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ORTHONORMAL POLYNOMIAL BASIS IN LOCAL DIRICHLET SPACES

EMMANUEL FRICAIN AND JAVAD MASHREGHI

Abstract. We provide an orthogonal basis of polynomials for the local Dirichlet space $\mathcal{D}_\zeta$. These polynomials have numerous interesting features and a very unique algebraic pattern. We obtain the recurrence relation, the generating function, a simple formula for their norm, and explicit formulae for the distance and the orthogonal projection onto the subspace of polynomials of degree at most n. The latter implies a new polynomial approximation scheme in local Dirichlet spaces. Orthogonal polynomials in a harmonically weighted Dirichlet space, created by a finitely supported singular measure, are also studied.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $\mathbb{T}$ denote its boundary. For $f \in \mathrm{Hol}(\mathbb{D})$, the family of holomorphic functions on $\mathbb{D}$, and $\zeta \in \mathbb{T}$, the local Dirichlet integral is defined by

$$\mathcal{D}_\zeta(f) = \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, \frac{1-|z|^2}{|\zeta - z|^2}\, dA(z), \tag{1.1}$$

where $dA(z) = dx\, dy$ is the planar Lebesgue measure. The local Dirichlet space $\mathcal{D}_\zeta$ consists of all functions $f \in \mathrm{Hol}(\mathbb{D})$ with $\mathcal{D}_\zeta(f) < \infty$. Our focus in this paper is mainly on local Dirichlet spaces. However, we also need to consider a generalization given by

$$\mathcal{D}_\mu(f) = \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, P\mu(z)\, dA(z), \tag{1.2}$$

where $P\mu$ is the Poisson integral of μ. A detailed description of these spaces is available at [10, Ch. 7]. A short account is provided in Section 3. Note that $\mathcal{D}_\mu(f)$ is written $\mathcal{D}_\zeta(f)$ when $\mu = \delta_\zeta$, the Dirac measure at point ζ. These spaces have been the focus of numerous

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studies, e.g., invariant subspaces for the shift operator [11, 14, 24, 22], multipliers [18, 23], superharmonic weights [2, 3], and polynomial approximation and connections to de Branges–Rovnyak spaces [8, 9, 25], cyclicity [1, 4, 13, 15, 27], Carleson measures [7, 6, 5].

S. Richter showed that polynomials are dense in $\mathcal{D}_\mu$ [21]. The estimation needed in this approach for the Dirichlet integral of dilations f_r were further improved by Richter and Sundberg [22], Aleman [2] and Sarason [25]. Knowing that polynomials are dense in $\mathcal{D}_\mu$, we may apply the classical Gram–Schmidt procedure to create an orthogonal basis of polynomials for $\mathcal{D}_\mu$. Direct application of the Gram-Schmidt formula is usually difficult and does not lead to satisfactory results. However, if via other means, we come up with a sequence of orthogonal polynomials p_n, with $\deg(p_n) = n$, $n \geq 0$, then up to a multiplicative constant this is precisely the orthogonal sequence which is promised by the Gram-Schmidt process. This is how we succeeded to present the new family of orthogonal polynomials for $\mathcal{D}_\mu$ and study further their properties.

The outline of this note is as follows. A brief description of the main results is provided in the next section. In Section 3, the background needed on weighted Dirichlet spaces is presented. In Section 4, the promissed orthogonal polynomials are obtained and their projection properties are discussed in Section 5. In Section 6, we consider harmonically weighted Dirichlet spaces generated by a finite sum of Dirac measures. No wonder the formulas are more complicated in this setting. However, in Section 7, we show that if the distribution of measures has a symmetric pattern, then the life is easier and the orthogonal polynomials have a simpler description.

2. A brief description of results

Our main observation is that the family of polynomials

$$p_n(z) := 1 + (z-1)(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}), \qquad n \geq 0,$$

forms an orthogonal basis for $\mathcal{D}_1$, where the coefficients a_n are positive integers which satisfy the recursive equation

$$a_n = 3a_{n-1} - a_{n-2}, \qquad n \geq 2,$$

with the initial conditions $a_0 = 1$ and $a_1 = 2$. More explicitly, they are given by the formula

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{2n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{2n+1}, \qquad n \geq 0.$$

The norm of polynomials are given by the simple formula

$$\|p_n\|_{\mathcal{D}_1} = \sqrt{a_n a_{n-1}}, \qquad n \geq 1.$$

Hence, asymptotically,

$$\|p_n\|_{\mathcal{D}_1} \sim \frac{1}{\sqrt{5}} \left(\frac{3+\sqrt{5}}{2} \right)^n, \qquad n \to \infty.$$

They satisfy the recurrence relation

$$p_n = (1+3x)p_{n-1} - (x^2+3x)p_{n-2} + x^2 p_{n-3}, \qquad n \geq 3,$$

and have

$$g(z,t) = \frac{1-(1+2z)t+zt^2}{(1-t)(1-3zt+z^2t^2)}$$

as the generating function.

These polynomials provide the orthogonal decomposition

$$f = \hat{f}(0)p_0 + \sum_{n=1}^{\infty} \left(a_n \hat{f}(n) + \sum_{k=n+1}^{\infty} (a_k - a_{k-1}) \hat{f}(k) \right) \frac{p_n}{a_n a_{n-1}},$$

for each $f \in \mathcal{D}_1$. Therefore, we immediately obtain the projection

$$\mathbf{P}_n f = \hat{f}(0)p_0 + \sum_{j=1}^{n} \left(a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1}) \hat{f}(k) \right) \frac{p_j}{a_j a_{j-1}}.$$

onto $\mathcal{P}_n$, the space of polynomials of degree at most n, and consequently

$$\operatorname{dist}_{\mathcal{D}_1}(f, \mathcal{P}_n) = \left\{ \sum_{j=n+1}^{\infty} \frac{\left| a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1}) \hat{f}(k) \right|^2}{a_j a_{j-1}} \right\}^{1/2}.$$

The sequence of polynomials $(\mathbf{P}_n f)_{n \geq 0}$ converges to f in $\mathcal{D}_1$. This result complements polynomial approximation schemes obtained in [17, 18, 19].

We study the same phenomenon for harmonically weighted Dirichlet spaces which are created by a finite discrete measure $\mu = \sum_{i=1}^{s} c_i \delta_{\zeta_i}$, $c_i > 0$, $\zeta_i \in \mathbb{T}$. In this case, the polynomials satisfy the property

$$p_m = z^{m-s} q + a_1^{(m)} p_{m-1} + \cdots + a_s^{(m)} p_{m-s},$$

where $q(z) = \prod_{i=1}^{s} (z - \zeta_i)$. When the Dirac measures are equally distributed, we show that the orthogonal polynomials are closely related to the orthogonal polynomials of a single local Dirichlet space.

3. Background on harmonically weighted Dirichlet spaces

The local Dirichlet space is originally defined via the growth restriction (1.1). However, one of the most important features of these spaces is the following characterization [22]: a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $\mathcal{D}_\zeta$ if and only if

$$f(z) = a + (z-\zeta)g(z), \tag{3.1}$$

where $g \in H^2$ and $a \in \mathbb{C}$. Moreover, we have

$$\mathcal{D}_\zeta(f) = \|g\|_{H^2}^2. \tag{3.2}$$

See also [10, page 111]. We usually equip $\mathcal{D}_\zeta$ with the norm

$$\|f\|_{\mathcal{D}_\zeta}^2 := \|f\|_{H^2}^2 + \mathcal{D}_\zeta(f). \tag{3.3}$$

Note that $\sqrt{\mathcal{D}_\zeta(f)}$ is a semi-norm and that is why the term $\|f\|_{H^2}^2$ is added in (3.3) to create a genuine norm. This characterization has numerous essential applications and, in particular, enables us to present an orthonormal basis for $\mathcal{D}_\zeta$.

Using that $|g(r\zeta)| = o((1-r^2)^{-1/2})$, the relation (3.1) immediately implies

$$f(\zeta) := \lim_{r\to 1^-} f(r\zeta) = a. \tag{3.4}$$

Moreover if $f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k$ is in $\mathcal{D}_\zeta$, then it is proved in [19, Corollary 2] that the series $\sum_{k=0}^\infty \hat{f}(k)\zeta^k$ converges and by Abel's theorem on radial limits, we have

$$a = f(\zeta) = \sum_{k=0}^\infty \hat{f}(k)\zeta^k.$$

As a matter of fact, f approaches the boundary value $f(\zeta)$ in an oricyclic region from within $\mathbb{D}$ and tangent to $\mathbb{T}$ at the point ζ. This observation enables us to define the difference quotient operator $Q_\zeta : \mathcal{D}_\zeta \to H^2$,

$$(Q_\zeta f)(z) := \frac{f(z) - f(\zeta)}{z-\zeta}, \qquad z \in \mathbb{D}.$$

By (3.1) and (3.2) we see that $\|Q_\zeta f\|_{H^2}^2 = \mathcal{D}_\zeta(f)$, and a further look at (3.3) shows $\|Q_\zeta f\|_{H^2}^2 \le \|f\|_{\mathcal{D}_\zeta}^2$. Therefore, we can rewrite the definition of norm as

$$\|f\|_{\mathcal{D}_\zeta}^2 = \|f\|_{H^2}^2 + \|Q_\zeta f\|_{H^2}^2, \tag{3.5}$$

which, using polarization identity, turns into

$$\langle f_1, f_2\rangle_{\mathcal{D}_\zeta} = \langle f_1, f_2\rangle_{H^2} + \langle Q_\zeta f_1, Q_\zeta f_2\rangle_{H^2}. \tag{3.6}$$

Let us compute the function $Q_\zeta f$ explicitly. If we write $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ and $(Q_\zeta f)(z) = \sum_{k=0}^{\infty} b_k z^k$, with $(b_k)_k \in \ell^2$, and equate coefficients of z^k in the relation

$$f(z) = a + (z - \zeta)(Q_\zeta f)(z),$$

then we obtain

$$\begin{cases} \hat{f}(0) &= a - \zeta b_0 \\ \hat{f}(k) &= b_{k-1} - \zeta b_k, \qquad (k \geq 1). \end{cases}$$

Now, note that for $N \geq n \geq 1$,

$$\begin{aligned} \sum_{k=n}^{N} \hat{f}(k) \zeta^{k-n} =& \sum_{k=n}^{N} \left(\zeta^{k-n} b_{k-1} - \zeta^{k-n+1} b_k \right) \\ =& \zeta^{1-n} \sum_{k=n}^{N} \left(b_{k-1} \zeta^{k-1} - b_k \zeta^k \right) \\ =& b_{n-1} - b_N \zeta^{N+1-n}. \end{aligned}$$

Since $(b_n)_n$ is square summable, then $b_N \to 0$, as $N \to \infty$. Hence,

$$b_{n-1} = \sum_{k=n}^{\infty} \hat{f}(k) \zeta^{k-n}.$$

Therefore,

$$(Q_\zeta f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} \hat{f}(k) \zeta^{k-n-1} \right) z^n. \tag{3.7}$$

Let μ be a positive and finite Borel measure on $\mathbb{T}$, and extend it to the open unit disc via the Poisson integral formula

$$P\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} \, d\mu(\zeta), \qquad z \in \mathbb{D}.$$

Note that $P\mu$ is a positive harmonic function on $\mathbb{D}$. Moreover, by Herglotz theorem [16], every positive harmonic function on $\mathbb{D}$ has such a representation. Then the harmonically weighted Dirichlet integral is defined by (1.2).

4. An orthonormal basis of polynomials in $\mathcal{D}_\zeta$

A separable Hilbert space has a countable basis of orthonormal elements. Since polynomials are dense in $\mathcal{D}_\zeta$, from the theoretical point of view, the Gram-Schmidt orthogonalization process can be applied to the sequence $(z^n)_{n \geq 0}$ to create an orthogonal basis for $\mathcal{D}_\zeta$. In this

section we provide an explicit formula for these polynomials. To simplify the presentation, we do the calculation for $\mathcal{D}_1$. However, either by modifying the proof below or by applying the rotation $z \longmapsto \bar{\zeta} z$, one can easily obtain the corresponding family for $\mathcal{D}_\zeta$.

Theorem 4.1. *Let $p_0(z) = 1$, $p_1(z) = z$ and*

$$p_n(z) = 1 + (z-1)(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}), \qquad n \geq 2,$$

where

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{2n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{2n+1}, \qquad n \geq 0. \tag{4.2}$$

Then a_n, $n \geq 0$, are (unique) positive integers such that $a_0 = 1$, $a_1 = 2$,

$$a_n = 3a_{n-1} - a_{n-2}, \qquad n \geq 2, \tag{4.3}$$

and that $(p_n)_{n\geq 0}$ forms an orthogonal basis for $\mathcal{D}_1$.

Proof. Assume that $a_0 = 1$ and the other coefficients are not known at the beginning. To achieve the orthogonality, we show by induction that they are uniquely determined. By direct verification, p_0 and p_1 are orthogonal. For a fixed $n \geq 2$, assume that the unique positive integers $a_0, \ldots, a_{n-1}$ exist such that the polynomials $p_0, \ldots, p_n$ are pairwise orthogonal, i.e.,

$$\langle p_k, p_j \rangle_{\mathcal{D}_1} = 0, \qquad 0 \leq k < j \leq n.$$

We proceed to establish the existence of a positive integer a_n such that

$$p_{n+1}(z) := 1 + (z-1)(a_0 + a_1 z + \cdots + a_n z^n)$$

is orthogonal to all the previous polynomials $p_0, \ldots, p_{n-1}$, i.e.,

$$\langle p_k, p_{n+1} \rangle_{\mathcal{D}_1} = 0, \qquad 0 \leq k \leq n.$$

According to (3.6) and that $Q_1 p_0 = 0$,

$$\langle p_0, f \rangle_{\mathcal{D}_1} = \langle p_0, f \rangle_{H^2} = \overline{f(0)}, \qquad f \in \mathcal{D}_1.$$

Since for any choice of the coefficients a_k, $k \geq 1$, we have $p_m(0) = 0$, $m \geq 1$, we also automatically have

$$\langle p_0, p_m \rangle_{\mathcal{D}_1} = 0, \qquad m \geq 1.$$

In fact, this happens due to fixing $a_0 = 1$ at the very beginning.

For any choice of a_n, the polynomial p_{n+1} can be written as

$$\begin{aligned} p_{n+1}(z) &= 1 + (z-1)(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n) \\ &= p_n(z) + a_n (z-1) z^n. \end{aligned}$$

Hence, by (3.6), for $1 \leq k \leq n-1$ and for any a_n,

$$\begin{aligned}\langle p_k, p_{n+1}\rangle_{\mathcal{D}_1} &= \langle p_k, p_n + a_n(z-1)z^n\rangle_{\mathcal{D}_1} \\ &= \langle p_k, p_n\rangle_{\mathcal{D}_1} + \overline{a_n}\langle p_k, (z-1)z^n\rangle_{\mathcal{D}_1} \\ &= 0 + \overline{a_n}\langle p_k, (z-1)z^n\rangle_{H^2} \\ &\quad + \overline{a_n}\langle a_0 + \cdots + a_{k-1}z^{k-1}, z^n\rangle_{H^2} \\ &= 0.\end{aligned}$$

Note that $Q_1 p_k = a_0 + \cdots + a_{k-1}z^{k-1}$ and $Q_1(z-1)z^n = z^n$. Again, up to this point, a_n plays no role!

It remains to treat the case $k = n$. Thus, once more by (3.6),

$$\begin{aligned}\langle p_n, p_{n+1}\rangle_{\mathcal{D}_1} &= \langle p_n, p_n + a_n(z-1)z^n\rangle_{\mathcal{D}_1} \\ &= \langle p_n, p_n\rangle_{\mathcal{D}_1} + \overline{a_n}\langle p_n, (z-1)z^n\rangle_{\mathcal{D}_1} \\ &= \langle p_n, p_n\rangle_{\mathcal{D}_1} + \overline{a_n}\langle p_n, (z-1)z^n\rangle_{H^2} \\ &\quad + \overline{a_n}\langle a_0 + \cdots + a_{n-1}z^{n-1}, z^n\rangle_{H^2} \\ &= \langle p_n, p_n\rangle_{\mathcal{D}_1} - \overline{a_n}\langle p_n, z^n\rangle_{H^2}.\end{aligned}$$

From here we see that in order to obtain $\langle p_n, p_{n+1}\rangle_{\mathcal{D}_1} = 0$, the only possibility is to choose

$$a_n = \frac{\langle p_n, p_n\rangle_{\mathcal{D}_1}}{\langle z^n, p_n\rangle_{H^2}}, \qquad n \geq 1. \tag{4.4}$$

Therefore, a_n is uniquely determined. Note that $\langle z^n, p_n\rangle_{H^2} = a_{n-1} \neq 0$. Moreover, since $\deg(p_n) = n$, $n \geq 0$, and the polynomials are dense in $\mathcal{D}_1$, the system $\{p_n : n \geq 0\}$ is complete.

The identity (4.4), while establishes the uniqueness of a_n, does not show that they are positive integers. We need to explore a bit more and derive a recursive formula for a_n to achieve this goal. According to (3.6), we have

$$\langle p_n, p_n\rangle_{\mathcal{D}_1} = \sum_{k=0}^{n-2}\left(a_k^2 + (a_k - a_{k+1})^2\right) + 2a_{n-1}^2. \tag{4.5}$$

Therefore, by (4.4), the coefficients are inductively given by

$$a_n = \frac{\sum_{k=0}^{n-2}\left(a_k^2 + (a_k - a_{k+1})^2\right) + 2a_{n-1}^2}{a_{n-1}}, \qquad n \geq 1. \tag{4.6}$$

Since $a_0 = 1$, plugging $n = 1$ in the above formula gives $a_1 = 2$. In the formula (4.6), replace n by $n-1$ to get

$$\sum_{k=0}^{n-3}\left(a_k^2 + (a_k - a_{k+1})^2\right) = a_{n-1}a_{n-2} - 2a_{n-2}^2, \qquad n \geq 2. \tag{4.7}$$

Plugging this identity back to the formula (4.6) gives us the remarkably simpler identity (4.3). This relation shows that all coefficients are positive integers. If we put $a_n = r^n$ in (4.3), we see that the characteristic equation of the difference equation is $r^2 - 3r + 1 = 0$, whose roots are $(3 \pm \sqrt{5})/2$. Therefore, the explicit solution of the difference equation (4.3), with initial conditions $a_0 = 1$ and $a_1 = 2$, is

$$a_n = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{3+\sqrt{5}}{2} \right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{3-\sqrt{5}}{2} \right)^n, \qquad n \geq 0. \tag{4.8}$$

This representation shows that the all coefficients are strictly positive too. Finally, after some easy calculation, this identity can be rewritten as (4.2). □

The first few polynomials which form an orthogonal basis for $\mathcal{D}_1$ are

$$\begin{aligned} p_0(z) &= 1, \\ p_1(z) &= 1 + (z-1), \\ p_2(z) &= 1 + (z-1)(1+2z), \\ p_3(z) &= 1 + (z-1)(1+2z+5z^2), \\ p_4(z) &= 1 + (z-1)(1+2z+5z^2+13z^3). \end{aligned}$$

Using the rotation $z \longmapsto \bar{\zeta} z$, which is a unitary operator from $\mathcal{D}_1$ onto $\mathcal{D}_\zeta$, we see that the family $\big(p_n(\bar{\zeta} z)\big)_{n \geq 0}$ is an orthogonal (or correspondingly, orthonormal if we normalize them) basis for $\mathcal{D}_\zeta$.

Corollary 4.9. *The norm of p_n is*

$$\|p_n\|_{\mathcal{D}_1} = \sqrt{a_n a_{n-1}}, \qquad (n \geq 1).$$

In particular,

$$\|p_n\|_{\mathcal{D}_1} \sim \frac{1}{\sqrt{5}} \left(\frac{3+\sqrt{5}}{2} \right)^n, \qquad n \to \infty.$$

Proof. By (4.4)

$$\|p_n\|^2_{\mathcal{D}_1} = \langle p_n, p_n \rangle_{\mathcal{D}_1} = a_n a_{n-1}.$$

The asymptotic value is a direct consequence of (4.8). □

The above formula for the norm of p_n can be used to normalize the polynomials and create an orthonormal basis for $\mathcal{D}_1$.

According to Theorem 4.1, the polynomials p_n satisfy the relation

$$p_n(z) = a_{n-1} z^{n-1}(z-1) + p_{n-1}(z), \qquad n \geq 1. \tag{4.10}$$

By the Shohat–Favard theorem [12, 26], a sequence of polynomials satisfying a recurrence relation with 3 terms is a sequence of orthogonal

polynomials. A similar result had also been used by Stieltjes in the theory of continued fractions earlier than Favard and Shohat works [20]. Due to the special inner product which is used in local Dirichlet spaces, we show that the sequence $(p_n)_{n\geq 0}$ satisfies a recurrence relation with four terms.

Corollary 4.11. *The sequence of orthogonal polynomials $(p_n)_{n\geq 0}$ satisfies the recurrence relation*

$$p_n = (1+3x)p_{n-1} - (x^2+3x)p_{n-2} + x^2 p_{n-3}, \qquad n \geq 3.$$

Proof. By (4.10), applied three times,

$$\begin{aligned} p_n(z) - p_{n-1}(z) &= a_{n-1}z^{n-1}(z-1), \\ z\big[p_{n-1}(z) - p_{n-2}(z)\big] &= a_{n-2}z^{n-1}(z-1), \\ z^2\big[p_{n-2}(z) - p_{n-3}(z)\big] &= a_{n-3}z^{n-1}(z-1). \end{aligned}$$

Hence, by (4.3) (replace n by $n-1$), we see

$$\big[p_n(z) - p_{n-1}(z)\big] - 3z\big[p_{n-1}(z) - p_{n-2}(z)\big] + z^2\big[p_{n-2}(z) - p_{n-3}(z)\big] = 0.$$

Now, simplify. □

A generating function $g(z,t)$ is such that, at least, the formal development

$$g(z,t) = \sum_{n=0}^{\infty} p_n(z)t^n$$

is valid. The explicit formula for a_n enables us to find the generating function for these polynomials.

Corollary 4.12. *The generating function of $(p_n)_{n\geq 0}$ is*

$$g(z,t) = \frac{1-(1+2z)t+zt^2}{(1-t)(1-3zt+z^2t^2)}.$$

Proof. The main definition says

$$g(z,t) \;=\; \sum_{n=0}^{\infty} p_n(z)t^n = \sum_{n=0}^{\infty}\left(1+(z-1)\sum_{k=0}^{n-1} a_k z^k\right)t^n.$$

By (4.8),

$$a_n = \frac{\sqrt{5}+1}{2\sqrt{5}}w^n + \frac{\sqrt{5}-1}{2\sqrt{5}}w^{-n}, \qquad n \geq 0,$$

where $w = (3+\sqrt{5})/2$. Therefore,

$$\begin{aligned}
g(z,t) &= \sum_{n=0}^{\infty}\left(1+(z-1)\sum_{k=0}^{n-1}\left(\frac{\sqrt{5}+1}{2\sqrt{5}}w^k+\frac{\sqrt{5}-1}{2\sqrt{5}}w^{-k}\right)z^k\right)t^n \\
&= \sum_{n=0}^{\infty}\left(1+(z-1)\left(\frac{\sqrt{5}+1}{2\sqrt{5}}\frac{1-w^nz^n}{1-wz}+\frac{\sqrt{5}-1}{2\sqrt{5}}\frac{1-w^{-n}z^n}{1-w^{-1}z}\right)\right)t^n \\
&= \frac{1}{1-t}+(z-1)\left(\frac{\sqrt{5}+1}{2\sqrt{5}}\frac{\frac{1}{1-t}-\frac{1}{1-twz}}{1-wz}+\frac{\sqrt{5}-1}{2\sqrt{5}}\frac{\frac{1}{1-t}-\frac{1}{1-tw^{-1}z}}{1-w^{-1}z}\right) \\
&= \frac{1}{1-t}+\frac{t(z-1)}{1-t}\left(\frac{\sqrt{5}+1}{2\sqrt{5}}\frac{1}{1-twz}+\frac{\sqrt{5}-1}{2\sqrt{5}}\frac{1}{1-tw^{-1}z}\right) \\
&= \frac{1}{1-t}+\frac{t(z-1)(1-tz)}{(1-t)(1-3tz+t^2z^2)} \\
&= \frac{1-(1+2z)t+zt^2}{(1-t)(1-3zt+z^2t^2)}.
\end{aligned}$$

□

Note that how the normalizations $p_n(1)=1$ for all $n\geq 1$, and $p_0=1$ while $p_n(0)=0$ for all $n\geq 1$, are reflected in the explicit formula for $g(z,t)$. Due to these facts, on one hand,

$$g(1,t)=\sum_{n=0}^{\infty}p_n(1)t^n=\sum_{n=0}^{\infty}t^n=\frac{1}{1-t}=\frac{1-(1+2z)t+zt^2}{(1-t)(1-3zt+z^2t^2)}\bigg|_{z=1},$$

and, on the other hand,

$$g(0,t)=\sum_{n=0}^{\infty}p_n(0)t^n=p_0(0)=1=\frac{1-(1+2z)t+zt^2}{(1-t)(1-3zt+z^2t^2)}\bigg|_{z=0}.$$

5. The projection onto $\mathcal{P}_n$

We write

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$$

for the Taylor expansion of a function $f\in \mathrm{Hol}(\mathbb{D})$. Apart from indicating the nth Taylor coefficient of f, the notation $\hat{f}(n)$ has no other specific meaning here. However, in the Hardy space H^2, it can also be interpreted as the nth Fourier coefficient of the boundary function which lives on $\mathbb{T}$. We avoid using a_n for the coefficients of f, since they are already exploited in the definition of polynomials p_n.

Let $\mathcal{P}_n$ denote the family of all polynomials of degree at most n. Given $f \in \mathcal{D}_1$, what is the best polynomial approximation in $\mathcal{P}_n$ to f, and at the same token what is the distance between f and $\mathcal{P}_n$. To answer these questions, we start with the following result. Recall that for $f \in \mathcal{D}_1$, the series $\sum_{k=0}^{\infty} \hat{f}(k)$ is convergent and is equal to $f(1)$.

Lemma 5.1. *Let $f \in \mathcal{D}_1$. Then the decomposition of f with respect to the orthogonal basis $(p_n)_{n\geq 0}$ is*

$$f = \hat{f}(0)p_0 + \sum_{n=1}^{\infty} \left(a_n \hat{f}(n) + \sum_{k=n+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right) \frac{p_n}{a_n a_{n-1}},$$

Proof. Recall that

$$\begin{aligned} p_n(z) &= 1 + (z-1)\sum_{k=0}^{n-1} a_k z^k \\ &= \sum_{k=1}^{n-1} (a_{k-1} - a_k) z^k + a_{n-1} z^n. \end{aligned}$$

Given

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in \mathcal{D}_1,$$

let us write

$$f = \sum_{n=0}^{\infty} \alpha_n p_n.$$

Then, by (3.6),

$$\begin{aligned} \alpha_0 &= \langle f, p_0 \rangle_{\mathcal{D}_1} / \|p_0\|^2_{\mathcal{D}_1} \\ &= \langle f, p_0 \rangle_{H^2} + \langle Q_1 f, Q_1 p_0 \rangle_{H^2} \\ &= \langle f, 1 \rangle_{H^2} = \hat{f}(0). \end{aligned}$$

Moreover, for $n \geq 1$, using (3.6) and (3.7), we have

$$\begin{aligned} \alpha_n \|p_n\|^2_{\mathcal{D}_1} &= \langle f, p_n \rangle_{\mathcal{D}_1} \\ &= \langle f, p_n \rangle_{H^2} + \langle Q_1 f, Q_1 p_n \rangle_{H^2} \\ &= \sum_{k=1}^{n-1} (a_{k-1} - a_k)\hat{f}(k) + a_{n-1}\hat{f}(n) + \sum_{k=0}^{n-1} a_k \left(\sum_{j=k+1}^{\infty} \hat{f}(j) \right) \\ &= a_n \hat{f}(n) + \sum_{k=n+1}^{\infty} (a_k - a_{k-1})\hat{f}(k). \end{aligned}$$

In the above simplification, we used the relation

$$a_0 + a_1 + \cdots + a_k = -2a_{k+1} + a_{k+2}, \qquad k \geq 0,$$

which can be easily proved by induction and (4.3). Therefore, with Corollary 4.9, we conclude that for each $f \in \mathcal{D}_1$, we have the orthonormal decomposition

$$f = \hat{f}(0)p_0 + \sum_{n=1}^{\infty} \left(a_n \hat{f}(n) + \sum_{k=n+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right) \frac{p_n}{a_n a_{n-1}},$$

where the above series converges in the norm (3.3). □

The representation formula in Lemma 5.1 is the key ingredient to find the orthogonal projection onto $\mathcal{P}_n$, as well as a formula for $\operatorname{dist}(f, \mathcal{P}_n)$.

Theorem 5.2. *Let $f \in \mathcal{D}_1$. Then the orthogonal projection of f onto $\mathcal{P}_n$ is*

$$\mathbf{P}_n f = \hat{f}(0)p_0 + \sum_{j=1}^{n} \left(a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right) \frac{p_j}{a_j a_{j-1}}.$$

Moreover,

$$dist_{\mathcal{D}_1}(f, \mathcal{P}_n) = \left\{ \sum_{j=n+1}^{\infty} \frac{\left| a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right|^2}{a_j a_{j-1}} \right\}^{1/2}.$$

Proof. The formula for $\mathbf{P}_n f$ is an easy consequence of Lemma 5.1. Moreover, by orthogonality and Corollary 4.9,

$$\begin{aligned} \operatorname{dist}_{\mathcal{D}_1}(f, \mathcal{P}_n) &= \|f - \mathbf{P}_n f\|_{\mathcal{D}_1} \\ &= \left\| \sum_{j=n+1}^{\infty} \left(a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right) \frac{p_j}{a_j a_{j-1}} \right\|_{\mathcal{D}_1} \\ &= \left\{ \sum_{j=n+1}^{\infty} \frac{\left| a_j \hat{f}(j) + \sum_{k=j+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) \right|^2}{a_j a_{j-1}} \right\}^{1/2}. \end{aligned}$$

□

Theorem 5.2 reveals a surprising uniqueness property of the orthogonal projections $\mathbf{P}_n$. It says that the orthogonal projection onto $\mathbf{P}_n f$ is completely determined by the first $n+1$ Taylor coefficients, as well as an average of the remaining coefficients.

Corollary 5.3. *Let $f, g \in \mathcal{D}_1$. Suppose that*

$$\hat{f}(j) = \hat{g}(j), \qquad 0 \le j \le n,$$

and that

$$\sum_{k=n+1}^{\infty} (a_k - a_{k-1})\hat{f}(k) = \sum_{k=n+1}^{\infty} (a_k - a_{k-1})\hat{g}(k).$$

Then $\mathbf{P}_n f = \mathbf{P}_n g$.

As a special case, we see that for any $m > n$,

$$\mathbf{P}_n z^m = (a_m - a_{m-1}) \sum_{j=1}^{n} \frac{p_j}{a_j a_{j-1}}.$$

Theorem 5.2 also provides a sequence of polynomials $(\mathbf{P}_n f)_{n\ge 0}$ which converges to f in $\mathcal{D}_1$. In [19], it was shown that Taylor polynomials of f might fail to converge. However, Fejér averages of Taylor polynomials converge to f. More sophisticated summability methods are discussed in [17, 18].

6. Measures with finite support

Let $\zeta_1, \dots, \zeta_s$ be s distinct points on $\mathbb{T}$. Let $c_j > 0$ and consider the discrete measure

$$d\mu = c_1 d\delta_{\zeta_1} + \cdots + c_s d\delta_{\zeta_s}.$$

Our goal is to explore the orthogonal basis of polynomials in the harmonically weighted Dirichlet space $\mathcal{D}_\mu$. As a straightforward generalization of (3.1), each $f \in \mathcal{D}_\mu$ has the unique representation

$$f(z) = a_0 + a_1 z + \cdots + a_{s-1} z^{s-1} + (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_s) g(z), \tag{6.1}$$

where $g \in H^2$. It is easy to see that

$$g := Q_{\zeta_1} Q_{\zeta_2} \cdots Q_{\zeta_s} f, \qquad f \in \mathcal{D}_\mu.$$

Note that the operators Q_{ζ_j} do not commute. However, and surprisingly enough, the action of $Q_{\zeta_1} Q_{\zeta_2} \cdots Q_{\zeta_s}$ on $\mathcal{D}_\mu$ does not depend on the ordering of Q_{ζ_j}. Therefore, we may define the operator

$$Q := Q_{\zeta_1} Q_{\zeta_2} \cdots Q_{\zeta_s}$$

for which the ordering of Q_{ζ_j} is irrelevant for our purpose. For simplicity of notations, put

$$q(z) := (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_s),$$

Therefore, we can rewrite (6.1) as

$$f(z) = a_0 + a_1 z + \cdots + a_{s-1} z^{s-1} + q(z)(Qf)(z). \tag{6.2}$$

As a generalization of (3.5), let

$$\|f\|^2_{\mathcal{D}_\mu} := \|f\|^2_{H^2} + \|Qf\|^2_{H^2}. \tag{6.3}$$

The inner product is

$$\langle f, g\rangle_{\mathcal{D}_\mu} := \langle f, g\rangle_{H^2} + \langle Qf, Qg\rangle_{H^2}. \tag{6.4}$$

In the first place, since

$$Qz^j = 0, \qquad 0 \le j \le s-1,$$

and $1, z, z^2, \dots, z^{s-1}$ are orthonormal in H^2, our first s orthonormal polynomials are already given by

$$p_j(z) := z^j, \qquad 0 \le j \le s-1.$$

We apply the Gram–Schmidt process and find the remaining terms of the sequence. But, for the new terms we encounter an interesting phenomenon. In the most general setting, since p_m is a monic polynomial of order m, there are unique coefficients $a_k^{(m)}$ such that

$$p_m = z^m + a_1^{(m)} p_{m-1} + + a_2^{(m)} p_{m-2} + \cdots + a_m^{(m)} p_0.$$

In an arbitrary orthogonalization process, all the m coefficients $a_j^{(m)}$ are present. However, in this case, we show that just s coefficients, a fixed number, are needed. The rest are zero! The following result is a generalization of (4.10).

Theorem 6.5. *Let $c_j > 0$ and consider the discrete measure*

$$d\mu = c_1 d\delta_{\zeta_1} + \cdots + c_s d\delta_{\zeta_s}.$$

Let $(p_m)_{m\ge 0}$ be the unique sequence of monic orthogonal polynomials in $\mathcal{D}_\mu$. Then, for each $m \ge s$, there are constants $a_1^{(m)}, \dots, a_s^{(m)}$ such that

$$p_m = z^{m-s} q + a_1^{(m)} p_{m-1} + \cdots + a_s^{(m)} p_{m-s}.$$

Proof. In this proof, for the simplicity of notations, we write a_j for $a_j^{(m)}$. The proof is by induction. Assume that p_j, for $n \le j \le m-1$, satisfy the above mentioned functional equation. We define p_m by the required formula, where the coefficients a_k are unknown. We show that the coefficients can be found such that p_m is orthogonal to p_j, $0 \le j \le m-1$.

First, for $0 \le j \le m-s-1$, we have

$$\langle p_m, z^j\rangle_\mu = \langle z^{m-s} q, z^j\rangle_\mu + \sum_{k=1}^{s} a_k \langle p_{m-k}, z^j\rangle_\mu.$$

The monomials z^j, $0 \le j \le m-s-1$, are in the linear combination of $p_0, \dots, p_{m-s-1}$. Hence, by the induction hypothesis,

$$\langle p_{m-k}, z^j \rangle_\mu = 0, \qquad 1 \le k \le s.$$

Moreover, by (6.4),

$$\langle z^{m-s}q, z^j \rangle_\mu = \langle z^{m-s}q, z^j \rangle_2 + \langle Q(z^{m-s}q), Q(z^j) \rangle_2.$$

Since $0 \le j \le m-s-1$, we certainly have $\langle z^{m-s}q, z^j \rangle_2 = 0$. Also, since $q(\zeta_k) = 0$, we have $Q(z^{m-s}q) = z^{m-s}$, and $Q(z^j) \in \mathcal{P}_{m-2s-1}$. Thus, $\langle Q(z^{m-s}q), Q(z^j) \rangle_2 = 0$ (this latter relation holds even for $j \le m-1$). Therefore, for any choice of the coefficients a_k, we automatically have

$$\langle p_m, z^j \rangle_\mu = 0, \qquad (0 \le j \le m-s-1).$$

The remaining requirement consists of the s equations

$$\langle p_m, z^j \rangle_\mu = 0, \qquad (m-s \le j \le m-1).$$

More explicitly, these equations form the system $Ax = b$, where A is the lower triangular matrix

$$A = \begin{pmatrix} \langle p_{m-s}, z^{m-s} \rangle_\mu & 0 & \cdots & 0 \\ \langle p_{m-s}, z^{m-s+1} \rangle_\mu & \langle p_{m-s+1}, z^{m-s+1} \rangle_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle p_{m-s}, z^{m-1} \rangle_\mu & \langle p_{m-s+1}, z^{m-1} \rangle_\mu & \cdots & \langle p_{m-1}, z^{m-1} \rangle_\mu \end{pmatrix},$$

and

$$x = \begin{pmatrix} a_s \\ a_{s-1} \\ \vdots \\ a_1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} \langle z^{m-s}q, z^{m-s} \rangle_\mu \\ \langle z^{m-s}q, z^{m-s+1} \rangle_\mu \\ \vdots \\ \langle z^{m-s}q, z^{m-1} \rangle_\mu \end{pmatrix}.$$

By induction, the elements on the diagonal of A are non-zero. Hence, a_k are uniquely determind. □

As another possibility, we may modify (6.3) and define the new norm

$$\|f\|^2_{\mathcal{D}_\mu} := \|f\|^2_{H^2} + \sum_{k=1}^{s} c_k \|Q_{\zeta_k} f\|^2_{H^2}. \tag{6.6}$$

The calculations with respect to this norm are more delicate. Using the polarization identity, the inner product becomes

$$\langle f, g \rangle_{\mathcal{D}_\mu} := \langle f, g \rangle_{H^2} + \sum_{k=1}^{s} c_k \langle Q_{\zeta_k} f, Q_{\zeta_k} g \rangle_{H^2}. \tag{6.7}$$

All these norms on $\mathcal{D}_\mu$ are equivalent. However, under certain specific cases, they might even coincide. One such case, which will be needed in Section 7, is presented below.

Let the points ζ_k be uniformly distributed on $\mathbb{T}$, e.g.,

$$\zeta_k = e^{i2k\pi/s}, \qquad 1 \le k \le s, \tag{6.8}$$

and assume that the masses at each point are also equal and normalized so that

$$c_1 = c_2 = \cdots = c_s = 1/s^2.$$

The important effect of weight $1/s^2$ is seen below in (6.10). Then

$$q(z) = (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_s) = z^s - 1$$

and (6.2) becomes

$$f(z) = a_0 + a_1 z + \cdots + a_{s-1}z^{s-1} + (z^s-1)Qf(z). \tag{6.9}$$

Let

$$\begin{aligned} \mathcal{D}_\mu(0) &= \{f \in \mathcal{D}_\mu : f(\zeta_1) = f(\zeta_2) = \cdots = f(\zeta_s) = 0\} \\ &= \{f \in \mathcal{D}_\mu : a_0 = a_1 = \cdots = a_{s-1} = 0\}. \end{aligned}$$

Hence, for each $f \in \mathcal{D}_\mu(0)$, the norm (6.6) becomes

$$\begin{aligned} \|f\|^2_{\mathcal{D}_\mu} &= \|f\|^2_{H^2} + \frac{1}{s^2}\sum_{k=1}^{s} \|Q_{\zeta_k}(f)\|_2^2 \\ &= \|f\|^2_{H^2} + \frac{1}{s^2}\sum_{k=1}^{s} \int_{\mathbb{T}} \left|\frac{z^s-1}{z-\zeta_k}\right|^2 |Qf(z)|^2 dm(z) \\ &= \|f\|^2_{H^2} + \int_{\mathbb{T}} \left(\frac{1}{s^2}\sum_{k=1}^{s} \left|\frac{z^s-1}{z-\zeta_k}\right|^2\right) |Qf(z)|^2 dm(z) \\ &= \|f\|^2_{H^2} + \int_{\mathbb{T}} |Qf(z)|^2 dm(z) \\ &= \|f\|^2_{H^2} + \|Qf\|^2_{H^2}. \end{aligned} \tag{6.10}$$

This is precisely the norm (6.3).

7. The uniform distribution

Let the points ζ_k be uniformly distributed on $\mathbb{T}$ as in (6.8), and equip D_μ with the norm (6.3). Note that under this norm the precise value of c_j are not important. The only restriction is $c_j > 0$. Due to its simplified form and resemblance to (3.5), we can now produce an orthogonal basis for $\mathcal{D}_\mu$.

Let $f(z) = a + (z-1)g(z) \in \mathcal{D}_1$. Then

$$z^r f(z^s) = az^r + (z^s-1)z^r g(z^s) \in \mathcal{D}_\mu, \qquad (r = 0, 1, \ldots, s-1).$$

Moreover, for any two functions of this type,

$$\begin{aligned} & \langle z^r f_1(z^s), z^\rho f_2(z^s)\rangle_{\mathcal{D}_\mu} \\ = \; & \big\langle az^r + (z^s-1)z^r g(z^s),\, bz^\rho + (z^s-1)z^\rho h(z^s)\big\rangle_{\mathcal{D}_\mu} \\ = \; & \big\langle az^r + (z^s-1)z^r g(z^s),\, bz^\rho + (z^s-1)z^\rho h(z^s)\big\rangle_{H^2} \\ & + \big\langle z^r g(z^s),\, z^\rho h(z^s)\big\rangle_{H^2}. \end{aligned}$$

It is trivial that the last two inner products are zero if $r \neq \rho$, and if $r = \rho$, we have

$$\begin{aligned} & \langle z^r f_1(z^s), z^r f_2(z^s)\rangle_{\mathcal{D}_\mu} \\ = \; & \big\langle a + (z^s-1)g(z^s),\, b + (z^s-1)h(z^s)\big\rangle_{H^2} + \big\langle g(z^s), h(z^s)\big\rangle_{H^2} \\ = \; & \big\langle a + (z-1)g(z),\, b + (z-1)h(z)\big\rangle_{H^2} + \big\langle g(z), h(z)\big\rangle_{H^2} \\ = \; & \langle f_1, f_2\rangle_{H^2} + \langle g, h\rangle_{H^2} \\ = \; & \langle f_1, f_2\rangle_{H^2} + \langle Q_1 f_1, Q_1 f_2\rangle_{H^2} \\ = \; & \langle f_1, f_2\rangle_{\mathcal{D}_1}. \end{aligned}$$

In short, we may write

$$\langle z^r f_1(z^s), z^\rho f_2(z^s)\rangle_{\mathcal{D}_\mu} = \delta_{r\rho}\langle f_1, f_2\rangle_{\mathcal{D}_1}.$$

We are now ready to present the orthogonal family of polynomials.

Theorem 7.1. *Let $c_k > 0$, $\zeta_k = e^{i2k\pi/s}$, $1 \le k \le s$, and consider the discrete measure*

$$d\mu = c_1 d\delta_{\zeta_1} + \cdots + c_s d\delta_{\zeta_s}.$$

Let

$$f_m(z) := p_m(z^s), \qquad (m \ge 0),$$

where p_m are the polynomials obtained in Theorem 4.1. Then the family

$$\{f_m,\, zf_m,\, \ldots,\, z^{s-1}f_m : m \ge 0\}$$

is an orthogonal basis for $\mathcal{D}_\mu$.

Since for each k,

$$f_m(\zeta_k) = p_m(\zeta_k^s) = p_m(1) = 1, \qquad (m \ge 0),$$

the above calculations also show that the family

$$\{f_m - f_0,\, z(f_m - f_0),\, \ldots,\, z^{s-1}(f_m - f_0) : m \ge 1\}$$

is an orthogonal basis for $\mathcal{D}_\mu(0)$. In this case, the orthogonality holds even under the inner product (6.7).

8. Concluding remarks

We end this note with some remarks and questions which are not fully understood yet.

(i) The classical orthogonal polynomials satisfy a differential equation, which usually stems from physical applications. Is there a differential equation for the orthogonal polynomials p_n obtained in Theorem 4.1?

(ii) We saw that

$$f_N = \hat{f}(0)p_0 + \sum_{n=1}^{N} \left(a_n \hat{f}(n) + \sum_{k=n+1}^{\infty} (a_k - a_{k-1}) \hat{f}(k) \right) \frac{p_n}{a_n a_{n-1}}$$

converges to f in $\mathcal{D}_1$-norm. Is it possible simplify the expression of f_N and write it more directly in terms of Taylor polynomials of f?

(iii) The properties of orthogonal polynomials given in Theorem 6.5 are not fully explored yet. It remains to find the recursive equation, differential equation (if any), the generating function, projection onto $\mathcal{P}_n$, etc.

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