This allowed Krajinovic to find a way to analytically integrate the three Euler-Lagrange equations governing the problem. The description of the zigzag warping of Fig. 1, which is inspired by the earlier models of Allen [1], has instead the advantage to turn out to naturally display, as static quantities, the bending moments related to the *variations* of  $\phi_s(x)$  and  $\phi_c(x)$ , along with their derivatives with respect to the beam axis, which result to be the shear forces developed by the skins and the core.

The considered Krajinovic model disregards the through-the-thickness strain  $\varepsilon_y$ , such as the stress component  $\sigma_y$  remains indeterminate if computed consistently with the assumed kinematics.

In sandwich beams, the ratios

$$\epsilon = \frac{t}{c}, \quad n = \frac{E_s}{E_c}, \quad \varphi = \frac{G_s}{G_c} \quad (3)$$

are particularly relevant to select the most appropriate model for a specific problem [28].

In Eq. (3)  $E$  and  $G$  are the longitudinal modulus and the shear modulus in the  $xy$  plane, respectively, where the indices  $c$  and  $s$  denote their values assumed in the core and in the skins, respectively. Differently from other models for sandwich panels [1], Krajinovic theory does not impose any particular restriction on the parameters that define the *relative stiffness between skins and core*, the latter governing the sandwich flexure. Therefore, the application of Krajinovic theory is not limited to *thin* skins (i.e.  $\epsilon$  sufficiently close to zero), nor to

*antiplane* sandwiches (i.e.  $n \rightarrow \infty$ ).

By following [3] and the warping description of Fig. 1, the balance equations governing Krajinovic model read

$$V'(x) = -q(x), \quad M'_c(x) = V_c(x), \quad M'_s(x) = V_s(x), \quad (4)$$

in which  $'$  denotes the derivative with respect to  $x$ ,  $q(x)$  is the distributed transverse load (acting along the  $y$  direction) and  $M(x) = M_s(x) + M_c(x)$  and  $V(x) = V_s(x) + V_c(x)$  are, respectively, the total bending moment and the total shear force, whose contributions result to be defined by the structural constitutive equations

$$M_c = -\frac{E_s b t c}{2}(c\phi'_c + t\phi'_s) - \frac{E_c b c^3}{12}\phi'_c, \quad M_s = -\frac{E_s b t^3}{6}\phi'_s - \frac{E_s b t^2}{2}(c\phi'_c + t\phi'_s), \quad (5)$$

$$V_c = G_c b c(v' - \phi_c), \quad V_s = 2G_s b t(v' - \phi_s), \quad (6)$$

where  $b$  is the sandwich width. While  $V_c$  and  $V_s$  are the contributions to the shear force acting in the core and in the skins, respectively, a similar connection does not hold for the moments  $M_c$  and  $M_s$ , which are conjugated to the *variations* of  $\phi_c$  and  $\phi_s$ . The three coupled Euler-Lagrange equations governing the theory, to be solved for  $v(x)$ ,  $\phi_c(x)$ , and  $\phi_s(x)$ , are obtained by substituting Eqs. (5) and (6) into Eq. (4).

Let us remark that the natural boundary conditions leave the freedom to separately impose  $M_s$  and  $M_c$  as related to the *variations* of  $\phi_s$  and  $\phi_c$ , respectively. Hence, the same bending moment  $M$  results from any combination of  $M_s$  and  $M_c$  such that  $M = M_s + M_c$ , each choice of  $M_s$  and  $M_c$  corresponding to a different solution in terms of stresses. Because of this fundamental reason, this model is not “à la De Saint Venant” [28].

The Krajinovic solution, as a first step, requires the integration of the differential equation

$$\frac{d^6 f(\xi)}{d\xi^6} - k^2 \frac{d^4 f(\xi)}{d\xi^4} = -\frac{\beta_2 L^6}{\beta_1 \alpha_2} q(\xi) \quad (7)$$

to determine the auxiliary structural function  $f(x)$ . In Eq. (7),  $L$  is the beam length,

$$\xi = \frac{x}{L} \quad (8)$$

is the non-dimensional coordinate along the beam axis, and  $k$ ,  $\beta_1$ ,  $\beta_2$ ,  $\alpha_2$  are constant coefficients conveniently provided below.

Once  $f(x)$  is known, the structural functions can be obtained, by combining the relations presented in [15, 3], leading to

$$\phi_c(\xi) = \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{1}{L^3} \frac{d^3 f(\xi)}{d\xi^3} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{1}{L} \frac{df(\xi)}{d\xi}, \quad (9)$$

$$\phi_s(\xi) = -\left( \frac{c(1+m)}{2} + \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{1}{L^3} \frac{d^3 f(\xi)}{d\xi^3} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{1}{L} \frac{df(\xi)}{d\xi}, \quad (10)$$

$$v(\xi) = \frac{\alpha_2}{\beta_2} \frac{1}{L^4} \frac{d^4 f(\xi)}{d\xi^4} - \left( \beta_3 + \beta_1 \frac{\alpha_2}{\alpha_1} \right) \frac{1}{\beta_2} \frac{1}{L^2} \frac{d^2 f(\xi)}{d\xi^2} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} f(\xi). \quad (11)$$

In Eqs. (7), (9), (10), and (11), the coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ ,  $m$ ,  $k$  turn out to depend on the heterogeneity coefficients (3) and on other sandwich parameters as

$$\begin{aligned} \alpha_1 &= \frac{bc^3 E_c}{12} \omega_1, & \alpha_2 &= \frac{bc^5 E_c}{12} \left( \frac{2 + 3n\epsilon(1 + \omega_1)}{n(3 + 4\epsilon)^2} + \epsilon \right) \epsilon, \\ \beta_1 &= bcG_c(1 + 2\varphi\epsilon), & \beta_2 &= bc^2 G_c \left( \frac{\varphi\omega_2}{n(3 + 4\epsilon)} - \epsilon \right), & \beta_3 &= bc^3 G_c \left( \frac{\varphi\omega_2^2}{2n^2\epsilon(3 + 4\epsilon)^2} + \epsilon^2 \right), \\ m &= \frac{1 + n\epsilon(3 + 2\epsilon)}{n\epsilon(3 + 4\epsilon)}, & k &= \sqrt{\frac{(\beta_1 \beta_3 - \beta_2^2)L^2}{\beta_1 \alpha_2}}, \end{aligned} \quad (12)$$

where

$$\omega_1 = 1 + 2n\epsilon(3 + 6\epsilon + 4\epsilon^2), \quad \omega_2 = 1 + 6n\epsilon + 6n\epsilon^2.$$



Let us mention that the coefficient  $m$  in Eq. (12) enters the warping shape function deduced by Krajinovic in order to devise a way to analytically integrate the Euler-Lagrange governing equations [15].

### 3 The full stress field: analytical expression for the through-the-thickness normal stress *via* double stress recovery

In the absence of issues concerned with *severe* boundary conditions [22] and *soft* cores [9, 24, 17], the sole stress component that is accurate if computed consistently with the assumed zigzag kinematics in Krajinovic theory [3, 2] is normal longitudinal stress, which reads

$$\sigma_x(x, y) = \begin{cases} -E_s \left[ \phi'_c(x) \frac{c}{2} \text{sgn}(y) + \phi'_s(x) \left( y - \frac{c}{2} \text{sgn}(y) \right) \right] & \text{if } y \in A_s \\ -E_c \phi'_c(x) y & \text{if } y \in A_c \end{cases} \quad (13)$$

where  $A_s$  and  $A_c$  are the sandwich domains occupied by the skins and the core, respectively.

For what concern the other stress components, consistently with the kinematics, one has the following piecewise uniform shear stress

$$\tau_{xy}^{\text{uni}}(x) = \begin{cases} G_s [v'(x) - \phi_s(x)] & \text{if } y \in A_s \\ G_c [v'(x) - \phi_c(x)] & \text{if } y \in A_c \end{cases} \quad (14)$$

which obviously violate equilibrium at the interfaces, whereas the through-the-thickness normal stress  $\sigma_y$  is even indeterminate because of the assumed rigidity of the cross-section along the  $y$  axis.

Estimate (14) can be largely improved as proposed in [3], by resorting to the application of the Jourawski approach [14] to the zigzag kinematics, that is the *first* the stress recovery discussed about Eq. (1). Hence, the average over the beam width  $b$  of the shear stress

$\tau_{xy}(x, y, z)$ , relying on the zigzag kinematics, reads [3]

$$\bar{\tau}_{xy}^{zz}(x, y) = \begin{cases} E_s \left\{ \phi_s''(x) \frac{1}{2} \left[ \left( |y| - \frac{c}{2} \right)^2 - t^2 \right] - \phi_c''(x) \frac{c}{2} \left( \frac{c}{2} + t - |y| \right) \right\} & \text{if } y \in A_s \\ E_c \phi_c''(x) \frac{1}{2} \left( y^2 - \frac{c^2}{4} \right) - \underbrace{E_s \frac{t}{2} [\phi_s''(x)t + \phi_c''(x)c]}_{\bar{\tau}_{xy}^{\text{int}}(x)} & \text{if } y \in A_c \end{cases} \quad (15)$$

where  $\bar{\tau}_{xy}^{\text{int}}(x) = \bar{\tau}_{xy}^{zz}(x, y = |c/2|)$  is the interfacial shear stress.

As a main novelty of this investigation, we follow a *second* stress recovery procedure to determine  $\sigma_y$  by substituting Eq. (15) into Eq. (2). By also assuming, for simplicity, that the transverse load  $q(x)$  is applied either at  $y = -t - c/2$  or at  $y = t + c/2$  (or even split in two contributions on these two external boundaries), one can estimate

$$\sigma_y^{\text{rec}}(x, y) = \sigma_y(x, -t - c/2) - \int_{-t-c/2}^y \frac{\partial \bar{\tau}_{xy}^{zz}(x, y^*)}{\partial x} dy^*. \quad (16)$$

Then, for the three sandwich layers, we obtain

$$\begin{aligned} \sigma_y^{\text{rec}}(x, y) = & \sigma_y(x, -t - c/2) + E_s \left\{ \phi_s'''(x) \frac{1}{2} \left[ t^2 \left( y + \frac{c}{2} + \frac{2t}{3} \right) - \frac{1}{3} \left( y + \frac{c}{2} \right)^3 \right] \right. \\ & \left. + \phi_c'''(x) \frac{c}{2} \left[ y \left( \frac{c}{2} + t + \frac{y}{2} \right) + \frac{1}{2} \left( \frac{c}{2} + t \right)^2 \right] \right\}, \quad \text{for } y \in [-t - c/2, -c/2], \end{aligned} \quad (17a)$$

$$\begin{aligned} \sigma_y^{\text{rec}}(x, y) = & \sigma_y^{\text{rec}}(x, -c/2) + E_c \phi_c'''(x) \frac{1}{2} \left( \frac{c^2 y}{4} - \frac{y^3}{3} + \frac{c^3}{12} \right) \\ & + E_s \frac{t}{2} \left( y + \frac{c}{2} \right) [\phi_s'''(x)t + \phi_c'''(x)c], \quad \text{for } y \in [-c/2, c/2], \end{aligned} \quad (17b)$$

$$\begin{aligned} \sigma_y^{\text{rec}}(x, y) = & \sigma_y^{\text{rec}}(x, c/2) + E_s \left\{ \phi_s'''(x) \frac{1}{2} \left[ t^2 \left( y - \frac{c}{2} \right) - \frac{1}{3} \left( y - \frac{c}{2} \right)^3 \right] \right. \\ & \left. + \phi_c'''(x) \frac{c}{2} \left[ y \left( \frac{c}{2} + t - \frac{y}{2} \right) - \frac{c}{2} \left( \frac{c}{4} + t \right) \right] \right\} \quad \text{for } y \in [c/2, t + c/2]. \end{aligned} \quad (17c)$$

In the above relations  $\sigma_y(x, -t - c/2)$  is a known value, while, by evaluating (17a) at  $y = -c/2$ ,

(17b) at  $y = c/2$ , and (17c) at  $y = t + c/2$ , it results

$$\sigma_y^{\text{rec}}(x, -c/2) = \sigma_y(x, -t - c/2) + E_s \left( \phi_s'''(x) \frac{t^3}{3} + \phi_c'''(x) \frac{ct^2}{4} \right), \quad (18)$$

$$\sigma_y^{\text{rec}}(x, c/2) = \sigma_y^{\text{rec}}(x, -c/2) + E_c \phi_c'''(x) \frac{c^3}{12} + E_s \frac{tc}{2} [\phi_s'''(x)t + \phi_c'''(x)c], \quad (19)$$

$$\sigma_y^{\text{rec}}(x, t + c/2) = \sigma_y^{\text{rec}}(x, c/2) + E_s \left( \phi_s'''(x) \frac{t^3}{3} + \phi_c'''(x) \frac{ct^2}{4} \right). \quad (20)$$

Hence, from the equilibrium of the forces along the  $y$  direction for each layer, we deduce

$$E_s \left( \phi_s'''(x) \frac{t^3}{3} + \phi_c'''(x) \frac{ct^2}{4} \right) = -\frac{V_s'}{2b}, \quad (21)$$

$$E_c \phi_c'''(x) \frac{c^3}{12} + E_s \frac{tc}{2} [\phi_s'''(x)t + \phi_c'''(x)c] = -\frac{V_c'}{b}, \quad (22)$$

which perfectly agree with the expressions that one can obtain by deriving the second and third relations of Eq. (4) and substituting in the obtained relations the definitions of the bending moments in Eq. (5). This check shows that the double stress recovery provides interfacial  $\sigma_y$  that can be obtained also by imposing the equilibrium of the free-body diagram of the layer elements, which delivers balance equations equivalent to those obtained by minimising the Total Potential Energy governing the problem [15, 3, 26]. However, the double stress recovery provides the  $\sigma_y$  field all over the sandwich, which is worth to be explored. In particular, on the neutral axis we obtain

$$\sigma_y^{\text{rec}}(x, 0) = \sigma_y^{\text{rec}}(x, -c/2) + E_c \phi_c'''(x) \frac{c^3}{24} + E_s \frac{tc}{4} [\phi_s'''(x)t + \phi_c'''(x)c].$$

By setting  $t = 0$ , we particularise Eq. (17b) to a homogeneous beam with rectangular cross-section of height  $c$ , thus obtaining

$$\begin{aligned} \sigma_y^{\text{rec}}(x, y) &= \sigma_y^{\text{rec}}(x, -c/2) + E_c \phi_c'''(x) \frac{1}{2} \left( \frac{c^2 y}{4} - \frac{y^3}{3} + \frac{c^3}{12} \right) \\ &= \sigma_y^{\text{rec}}(x, -c/2) + q(x) \frac{12}{bc^3} \frac{1}{2} \left( \frac{c^2 y}{4} - \frac{y^3}{3} + \frac{c^3}{12} \right) \\ &= \sigma_y^{\text{rec}}(x, -c/2) + \frac{q(x)}{b} \left( \frac{3y}{2c} - \frac{2y^3}{c^3} + \frac{1}{2} \right). \end{aligned}$$

This result shows that on the neutral axis the through-the-thickness stress on the top surface is varied by half of the applied load only. Moreover, if  $q(x)$  is applied as a compressive load on the top surface, one has

$$\sigma_y^{\text{rec}}(x, y) = -\frac{q(x)}{b} + \frac{q(x)}{b} \left( \frac{3y}{2c} - \frac{2y^3}{c^3} + \frac{1}{2} \right),$$

which correctly vanishes at  $y = c/2$ . As expected and well-known in classical structural mechanics, this component of stress is negligible in homogeneous beams, consistently with  $\sigma_x/\sigma_y \sim (L/c)^2$ . Hence, the main point of this investigation is to demonstrate that this may not be the case in laminated structures. In other words, it would be sufficient to show that the contribution  $V'_s/(2b)$  in Eq. (21) can be large enough to require the evaluation of  $\sigma_y$  for a thorough analysis of the stress state in sandwich panels.

In particular, when one has to apply an appropriate criterion to establish whether delamination may occur,  $\sigma_y$  should be accounted for if it is on the order of magnitude of the shear stress at the interfaces, that is the ratio

$$\frac{\bar{\tau}_{xy}^{\text{int}}(x)}{\sigma_y^{\text{rec}}(x, -c/2)} = \frac{-E_s \frac{t}{2} [\phi_s''(x)t + \phi_c''(x)c]}{\sigma_y(x, -t - c/2) - \frac{V'_s}{2b}} = \frac{\phi_s''(x)t + \phi_c''(x)c}{\frac{2q(x)}{E_s t b} - \phi_s'''(x) \frac{2t^2}{3} - \phi_c'''(x) \frac{ct}{2}},$$

obtained from Eqs. (15) and (21), should, in modulus, not be much larger than 1. In Sect. 4 we show that this is surely the case, as  $\sigma_y^{\text{rec}}(x, -c/2)$  may even be a tensile stress significantly larger than  $\bar{\tau}_{xy}^{\text{int}}(x)$ . This is of course a further interesting effect due to the heterogeneity of laminated structures. With this goal in mind, we also provide general relations for the interfacial shear stress, which can be easily obtained from the relations presented in [3, 2]. Note that, differently from  $\sigma_y$ , in Krajinovic model the shear stress at the two interfaces is

identical. From Eqs. (15), (9), and (10) we obtain

$$\bar{\tau}_{xy}^{\text{int}}(x) = -E_s \frac{t}{2} \left\{ \left[ -t \left( \frac{c(1+m)}{2} + \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) + c \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \right] \frac{1}{L^5} \frac{d^5 f(\xi)}{d\xi^5} + (t+c) \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{1}{L^3} \frac{d^3 f(\xi)}{d\xi^3} \right\}. \quad (23)$$

The evaluation of  $\sigma_y$  at the interfaces basically corresponds to evaluate  $V'_s/(2b)$  in Eq. (21).

From Eqs. (9) and (10), we obtain

$$\phi_c''' = \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{1}{L^6} \frac{d^6 f(\xi)}{d\xi^6} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{1}{L^4} \frac{d^4 f(\xi)}{d\xi^4}, \quad (24)$$

$$\phi_s''' = - \left( \frac{c(1+m)}{2} + \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{1}{L^6} \frac{d^6 f(\xi)}{d\xi^6} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{1}{L^4} \frac{d^4 f(\xi)}{d\xi^4}, \quad (25)$$

which, if substituted in Eq. (21), deliver

$$-\frac{V'_s}{2b} = E_s t^2 \left\{ \left[ -\frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \left( \frac{t}{3} + \frac{c}{4} \right) - \frac{ct(1+m)}{6} + \frac{ct}{4} \right] \frac{1}{L^6} \frac{d^6 f(\xi)}{d\xi^6} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \left( \frac{t}{3} + \frac{c}{4} \right) \frac{1}{L^4} \frac{d^4 f(\xi)}{d\xi^4} \right\}. \quad (26)$$

Before delving into the numerical results, we provide relations that focus on the important case of a uniform transverse load  $q$ .

### 3.1 Particularisation to the case of uniform load

For a uniform transverse load  $q$ , the general solution of Krajinovic differential equation (7)

reads

$$f(\xi) = C_1 \sinh(k\xi) + C_2 \cosh(k\xi) + C_0 \xi^4 + C_3 \xi^3 + C_4 \xi^2 + C_5 \xi + C_6, \quad (27)$$

where

$$C_0 = \frac{\beta_2 q L^4}{24(\beta_1 \beta_3 - \beta_2^2)} = \frac{q L^4}{12 b c^2 G_c} \frac{n \epsilon (3 + 4 \epsilon) [\varphi \omega_2 - n \epsilon (3 + 4 \epsilon)]}{\varphi \omega_1^2} \quad (28)$$

and the integration constants  $C_i$  with  $i = 1, \dots, 6$  must be determined by imposing the essential and natural boundary conditions of the boundary value problem considered [16, 3, 2].

By substituting Eqs. (27) and (28) into Eqs. (23) and (26), we obtain

$$\begin{aligned} \bar{\tau}_{xy}^{\text{int}} = -E_s \frac{t}{2} \left\{ 6(t+c) \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{k^3}{L^3} [4C_0 \xi + C_3] + [C_1 \cosh(k\xi) + C_2 \sinh(k\xi)] \right. \\ \left. \times \left\{ \left[ -t \left( \frac{c(1+m)}{2} + \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) + c \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \right] \frac{k^5}{L^5} + (t+c) \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{k^3}{L^3} \right\} \right\}, \quad (29) \end{aligned}$$

$$\begin{aligned} -\frac{V_s'}{2b} = \frac{q}{\alpha_1} \left( \frac{t}{3} + \frac{c}{4} \right) + [C_1 \sinh(k\xi) + C_2 \cosh(k\xi)] \\ \times E_s t^2 \left\{ \left[ -\frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \left( \frac{t}{3} + \frac{c}{4} \right) - \frac{ct(1+m)}{6} + \frac{ct}{4} \right] \frac{k^6}{L^6} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \left( \frac{t}{3} + \frac{c}{4} \right) \frac{k^4}{L^4} \right\}. \quad (30) \end{aligned}$$

The application of Eqs. (29) and (30) to various benchmarks, corroborated with the results of accurate FE analyses in which the sandwich is modelled as a 2D plane-stress continuum, has shown that a very interesting boundary value problem is that of the cantilever beam, where  $\sigma_y$  may play a very important role. This does not seem to be the case, for instance, in the propped-cantilever and simply-supported beams, although there might be room for different conclusions for sandwich parameters outside the ranges that we have explored in our preliminary analyses. In the following we deal with the cantilever case, while in Appendix A we provide the analytical results for the simply-supported sandwich beam and skip other boundary value problems for the sake of brevity.

## 4 Cantilever sandwich subjected to uniform transverse load

With the fully-clamped section at  $x = 0$ , the integration constants  $C_i$  with  $i = 1, \dots, 6$  in Eq. (27) are obtained by imposing the essential boundary conditions

$$v(\xi = 0) = 0, \quad \phi_c(\xi = 0) = 0, \quad \phi_s(\xi = 0) = 0,$$

along the natural boundary conditions  $V(\xi = 1) = 0$ ,  $M_c(\xi = 1) = 0$ ,  $M_s(\xi = 1) = 0$ , corresponding to

$$v'(\xi = 1) - \varphi_c(\xi = 1) + 2\varphi\epsilon[v'(\xi = 1) - \phi_s(\xi = 1)] = 0, \quad \phi'_c(\xi = 1) = 0, \quad \phi'_s(\xi = 1) = 0.$$

Solving the system results in the expression for the Krajinovic auxiliary function [2]

$$f(\xi) = \frac{24C_0}{k^4} \frac{k(e^{-k(2-\xi)} - e^{-k\xi}) - e^{-k(1-\xi)} - e^{-k(1+\xi)}}{1 + e^{-2k}} + C_0\xi^4 - 4C_0\xi^3 + \frac{6C_0}{k^2}(k^2 + 2)\xi^2 - \frac{24C_0}{k^2}\xi + 12C_0 \left( \frac{1}{k^2} + \frac{\alpha_1\beta_3}{k^2\alpha_2\beta_1} + \frac{2}{k^4} + \frac{2\alpha_1\beta_2^2}{k^4\alpha_2\beta_1^2} \frac{1 - k + (1+k)e^{-2k} - 2e^{-k}}{1 + e^{-2k}} \right). \quad (31)$$

Then, by using Eqs. (23), (26), (28), and (31), we obtain

$$\begin{aligned} \bar{\tau}_{xy}^{\text{int}}(\xi) &= -E_s \frac{t}{2} \left\{ \left[ -t \left( \frac{c(1+m)}{2} + \frac{\beta_1\alpha_2}{\beta_2\alpha_1} \right) + c \left( t - \frac{\beta_1\alpha_2}{\beta_2\alpha_1} \right) \right] \frac{k^2}{L^5} \right. \\ &\quad \left. + (t+c) \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2L^3} \right\} \frac{24C_0}{k} \frac{k(e^{-k(2-\xi)} + e^{-k\xi}) - e^{-k(1-\xi)} + e^{-k(1+\xi)}}{1 + e^{-2k}} \\ &\quad \left. + 24C_0(\xi - 1)(t+c) \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2L^3} \right\} \\ &= -qE_s \frac{t}{2} L \left[ \frac{1-m}{2k} ct \frac{\beta_2}{\beta_1\alpha_2} \frac{k(e^{-k(2-\xi)} + e^{-k\xi}) - e^{-k(1-\xi)} + e^{-k(1+\xi)}}{1 + e^{-2k}} + \frac{(\xi - 1)(t+c)}{\alpha_1} \right], \quad (32) \end{aligned}$$

$$\begin{aligned} -\frac{V'_s}{2b} &= qE_s t^2 \left\{ \left( \frac{t}{3} + \frac{c}{4} \right) \frac{1}{\alpha_1} + \frac{tc}{2} \left( -\frac{1+m}{3} + \frac{1}{2} \right) \right. \\ &\quad \left. \times \frac{\beta_2}{\beta_1\alpha_2} \frac{k(e^{-k(2-\xi)} - e^{-k\xi}) - e^{-k(1-\xi)} - e^{-k(1+\xi)}}{1 + e^{-2k}} \right\}. \quad (33) \end{aligned}$$

Given that the displacement-based FE method in which the sandwich beam is modelled as a 2D continuum cannot provide reliable results at the fully-clamped cross-section [11, 29, 2], below we provide explicit analytical expressions for the recovered stresses therein.

#### 4.1 The recovered stresses at the fully-clamped cross-section

Evaluation of the relevant derivatives of Eq. (31) in  $x = 0$  gives

$$\frac{d^4 f(\xi)}{d\xi^4} \Big|_{\xi=0} = 24C_0 \left( \frac{k(e^{-2k} - 1) - 2e^{-k}}{1 + e^{-2k}} + 1 \right), \quad \frac{d^6 f(\xi)}{d\xi^6} \Big|_{\xi=0} = 24C_0 k^2 \frac{k(e^{-2k} - 1) - 2e^{-k}}{1 + e^{-2k}},$$

such as, from Eqs. (24) and (25), one obtains

$$\phi_c'''(0) = \frac{24C_0}{L^4} \left\{ \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2} + \frac{k(e^{-2k} - 1) - 2e^{-k}}{1 + e^{-2k}} \left[ \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2} + \left( t - \frac{\beta_1\alpha_2}{\beta_2\alpha_1} \right) \frac{k^2}{L^2} \right] \right\}, \quad (34)$$

$$\begin{aligned} \phi_s'''(0) = \frac{24C_0}{L^4} \left\{ \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2} + \frac{k(e^{-2k} - 1) - 2e^{-k}}{1 + e^{-2k}} \right. \\ \left. \times \left[ \frac{\beta_1\beta_3 - \beta_2^2}{\alpha_1\beta_2} - \left( \frac{\beta_1\alpha_2}{\beta_2\alpha_1} + \frac{c(1+m)}{2} \right) \frac{k^2}{L^2} \right] \right\}, \quad (35) \end{aligned}$$

to be substituted in Eq. (17) to obtain  $\sigma_y^{\text{rec}}(0, y)$ .

About the recovered shear stress, we report the results in [2], adopting the notation

$\bar{\tau}_{xy}^{\text{fc}}(y) = \bar{\tau}_{xy}^{\text{zz}}(\xi = 0, y)$ . Hence, the shear stress reads

$$\bar{\tau}_{xy}^{\text{fc}}(y) = \begin{cases} \frac{qL}{bcD_1} \left[ D_4 + 12\theta_0 \frac{|y|}{c} + 12D_2 \left( \frac{y}{c} \right)^2 \right] & \text{if } y \in A_s \\ \frac{qL}{bcD_1} \left[ D_5 + D_3 \left( \frac{y}{c} \right)^2 \right] & \text{if } y \in A_c \end{cases}$$

where

$$\theta_0 = \varphi\omega_2 - n\epsilon(3 + 4\epsilon), \quad D_1 = 4\epsilon^2(2 + 3n\epsilon)(1 + 2\varphi\epsilon), \quad D_2 = 3n\epsilon(1 - 2\varphi) - \varphi,$$

$$D_3 = 24\epsilon^2(3\varphi - 2), \quad D_4 = 3(1 + 2\epsilon)[D_2 + 2\epsilon(\varphi + n\epsilon)], \quad D_5 = 6\epsilon^2(2 - \varphi + 2n\epsilon).$$

## 4.2 Results and discussion

We consider the following set of parameters, corresponding to a quite large *relative stiffness between skins and core*,

$$n = 1000, \quad \epsilon = 9/140, \quad \nu_s = 0. \text{ and } \nu_c = 0.49 \Rightarrow \varphi = 1490, \quad (c + 2t)/L = 79/800 \quad (36)$$

and compare the proposed analytical estimates with the results of FE simulations where the sandwich beam is modelled as a 2D plane-stress continuum. Next, we illustrate this FE model.



#### 4.2.1 The FE discretisation of the reference two-dimensional plane-stress continuum model

The FE model, which is developed within the commercial FE code ABAQUS [6], consists of 1000 columns of bi-quadratic (8-noded) FEs with reduced integration (4 Gauss points for each FE), denoted as CPS8R in ABAQUS. In each column, the core is discretised along its thickness with 70 FEs, and each skin is discretised with 25 FEs. All FEs have the same side along the  $x$  axis, whereas the sides along the  $y$  direction are uniform within each layer, thus being different for core and skins. Overall, the mesh consists of 120,000 8-noded FEs.

The fully-clamped cross-section is realised by setting to zero both displacement components of all the nodes belonging to the left side (located at  $x = 0$ ) and the uniform load is applied as a pressure to the external surface of the top skin (i.e. at  $y = -t - c/2$ ).

#### 4.2.2 Comparison between analytical and numerical results

For the sake of brevity we disregard the longitudinal stress  $\sigma_x$  as given by Eq. (13), which is well-known to be very well estimated by the considered structural theory [3, 28, 2].

In Fig. (2) we present the analytical results for the cross-sectional profiles of  $\sigma_y^{\text{rec}}$  and  $\bar{\tau}_{xy}^{\text{zz}}$  at three different positions in the clamped sandwich region,  $\xi = 0., 0.03, 0.06$ . While the behaviour of the shear stress is quite well-known [2], that of the through-the-thickness normal stress constitutes an interesting novelty: after attaining the assigned value  $-q/b$  at  $y = -t - c/2$ , it experiences a quite large gradient and assumes a relatively large tensile value at the interface ( $y = -c/2$ ). Then, it is subject to an opposite gradient all along the core thickness, such that at the lower interface ( $y = c/2$ ) it assumes a compressive value

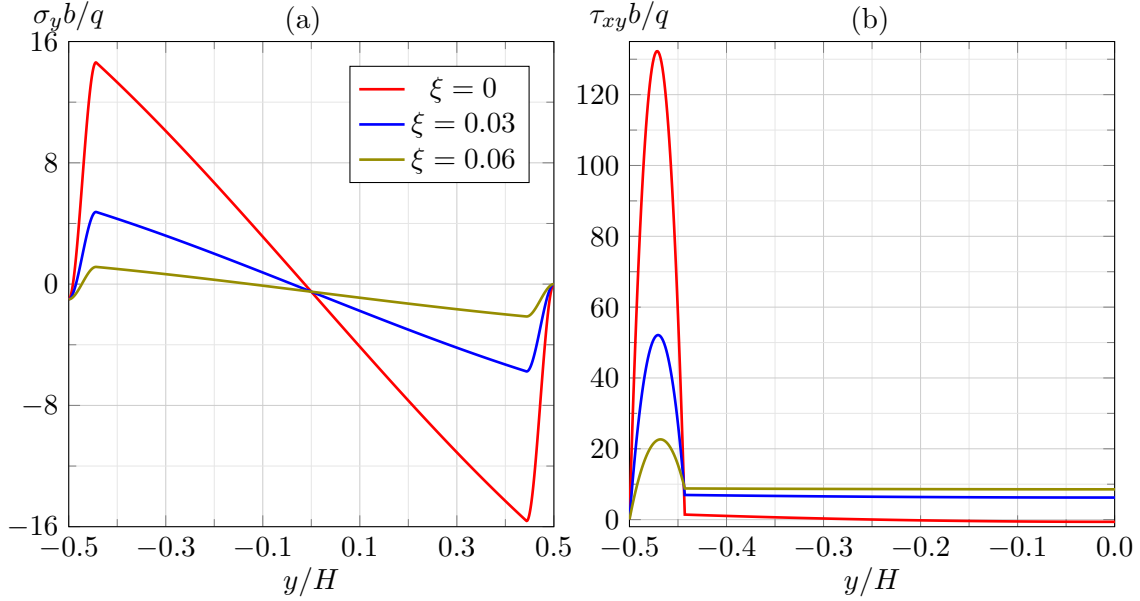


Figure 2: Non-dimensional through-the-thickness normal stress (a) and shear stress (b) at different cross-sections: analytical estimates

of magnitude similar to that at the upper interface. Of course, it finally goes to zero at  $y = t + c/2$ . This type of profile for  $\sigma_y$  is due to the fast variation of  $\tau_{xy}$  along the beam axis in the clamped sandwich region, where the shear stress rapidly migrates from the core to the skins [2]. Hence, in this benchmark, at the upper interface ( $y = -t - c/2$ ) in the clamped region,  $\sigma_y$  counter-intuitively results in a peel stress that may prevent the interface from developing its intrinsic shear strength.

The foregoing predictions have been validated against the results obtained with the FE model described in Sect. 4.2.1. Fig. 3 reports the deformed shape, which is characterised by a change of curvature already observed in [2] for parameters different from those selected here, as in Eq. (36). Also, in Fig. 3 we display the contour of  $\sigma_y$ , such contour being limited in such a way as to give an indication of the behaviour over the whole sandwich. By the way, the sandwich region where the contour turns out to be cut is that next to the fully-clamped

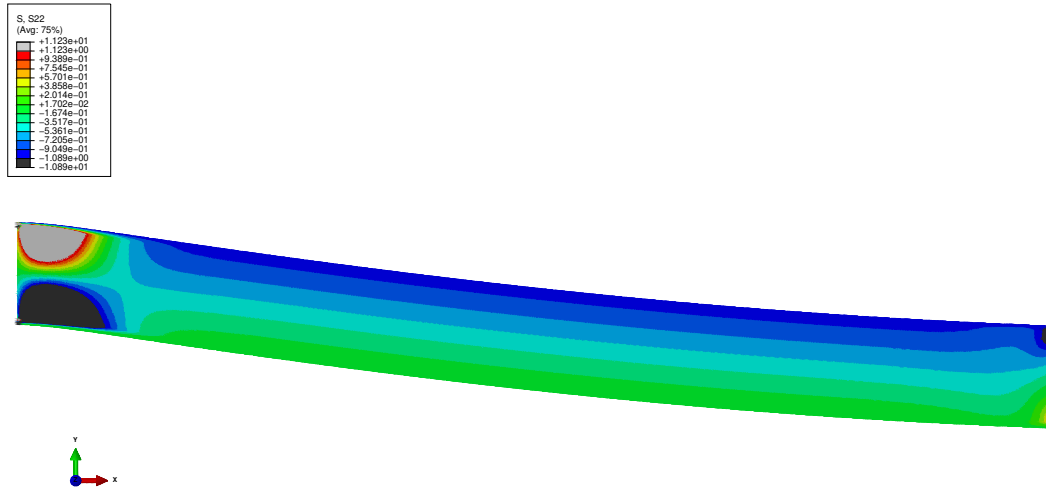


Figure 3: Amplified deformed configuration and contour of the through-the-thickness normal stress for the cantilever sandwich modelled as a plane-stress continuum with the FE code ABAQUS

sandwich end, in which it is well-known that the adopted displacement-based FEs cannot capture some important aspects of the exact linear elastic solution of this 2D boundary value problem, even if one resorts to extremely refined meshes. This is because the longitudinal stress,  $\sigma_x$ , turns out to be unbounded at the corner points of the fully-clamped cross-section [11, 29]. The FE solution is unable to describe this singularity and, as a consequence, it even does not match the static boundary conditions on a portion of the unconstrained sides ending in the corner points of the clamped section. This problem is common to other structures modelled as 2D continua and having a fully-clamped side with corners, as in axisymmetric functionally-graded plates [25].

Figs. 4 and 5 display the comparison between the analytical and FE results. It is interesting to observe that, because of the extremely fine FE mesh implemented in ABAQUS, the match is already quite good at  $\xi = 0.03$ , that is very close to the encastre. Instead, for the

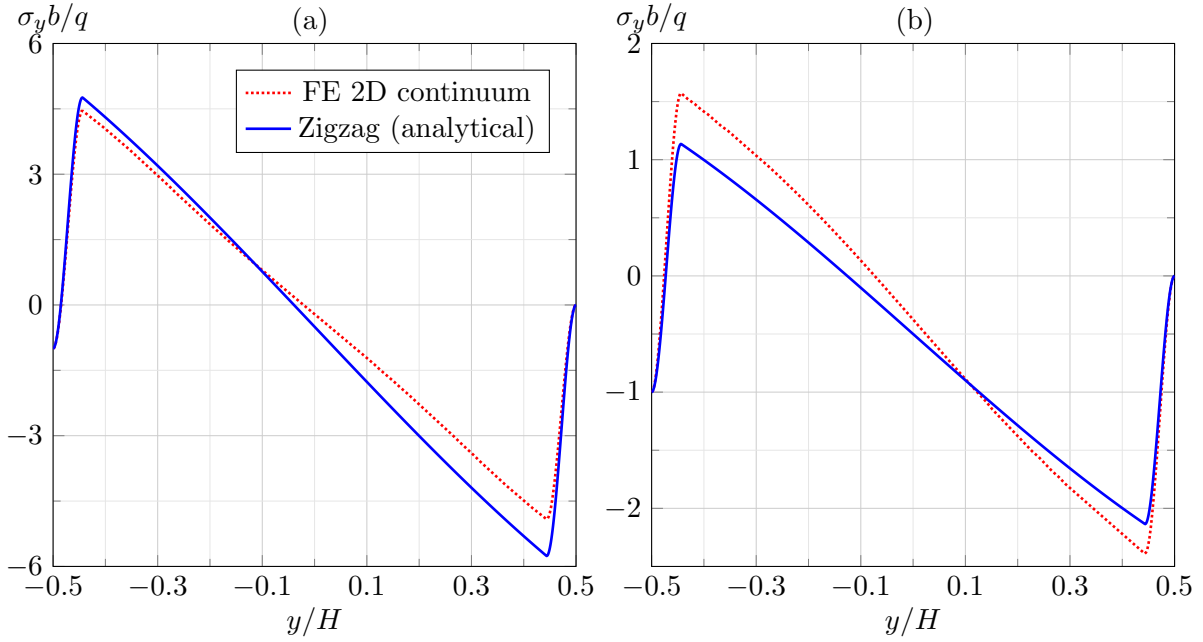


Figure 4: Non-dimensional through-the-thickness normal stress at  $\xi = 0.03$  (a) and  $\xi = 0.06$  (b): comparison between the results of the FE simulation as a 2D plane-stress continuum and the stress recovery in the structural theory relying on the zigzag warping

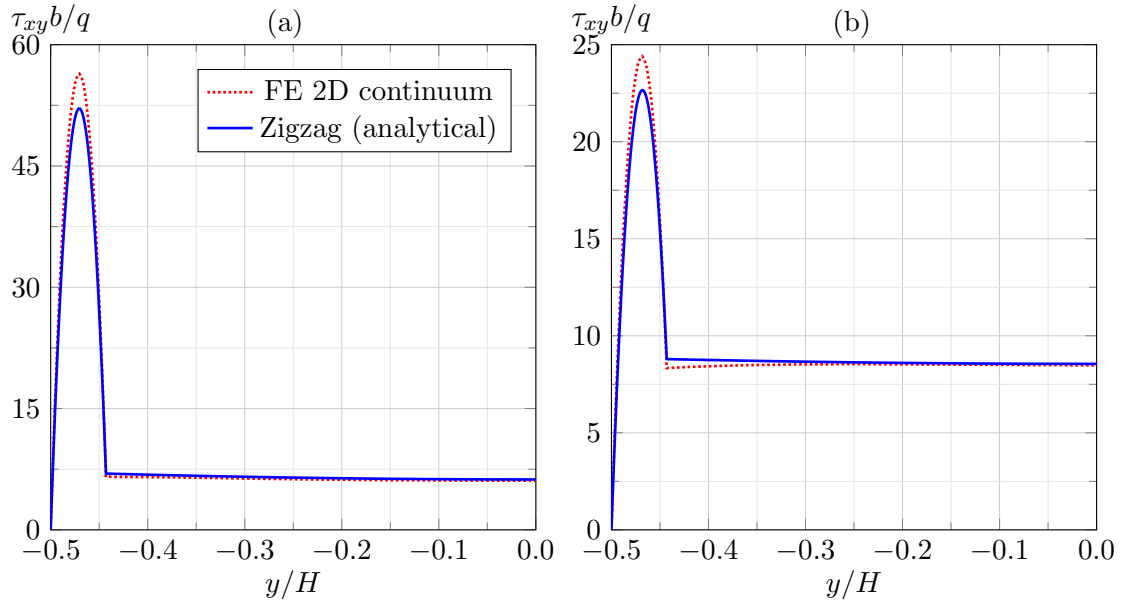


Figure 5: Non-dimensional shear stress at  $\xi = 0.03$  (a) and  $\xi = 0.06$  (b): comparison between the results of the FE simulation as a 2D plane-stress continuum and the stress recovery in the structural theory relying on the zigzag warping. Note that, for the sake of graphical clarity, we plot only half of the  $\tau_{xy}(y)$  profiles even though  $\tau_{xy}$  is not perfectly symmetric with respect to  $y = 0$  in the 2D continuum analysis, because of the load applied at  $y = -t - c/2$

reasons above explained, at the fully-clamped cross-section ( $\xi = 0.$ ), the 2D plane-stress continuum FE model is totally unreliable in terms of all the three stress components, for instance providing  $\tau_{xy}(y/H = 0.5)b/q \approx 25.$  and  $\tau_{xy}(y/H = -0.5)b/q \approx 28.,$  instead of vanishing values. Given the good match in Figs. 4 and 5 at  $\xi = 0.03,$  we infer that the structural theory relying on the zig-zag warping with the double stress recovery provides useful estimates also at the encastre.

The most important result of this investigation is that we observe that there are portions of the interfaces where the through-the-thickness normal stress is comparable to, or or even much larger than, the shear stress, the former remarkably being a tensile stress. We think that this might important to study delamination, which may be of crucial importance for the analysis and design of sandwich panels. By entering into further details of the considered benchmark, at  $(\xi, y) = (0.03, -c/2)$  we obtain  $\bar{\tau}_{xy}^{\text{int}}b/q = 6.96941$  and  $\sigma_y^{\text{rec}}b/q = 5.7537;$  at the clamped section, the zigzag model even predicts much larger  $\sigma_y^{\text{rec}}(\xi = 0, y = -c/2)b/q$  than  $\bar{\tau}_{xy}^{\text{int}}(\xi = 0)b/q,$  which are equal to 15.6 and 1.42, respectively.

Fig. 6 displays the two stress components all along the upper interface ( $y = -c/2,$ ), where  $\sigma_y$  is a tensile stress. Note that the particular behaviour of the shear stress is known since [2]: from the free end to the clamped section it increases almost linearly with the shear force, until the effect of hampering warping at the clamped section requires a sudden decrease of  $\tau_{xy}$  in the core, which has then to migrate to the skins.

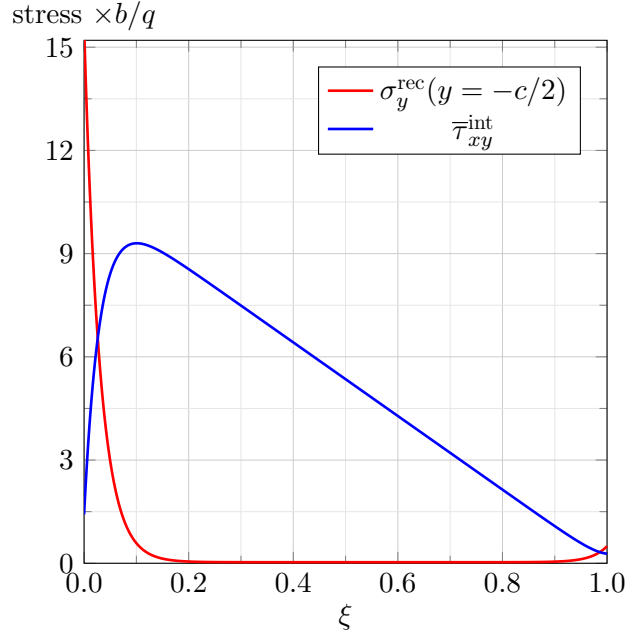


Figure 6: Non-dimensional through-the-thickness normal stress and shear stress all along the upper interface ( $y = -c/2$ ): analytical estimates relying on Krajinovic theory and stress recovery

## 5 Concluding remarks

In this contribution, we have provided explicit analytical estimates for the through-the-thickness normal stress component,  $\sigma_y$ , of sandwich beams with identical skins and governed by the Krajinovic theory, which relies on zigzag warping under flexure [15].

The new analytical estimates for  $\sigma_y$  complement those already established for the longitudinal stress  $\sigma_x$  [15, 16] and for the shear stress  $\tau_{xy}$  [3, 2] and rely on a double stress recovery procedure, already followed in the literature on numerical schemes for equivalent single layer models (ESLMs) of multilayered structures [4, 18, 7]. In the case of beams, after  $\tau_{xy}$  is obtained from a first Jourawski-like stress recovery [14, 18, 3], one resorts to the balance equation  $\partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = 0$  to obtain  $\sigma_y$ . While this procedure gives no problems if one

can obtain analytical solutions, it instead requires care in numerical approaches, where the functions approximating the primal structural variables may need high continuity conditions.

We have validated the obtained analytical results against finite element simulations in which the sandwich beam is modelled as a two-dimensional plane-stress continuum. The main practical outcome of this investigation is that there are boundary value problems in which  $\sigma_y$  at the interfaces is a tensile stress of magnitude larger than  $\tau_{xy}$ , such as  $\sigma_y$  should surely be accounted for when studying delamination. In our study, this happens for a cantilever sandwich subjected to a uniform transverse load in a region close to the fully-clamped cross-section.

Alternatively and complementarily to the use of more complex exact linear elastic solutions for the sandwich panel modelled as a continuum [21], the proposed analytical solutions could be used to test various higher-order models for laminated composite panels, with the purpose of assessing an adopted numerical method, a new proposal for a warping function, or both. Although here we have focused on sandwich beams with identical skins, in this regard, let us also mention the ESLMs, which can be particularised to our layout even if they are typically developed to deal with structures constituted by several layers. ESLMs aim at accurate solutions while maintaining the number of unknown structural variables as limited as possible and independent of the number of layers, thus opening up to a large number of proposals in the literature and usually requiring numerical solutions. Within this realm, we mention both laminated beams [32, 10, 27, 5, 13] and laminated plates (see, e.g., [4, 7, 23], dealing with the double stress recovery adopted here, *and references therein*), in which the test could be

accomplished by subjecting the plate to cylindrical bending (along with substituting, in our equations,  $E$  with  $E/(1 - \nu^2)$  for the longitudinal moduli of the layers).

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### Statements and Declarations

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**Data availability** Data will be made available on request.

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## A Analytical solution at the beam end in a simply-supported sandwich subjected to uniform transverse load

In this boundary value problem the Krajinovic auxiliary function results [3]

$$f(\xi) = \frac{24C_0}{k^4} \left[ \frac{e^{-k} - 1}{1 - e^{-2k}} e^{k(\xi-1)} - \frac{1}{2}(\coth k - \operatorname{csch} k + 1) e^{-k\xi} \right] + C_0 \xi^4 - 2C_0 \xi^3 + \frac{12C_0}{k^2} \xi^2 + C_0 \left( 1 - \frac{12}{k^2} \right) \xi + \frac{24C_0}{k^4}. \quad (37)$$

Evaluation of the relevant derivatives of Eq. (37) in  $x = 0$  gives

$$\left. \frac{d^4 f(\xi)}{d\xi^4} \right|_{\xi=0} = 24C_0 \left( \frac{e^{-k} - 1}{1 - e^{-2k}} e^{-k} - \frac{1}{2}(\coth k - \operatorname{csch} k + 1) + 1 \right) = 24C_0 \frac{e^{-2k}}{1 - e^{-2k}}, \quad (38)$$

$$\left. \frac{d^6 f(\xi)}{d\xi^6} \right|_{\xi=0} = 24C_0 k^2 \left( \frac{e^{-k} - 1}{1 - e^{-2k}} e^{-k} - \frac{1}{2}(\coth k - \operatorname{csch} k + 1) \right) = 24C_0 k^2 \frac{2e^{-2k} - 1}{1 - e^{-2k}}. \quad (39)$$

Therefore, from Eqs. (24) and (25) one has

$$\phi_c'''(0) = \frac{24C_0}{L^4} \left\{ \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{k^2}{L^2} \frac{2e^{-2k} - 1}{1 - e^{-2k}} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{e^{-2k}}{1 - e^{-2k}} \right\}, \quad (40)$$

$$\phi_s'''(0) = \frac{24C_0}{L^4} \left\{ - \left( \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} + \frac{c(1+m)}{2} \right) \frac{k^2}{L^2} \frac{2e^{-2k} - 1}{1 - e^{-2k}} + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} \frac{e^{-2k}}{1 - e^{-2k}} \right\}. \quad (41)$$

Then, by substituting Eqs. (40) and (41) into Eq. (21), one has

$$-\frac{V_s'(0)}{2b} = \frac{24C_0 E_s t^2}{L^4 (1 - e^{-2k})} \left\{ \left[ - \left( \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} + \frac{c(1+m)}{2} \right) \frac{k^2}{L^2} (2e^{-2k} - 1) + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} e^{-2k} \right] \frac{t}{3} \right. \\ \left. + \left[ \left( t - \frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \right) \frac{k^2}{L^2} (2e^{-2k} - 1) + \frac{\beta_1 \beta_3 - \beta_2^2}{\alpha_1 \beta_2} e^{-2k} \right] \frac{c}{4} \right\}$$

and, by also accounting for Eqs. (12) and (28), we finally obtain

$$-\frac{V_s'(0)}{2b} = \frac{q E_s t^2}{1 - e^{-2k}} \left\{ \left[ \left( \frac{1}{\alpha_1} + \frac{c(1+m)\beta_2}{2\beta_1 \alpha_2} \right) (1 - 2e^{-2k}) + \frac{e^{-2k}}{\alpha_1} \right] \frac{t}{3} \right. \\ \left. + \left[ \left( \frac{1}{\alpha_1} - \frac{t\beta_2}{\beta_1 \alpha_2} \right) (1 - 2e^{-2k}) + \frac{e^{-2k}}{\alpha_1} \right] \frac{c}{4} \right\}, \quad (42)$$

which can be studied in order to establish whether, for appropriate sets of parameters,  $\sigma_y^{\text{rec}}(x = 0, y)$  can result, in modulus, significantly larger than  $|q/b|$ .

As already mentioned, both our 2D FE analyses of the sandwich as a plane-stress continuum and the application of the foregoing formulæ suggest that the simply-supported case is not very much interesting as  $\sigma_y$  hardly is ever larger than  $|q/b|$ .

However, this benchmark unexpectedly displays a feature that, in our opinion, would deserve to be further studied. It consists in the fact that the 2D FE analysis gives a  $\sigma_y$  profile that is in remarkable disagreement with that predicted by Eqs. (17) and (42) in a quite large sandwich region at the support, the latter being modelled by setting to zero the transversal displacement component of all the nodes at the beam-end side. In this respect, contrary to our experience with the longitudinal and shear stress components, the comparison between analytical estimates and FE results in terms of through-the-thickness stress is “more difficult” for the simply-supported cross-section than for the fully-clamped cross-section. In practice,

this means that, in the simply-supported benchmark, in order to observe agreement between the results of the analytical and FE models we need to select a cross-section significantly farther from the beam end than in the case of a fully-clamped cross-section.