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# REPRESENTING SYSTEMS GENERATED BY REPRODUCING KERNELS

EMMANUEL FRICAIN, HAI KHOI LE, AND PASCAL LEFÈVRE

ABSTRACT. In that paper, we study the representing and absolutely representing systems in the context of reproducing kernel Hilbert spaces. We prove in particular that, in many classical spaces including weighted Hardy and Dirichlet spaces and de Branges–Rovnyak spaces, there cannot exist absolutely representing systems of reproducing kernels.

## 1. INTRODUCTION

The theory of representing systems in topological vector spaces has been developed intensively during the last three decades. It has many important applications in different areas of mathematics, in particular to solve certain functional equations by representing the solutions in series of simpler functions, like exponential functions or rational functions.

It should be noted that in a study of representing systems and absolutely representing systems, several important criteria for a countable system to be representing (respectively, absolutely representing) in Fréchet–Schwartz or dual Fréchet–Schwartz spaces have been obtained. In particular, the notion of sufficient sets has been developed and proved to be a powerful tool. Roughly speaking, a subset  $S$  of a set  $X$  is a sufficient set for a locally convex inductive limit of weighted spaces of functions on  $X$  if a (*a-priori*) weaker topology given by a system of semi-norms induced by  $S$  coincide with the original topology. Sufficient sets for some spaces of entire functions and infinitely differentiable functions were considered by Ehrenpreis [4], Taylor [14, 15], Schneider [13], Baernstein [1], Napalkov [10] and others. A functional analytic treatment of inductive limits of weighted function spaces has also been given by K.D. Bierstedt, R.G. Meise and W.H. Summers [3]. We refer the reader to the work by Yu.F. Korobeiknik, especially the survey article [8] as well as [9] and references therein for more details.

In that paper, we study the representing and absolutely representing systems in the context of reproducing kernel Hilbert spaces. In particular, we are interested in the following question.

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**Question 1.1.** *Let  $\mathcal{H}$  be a reproducing kernel Hilbert space on a set  $X$  and let  $k_x^{\mathcal{H}}$  be the reproducing kernel of  $\mathcal{H}$  at point  $x \in X$ . Can we characterize the sequence of points  $(x_n)_{n \geq 1}$  in  $X$  such that  $(k_{x_n}^{\mathcal{H}})_{n \geq 1}$  is a representing system (respectively an absolutely representing system) for its closed linear span?*

In Section 2, we give some general preliminaries on representing systems in Banach spaces setting. In Section 3, we prove that there cannot exist absolutely representing systems of reproducing kernels in the Hardy space  $H^2$  and we also give a necessary condition for representing systems. In the last section, we generalize the non-existence of absolutely representing systems of reproducing kernels in various Hilbert spaces, including weighted Hardy spaces, weighted Dirichlet spaces and de Branges–Rovnyak spaces.

## 2. REPRESENTING SYSTEM AND ABSOLUTELY REPRESENTING SYSTEMS IN BANACH SPACES

**Definition 2.1.** *Let  $\mathfrak{X}$  be a complex and separable Banach space of infinite dimension. A sequence of non-zero elements  $(x_n)_{n \geq 1}$  is said to be a representing system for  $\mathfrak{X}$  if, for every element  $x \in \mathfrak{X}$ , there exists a sequence of complex numbers  $(a_n)_{n \geq 1}$  such that*

$$(2.2) \quad x = \sum_{n=1}^{\infty} a_n x_n,$$

where the series converges in norm of  $\mathfrak{X}$ .

It should be noted that in the definition, we do not ask that the sequence  $(a_n)_{n \geq 1}$  is unique. If we impose the uniqueness, then it coincides with the notion of Schauder basis. In other words, the sequence  $(x_n)_{n \geq 1}$  is a (Schauder) basis if and only if it is a representing system which is minimal.

**Definition 2.3.** *We say that a sequence  $\mathfrak{X} = (x_n)_{n \geq 1}$  is an absolutely representing system for  $\mathfrak{X}$  if we have (2.2) and moreover*

$$\sum_{n \geq 1} |a_n| \|x_n\| < \infty.$$

It should be noted that every separable Banach space admits an absolutely representing system. That follows directly from Banach–Mazur theorem (see for instance [2]). Nevertheless we give the proof for completeness.

**Proposition 2.4.** *Let  $\mathfrak{X}$  be a separable Banach space. Then there exists a sequence  $(x_n)_{n \geq 1}$  which forms an absolutely representing system for  $\mathfrak{X}$ .*

*Proof.* Since  $\mathfrak{X}$  is separable, we can find a sequence of vectors  $(x_n)_{n \geq 1}$  dense in the unit ball of  $\mathfrak{X}$ . We shall prove that  $(x_n)_{n \geq 1}$  is an absolutely representing system for  $\mathfrak{X}$ , which means

that, for every  $x \in \mathfrak{X}$ , there is a sequence  $(a_n)_{n \geq 1} \in \ell^1$  such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Let  $x \in \mathfrak{X}$ . Without loss of generality, we can of course assume that  $\|x\| = 1$ . Then we can choose  $n_1 \geq 1$  such that

$$\|x - x_{n_1}\| \leq \frac{1}{2}.$$

Assume that we have chosen  $n_1 < n_2 < \dots < n_j$  satisfying

$$(2.5) \quad \left\| x - x_{n_1} - \frac{1}{2}x_{n_2} - \dots - \frac{1}{2^{j-1}}x_{n_j} \right\| \leq \frac{1}{2^j}.$$

Since  $2^j(x - x_{n_1} - \dots - \frac{1}{2^{j-1}}x_{n_j})$  lies in the unit ball of  $\mathfrak{X}$ , we can choose  $n_{j+1} > n_j$  such that

$$\left\| 2^j(x - x_{n_1} - \dots - \frac{1}{2^{j-1}}x_{n_j}) - x_{n_{j+1}} \right\| < \frac{1}{2}.$$

Hence

$$\left\| x - x_{n_1} - \frac{1}{2}x_{n_2} - \dots - \frac{1}{2^j}x_{n_{j+1}} \right\| \leq \frac{1}{2^{j+1}},$$

and by induction we obtain (2.5) for all  $j \geq 1$ . Take now

$$a_n = \begin{cases} \frac{1}{2^{j-1}} & \text{if } n = n_j, j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, of course  $(a_n)_{n \geq 1} \in \ell^1$  and letting  $j \rightarrow \infty$  in (2.5) gives

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

□

The following characterization of absolutely representing systems will be useful for us. As usual, given a Banach space  $\mathfrak{X}$ , we denote by  $\mathfrak{X}^*$  the dual space of  $\mathfrak{X}$ .

**Lemma 2.6.** *Let  $\mathfrak{X}$  be a Banach space,  $(x_n)_{n \geq 1}$  be a sequence of vectors in  $\mathfrak{X}$ ,  $\|x_n\| = 1$ ,  $n \geq 1$ . The following are equivalent:*

- (i) *the sequence  $(x_n)_{n \geq 1}$  is an absolutely representing system for  $\mathfrak{X}$ ;*
- (ii) *there exists  $C > 0$  such that for every  $x^* \in \mathfrak{X}^*$ ,  $\|x^*\| = 1$ , there is an integer  $n \geq 1$  such that  $|x^*(x_n)| \geq C$ .*

*Proof.* First note that the map  $T$  defined, for  $a = (a_n)_{n \geq 1} \in \ell^1$ , by

$$(2.7) \quad T(a) = \sum_{n=1}^{\infty} a_n x_n$$

is linear and bounded from  $\ell^1$  into  $\mathfrak{X}$ . Moreover, an easy computation shows that its adjoint  $T^* : \mathfrak{X}^* \rightarrow \ell^\infty$  is given by

$$T^*(x^*) = (x^*(x_n))_{n \geq 1}, \quad x^* \in \mathfrak{X}^*.$$

It remains to note that (i)  $\iff T$  onto  $\iff T^*$  bounded below  $\iff$  (ii). □

As already mentioned, a Schauder basis is a representing system which is minimal. For a class of Banach spaces, including Hilbert spaces, the situation for absolutely representing systems changes dramatically.

**Proposition 2.8.** *Let  $\mathfrak{X}$  be a Banach space which is not isomorphic to  $\ell^1$  and let  $(x_n)_{n \geq 1}$  be a sequence of unit vectors in  $\mathfrak{X}$  which is a basis for  $\mathfrak{X}$ . Then  $(x_n)_{n \geq 1}$  cannot be an absolutely representing system for  $\mathfrak{X}$ .*

*Proof.* Let  $T : \ell^1 \rightarrow \mathfrak{X}$  be the map defined by (2.7). It is clear that  $T$  is a linear and bounded map from  $\ell^1$  into  $\mathfrak{X}$ . Since  $(x_n)_{n \geq 1}$  is a basis,  $T$  is also one-to-one. Assume that  $(x_n)_{n \geq 1}$  is an absolutely representing system for  $\mathfrak{X}$ . Then the operator  $T$  is also onto and thus an isomorphism, which contradicts the fact that  $\mathfrak{X}$  is not isomorphic to  $\ell^1$ . □

Note that Proposition 2.8 applies in particular to Hilbert spaces.

### 3. REPRESENTING SYSTEMS IN HARDY SPACES

Let  $H^2$  be the Hardy space on the open unit disc  $\mathbb{D}$ . It is a reproducing kernel Hilbert space with reproducing kernel given by  $k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$  where  $\lambda, z \in \mathbb{D}$ . That means that

$$f(\lambda) = \langle f, k_\lambda \rangle_2, \quad \text{where } f \in H^2.$$

We denote by  $\hat{k}_\lambda$  the normalized reproducing kernel,  $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|_2$ . Let us recall that if  $(\lambda_n)_{n \geq 1}$  is a sequence of distinct points in the unit disc, then we have the following dichotomy: if  $(\lambda_n)_{n \geq 1}$  satisfies the Blaschke condition, that is if

$$\sum_{n \geq 1} (1 - |\lambda_n|) < \infty,$$

then the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is minimal and

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = K_B,$$

where  $\text{span}(\dots)$  denotes the closed linear span generated by  $(\dots)$  and  $B$  is the associated Blaschke product  $B = \prod_{n \geq 1} b_{\lambda_n}$  and  $b_{\lambda_n}$  is the single Blaschke factor defined by  $b_{\lambda_n}(z) =$

$\frac{|\lambda_n|}{\lambda_n}(\lambda_n - z)(1 - \overline{\lambda_n}z)^{-1}$ . On the contrary, if  $(\lambda_n)_{n \geq 1}$  does not satisfy the Blaschke condition, then the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is not minimal and

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = H^2.$$

See for instance [6, Lemma 14.16].

In the case when the sequence  $(\lambda_n)_{n \geq 1}$  is a Blaschke sequence, we can easily characterize the representing systems formed by reproducing kernels.

**Proposition 3.1.** *Let  $(\lambda_n)_{n \geq 1}$  be a Blaschke sequence of distinct points in  $\mathbb{D}$  and  $B$  the associated Blaschke product. The following are equivalent:*

- (i) *the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is a representing system for  $K_B$ ;*
- (ii) *the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is a Riesz basis for  $K_B$ ;*
- (iii) *the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is uniformly minimal;*
- (iv) *the sequence  $(\lambda_n)_{n \geq 1}$  satisfies the Carleson condition, that is*

$$\inf_{n \geq 1} |B_n(\lambda_n)| > 0,$$

where  $B_n = B/b_{\lambda_n}$ .

*Proof.* The equivalence between (ii), (iii) and (iv) is well-known (see [6, 11]). The implication (ii)  $\implies$  (i) is obvious. It remains to prove (i)  $\implies$  (iii). Since  $(\lambda_n)_{n \geq 1}$  is a Blaschke sequence, we know that the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is minimal. If it is furthermore assumed to be a representing system, then it should be a (Schauder) basis. Hence, in particular, it should be uniformly minimal. □

It is quite easy to prove that there does not exist absolutely representing systems formed by reproducing kernels of  $H^2$ .

**Theorem 3.2.** *Let  $(\lambda_n)_{n \geq 1}$  be a sequence of distinct points in  $\mathbb{D}$ . Then the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  cannot be an absolutely representing system for its closed linear span in  $H^2$ .*

*Proof.* Assume that the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  is an absolutely representing system for its closed linear span. We shall distinguish two cases.

**First case:**  $(\lambda_n)_{n \geq 1}$  is a Blaschke sequence. Let  $B$  be the associated Blaschke product. Then we know that the sequence  $(\hat{k}_{\lambda_n})_{n \geq 1}$  generates the space  $K_B$  and according to Proposition 3.1, it should be a Riesz basis for  $K_B$ . Then apply Proposition 2.8 to get a contradiction.

**Second case:**  $(\lambda_n)_{n \geq 1}$  is not a Blaschke sequence. In that case, we know that

$$\text{span}(\hat{k}_{\lambda_n} : n \geq 1) = H^2.$$

According to Lemma 2.6, there is a constant  $C > 0$  such that for any  $h \in H^2$ ,  $\|h\|_2 = 1$ , we can find  $n \geq 1$  such that  $|h(\lambda_n)|(1 - |\lambda_n|^2)^{1/2} \geq C$ . Choose  $N$  sufficiently large so that  $2 \leq (N + 1)C^2$ . Consider  $\varphi(t) = (1 - t)t^N$  and point out that

$$\sup_{t \in [0,1]} \varphi(t) \leq \frac{1}{N + 1}.$$

For  $h(z) = z^N$ , there is an integer  $n$  such that

$$C^2 \leq |h(\lambda_n)|^2(1 - |\lambda_n|^2) = |\lambda_n|^{2N}(1 - |\lambda_n|^2) = \varphi(|\lambda_n|^2) \leq \frac{1}{N + 1} \leq \frac{C^2}{2},$$

which gives the desired contradiction in that second case. □

We give now a necessary condition so that a sequence of reproducing kernels forms a representing system for  $H^2$ .

**Theorem 3.3.** *Let  $(\lambda_n)_{n \geq 1}$  be a sequence in  $\mathbb{D}$  which is not a Blaschke sequence. Assume that*

$$(3.4) \quad \sum_n |\lambda_n - \lambda_{n+1}| < \infty.$$

*Then the sequence  $(k_{\lambda_n})_{n \geq 1}$  is not a representing system for  $H^2$ .*

*Proof.* Condition (3.4) obviously implies that  $(\lambda_n)_{n \geq 1}$  is convergent to some point in  $\overline{\mathbb{D}}$ . In particular, there exists  $\zeta_0 \in \mathbb{T} = \partial\mathbb{D}$  such that

$$(3.5) \quad \inf_{n \geq 1} |\lambda_n - \zeta_0| > 0.$$

Assume now that the sequence  $(k_{\lambda_n})_{n \geq 1}$  is a representing system for  $H^2$ . Then there exists a sequence of complex numbers  $(a_n)_{n \geq 1}$  such that

$$\frac{1}{(1 - \overline{\zeta_0}z)^{1/4}} = \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n}z},$$

where the series converges in the  $H^2$ -norm. In particular, the series is also pointwise convergent and taking  $z = 0$ , we get that the series  $\sum_n a_n$  is convergent. Moreover, using (3.5), we have

$$\left| \frac{1}{1 - \overline{\lambda_n}\zeta_0} - \frac{1}{1 - \overline{\lambda_{n+1}}\zeta_0} \right| = \frac{|\lambda_n - \lambda_{n+1}|}{|1 - \overline{\lambda_n}\zeta_0||1 - \overline{\lambda_{n+1}}\zeta_0|} \lesssim |\lambda_n - \lambda_{n+1}|.$$

Using now (3.4), we can apply Abel's theorem to get that the series

$$\sum_n a_n \frac{1}{1 - \overline{\lambda_n}\zeta_0}$$

is convergent and

$$\lim_{z \xrightarrow{\triangleleft} \zeta_0} \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n} z} = \sum_{n=1}^{\infty} a_n \frac{1}{1 - \overline{\lambda_n} \zeta_0},$$

where  $\lim_{z \xrightarrow{\triangleleft} \zeta_0}$  denotes the non-tangential limit at  $\zeta_0$ . Hence  $(1 - \overline{\zeta_0} z)^{-1/4}$  has a non-tangential limit at  $\zeta_0$  which gives a contradiction.  $\square$

It would be interesting to answer the following open question.

**Question 3.6.** *can we construct a sequence of points  $(\lambda_n)_n$  in  $\mathbb{D}$  which is not a Blaschke sequence and such that  $(k_{\lambda_n})_n$  forms a representing system for  $H^2$ ?*

#### 4. ABSOLUTELY REPRESENTING SYSTEMS IN VARIOUS REPRODUCING KERNEL HILBERT SPACES

We can generalize Theorem 3.2 for a large class of reproducing kernel Hilbert spaces. Let  $\mathcal{H}$  be a reproducing kernel Hilbert space of analytic functions on the unit disc  $\mathbb{D}$ . We say that  $\mathcal{H}$  is *invariant with respect to the difference quotients* if, for every  $\lambda \in \mathbb{D}$ , we have  $f \in \mathcal{H} \implies Q_\lambda f \in \mathcal{H}$ , where

$$(Q_\lambda f)(z) = \frac{f(z) - f(\lambda)}{z - \lambda} = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(\lambda)}{(k+1)!} (z - \lambda)^k, \quad z \in \mathbb{D}.$$

Many classical spaces are invariant with respect to the difference quotients, e.g. the Hardy space  $H^2$ , the Dirichlet space  $\mathcal{D}$ , the Bergman space  $\mathcal{B}$ ,...

**Lemma 4.1.** *Let  $\mathcal{H}$  be a reproducing kernel Hilbert space which is invariant with respect to the difference quotients. Let  $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$  be a sequence of reproducing kernels of  $\mathcal{H}$ . Then we have the following dichotomy:*

- (a) *either  $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$  is minimal;*
- (b) *or  $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$  is complete in  $\mathcal{H}$ .*

*Proof.* Assume that  $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$  is not complete in  $\mathcal{H}$  and let us prove that the sequence is minimal. There exists  $f \in \mathcal{H}$ ,  $f \neq 0$ , such that  $f(\lambda_n) = 0$ ,  $n \geq 1$ . Fix an integer  $N \geq 1$  and denote by  $\ell$  the smallest integer such that  $f^{(\ell)}(\lambda_N) \neq 0$ . Then define  $g = Q_{\lambda_N}^\ell f$  where  $Q_{\lambda_N}^\ell = Q_{\lambda_N} \circ Q_{\lambda_N} \circ \dots \circ Q_{\lambda_N}$  ( $\ell$ -times). By hypothesis, the function  $g$  belongs to  $\mathcal{H}$ . Moreover, since

$$(Q_\lambda^s f)(z) = \sum_{k=0}^{\infty} \frac{f^{(k+s)}(\lambda)}{(k+s)!} (z - \lambda)^k,$$

we have  $g(\lambda_N) \neq 0$  and, by induction, it is easy to check that  $g(\lambda_n) = 0$ ,  $n \geq 1$ ,  $n \neq N$ . Hence

$$k_{\lambda_N}^{\mathcal{H}} \notin \text{span} (k_{\lambda_n}^{\mathcal{H}} : n \neq N),$$



which proves the minimality of  $(k_{\lambda_n}^{\mathcal{H}})_{n \geq 1}$ .

□

**4.1. Weighted Hilbert spaces of analytic functions and absolutely representing systems.** Given a positive integrable function  $\omega$  on  $[0, 1]$  which is not zero almost everywhere on a neighborhood of 1, we extend it by  $\omega(z) = \omega(|z|)$ ,  $z \in \mathbb{D}$ , and call such  $\omega$  a weight function. We denote by  $\mathfrak{D}_\omega$  the space consisting of analytic functions  $f$  on  $\mathbb{D}$  such that

$$\|f'\|_\omega^2 = \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty,$$

where  $dA(z) = dx dy / \pi$  is normalized area measure in  $\mathbb{D}$ . The Hilbert space  $\mathfrak{D}_\omega$  is endowed with the norm

$$\|f\|_{\mathfrak{D}_\omega}^2 = |f(0)|^2 + \|f'\|_\omega^2.$$

A simple computation shows that an analytic function  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  belongs to  $\mathfrak{D}_\omega$  if and only if

$$\|f\|_{\mathfrak{D}_\omega}^2 = \sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,$$

where  $\omega_0 = 1$  and

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \geq 1.$$

The following result is probably known but we couldn't find reference so we include the proof.

**Lemma 4.2.** *Let  $\omega$  be a weight function. Then  $\mathfrak{D}_\omega$  is invariant with respect to the difference quotients and is a reproducing kernel Hilbert space whose kernel is given by*

$$k_\lambda^\omega(z) = \sum_{k=0}^{\infty} \frac{\bar{\lambda}^k}{\omega_k} z^k.$$

*Proof.* To prove that  $\mathfrak{D}_\omega$  is invariant with respect to the difference quotients, let  $f \in \mathfrak{D}_\omega$  and  $\lambda \in \mathbb{D}$ . Fix  $\varepsilon > 0$  such that the closed disc  $K = \overline{D(\lambda, \varepsilon)}$  is contained in  $\mathbb{D}$ . If  $g = Q_\lambda f$ , then write

$$\int_{\mathbb{D}} |g'(z)|^2 \omega(z) dA(z) = \int_K |g'(z)|^2 \omega(z) dA(z) + \int_{\mathbb{D} \setminus K} |g'(z)|^2 \omega(z) dA(z).$$

On one hand, note that

$$\int_K |g'(z)|^2 \omega(z) dA(z) \leq \sup_{z \in K} |g'(z)|^2 \int_{\mathbb{D}} \omega(z) dA(z) = 2 \|\omega\|_{L^1[0,1]} \sup_{z \in K} |g'(z)|^2 < \infty.$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{D} \setminus K} |g'(z)|^2 \omega(z) dA(z) &= \int_{\mathbb{D} \setminus K} \frac{|f(z) - f(\lambda) - (z - \lambda)f'(\lambda)|^2}{|z - \lambda|^4} \omega(z) dA(z) \\ &\lesssim \int_{\mathbb{D}} (|f(z)|^2 + |f(\lambda)|^2 + |f'(\lambda)|^2) \omega(z) dA(z). \end{aligned}$$

Since  $f \in \mathfrak{D}_\omega$ , we have

$$\int_{\mathbb{D}} (|f(\lambda)|^2 + |f'(\lambda)|^2) \omega(z) dA(z) \lesssim \|\omega\|_{L^1[0,1]} + \|f'\|_\omega^2 < \infty.$$

Writing  $f(z) = \sum_{n \geq 0} a_n z^n$ , it remains to note that

$$\begin{aligned} \int_{\mathbb{D}} |f(\lambda)|^2 \omega(z) dA(z) &= 2 \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} \omega(r) dr \\ &\leq 2 \sum_{n=0}^{\infty} |a_n|^2 n^2 \int_0^1 r^{2n-1} \omega(r) dr \\ &= \sum_{n=0}^{\infty} |a_n|^2 \omega(n) = \|f\|_{\mathfrak{D}_\omega}^2 < \infty. \end{aligned}$$

Hence

$$\int_{\mathbb{D}} |g'(z)|^2 \omega(z) dA(z) < \infty,$$

and  $g = Q_\lambda f \in \mathfrak{D}_\omega$ .

Using a duality argument and Hadamard's formula, we see that  $\mathfrak{D}_\omega$  is a reproducing kernel Hilbert space (that is it has bounded evaluation at point  $\lambda \in \mathbb{D}$ ) if and only if

$$(4.3) \quad \liminf_{n \rightarrow \infty} \omega_n^{1/n} \geq 1.$$

By hypothesis, there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \leq \varepsilon_0$ , we have

$$C_\varepsilon := \int_{1-\varepsilon}^1 \omega(r) dr > 0,$$

which gives

$$\omega_n \geq 2n^2 \int_{1-\varepsilon}^1 r^{2n-1} \omega(r) dr \geq 2n^2 C_\varepsilon (1 - \varepsilon)^{2n-1}$$

Hence we get

$$\liminf_{n \rightarrow \infty} \omega_n^{1/n} \geq (1 - \varepsilon)^2.$$

Letting  $\varepsilon$  go to zero, we obtain (4.3) and  $\mathfrak{D}_\omega$  is a reproducing kernel Hilbert space.

Finally, let  $\lambda \in \mathbb{D}$  and  $f(z) = \sum_{n \geq 0} a_n z^n \in \mathfrak{D}_\omega$ . By (4.3), the function

$$k_\lambda^\omega(z) = \sum_{k=0}^{\infty} \frac{\overline{\lambda}^k}{\omega_k} z^k.$$

is an analytic function on  $\mathbb{D}$  which belongs to  $\mathfrak{D}_\omega$ . Moreover, we have

$$\langle f, k_\lambda^\omega \rangle_{\mathfrak{D}_\omega} = \sum_{n=0}^{\infty} a_n \frac{\lambda^n}{\omega_n} \omega_n = \sum_{n=0}^{\infty} a_n \lambda^n = f(\lambda),$$

which proves that  $k_\lambda^\omega$  is the reproducing kernel for  $\mathfrak{D}_\omega$ .  $\square$

Note that

$$\|k_\lambda^\omega\|_{\mathfrak{D}_\omega}^2 = \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{\omega_k}.$$

We denote by  $\widehat{k}_\lambda^\omega$  the normalized reproducing kernel and

$$S_N = \sum_{k=0}^N \frac{1}{\omega_k}.$$

**Lemma 4.4.** *Let  $\omega$  be a weight function. Then*

$$\limsup_{n \rightarrow \infty} \omega_n S_n = \infty.$$

*Proof.* Assume that  $\limsup_{n \rightarrow \infty} \omega_n S_n < \infty$ . Then there exists a constant  $C$  such that for every  $n \geq 1$ , we have  $1 \leq \omega_n S_n \leq C$ . Writing

$$S_n \leq \frac{C}{\omega_n} = C(S_n - S_{n-1}),$$

we get  $qS_{n-1} \leq S_n$ , where  $q = C/(C-1) > 1$ . Hence  $S_n \geq q^n S_0$ , which gives

$$\omega_n^{1/n} \leq \left(\frac{C}{S_0}\right)^{1/n} q^{-1},$$

and that contradicts (4.3).  $\square$

**Theorem 4.5.** *Let  $\omega$  be a weight function,  $(\lambda_n)_{n \geq 1}$  a sequence of distinct points in the unit disc. Then the sequence  $(\widehat{k}_{\lambda_n}^\omega)_{n \geq 1}$  cannot be an absolutely representing system of its closed linear span.*

*Proof.* Assume that the sequence  $(\widehat{k}_{\lambda_n}^\omega)_{n \geq 1}$  is an absolutely representing system of its closed linear span. We shall distinguish two cases.

**First case:** the sequence  $(\widehat{k}_{\lambda_n}^\omega)_{n \geq 1}$  is minimal. Thus, it should be a (Schauder) basis for its closed linear span. Then apply Proposition 2.8 to get a contradiction.

**Second case:** the sequence  $(\widehat{k}_{\lambda_n}^\omega)_{n \geq 1}$  is not minimal. Then, according to Lemma 4.2 and Lemma 4.1, we have

$$\text{span}(\widehat{k}_{\lambda_n}^\omega : n \geq 1) = \mathfrak{D}_\omega.$$

It follows from Lemma 2.6 that there is a constant  $C > 0$  such that for every  $h \in \mathfrak{D}_\omega$ , we can find  $n \geq 1$  such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega} \|h\|_{\mathfrak{D}_\omega}.$$

Applying this to  $h(z) = z^{N+1}$  for some  $N \in \mathbb{N}$ , there is  $n \geq 1$  such that

$$|\lambda_n|^{2(N+1)} \geq C^2 \|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega}^2 \|z^{N+1}\|_{\mathfrak{D}_\omega}^2.$$

Note that

$$\|z^{N+1}\|_{\mathfrak{D}_\omega}^2 = \omega_{N+1},$$

and

$$\|k_{\lambda_n}^\omega\|_{\mathfrak{D}_\omega}^2 \geq \sum_{k=0}^{N+1} \frac{|\lambda_n|^{2k}}{\omega_k} \geq |\lambda_n|^{2(N+1)} S_{N+1}.$$

Hence we obtain  $|\lambda_n|^{2(N+1)} \geq C^2 |\lambda_n|^{2(N+1)} S_{N+1} \omega_{N+1}$ , that is

$$1 \geq C^2 S_{N+1} \omega_{N+1}.$$

According to Lemma 4.4, we get the desired contradiction in that second case. □

**Example 4.6.** Let  $\omega(r) = (1 - r^2)^\alpha$  with  $\alpha > -1$ . Then  $\omega$  is a weight function and we can apply Theorem 4.5 to  $\mathfrak{D}_\omega$ .

For  $\alpha = 0$ , we get that  $\omega_n = n$  and  $\mathfrak{D}_\omega$  is the classical Dirichlet space  $\mathfrak{D}$ . Therefore we can apply Theorem 4.5 to the Dirichlet space.

For  $\alpha = 1$ , we get that  $\omega_n = 1$ ,  $n \geq 0$ , and  $\mathfrak{D}_\omega = H^2$ . Thus Theorem 4.5 is a generalization of Theorem 3.2.

For  $\alpha = 2$ , we get that  $\omega_n = 1/n$ ,  $n \geq 1$ , and  $\mathfrak{D}_\omega$  is the classical Bergman space  $\mathfrak{B}$  and we can apply Theorem 4.5 to the Bergman space.

**4.2. Dirichlet type spaces and absolutely representing systems.** We now study an example when the weight  $\omega$  is not (necessarily) radial but corresponds to the Poisson transform of a positive finite Borel measure  $\mu$  on  $\mathbb{T}$ . Let

$$\omega(z) = P[\mu](z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} d\mu(\zeta)$$

and let  $\mathfrak{D}_\mu = \mathfrak{D}_\omega$  be the space consisting of analytic functions  $f$  on  $\mathbb{D}$  satisfying

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty.$$

It is known (see [5, Theorem 7.1.2]) that  $\mathfrak{D}_\mu \subset H^2$  and  $\mathcal{D}_\mu$  is a reproducing kernel Hilbert space whose norm is given by

$$\|f\|_\mu^2 = \|f\|_2^2 + \mathcal{D}_\mu(f), \quad f \in \mathfrak{D}_\mu.$$

It is also known that  $\mathfrak{D}_\mu$  is invariant with respect to the difference quotients. Denote by  $\widehat{k}_\lambda^\mu$  the normalized reproducing kernel for  $\mathfrak{D}_\mu$  at point  $\lambda \in \mathbb{D}$ .

**Theorem 4.7.** *Let  $\mu$  be a positive finite Borel measure on  $\mathbb{T}$ , let  $(\lambda_n)_{n \geq 1}$  a sequence of distinct points in the unit disc. Then the sequence  $(\widehat{k}_{\lambda_n}^\mu)_{n \geq 1}$  cannot be an absolutely representing system of its closed linear span.*

*Proof.* The proof is along the same lines as the proof of Theorem 4.5. First note that a simple computation shows that for every integer  $N \geq 0$ , we have

$$(4.8) \quad \|z^N\|_\mu^2 = 1 + N\|\mu\|.$$

Assume that the sequence  $(\widehat{k}_{\lambda_n}^\mu)_{n \geq 1}$  is an absolutely representing system of its closed linear span. We shall distinguish two cases.

**First case:** the sequence  $(\widehat{k}_{\lambda_n}^\mu)_{n \geq 1}$  is minimal. Thus, it should be a (Schauder) basis for its closed linear span. Then apply Proposition 2.8 to get a contradiction.

**Second case:** the sequence  $(\widehat{k}_{\lambda_n}^\mu)_{n \geq 1}$  is not minimal. Then, since  $\mathfrak{D}_\mu$  is invariant with respect to the difference quotients, we can apply Lemma 4.1, and we get

$$\text{span}(\widehat{k}_{\lambda_n}^\mu : n \geq 1) = \mathfrak{D}_\mu.$$

It follows from Lemma 2.6 that there is a constant  $C > 0$  such that for every  $h \in \mathfrak{D}_\mu$ , we can find  $n \geq 1$  such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^\mu\|_{\mathfrak{D}_\mu} \|h\|_{\mathfrak{D}_\mu}.$$

Applying this to  $h(z) = z^N$  for  $N \in \mathbb{N}$ , there is  $n \geq 1$  such that

$$|\lambda_n|^{2N} \geq C^2 \|k_{\lambda_n}^\mu\|_{\mathfrak{D}_\mu}^2 \|z^N\|_{\mathfrak{D}_\omega}^2.$$

Since  $1 \in \mathfrak{D}_\mu$ , note that

$$1 = \langle 1, k_{\lambda_n}^\mu \rangle_\mu \leq \|1\|_\mu \|k_{\lambda_n}^\mu\|_\mu,$$

and we deduce

$$1 \geq |\lambda_n|^{2N} \geq C^2 \|1\|_\mu^{-2} \|z^N\|_\mu^2.$$

If we let  $N$  tend to  $\infty$  we obtain a contradiction with (4.8) in that second case.  $\square$

**4.3. De Branges–Rovnyak spaces and absolutely representing systems.** In this section we study the case of de Branges–Rovnyak spaces. We denote by  $\mathcal{B}_{H^\infty}$  the unit ball of  $H^\infty$ . Recall that, given a function  $b \in \mathcal{B}_{H^\infty}$ , the de Branges–Rovnyak space  $\mathcal{H}(b)$  is the reproducing kernel Hilbert space of analytic functions on  $\mathbb{D}$  whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad z, \lambda \in \mathbb{D}.$$

Note that

$$\|k_\lambda^b\|_b^2 = \frac{1 - |b(\lambda)|^2}{1 - |\lambda|^2}.$$

We also denote by  $\widehat{k}_\lambda^b$  the normalized reproducing kernel of  $\mathcal{H}(b)$  at point  $\lambda$ .

The space  $\mathcal{H}(b)$  can also be defined as the range space  $(I - T_b T_{\bar{b}})^{1/2} H^2$  equipped with the norm which makes  $(I - T_b T_{\bar{b}})^{1/2}$  a partial isometry. Here  $T_\varphi$  is the Toeplitz operator on  $H^2$  with symbol  $\varphi \in L^\infty(\mathbb{T})$  defined by

$$T_\varphi f = P_+(\varphi f), \quad f \in H^2,$$

where  $P_+$  is the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2$ . The book [12] is the classic reference for  $\mathcal{H}(b)$  spaces (see also [7]).

For our result, we focus on the case when  $b$  is a non-extreme point of  $\mathbf{b}(H^\infty)$ , equivalently,  $\log(1 - |b|) \in L^1(\mathbb{T}, m)$  (where  $\mathbb{T} := \{\zeta \in \mathbf{C} : |\zeta| = 1\}$  and  $m$  is Lebesgue measure on  $\mathbb{T}$  normalized so that  $m(\mathbb{T}) = 1$ ). Then there exists a unique outer function  $a \in \mathbf{b}(H^\infty)$ , called the *Pythagorean mate* for  $b$ , such that  $a(0) > 0$  and  $|a|^2 + |b|^2 = 1$  almost everywhere on  $\mathbb{T}$ . The pair  $(a, b)$  is said to be a *Pythagorean pair*.

When  $b$  is non-extreme and  $a$  is its Pythagorean mate, an important vector space of functions is

$$\mathcal{M}(a) := T_a H^2 = a H^2.$$

We also equip this space with the range norm which makes  $T_a$  an isometry from  $H^2$  onto  $\mathcal{M}(a)$ . Then  $\mathcal{M}(a)$  is an Hilbert space which is included into  $\mathcal{H}(b)$ . Moreover, the inclusion  $\mathcal{M}(a) \subset \mathcal{H}(b)$  is contractive (see [12, p. 24]). Finally, we recall that  $\mathcal{H}(b)$  is invariant with respect to the difference quotients.

**Theorem 4.9.** *Let  $b$  be a non extreme point of  $\mathcal{B}_{H^\infty}$  and let  $(\lambda_n)_{n \geq 1}$  be a sequence of distinct points in the unit disc. Then the sequence  $(\widehat{k}_{\lambda_n}^b)_{n \geq 1}$  cannot be an absolutely representing system of its closed linear span.*

*Proof.* Assume that the sequence  $(\widehat{k}_{\lambda_n}^b)_{n \geq 1}$  is an absolutely representing system of its closed linear span. We shall distinguish two cases.

**First case:** the sequence  $(\widehat{k}_{\lambda_n}^b)_{n \geq 1}$  is minimal. Thus, it should be a basis for its closed linear span. Then apply Proposition 2.8 to get a contradiction.

**Second case:** the sequence  $(\widehat{k}_{\lambda_n}^b)_{n \geq 1}$  is not minimal. Then, according to Lemma 4.1, we have

$$\text{span}(\widehat{k}_{\lambda_n}^b : n \geq 1) = \mathcal{H}(b).$$

It follows from Lemma 2.6 that there is a constant  $C > 0$  such that for every  $h \in \mathcal{H}(b)$ , we can find  $n \geq 1$  such that

$$|h(\lambda_n)| \geq C \|k_{\lambda_n}^b\|_b \|h\|_b.$$

As in the proof in the  $H^2$  case, choose  $N$  sufficiently large so that  $2 \leq (N + 1)C^2$ . Consider  $h(z) = z^N a(z)$ . Since  $h \in \mathcal{M}(a) \subset \mathcal{H}(b)$  and

$$1 = \|z^N\|_2 = \|h\|_{\mathcal{M}(a)} \leq \|h\|_b,$$

there is an integer  $n$  such that

$$C^2(1 - |b(\lambda_n)|^2) \leq |a(\lambda_n)|^2 |\lambda_n|^{2N} (1 - |\lambda_n|^2).$$

Remind now that  $|a|^2 + |b|^2 = 1$  a.e. on  $\mathbb{T}$ . Hence a standard argument based on Poisson integral gives that  $|a(z)|^2 + |b(z)|^2 \leq 1$  for every  $z \in \mathbb{D}$ , which gives

$$C^2 \leq |\lambda_n|^{2N} (1 - |\lambda_n|^2) \leq \frac{1}{N + 1} \leq \frac{C^2}{2}.$$

Thus we get the desired contradiction in that second case. □

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