WEIGHTED HOLOMORPHIC DIRICHLET SERIES AND COMPOSITION OPERATORS WITH POLYNOMIAL SYMBOLS

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Abstract

In this paper, we introduce a general class of weighted spaces of holomorphic Dirichlet series (with real frequencies) analytic in some half-plane and study composition operators on these spaces. In the particular case when the symbol inducing the composition operator is an affine function, we give criteria for boundedness and compactness. We also study the cyclicity property and as a byproduct give a characterization so that the direct sum of the identity plus a weighted forward shift operator on ℓ^2 is cyclic.

Contents

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1. Introduction

1.1. *Dirichlet series*
Let $\Lambda = (\lambda_n)_{n\geq 1}$ be a strictly increasing sequence of non-negative real numbers tending to ∞. A *Dirichlet series of type* Λ (Dirichlet series in short) is a series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \tag{1.1}$$

where z is a complex variable, and the coefficients of the series are given by a sequence $(a_n)_{n\geq 1}$ of complex numbers.

If $\lambda_n = n$, $n \geq 1$, then (1.1) are power series in $\zeta = e^{-z}$. When $\lambda_n = \log n$, $n \geq 1$, we get the so-called classical Dirichlet series, which have many applications in analytic number theory (see, e.g., [1]). Classical Dirichlet series also relate to several problems in functional analysis (see, e.g., [15] and references therein). We refer the reader to the monograph [14] or [26] for more detailed information on Dirichlet series.

The properties of convergence of Dirichlet series depend on two specific quantities that we introduce now. Fix $\Lambda = (\lambda_n)_{n\geq 1}$ a strictly increasing sequence of non-negative real numbers tending to ∞, and define

$$L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n}. \tag{1.2}$$

Now to each Dirichlet series f of type Λ given by (1.1) we may associate the following quantity

$$D_f = \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n}. \tag{1.3}$$

Note that the quantity D_f is specific to each Dirichlet series of type Λ with coefficients $(a_n)_{n\geq 1}$. Nevertheless, most of the time there is no confusion and we omit the reference to the associated Dirichlet series and write generally D.

It is known that if a Dirichlet series converges at some $w \in \mathbb{C}$, it converges for all z with $\Re(z) > \Re(w)$, and more precisely, for every fixed non-negative real number M, it converges uniformly in the region $\{\Re(z) \geq \Re(w), |z - w| \leq M(\Re(z) - \Re(w))\}$, see [26, page 5]. Let us denote by σ_c the abscissa of convergence of a Dirichlet series, which is defined as

$$\sigma_c = \inf \left\{ r \in \mathbb{R} : \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \text{ converges on } \mathbb{C}_r \right\},$$

where $\mathbb{C}_r := \{z : \Re(z) > r\}$. We also need the following extension of the definition of $\mathbb{C}_r$ for $r = \pm\infty$. By convention $\mathbb{C}_\infty = \emptyset$ and $\mathbb{C}_{-\infty} = \mathbb{C}$.

The abscissa σ_u of uniform convergence and σ_a of absolute convergence are defined in a similar way. So a Dirichlet series converges (resp. uniformly, absolutely) in the right half-plane $\mathbb{C}_{\sigma_c}$ (resp. $\mathbb{C}_{\sigma_u}$, $\mathbb{C}_{\sigma_a}$) to a holomorphic function.

In the case $L < \infty$ the three abscissae are related by the Valiron formula (see, e.g., [28])

$$D \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq D + L. \tag{1.4}$$

In particular, the Dirichlet series (1.1) represents an entire function if and only if $D = -\infty$.

1.2. *Composition operators*
Let X be a functional Banach space and suppose that all $f \in X$ have a common domain $G \subseteq \mathbb{C}$. Let φ be an analytic self-map on G. Then φ induces the *composition operator* C_φ on X defined by

$$C_\varphi(f) = f \circ \varphi, \qquad \forall f \in X.$$

The first natural and non trivial question is to know if C_φ maps X into itself, and if this is the case, what can be said about standard spectral properties of C_φ as an operator on X. There is a rich literature on this topic when X is the Hardy space, the Bergman space or the Dirichlet space (see for instance [4, 25]). The situation of spaces of classical Dirichlet series received also much attention after the founding papers of Gordon–Hedenmalm [12] and Bayart [3] (see for instance [2, 22, 23]). The case of general Dirichlet series is less studied. Nevertheless, when X is some weighted space of **entire** Dirichlet series of type Λ satisfying a certain property, properties of composition operators acting on X are quite well understood (see [9, 18, 19]). In [8], using Liouville's theorem, it is proved that if C_φ defines a bounded composition operator on a weighted Hilbert space of entire Dirichlet series, then φ must be an affine function.

The aim of this paper is to focus on the situation when our space of Dirichlet series is formed by functions which are holomorphic in **some half-plane** but not necessarily the whole plane. In this context, we could not apply Liouville's

theorem. Nevertheless, we will still focus on the case when φ is a polynomial (and then necessarily φ must be affine), since, for general symbols, we could not really hope that C_φ maps a space of Dirichlet series into itself because the restrictions on the sequence Λ generating the space will be too severe.

When $\varphi(z) = az + b$, we will characterize boundedness (see Section 3) and compactness (see Section 4) of C_φ on a weighted Dirichlet space $\mathcal{H}(\beta, \Lambda)$ (see next section for the definition). Then we will also study the dynamics properties (cyclicity and supercyclicity) in Section 5 of C_φ on $\mathcal{H}(\beta, \Lambda)$. In Section 6 we give some results about complex symmetry. In the final section we end with a note of how our results relate for weighted Hilbert spaces of entire Dirichlet series.

2. Weighted spaces of holomorphic Dirichlet series

Fix $\Lambda = (\lambda_n)_{n\geq 1}$ a strictly increasing sequence of non-negative real numbers tending to ∞ and satisfying $L < \infty$. To perform the main object of our investigation, we need the following result from [19].

LEMMA 2.1. *$L < \infty$ if and only if $\displaystyle\sum_{n=1}^{\infty} e^{-r\lambda_n} < \infty$ for all $r > L$. Furthermore, $\displaystyle\sum_{n=1}^{\infty} e^{-r\lambda_n} = \infty$ for any $r < L$.*

Let $\beta = (\beta_n)_{n\geq 1}$ be a sequence of positive real numbers. The sequence space

$$\ell^2_\beta = \left\{ (a_n)_{n\geq 1} \subset \mathbb{C} : \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

is a Hilbert space with inner product defined for $a = (a_n)_{n\geq 1}, b = (b_n)_{n\geq 1} \in \ell^2_\beta$ by

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \overline{b}_n \beta_n^2. \tag{2.1}$$

We put

$$\beta_* = \liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n}. \tag{2.2}$$

The following result serves as an important motivation of our investigation.

PROPOSITION 2.2. *Suppose $\beta_* \neq -\infty$. If $f(z) = \displaystyle\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ is a Dirichlet series of type Λ whose coefficients $(a_n)_{n\geq 1} \in \ell^2_\beta$, then $\sigma_a \leq \frac{L}{2} - \beta_*$. In particular, f is holomorphic in $\mathbb{C}_{\frac{L}{2}-\beta_*}$.*

PROOF. Suppose that $\beta_* \neq \infty$. Fix a compact subset K of $\mathbb{C}_{\frac{L}{2}-\beta_*}$. Then, there exists $\eta > \frac{L}{2} - \beta^*$ such that for every $z \in K$, we have $\Re(z) \geq \eta$. Thus

$$\sum_{n=1}^{\infty} \sup_{z\in K} \left|a_n e^{-\lambda_n z}\right| \leq \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n \eta}.$$

Apply the Cauchy-Schwarz inequality to get

$$\sum_{n=1}^{\infty} \sup_{z\in K} \left|a_n e^{-\lambda_n z}\right| \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{e^{-2\lambda_n \eta}}{\beta_n^2}\right)^{1/2}.$$

Since $\eta > \frac{L}{2} - \beta_*$, we can take $0 < \varepsilon < \delta := \eta - \left(\frac{L}{2} - \beta_*\right)$. By definition of β_*, there exists N such that for all $n \geq N$,

$$\frac{1}{\beta_n^2} \leq e^{-2\lambda_n\left(\beta_* - \frac{\varepsilon}{2}\right)}.$$

Applying Lemma 2.1 with $r = L + \delta$, we obtain

$$\sum_{n\geq N} \frac{1}{\beta_n^2} e^{-2\lambda_n \eta} \leq \sum_{n\geq N} e^{-2\lambda_n\left(\beta_* - \frac{\varepsilon}{2}\right)} e^{-2\lambda_n \eta} \leq \sum_{n\geq N} e^{-\lambda_n (L+\delta)} < \infty,$$

which proves that

$$\sum_{n=1}^{\infty} \sup_{z\in K} \left|a_n e^{-\lambda_n z}\right| < \infty.$$

Thus, f converges uniformly on K. Since this is valid for every compact K in $\mathbb{C}_{\frac{L}{2}-\beta_*}$, we deduce that f is analytic on $\mathbb{C}_{\frac{L}{2}-\beta_*}$.

The case when $\beta_* = \infty$ follows along the same lines with some tiny adjustments left to the reader. This completes the proof.

Proposition 2.2 leads us to the following definition of weighted spaces of holomorphic Dirichlet series in $\mathbb{C}_{\frac{L}{2}-\beta_*}$:

$$\mathcal{H}(\beta, \Lambda) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n)_{n\geq 1} \in \ell^2_\beta \right\}.$$

This is a Hilbert space with inner product inherited from (2.1). More precisely, the inner product on $\mathcal{H}(\beta, \Lambda)$ is defined as

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \overline{b}_n \beta_n^2, \tag{2.3}$$

for every $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$. Note that the inner product (2.3) is well defined because of the uniqueness property on

coefficients for Dirichlet series of type Λ, namely if $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = 0$, then $a_n = 0$ for all $n \geq 1$, see [26, page 8].

REMARK 2.3. The assumption $\beta_* > -\infty$ in Proposition 2.2 is important, otherwise $\mathbb{C}_{\frac{L}{2}-\beta_*} = \emptyset$. It is also essential because we can prove that if $\beta_* = -\infty$ then for every $z_0 \in \mathbb{C}$ there exists a Dirichlet series of type Λ, $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ with $(a_n)_{n\geq 1} \in \ell^2_\beta$, which does not converge at z_0. Indeed, there exists $(n_p)_{p\geq 1} \uparrow \infty$ large enough, such that

$$\beta^2_{n_p} < e^{-2\lambda_{n_p}\Re(z_0)}, \text{ for all } p \geq 1.$$

Take $(a_n)_{n\geq 1}$ as follows

$$a_n = \begin{cases} \frac{1}{p} e^{\lambda_{n_p} z_0}, & n = n_p \ (p = 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \leq \sum_{p=1}^{\infty} \frac{1}{p^2} e^{2\lambda_{n_p}\Re(z_0)} e^{-2\lambda_{n_p}\Re(z_0)} = \sum_{p=1}^{\infty} \frac{1}{p^2} < \infty.$$

However, at z_0 we have

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z_0} = \sum_{p=1}^{\infty} \frac{1}{p} e^{\lambda_{n_p} z_0} e^{-\lambda_{n_p} z_0} = \sum_{p=1}^{\infty} \frac{1}{p} = \infty.$$

Note that for the case $\beta_* = \infty$, we have entire Dirichlet series which have been studied quite well (see, e.g., [8] and related references). **Therefore, in the sequel we assume that the condition $\beta_* \neq \pm\infty$ always holds.**

In the rest of the paper, we adopt the following notation. For $n \geq 1$, $z \in \mathbb{C}$,

$$q_n(z) = \frac{1}{\beta_n} e^{-\lambda_n z}. \tag{2.4}$$

PROPOSITION 2.4. *The sequence $(q_n)_{n\geq 1}$ forms an orthonormal basis of $\mathcal{H}(\beta, \Lambda)$.*

PROOF. It is immediate from (2.3).

Note that the proof of Proposition 2.2 shows that for every point $w \in \mathbb{C}_{\frac{L}{2}-\beta_*}$, the evaluation functional δ_w is continuous on $\mathcal{H}(\beta, \Lambda)$. Furthermore,

using Proposition 2.4, we can compute the kernel k_w at point $w \in \mathbb{C}_{\frac{L}{2}-\beta_*}$ by

$$\begin{aligned} k_w(z) &= \sum_{n=1}^{\infty} \langle k_w, q_n \rangle q_n(z) = \sum_{n=1}^{\infty} \overline{q_n(w)} q_n(z) \\ &= \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} e^{-\lambda_n(z+\overline{w})}. \end{aligned}$$

Thus we obtain the following result.

PROPOSITION 2.5. *The spaces $\mathcal{H}(\beta, \Lambda)$ are all reproducing kernel Hilbert spaces with reproducing kernel*

$$K(z, w) = k_w(z) = \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} e^{-\lambda_n(z+\overline{w})}, \quad z, w \in \mathbb{C}_{\frac{L}{2}-\beta_*}. \tag{2.5}$$

In particular, we deduce the norm of the kernel

$$\|k_w\|^2 = \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} e^{-2\lambda_n \Re(w)}, \quad w \in \mathbb{C}_{\frac{L}{2}-\beta_*}. \tag{2.6}$$

3. Bounded composition operators induced by a polynomial

Since $\mathcal{H}(\beta, \Lambda)$ is a functional Hilbert space (in which evaluations are continuous), it follows easily from the closed graph theorem that the space $\mathcal{H}(\beta, \Lambda)$ is *invariant* under a composition operator C_φ, i.e. $C_\varphi\big(\mathcal{H}(\beta, \Lambda)\big) \subseteq \mathcal{H}(\beta, \Lambda)$, if and only if C_φ is bounded on $\mathcal{H}(\beta, \Lambda)$.

The study of boundedness is based on the following two simple lemmas. The first lemma is similar to one that appears in [19] for the case of entire Dirichlet series.

LEMMA 3.1. *Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ be a Dirichlet series with $L < \infty$ and $D \neq \infty$. If a_m is the first non-zero coefficient, then*

$$\lim_{\Re(z)\to\infty} e^{\lambda_m z} f(z) = a_m.$$

PROOF. Let $f(z) = \sum_{n=m}^{\infty} a_n e^{-\lambda_n z}$ be a Dirichlet series where $a_m \neq 0$. Then we have

$$e^{\lambda_m z} f(z) = \sum_{n \geq m} a_n e^{-(\lambda_n - \lambda_m) z}. \tag{3.1}$$

Notice that $\lambda_n - \lambda_m \geq 0$ for all $n \geq m$, so that the right-hand side of (3.1) is a Dirichlet series of type $(\lambda_n - \lambda_m)_{n \geq m}$. We claim that the associated abscissa of uniform convergence σ'_u satisfies $\sigma'_u \neq \infty$.

We show that the associated L' and D' to the Dirichlet series above satisfy $L' = L$ and $D' = D$. Indeed, since $\lim\limits_{n\to\infty} \dfrac{\lambda_n}{\lambda_n - \lambda_m} = 1$, it follows that

$$L' = \limsup_{n\to\infty} \frac{\log n}{\lambda_n - \lambda_m} = \limsup_{n\to\infty} \left(\frac{\log n}{\lambda_n} \cdot \frac{\lambda_n}{\lambda_n - \lambda_m} \right) = L < \infty.$$

and similarly $D' = D$.

Since $L < \infty$ and $D \neq \infty$, it follows from (1.4) that $\sigma'_u \neq \infty$, so that $e^{\lambda_m z} f(z)$ is uniformly convergent in some half-plane. Therefore we may interchange limit and sum to obtain

$$\begin{aligned} \lim_{\Re(z)\to\infty} e^{\lambda_m z} f(z) &= \lim_{\Re(z)\to\infty} \sum_{n\geq m} a_n e^{-(\lambda_n - \lambda_m)z} \\ &= \sum_{n\geq m} \lim_{\Re(z)\to\infty} \left(a_n e^{-(\lambda_n - \lambda_m)z} \right) = a_m. \end{aligned}$$

This proves the lemma.

When we study the composition operator C_φ on a functional Hilbert space of analytic functions on a domain Ω, the first natural condition to require is that the symbol φ is an analytic self-map of Ω. The second lemma characterizes the polynomials which are self-map of $\mathbb{C}_\theta$.

LEMMA 3.2. *Let $\theta \in \mathbb{R}$ and let φ be a polynomial. Then φ is a self-map of the half-plane $\mathbb{C}_\theta$ if and only if $\varphi(z) = az + b$ and*

(i) *either $a \in \mathbb{R}^+$ and $\Re(b) \geq (1-a)\theta$*
(ii) *or $a = 0$ and $\Re(b) > \theta$.*

PROOF. Let us first prove that a polynomial of degree $d \geq 2$ is never a self-map of $\mathbb{C}_\theta$. Indeed, if $\varphi(z) = a_d z^d + \ldots$, with $d \geq 2$ and $a_d = \rho e^{it}$, $\rho \in \mathbb{R}_+$ and $t \in [0, 2\pi)$. Assume first that $t \in (0, 2\pi)$ and consider $z(R) = Re^{i\frac{\pi - t}{d}}$, $R > 0$. Since

$$-\frac{\pi}{d} < \frac{\pi - t}{d} < \frac{\pi}{d},$$

and $0 < \frac{\pi}{d} \leq \frac{\pi}{2}$, we get $\cos(\frac{\pi - t}{d}) > 0$ and then the point $z(R)$ belongs to $\mathbb{C}_\theta$ for R large enough. On the other hand, $\varphi(z(R)) \sim_{R\to\infty} a_d z(R)^d$ and

$$a_d z(R)^d = \rho R^d e^{it} e^{\pi - t} = -\rho R^d \to -\infty, \qquad \text{as } R \to \infty.$$

Hence, for R large enough, $\varphi(z(R))$ cannot belong to $\mathbb{C}_\theta$.

Now if $t = 0$, that is $a_d = \rho > 0$, then choose t_0, t_1, α such that

$$\frac{\pi}{2d} < t_0 < \alpha < t_1 < \min\left(\frac{\pi}{2}, \frac{3\pi}{2d} \right),$$

and consider $z(R) = Re^{i\alpha}$. As before, when R is large enough, $z(R)$ belongs to $\mathbb{C}_\theta$ (because $\cos(\alpha) > 0$), and $\varphi(z(R)) \sim_{R\to\infty} \rho R^d e^{id\alpha}$. Observe now that $\frac{\pi}{2} < d\alpha < \frac{3\pi}{2}$, whence $\cos(d\alpha) < 0$, and for R large enough, $\varphi(z(R))$ cannot belong to $\mathbb{C}_\theta$.

We now may suppose that $\varphi(z) = az + b$, with $a, b \in \mathbb{C}$. Suppose $az + b$ is a self-map on $\mathbb{C}_\theta$. If $a = 0$, then it clear that $\Re(b) > \theta$, because $b \in \mathbb{C}_\theta$. So we get (ii).

In case $a \neq 0$, we show that $a \in \mathbb{R}^+$. Assume to the contrary that $a \notin \mathbb{R}^+$, which means that $a = |a|e^{i\theta_1}$, with $\theta_1 \neq 0 \mod (2\pi)$. For every $z = x + iy$, $x, y \in \mathbb{R}$, we have $\Re(az) = x|a|\cos(\theta_1) - y|a|\sin(\theta_1)$.

Case 1: If $\theta_1 = \pi \mod (2\pi)$, then $\Re(az) = -x|a|$, and so letting $\Re(z) = x \to \infty$, we get that $\Re(az + b) \to -\infty$, which contradicts that $az + b$ is a self-map of $\mathbb{C}_\theta$.

Case 2: If $\theta_1 \neq \pi \mod (2\pi)$, then $\sin(\theta_1) \neq 0$. Thus if we fix $x > \theta$ and let $\Im(z) = y \to \infty$ (if $\sin(\theta_1) > 0$) or $\Im(z) = y \to -\infty$ (if $\sin(\theta_1) < 0$), we get that $\Re(az + b) \to -\infty$, which also contradicts the assumption.

Thus we have $a \in \mathbb{R}^+$. Then for any $\varepsilon > 0$, a point $z \in \mathbb{C}_\theta$ with $\Re(z) = \theta + \varepsilon$ satisfies $\Re(az + b) = a\theta + a\varepsilon + \Re(b) > \theta$. This means $\Re(b) > (1 - a)\theta - a\varepsilon$. Letting $\varepsilon \to 0$, we obtain (i).

Conversely, if (ii) holds, i.e. $a = 0$ and $\Re(b) > \theta$, then we are obviously done. If (i) holds, i.e. $a \in \mathbb{R}^+$ and $\Re(b) \geq (1 - a)\theta$, then for any $z \in \mathbb{C}_\theta$, we have $\Re(az + b) = a\Re(z) + \Re(b) > a\theta + \Re(b) \geq a\theta + (1 - a)\theta = \theta$, which shows that $az + b \in \mathbb{C}_\theta$. Thus $az + b$ is a self-map of $\mathbb{C}_\theta$. This proves the lemma.

3.1. *Necessary conditions for boundedness of C_φ: a polynomial symbol*
In this subsection, we obtain an important necessary condition which a polynomial symbol φ inducing a bounded composition operator C_φ must satisfy. See [8, 19] for analogous results in the entire case. First, note that in the case when $\beta_* \neq \pm\infty$, all $f \in \mathcal{H}(\beta, \Lambda)$ converge on the proper right half-plane $\mathbb{C}_{\frac{L}{2} - \beta_*}$ and in particular, satisfy $D_f \neq \infty$ (by Valiron's formulae). Hence Lemma 3.1 can be applied to elements of $\mathcal{H}(\beta, \Lambda)$.

THEOREM 3.3. *Let $\beta_* \neq \pm\infty$ and φ be a self-map polynomial of $\mathbb{C}_{\frac{L}{2} - \beta_*}$. Suppose that the operator C_φ is bounded on $\mathcal{H}(\beta, \Lambda)$. Then the following assertions hold:*

1. *If $\lambda_1 > 0$, then $\varphi(z) = az + b,\ a \geq 1,\ b \in \mathbb{C}$.*
2. *If $\lambda_1 = 0$, then either $\varphi(z) = b,\ b \in \mathbb{C}$ (that is $a = 0$), or $\varphi(z) = az + b,\ a \geq 1,\ b \in \mathbb{C}$.*

Moreover, in addition, a satisfies the condition

$$\lambda_{m_k} = a\lambda_k, \quad \text{for all } k \in \mathbb{N}, \tag{3.2}$$

where m_k is the index of the first non-zero term of $C_{az+b}(e^{-\lambda_k z}) = e^{-\lambda_k(az+b)}$ in the Dirichlet series representation (1.1)*, and b satisfies either of the following conditions:*

$$\Re(b) > \frac{L}{2} - \beta_*, \quad a = 0, \tag{3.3}$$

$$\Re(b) \geq (1-a)\Big(\frac{L}{2} - \beta_*\Big), \quad a \geq 1. \tag{3.4}$$

Proof. By Lemma 3.2, φ is an affine function $\varphi(z) = az + b$, with either $a \in \mathbb{R}_+$ and $\Re(b) \geq (1-a)\Big(\frac{L}{2} - \beta_*\Big)$, or $a = 0$ and $\Re(b) > \frac{L}{2} - \beta_*$.

- Case 1: $\boldsymbol{\lambda_1 > 0}$. For each $k \in \mathbb{N}$, note that

$$C_\varphi(e^{-\lambda_k z}) = e^{-\lambda_k \varphi(z)} = e^{-a\lambda_k z} e^{-b\lambda_k}, \tag{3.5}$$

and since $C_\varphi(e^{-\lambda_k z}) \in \mathcal{H}(\beta, \Lambda)$, we can represent it as

$$C_\varphi(e^{-\lambda_k z}) = \sum_{n \geq m_k} b_n^{(k)} e^{-\lambda_n z},$$

where, by assumption, m_k is the smallest integer such that $b_{m_k}^{(k)} \neq 0$, and $(b_n^{(k)})_n \in \ell^2_\beta$. By Lemma (3.1), we have

$$\lim_{\Re(z)\to\infty} e^{\lambda_{m_k} z} C_\varphi(e^{-\lambda_k z}) = b_{m_k}^{(k)}.$$

Thus, according to (3.5), we get

$$\lim_{\Re(z)\to\infty} e^{(\lambda_{m_k} - a\lambda_k)z} = b_{m_k}^{(k)} e^{b\lambda_k} \neq 0.$$

That necessarily implies $\lambda_{m_k} - a\lambda_k = 0$, which gives (3.2). In particular, for $k = 1$, we have $a = \dfrac{\lambda_{m_1}}{\lambda_1} \geq 1$, because $m_1 \geq 1$.

- Case 2: $\boldsymbol{\lambda_1 = 0}$. In this case, replacing $k \in \mathbb{N}$ by $k \geq 2$ in the proof for Case 1 above, we still have $a = \dfrac{\lambda_{m_k}}{\lambda_k}$, for all $k \geq 2$. In particular, for $k = 2$, we have

$$a = \frac{\lambda_{m_2}}{\lambda_2} = \begin{cases} 0, & \text{if } m_2 = 1 \text{ (because } \lambda_1 = 0), \\ \geq 1, & \text{if } m_2 \geq 2 \text{ (because } \lambda_{m_2} \geq \lambda_2). \end{cases}$$

Finally, in that case, we have $m_1 = 1$ and thus $\lambda_{m_1} = 0 = a\lambda_1$. This completes the proof.

Remark 3.4. If $a \geq 1$, it follows immediately from (3.2) that the map $k \longmapsto m_k$ is strictly increasing.

From now on, an affine symbol φ stated in Theorem 3.3 is supposed to be given.

3.2. *Two trivial cases of the affine symbols ($a = 0, 1$)*
In the following, we denote by $\|\cdot\|_{\mathrm{op}}$ the operator norm.

PROPOSITION 3.5. *Let $\beta_* \neq \pm\infty$ and $b \in \mathbb{C}_{\frac{L}{2}-\beta_*}$. Then C_b is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$ if and only if $\lambda_1 = 0$. Moreover, in that case, we have $\|C_b\|_{\mathrm{op}} = \beta_1 \|k_b\|$.*

PROOF. Suppose that C_b is a bounded on $\mathcal{H}(\beta, \Lambda)$. By Theorem 3.3, λ_1 must necessarily be zero.

Conversely, suppose that $\lambda_1 = 0$. It implies that the function $\beta_1 q_1(z) = e^{-\lambda_1 z} = 1$ belongs to $\mathcal{H}(\beta, \Lambda)$. (Recall that the functions q_k are given by (2.4).) Now, note that for all $f \in \mathcal{H}(\beta, \Lambda)$, we have

$$(C_b f)(z) = f(b) = \langle f, k_b \rangle \beta_1 q_1(z),$$

Hence $C_b = \beta_1 q_1 \otimes k_b$ is a rank one operator. In particular, it is bounded and $\|C_b\|_{\mathrm{op}} = \beta_1 \|q_1\| \|k_b\|$. But $\|q_1\| = 1$, which implies that $\|C_b\|_{\mathrm{op}} = \beta_1 \|k_b\|$, as claimed.

PROPOSITION 3.6. *Let $\beta_* \neq \pm\infty$. Then C_{z+b} is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$ if and only if $\Re(b) \geq 0$. Moreover, in that case, we have $\|C_{z+b}\|_{\mathrm{op}} = e^{-\lambda_1 \Re(b)}$.*

PROOF. Suppose C_{z+b} is bounded. Then by (3.4), we get immediately that $\Re(b) \geq 0$. Conversely, suppose that $\Re(b) \geq 0$. Note that for every $k \in \mathbb{N}$, $C_{z+b} q_k = e^{-\lambda_k b} q_k$. Thus C_{z+b} is a diagonal operator with a sequence of eigenvalues equal to $(e^{-\lambda_k b})_{k \geq 1}$. Using now that $\Re(b) \geq 0$, we see that this sequence is decreasing in modulus, and so it is well-known (and easy to see) that C_{z+b} is bounded and $\|C_{z+b}\|_{\mathrm{op}} = e^{-\lambda_1 \Re(b)}$, as claimed.

3.3. *Characterization for boundedness of C_{az+b} ($a \neq 0, 1$)*
Throughout this subsection, we always assume that an affine symbol $\varphi(z) = az + b$, with $a > 1$ and $b \in \mathbb{C}$, is given.

We need some supplementary notation.

DEFINITION 3.7. For a given sequence of real frequencies $\Lambda = (\lambda_n)_{n \geq 1}$, define the set

$$\mathcal{R}(\Lambda) = \{r \in [1, \infty) : \forall n \in \mathbb{N},\ \exists m = m_n \geq n,\ r\lambda_n = \lambda_m\}.$$

If there exists a nontrivial $\varphi(z) = az + b$ that induces a bounded composition operator C_φ on $\mathcal{H}(\beta, \Lambda)$, then a belongs to $\mathcal{R}(\Lambda)$.

REMARK 3.8. Notice that we always have $1 \in \mathcal{R}(\Lambda)$. Furthermore, since $(\lambda_n)_{n \geq 1}$ is a strictly increasing sequence, for a given $r \in \mathcal{R}(\Lambda)$, to each n there corresponds a unique $m_n \geq n$, such that $r\lambda_n = \lambda_{m_n}$.

We put $\mathcal{R}_1(\Lambda) = \mathcal{R}(\Lambda) \setminus \{1\}$. Depending on the given sequence $(\lambda_n)_{n\geq 1}$, it may happen that $\mathcal{R}(\Lambda) = \{1\}$, i.e. $\mathcal{R}_1(\Lambda) = \emptyset$, as well as $\mathcal{R}_1(\Lambda) \neq \emptyset$. The following examples are taken from [18].

EXAMPLE 3.9. Let $\lambda_n = n!$. Then $\mathcal{R}(\Lambda)$ is the singleton $\{1\}$.

EXAMPLE 3.10. (1) For $\lambda_n = \log n$ (the classical Dirichlet series) or $\lambda_n = n$, every $\ell \in \mathbb{N}$ belongs to $\mathcal{R}(\Lambda)$.

(2) Consider a geometric sequence (λ_n) with the ratio $q > 1$, given by $\lambda_1 > 0$ and $\lambda_n = \lambda_1 q^{n-1}$, $n \geq 1$. In this case any value q^ℓ ($\ell \in \mathbb{N}$) belongs to $\mathcal{R}(\Lambda)$.

Now let $a \in \mathcal{R}_1(\Lambda)$ be given. In principle, for each $n \in \mathbb{N}$, the index $m_n \geq n$ for which $\lambda_{m_n} = a\lambda_n$, depends on a, i.e. $m_n = m_n^{(a)}$ (note that by Theorem 3.3, if $\lambda_1 = 0$, then $m_1 = 1$). To simplify the expositions, in what follows, we skip the superscript (a) whenever there is no confusion in context.

For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we also define the quantity

$$r_n(a, z) = r_n(\Lambda, \beta, a, z) := e^{-\lambda_n z} \frac{\beta_{m_n}}{\beta_n}.$$

Now we are able to state and prove the following boundedness criterion for the case $a > 1$.

PROPOSITION 3.11. *Let $\beta_* \neq \pm\infty$, $a > 1$ and $b \in \mathbb{C}$. Then C_{az+b} is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$ if and only if the following conditions are satisfied*

(1) $a \in \mathcal{R}_1(\Lambda)$,

(2) $\Re(b) \geq (1-a)\Big(\dfrac{L}{2} - \beta_*\Big)$,

(3) *the sequence* $\Big(r_n(a, \Re(b))\Big)_{n\geq 1}$ *is bounded.*

Moreover, in this case,

$$\|C_{az+b}\|_{\mathrm{op}} = \sup_{n\in\mathbb{N}} r_n(a, \Re(b)).$$

PROOF. *Necessity.* Suppose C_{az+b} is a bounded operator on $\mathcal{H}(\beta, \Lambda)$. Then conditions (1) and (2) follow from Theorem 3.3. It remains to show (3).

There is some constant $M > 0$ such that $\|C_{az+b}f\| \leq M\|f\|$ for all $f \in \mathcal{H}(\beta, \Lambda)$. In particular, for probe functions $q_n(z) = \frac{1}{\beta_n} e^{-\lambda_n z}$, by (1) we have $C_{az+b}q_n = \frac{\beta_{m_n}}{\beta_n} e^{-\lambda_n b} q_{m_n}$, and thus

$$\|C_{az+b}q_n\| = e^{-\lambda_n \Re(b)} \cdot \frac{\beta_{m_n}}{\beta_n} \leq M\|q_n\| = M, \quad \text{for all } n \in \mathbb{N},$$

which gives (3).

Sufficiency. Conversely, suppose all three conditions (1) – (3) are satisfied. Note that (2) guarantees, by Lemma 3.2, that $z \mapsto az + b$ is a self-map of $\mathbb{C}_{\frac{L}{2}-\beta_*}$. Also, (3) shows that there exists $M > 0$ such that $0 < r_n(a, \Re(b)) \leq M$, for all n. Let $f(z) = \sum_{k=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$. Hence, by (1),

$$\begin{aligned} \|C_{az+b}f\|^2 &= \sum_{n=1}^{\infty} |a_n e^{-\lambda_n b}|^2 \beta_{m_n}^2 = \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 r_n(a, \Re(b))^2 \\ &\leq M^2 \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 = M^2 \|f\|^2, \end{aligned}$$

which shows that C_{az+b} is bounded on $\mathcal{H}(\beta, \Lambda)$.

Moreover, from proofs of both necessity and sufficiency it follows that $\|C_{az+b}\|_{\text{op}} = \sup_{n\in\mathbb{N}} r_n(a, \Re(b))$. This completes the proof.

Combining Propositions 3.5, 3.6 and 3.11, we obtain a characterization of boundedness for C_{az+b} on $\mathcal{H}(\beta, \Lambda)$.

THEOREM 3.12. *Let $\beta_* \neq \pm\infty$ and $\varphi(z) = az + b$ $(a, b \in \mathbb{C})$. Consider the following statements.*

(1) *$\varphi(z) = b$ for some $b \in \mathbb{C}$ with $\Re(b) > \dfrac{L}{2} - \beta_*$.*

(2) *$\varphi(z) = az + b$, where*

$$\begin{cases} a \in \mathcal{R}(\Lambda), \\ \Re(b) \geq (1-a)\Big(\dfrac{L}{2} - \beta_*\Big), \\ \text{the sequence } \Big(r_n(a, \Re(b))\Big)_{n\geq 1} \text{ is bounded.} \end{cases}$$

The following are true for a composition operator C_φ acting on the space $\mathcal{H}(\beta, \Lambda)$.

(i) *If $\lambda_1 = 0$, then C_φ is bounded if and only if either (1) or (2) holds.*

(ii) *If $\lambda_1 > 0$, then C_φ is bounded if and only if (2) holds.*

4. Essential norm, compactness, Schatten class and compact differences

4.1. *Essential norm and compactness*

Compactness of a bounded composition operator on a Hilbert space $\mathcal{H}(\beta, \Lambda)$ can be investigated in different ways. The first makes use of a compactness criterion which states that a bounded linear operator T on $\mathcal{H}(\beta, \Lambda)$ is compact if and only if for any sequence (f_n) from $\mathcal{H}(\beta, \Lambda)$ which is weakly convergent

to 0, the sequence (Tf_n) converges strongly to 0 in $\mathcal{H}(\beta,\Lambda)$. The other way is via the essential norm of T defined by

$$\|T\|_e = \inf\{\|T-K\|_{\rm op} : K \text{ is a compact operator on } \mathcal{H}(\beta,\Lambda)\}.$$

Clearly, T is compact if and only if $\|T\|_e = 0$.

For a bounded composition operator C_{az+b}, in case $a=0$, by Proposition 3.5, we have the following simple result.

PROPOSITION 4.1. *Let $\beta_* \neq \pm\infty$. For any $b \in \mathbb{C}_{\frac{L}{2}-\beta_*}$, a composition operator C_b is always compact on the space $\mathcal{H}(\beta,\Lambda)$.*

PROOF. As noted in Proposition 3.5, C_b has rank one and hence it is compact.

PROPOSITION 4.2. *Let $\beta_* \neq \pm\infty$ and let C_{z+b} be a bounded composition operator on $\mathcal{H}(\beta,\Lambda)$. Then*

$$\|C_{z+b}\|_e = \begin{cases} 1 & \text{if } \Re(b) = 0 \\ 0 & \text{if } \Re(b) > 0. \end{cases}$$

In particular, C_{z+b} is a compact operator on $\mathcal{H}(\beta,\Lambda)$ if and only if $\Re(b) > 0$.

PROOF. Proposition 3.6 shows that $\Re(b) \geq 0$. Furthermore, as already noticed, C_{z+b} is a diagonal operator with a sequence of eigenvalues equal to $(e^{\lambda_k b})_{k\geq 1}$. But then it is well-known (see for instance [13, Problem 171]) that $\|C_{z+b}\|_e = \lim\limits_{n\to\infty} e^{-\lambda_n \Re(b)}$, which gives the desired conclusion.

Proposition 4.2 can also be obtained from the following results for $a \geq 1$.

THEOREM 4.3. *Let $\beta_* \neq \pm\infty$, $a \geq 1$ and C_{az+b} be a bounded composition operator on the space $\mathcal{H}(\beta,\Lambda)$. Then $\|C_{az+b}\|_e = \limsup\limits_{n\to\infty} r_n(a,\Re(b))$. In particular, C_{az+b} is compact if and only if $\lim\limits_{n\to\infty} r_n(a,\Re(b)) = 0$.*

PROOF. We follow the standard technique (see, e.g., [6]).

Note that by Theorem 3.12, the sequence $\Big(r_n(a,\Re(b))\Big)_{n\geq 1}$ is bounded.

• **Upper bound**. We use compact (finite rank) operators on $\mathcal{H}(\beta,\Lambda)$ defined by

$$K_N\colon f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \longmapsto \sum_{n=1}^{N} a_n e^{-\lambda_n z} \quad (N \in \mathbb{N}).$$

As the $C_{az+b}K_N$ are also compact, we have

$$\|C_{az+b}\|_e \leq \inf_{N\geq 1} \|C_{az+b} - C_{az+b}K_N\|_{\rm op}.$$

Observe that for an arbitrary $f(z)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\in\mathcal{H}(\beta,\Lambda)$ we have

$$\begin{aligned}\|(C_{az+b}-C_{az+b}K_N)f\|^2 &= \Big\|\sum_{n=N+1}^{\infty} a_n e^{-\lambda_n(az+b)}\Big\|^2\\ &= \sum_{n=N+1}^{\infty}|a_n|^2 e^{-2\lambda_n\Re(b)}\beta_{m_n}^2 \le \sup_{n\ge N+1}\Big(e^{-\lambda_n\Re(b)}\frac{\beta_{m_n}}{\beta_n}\Big)^2\sum_{n=N+1}^{\infty}|a_n|^2\beta_n^2\\ &\le \|f\|^2\sup_{n\ge N+1} r_n(a,\Re(b)).\end{aligned}$$

Thus

$$\|C_{az+b}\|_e \le \sup_{n\ge N+1} r_n(a,\Re(b)),\quad \text{for every } N\ge 1.$$

Letting $N\to\infty$, we get

$$\|C_{az+b}\|_e \le \limsup_{n\to\infty} r_n(a,\Re(b)).$$

• **Lower bound**. Let K be an arbitrary compact operator on $\mathcal{H}(\beta,\Lambda)$. Consider the sequence of probe functions $(q_n)_{n\ge1}$, whose norms are all 1. Since it converges weakly to 0 (because it is an orthonormal basis), $\lim_{n\to\infty}\|Kq_n\|=0$. Hence

$$\|C_{az+b}-K\|_{\mathrm{op}} \ge \|(C_{az+b}-K)q_n\| \ge \|C_{az+b}q_n\| - \|Kq_n\| \quad (n\ge1),$$

from which it follows that

$$\begin{aligned}\|C_{az+b}-K\|_{\mathrm{op}} &\ge \limsup_{n\to\infty}\left(\|C_{az+b}q_n\|-\|Kq_n\|\right)=\limsup_{n\to\infty}\|C_{az+b}q_n\|\\ &= \limsup_{n\to\infty}\left(e^{-\lambda_n\Re(b)}\frac{\beta_{m_n}}{\beta_n}\right)=\limsup_{n\to\infty} r_n(a,\Re(b)).\end{aligned}$$

Taking the infimum over all compact operators K on $\mathcal{H}(\beta,\Lambda)$, we obtain

$$\|C_{az+b}-K\|_e \ge \limsup_{n\to\infty} r_n(a,\Re(b)).$$

This completes the proof.

4.2. *Schatten class*

Recall that a bounded linear operator T on $\mathcal{H}(\beta,\Lambda)$ is called a Hilbert–Schmidt operator if it has finite Hilbert–Schmidt norm $\|T\|_{\mathrm{HS}}$, which means that for some orthonormal basis $(e_n)_{n\ge1}$ of $\mathcal{H}(\beta,\Lambda)$, we have

$$\|T\|_{\mathrm{HS}} := \Big(\sum_{n\in\mathbb{N}}\|Te_n\|^2\Big)^{1/2} < +\infty.$$

It is well known that $\|T\|_{\mathrm{HS}}$ does not depend on the choice of the orthonormal basis and that if T is Hilbert–Schmidt, then it is compact.

Furthermore, for $0 < p < \infty$, the Schatten p-class consists of all bounded linear operators T on $\mathcal{H}(\beta, \Lambda)$ for which $(T^*T)^{p/4}$ is a Hilbert–Schmidt operator (here T^* is the adjoint operator of T). The set of Schatten p-class operators forms an ideal in the algebra of all bounded linear operators on $\mathcal{H}(\beta, \Lambda)$. If T is diagonal with respect to an orthonormal basis $(e_n)_{n\geq 1}$, that is, $Te_n = a_n\, e_n$ for all $n \geq 1$, then it is well known that T belongs to the Schatten p-class if and only if $\displaystyle\sum_{n=1}^{\infty} |a_n|^p < \infty$. Some developments on operators in Schatten classes can be found in [21].

We study a Schatten class membership of C_{az+b}. For a bounded operator C_{az+b} on $\mathcal{H}(\beta, \Lambda)$, since $C_{az+b}q_n = \frac{\beta_{m_n}}{\beta_n} e^{-\lambda_n b} q_{m_n}$, therefore $\|C_{az+b}q_n\| = e^{-\lambda_n \Re(b)}.\frac{\beta_{m_n}}{\beta_n} = r_n(a, \Re(b))$, and we get an immediate result about its Hilbert–Schmidt property.

Proposition 4.4. *Let $\beta_* \neq \pm\infty$. A bounded composition operator C_{az+b} on $\mathcal{H}(\beta, \Lambda)$ is a Hilbert–Schmidt operator if and only if*

$$\sum_{n=1}^{\infty} r_n(a, \Re(b))^2 < \infty.$$

Concerning the Schatten p-class membership, for two trivial cases $a = 0$ and $a = 1$, some results can be obtained easily.

Proposition 4.5. *Let $0 < p < \infty$. A bounded composition operator C_b on $\mathcal{H}(\beta, \Lambda)$ belongs to the Schatten p-class.*

Proof. Since C_b is of rank one, in particular it is of finite rank and so has finitely many non-zero singular values. Done.

Also for the case $a = 1$, as noted in Proposition 3.6, C_{z+b} is a diagonal operator with a sequence of eigenvalues $(e^{-\lambda_n b})_{n\geq 1}$ corresponding to eigenvectors $(q_n)_{n\geq 1}$, and thus we have the following result.

Proposition 4.6. *Let $0 < p < \infty$ and $\beta_* \neq \pm\infty$. A bounded composition operator C_{z+b} on $\mathcal{H}(\beta, \Lambda)$ belongs to the Schatten p-class if and only if*

$$\sum_{n=1}^{\infty} e^{-p\lambda_n \Re(b)} < \infty.$$

In particular, C_{z+b} belongs to the Schatten p-class if $p\Re(b) > L$, and does not belong if $p\Re(b) < L$, where L is defined in (1.2).

Note that the second statement of the proposition above follows from Lemma 2.1.

To go further and get a similar result for $a > 1$, we need to compute the adjoint of C_{az+b}, which can be done easily. Indeed, for any $g(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$, we write $(C^*_{az+b}g)(z) = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}$ and have

$$d_n\beta_n = \langle C^*_{az+b}g, q_n\rangle = \langle g, C_{az+b}q_n\rangle = \Big\langle \sum_{k=1}^{\infty} a_k e^{-\lambda_k z}, C_{az+b}q_n\Big\rangle. \tag{4.1}$$

The terms $\langle e^{-\lambda_k z}, C_{az+b}q_n\rangle$ are computed in two cases:

- Case 1: $a = 0$. By Proposition 3.5, $\lambda_1 = 0$ and hence

$$\langle e^{-\lambda_k z}, C_b q_n\rangle = \begin{cases} \frac{e^{-\lambda_n \bar{b}}}{\beta_n}\beta_1^2, & k = 1 \\ 0, & k > 1. \end{cases}$$

- Case 2: $a \geq 1$. By Theorem 3.3, we have

$$\langle e^{-\lambda_k z}, C_{az+b}q_n\rangle = \begin{cases} \frac{e^{-\lambda_n \bar{b}}}{\beta_n}\beta_{m_n}^2, & m_n = k \\ 0, & m_n \neq k. \end{cases}$$

Substituting back these equations into (4.1) yields the following result.

Proposition 4.7. *Let $\beta_* \neq \pm\infty$, and let C_{az+b} be a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$. If $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$, then*

$$(C^*_{az+b}f)(z) = \begin{cases} a_1\beta_1^2 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n \bar{b}}}{\beta_n^2} e^{-\lambda_n z}, & a = 0 \\ \sum_{n=1}^{\infty} a_{m_n} \frac{\beta_{m_n}^2 e^{-\lambda_n \bar{b}}}{\beta_n^2} e^{-\lambda_n z}, & a \geq 1. \end{cases} \tag{4.2}$$

Proposition 4.8. *Let $a \geq 1$. The eigenvalues of $C^*_{az+b}C_{az+b}$ are precisely $r_k(a, \Re(b))^2$, $k \in \mathbb{N}$.*

Proof. We appeal to Proposition 4.7 which provides an explicit formula for the adjoint. Let $g_k = \beta_k q_k$ for all $k \in \mathbb{N}$. We have $(C_{az+b}g_k)(z) =$

$e^{-\lambda_k b}e^{-\lambda_{m_k} z}$. It follows that

$$(C^*_{az+b}C_{az+b}g_k)(z) = e^{-\lambda_k b}\frac{\beta^2_{m_k}}{\beta^2_k}e^{-\lambda_k \overline{b}}e^{-\lambda_k z} = e^{-2\lambda_k \Re(b)}\frac{\beta^2_{m_k}}{\beta^2_k}g_k(z)$$
$$= r_k(a,\Re(b))^2 g_k(z).$$

Now, recall that $(g_k)_{k\geq 1}$ forms an orthogonal basis of $\mathcal{H}(\beta,\Lambda)$. It follows that the only eigenvalues are precisely those corresponding to these vectors, i.e. precisely $r_k(a,\Re(b))^2$, $k\in\mathbb{N}$, as claimed.

As $r_k(a,\Re(b))>0$ for every k, we have the following corollary.

COROLLARY 4.9. *The eigenvalues of $|C_{az+b}|$ are precisely the values $r_k(a,\Re(b))$, $k\in\mathbb{N}$.*

As a consequence by direct substitution, we thus have the following theorem.

THEOREM 4.10. *Let $0<p<\infty$ and $a\geq 1$. Then, C_{az+b} is a pth Schatten class operator if and only if*

$$\sum_{k=1}^{\infty} r_k(a,\Re(b))^p < \infty.$$

REMARK 4.11. Note that the series

$$h_p(z) = \sum_{k=1}^{\infty}\frac{\beta^p_{m_k}}{\beta^p_k}e^{-p\lambda_k z},$$

is a Dirichlet series of type $(p\lambda_n)$, which can be considered as a "complex version" of the series $\displaystyle\sum_{k=1}^{\infty} r_k(a,x)^p$ when we replace $x\in\mathbb{R}$ by $z\in\mathbb{C}$. By [14, Chapter II, Theorem 8] the computation of σ_c is

$$\sigma_c = \limsup_{n\to\infty}\frac{\log\left(\displaystyle\sum_{k=1}^{n}\frac{\beta^p_{m_k}}{\beta^p_k}\right)}{p\lambda_n}.$$

Thus, if $\Re(b)$ is larger than this value then $h_p(\Re(b))$ converges and C_{az+b} is Schatten p-class. If $\Re(b)$ is smaller, then C_{az+b} is not Schatten p-class.

REMARK 4.12. Note also that it is possible that the membership to the p-th Schatten class of C_{az+b} does not rely on p at all. Pick $a>1$ and $\Lambda=(a^k)_{k\geq 1}$. In this case, $L=0$. Then pick $\beta=\left(\displaystyle\prod_{i=1}^{k-1}e^{a^i}\right)_{k\geq 1}$, for which $\beta_*=\dfrac{1}{a^2-a}$ and C_{az+b} is a well-defined bounded composition operator on $\mathcal{H}(\beta,\Lambda)$. The real

function $h_p(x) = \sum_{k=1}^{\infty} e^{pa^k} e^{-pa^k x} = \sum_{k=1}^{\infty} e^{-pa^k(x-1)}$ converges for all $x > 1$ and diverges everywhere else, regardless of the value of p. It follows that C_{az+b} is a Schatten p-class operator if and only if $\Re(b) > 1$.

4.3. *Compact differences*
In this section we determine when a difference of two bounded composition operators $C_{\varphi_1} - C_{\varphi_2}$ is a compact operator. Recall that the set of compact operators form a vector space. Hence by Proposition 4.1 and Theorem 4.3, it suffices to consider only the case when φ_1 and φ_2 are non-constant and the associated sequences (r_n) do not have limit 0.

LEMMA 4.13. *Let $\beta_* \neq \pm\infty$, $a, a' \geq 1$. Let C_{az+b} and $C_{a'z+b'}$ be bounded composition operators on $\mathcal{H}(\beta, \Lambda)$. Assume that*

$$\limsup_{n\to\infty} r_n(a, \Re(b)) > 0 \quad \textit{and} \quad \limsup_{n\to\infty} r_n(a', \Re(b')) > 0.$$

If $C_{az+b} - C_{a'z+b'}$ is compact, then $a = a'$.

PROOF. Consider the sequence of probe functions $(q_k)_{k\geq 1}$. By Proposition 2.4, $q_k \rightharpoonup 0$. Since $C_{az+b} - C_{a'z+b'}$ is compact, therefore $\|(C_{az+b} - C_{a'z+b'})q_k\| \to 0$. Assume to the contrary that $a \neq a'$. Now, since for each k we have $m_k^{(a)} \neq m_k^{(a')}$, therefore we have

$$\begin{aligned}
\|(C_{az+b} - C_{a'z+b'})q_k\|^2 &= \left\| \frac{1}{\beta_k} e^{-\lambda_k b} e^{-\lambda_{m_k^{(a)}} z} - \frac{1}{\beta_k} e^{-\lambda_k b'} e^{-\lambda_{m_k^{(a')}} z} \right\|^2 \\
&= \frac{1}{\beta_k^2} e^{-2\lambda_k \Re(b)} \beta^2_{m_k^{(a)}} + \frac{1}{\beta_k^2} e^{-2\lambda_k \Re(b')} \beta^2_{m_k^{(a')}} \\
&= r_k(a, \Re(b))^2 + r_k(a', \Re(b'))^2,
\end{aligned}$$

which gives the desired contradiction.

PROPOSITION 4.14. *Let $\beta_* \neq \pm\infty$, $a \geq 1$. Suppose that C_{az+b} and $C_{az+b'}$ are bounded composition operators on $\mathcal{H}(\beta, \Lambda)$. Then*

$$\|C_{az+b} - C_{az+b'}\|_e = \limsup_{k\to\infty} \frac{\beta_{m_k}}{\beta_k} \left| e^{-\lambda_k b} - e^{-\lambda_k b'} \right|.$$

PROOF. As before we define the finite-rank (and compact) partial sum operator K_N (see the proof of Theorem 4.3). Then $(C_{az+b} - C_{az+b'})K_N$ is compact and

$$\|C_{az+b} - C_{az+b'}\|_e \leq \|(C_{az+b} - C_{az+b'})(I - K_N)\|_{\mathrm{op}},$$

where I is the identity operator on $\mathcal{H}(\beta, \Lambda)$.

Let $f(z)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\in\mathcal{H}(\beta,\Lambda)$. We then have

$$\begin{aligned}\|(C_{az+b}-C_{az+b'})(I-K_N)(f)\|^2 &= \left\|\sum_{n=N+1}^{\infty} a_n e^{-\lambda_n b}e^{-\lambda_{m_n} z}-\sum_{n=N+1}^{\infty} a_n e^{-\lambda_n b'}e^{-\lambda_{m_n} z}\right\|^2\\ &= \sum_{n=N+1}^{\infty}|a_n|^2\left|e^{-\lambda_n b}-e^{-\lambda_n b'}\right|^2\beta_{m_n}^2\\ &\leq \sup_{n\geq N+1}\frac{\beta_{m_n}^2}{\beta_n^2}\left|e^{-\lambda_n b}-e^{-\lambda_n b'}\right|^2\|f\|^2.\end{aligned}$$

It therefore follows that $\|(C_{az+b}-C_{az+b'})(I-K_N)\|\leq \sup_{n\geq N+1}\frac{\beta_{m_n}}{\beta_n}\left|e^{-\lambda_n b}-e^{-\lambda_n b'}\right|$.

Taking limits as $N\to\infty$ gives $\|C_{az+b}-C_{az+b'}\|_e\leq \limsup_{n\to\infty}\frac{\beta_{m_n}}{\beta_n}\left|e^{-\lambda_n b}-e^{-\lambda_n b'}\right|$.

On the other hand, consider the probe functions q_k. Let K be a compact operator on $\mathcal{H}(\beta,\Lambda)$. As before, we have $\|q_k\|=1$ for all k and $\|Kq_k\|\to 0$. We have

$$\begin{aligned}\|(C_{az+b}-C_{az+b'})-K\| &\geq \limsup_{k\to\infty}\left(\|(C_{az+b}-C_{az+b'})q_k\|-\|Kq_k\|\right)\\ &\geq \limsup_{k\to\infty}\|(C_{az+b}-C_{az+b'})q_k\|\\ &= \limsup_{k\to\infty}\frac{\beta_{m_k}}{\beta_k}\left|e^{-\lambda_k b}-e^{-\lambda_k b'}\right|.\end{aligned}$$

Taking infimum over all compact operators K gives

$$\|C_{az+b}-C_{az+b'}\|_e\geq \limsup_{n\to\infty}\frac{\beta_{m_n}}{\beta_n}\left|e^{-\lambda_n b}-e^{-\lambda_n b'}\right|,$$

as needed. This completes the proof.

THEOREM 4.15. *Let $\beta_*\neq\pm\infty$, and let C_{φ_1} and C_{φ_2} be bounded composition operators on $\mathcal{H}(\beta,\Lambda)$. Then the difference $C_{\varphi_1}-C_{\varphi_2}$ is compact if and only if*

(1) both C_{φ_1} and C_{φ_2} are compact

or

(2) $a\geq 1$, $\varphi_1(z)=az+b$ and $\varphi_2(z)=az+b'$, where

(i) *we have*

$$\limsup_{n\to\infty} r_n(a,\Re(b))>0 \quad \text{and} \quad \limsup_{n\to\infty} r_n(a,\Re(b'))>0,$$

(ii) *and* $\lim_{k\to\infty}\frac{\beta_{m_k}}{\beta_k}\left|e^{-\lambda_k b}-e^{-\lambda_k b'}\right|=0.$

PROOF. Assume first that $C_{\varphi_1} - C_{\varphi_2}$ is compact but one of the operators C_{φ_1} and C_{φ_2} is not compact. Since the set of compact operators is a vector space, it implies that indeed both operators C_{φ_1} and C_{φ_2} are not compact. Now, according to Theorem 4.3, it means that condition (i) is satisfied (with the second inequality in a' instead of a). Then, we can apply Lemma 4.13 to get that $a = a'$, i.e. $\varphi_1(z) = az + b$ and $\varphi_2(z) = az + b'$. Condition (ii) now follows immediately from Proposition 4.14.

Conversely, if both operators C_{φ_1} and C_{φ_2} are compact, then their difference is compact. Suppose now that $\varphi_1(z) = az+b$ and $\varphi_2(z) = az+b'$, and (i) and (ii) are satisfied. Proposition 4.14 implies that $\|C_{\varphi_1} - C_{\varphi_2}\|_e = 0$, which gives that $C_{\varphi_1} - C_{\varphi_2}$ is compact. This completes the proof.

COROLLARY 4.16. *Let $\beta_* \neq \pm\infty$, and let C_{z+ci} and $C_{z+c'i}$ ($c, c' \in \mathbb{R}$) be bounded composition operators on $\mathcal{H}(\beta, \Lambda)$. The operator $C_{z+ci} - C_{z+c'i}$ is compact if and only if $\lim_{k\to\infty} \cos(\lambda_k(c - c')) = 1$.*

PROOF. We see that $m_k = k$ and so $\beta_{m_k} = \beta_k$, $k \in \mathbb{N}$. Moreover, since $\Re(ci) = \Re(c'i) = 0$, we have $r_n(a, \Re(ci)) = r_n(a', \Re(c'i)) = 1$. Then, according to Proposition 4.2 and Theorem 4.15, we see that $C_{z+ci} - C_{z+c'i}$ is compact if and only if

$$\lim_{k\to\infty} |e^{-\lambda_k ci} - e^{-\lambda_k c'i}| = 0.$$

An easy computation shows that

$$|e^{-\lambda_k ci} - e^{-\lambda_k c'i}|^2 = 2 - 2\cos(\lambda_k(c - c')),$$

which gives the result.

Corollary 4.16 is an analogue of [19, Theorem 4.12] corresponding to the case when $\beta_* = \infty$.

5. Closed range and cyclicity

In this section, we assume that C_{az+b} is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$, which means that a and b satisfy conditions of Theorem 3.12.

5.1. *Closed range*

We denote by $R(C_\varphi)$ the range of C_φ, i.e. $C_\varphi(\mathcal{H}(\beta, \Lambda))$. In this section we determine when $R(C_\varphi)$ is closed. Note that when $\varphi(z) = b$ (and $\lambda_1 = 0$), then $R(C_\varphi)$ is a one dimensional space (generated by the constant function $1 = e^{-\lambda_1 z}$) and so it is closed.

PROPOSITION 5.1. *Let $\beta_* \neq \pm\infty$, $a \geq 1$. Then, $R(C_{az+b})$ is closed if and only if $\inf_{n\in\mathbb{N}} r_n(a, \Re(b)) > 0$.*

PROOF. First note that it follows from the open mapping theorem and uniqueness principle for analytic functions that, since φ is a non-constant analytic function, then C_φ is injective. Now, suppose $B := \inf_{n\in\mathbb{N}} r_n(a, \Re(b)) > 0$. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$. Then,

$$\frac{\|C_{az+b}f\|^2}{\|f\|^2} = \frac{\sum_{n=1}^{\infty} |a_n|^2 \beta^2_{m_n} e^{-2\lambda_n \Re(b)}}{\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2} = \frac{\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 r_n(a, \Re(b))^2}{\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2} \geq B^2.$$

Thus C_{az+b} is bounded from below and $R(C_{az+b})$ is hence closed.

On the other hand, suppose $B := \inf_{n\in\mathbb{N}} r_n(a, \Re(b)) = 0$. Let (n_k) be a subsequence of $\mathbb{N}$ such that $r_{n_k}(a, \Re(b)) \to 0$. For each probe function q_{n_k}, we then have

$$\frac{\|C_{az+b}q_{n_k}\|^2}{\|q_{n_k}\|^2} = \frac{\frac{1}{\beta^2_{n_k}} \beta^2_{m_{n_k}} e^{-2\lambda_n \Re(b)}}{1} = r_{n_k}(a, \Re(b))^2 \to 0.$$

It follows that C_{az+b} cannot be bounded from below, and so $R(C_{az+b})$ is not closed. This completes the proof.

Proposition 5.1 is an analogue of [7, Theorem 6].

REMARK 5.2. In the non-constant case, a compact C_{az+b} cannot have closed range, and vice-versa if C_{az+b} has closed range, it is not compact.

5.2. *Cyclicity*

Let X be a Banach space and $T : X \to X$ be a bounded operator. We define the orbit of a vector $x \in X$ (w.r.t. T) as the set

$$\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}.$$

Furthermore, we recall that T is said to be

- cyclic if there exists $x \in X$ such that
$$\overline{\mathrm{Span}(\mathrm{Orb}(T, x))} = X,$$
- supercyclic if there exists $x \in X$ such that
$$\overline{\{\mu y : y \in \mathrm{Orb}(T, x), \mu \in \mathbb{C}\}} = X.$$

Note that for a given operator on an Hilbert space $\mathcal{H}$, if T is supercyclic, then it is of course cyclic. We will discuss in this section the cyclicity and supercyclicity of the operators C_φ on $\mathcal{H}(\beta, \Lambda)$. As we will see, C_φ is never supercyclic but cyclicity will depend on the arithmetic properties of $(\lambda_n)_{n\geq 1}$.

It is trivial that if $\lambda_1 = 0$, then of course the operators C_b and C_z are not cyclic. Indeed, in both cases the orbit of f (for every $f \in \mathcal{H}(\beta, \Lambda)$) contains only one function and so the orbit cannot generate a dense subspace.

We now split our study in two cases, depending whether $a = 1$ or $a > 1$.

- The case $a = 1$.

To study this case, we need the following two general results. The first one is quite classical and can be found for instance in [13, Chap. 18] for the finite dimensional case and in [24, Lemma 1] for the general case. The proof of the second one can be found in [17] and uses the spectral mapping theorem.

LEMMA 5.3. *Let D be a diagonal operator on an Hilbert space $\mathcal{H}$, given by $De_n = s_n e_n$, $n \geq 1$, where $(e_n)_{n\geq 1}$ is an orthonormal basis of $\mathcal{H}$. Then D is cyclic if and only if $s_n \neq s_m$, $n \neq m$.*

LEMMA 5.4. *Let T be a normal operator on a Hilbert space $\mathcal{H}$ of dimension greater than 1. Then T is not supercyclic.*

PROPOSITION 5.5. *Let $\beta_* \neq \pm\infty$, $b \neq 0$. Then,*

1. *C_{z+b} is cyclic if and only if $(\lambda_n - \lambda_m)b \in \mathbb{C} \setminus 2\pi\mathbb{Z}$ whenever $n \neq m$, and*
2. *C_{z+b} is not supercyclic.*

PROOF. (1) Recall that the set of probe functions $q_k(z) = \dfrac{1}{\beta_k} e^{-\lambda_k z}$ form a basis for $\mathcal{H}(\beta, \Lambda)$. Note also that for all $k \in \mathbb{N}$,

$$(C_{z+b} q_k)(z) = e^{-\lambda_k b} q_k(z).$$

That means that C_{z+b} is a diagonal operator with eigenvalue corresponding to $e^{-\lambda_k b}$. Thus (1) follows immediately from Lemma 5.3.

(2) Since C_{z+b} is diagonal, it is in particular normal. So (2) follows from Lemma 5.4. This completes the proof.

- The case $a > 1$.

The situation in this case is more interesting and the behavior of the iterates of C_{az+b} will depend on the following notion.

Given $a \in \mathcal{R}(\Lambda)$, $\Lambda = (\lambda_n)_{n\geq 1}$, we define an *initial point with respect to* a to be a term λ_k such that no $n < k$ exists such that $a^s \lambda_n = \lambda_k$ for some $s \in \mathbb{N}$. This is equivalent to say that, for $s \in \mathbb{N}_0$,

$$a^s \lambda_n = \lambda_k \Longrightarrow s = 0 \text{ and } n = k.$$

Note that λ_1 is always an initial point and if $\lambda_1 = 0$, then λ_2 is an initial point. Therefore, three cases may happen:

1. $(\lambda_n)_{n\geq 1}$ has precisely one initial point with respect to a. In that case, it must be the case that $\lambda_n = \lambda_1 a^{n-1}$ for every n, and $\lambda_1 \neq 0$.

(2) $(\lambda_n)_{n\geq 1}$ has one zero and one non-zero initial point. In that case, $\lambda_1 = 0$ and $\lambda_n = \lambda_2 a^{n-2}$, $n \geq 2$.
(3) $(\lambda_n)_{n\geq 1}$ has at least two non-zero initial points.

PROPOSITION 5.6. *Let $\beta_* \neq \pm\infty$, $a > 1$, and suppose $(\lambda_n)_{n\geq 1}$ has precisely one initial point with respect to a. Then, C_{az+b} is cyclic but not supercyclic.*

PROOF. Since $(\lambda_n)_{n\geq 1}$ has precisely one initial point, it must be the case that $\lambda_n = \lambda_1 a^{n-1}$ for every n, and $\lambda_1 \neq 0$. Let $f(z) = e^{-\lambda_1 z}$. By induction, we easily check that

$$(C^k_{az+b} f)(z) = \exp(-(\lambda_1 + \lambda_2 + \cdots + \lambda_k)b)e^{-\lambda_{k+1} z}.$$

In particular, $\mathrm{Orb}(C_{az+b}, f)$ contains the vectors of the basis $(q_k)_{k\geq 1}$ of $\mathcal{H}(\beta, \Lambda)$. Therefore C_{az+b} is cyclic.

To show C_{az+b} is not supercyclic, we show that for every $f \in \mathcal{H}(\beta, \Lambda)$ we can find a g such that $\mathrm{dist}(g, \{\mu y : y \in \mathrm{Orb}(C_{az+b}, f), \mu \in \mathbb{C}\})$ is bounded below by some non-zero constant. There are two cases.

- Case 1: $f(z) = a_1 e^{-\lambda_1 z}$ for some $0 \neq a_1 \in \mathbb{C}$. In this case, consider the function $g(z) = e^{-\lambda_1 z} + e^{-\lambda_2 z}$. It is then easy to see that

$$\begin{aligned} \|\mu f - g\|^2 &= |\mu a_1 - 1|^2 \beta_1^2 + \beta_2^2 \geq \beta_2^2, \\ \|\mu C_{az+b} f - g\|^2 &= \beta_1^2 + |\mu a_1 e^{-\lambda_1 b} - 1|^2 \beta_2^2 \geq \beta_1^2, \\ \|\mu C^k_{az+b} f - g\|^2 &\geq \beta_1^2 + \beta_2^2, \qquad \forall k \geq 2. \end{aligned}$$

  Therefore $\mathrm{dist}(g, \{\mu y : y \in \mathrm{Orb}(C_{az+b}, f), \mu \in \mathbb{C}\}) \geq \min\{\beta_1, \beta_2\}$.
- Case 2: $f(z) \neq a_1 e^{-\lambda_1 z}$ for any $a_1 \in \mathbb{C}$.

  Write $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$. Choose $\kappa \neq 0, -a_1$ and pick $g(z) = \kappa e^{-\lambda_1 z}$. Note that for all $k \geq 1$ we have that the coefficient of $e^{-\lambda_1 z}$ in the representation of $C^k_{az+b} f$ is 0, it follows then that for each $k \geq 1$, we have $\|\mu C^k_{az+b} f - g\|^2 \geq |\kappa|^2 \beta_1^2 > 0$.

  It remains to consider $k = 0$. Observe that g does not belong to $\mathrm{Span}(f)$ which is one-dimensional, therefore $\mathrm{dist}(\mathrm{span}(f), g) > 0$. Thus $\mathrm{dist}(g, \{\mu y : y \in \mathrm{Orb}(C_\varphi, f), \mu \in \mathbb{C}\}) > 0$.

This completes the proof.

PROPOSITION 5.7. *Let $\beta_* \neq \pm\infty$, $a > 1$, and suppose $(\lambda_n)_{n\geq 1}$ has at least two non-zero initial points with respect to a. Then, C_{az+b} is not cyclic.*

PROOF. Let λ_p, λ_q denote two non-zero initial points. Suppose for the sake of contradiction that C_{az+b} is cyclic. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ be a cyclic vector for C_{az+b}.

We claim that $a_p, a_q \neq 0$. Assume to the contrary that one of them is zero, say WLOG $a_p = 0$. Recall that $\mathrm{Orb}(C_{az+b}, f) = \left\{f, C_{az+b}f, C^2_{az+b}f, \ldots\right\}$. Note that

$$(C^k_{az+b}f)(z) = \sum_{n=1}^{\infty} a_n \exp\left(-b\frac{1-a^k}{1-a}\right)\exp\left(-a^k\lambda_n z\right).$$

Since λ_p is a non-zero initial point, $a^k\lambda_n \neq \lambda_p$ for all $k, n \geq 1$. It follows that the coefficient of λ_p in any function in $\mathrm{Orb}(C_{az+b}, f)$ is always 0. Hence for any $f_1 \in \mathrm{Span}(\mathrm{Orb}(C_{az+b}, f))$, one has

$$\|f_1 - q_p\|^2 \geq 1.$$

Therefore $\mathrm{Span}(\mathrm{Orb}(C_{az+b}, f))$ cannot be dense in $\mathcal{H}(\beta, \Lambda)$, a contradiction, completing the proof of the claim.

Hence $a_p, a_q \neq 0$. Since $a^k\lambda_n \neq \lambda_p, \lambda_q$ for all $k, n \geq 1$, the only function in $\mathrm{Orb}(C_{az+b}, f)$ with non-zero coefficients for $e^{-\lambda_p z}$ and $e^{-\lambda_q z}$ terms in $\mathrm{Span}(\mathrm{Orb}(C_{az+b}, f))$ is f itself.

Consider now the function $g(z) = a_p e^{-\lambda_p z} + 2a_q e^{-\lambda_q z} \in \mathcal{H}(\beta, \Lambda)$. Let $F \in \mathrm{Span}(\mathrm{Orb}(C_{az+b}, f))$ and fix a representation of F in elements of $\mathrm{Orb}(C_{az+b}, f)$. Let w the coefficient of f in this representation. Then we have

$$\|F - g\|^2 \geq |w-1|^2|a_p|^2\beta_p^2 + |w-2|^2|a_q|^2\beta_q^2.$$

Let $\xi = |a_p|^2\beta_p^2$ and $\eta = |a_q|^2\beta_q^2$. Write $w = w' + 1$. Then,

$$\begin{aligned}\|F - g\|^2 &\geq \xi|w'|^2 + \eta|w' - 1|^2 \\ &\geq \xi\Re(w')^2 + \eta(\Re(w') - 1)^2 \geq \frac{\xi\eta}{\xi + \eta},\end{aligned}$$

where we used the fact that the expression $\xi x^2 + \eta(x-1)^2$ has minimum $\frac{\xi\eta}{\xi+\eta}$ as x ranges over $\mathbb{R}$. It follows that $\mathrm{Span}(\mathrm{Orb}(C_{az+b}, f))$ cannot be dense in $\mathcal{H}(\beta, \Lambda)$. Contradiction.

It remains to study the case when $(\lambda_n)_{n\geq 1}$ has precisely one zero and one non-zero initial point with respect to a. In other words, $\lambda_1 = 0$ and $\lambda_n = \lambda_2 a^{n-2}$, $n \geq 2$. In that case, we will give a characterization of cyclicity in terms of $(r_n(a, b))_n$. Let us begin with the following result which can be easily checked.

PROPOSITION 5.8. *Let $\beta_* \neq \pm\infty$, $a > 1$, and suppose (λ_n) has precisely one zero and one non-zero initial point with respect to a. Then the matrix M*

of C_{az+b} with respect to the orthonormal basis (q_k) is

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & r_2(a,b) & 0 & \dots \\ 0 & 0 & r_3(a,b) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since cyclicity is invariant under unitary transformation, we will work for convenience with a transcription of our problem in the space of sequences ℓ^2. Recall that

$$\ell^2 = \left\{ a = (a_k)_{k\geq 0} : \|a\|_2^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}.$$

Now, it is clear that our problem is a particular case of the following more general problem.

QUESTION 5.9. Let $\alpha = (\alpha_n)$ be a bounded sequence of complex numbers and suppose that $T : \ell^2 \to \ell^2$ is the linear map whose matrix, with respect to the canonical orthonormal basis $\{e_0, e_1, e_2, \dots\}$, is given by

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & \alpha_1 & 0 & \dots \\ 0 & 0 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Is T cyclic?

Note that since (α_n) is supposed to be bounded, it is easy to see that T defines a bounded operator on ℓ^2.

In order to answer to Question 5.9, we need the following general result on cyclicity.

LEMMA 5.10. *Let $T : X \longrightarrow X$ be a bounded linear map on a Banach space X. Assume that there exists $\lambda \in \mathbb{C}$ such that*

$$\dim \ker(T^* - \lambda I)) \geq 2.$$

Then T is not cyclic.

PROOF. Let x_1, x_2 be two independent vectors in X^* (the dual space of X) such that $T^*(x_i^*) = \lambda x_i^*$, $i = 1, 2$. Define the linear map

$$\begin{array}{rccc} \Theta: & X & \longrightarrow & \mathbb{C}^2 \\ & v & \longmapsto & (\langle x_1^*, v\rangle, \langle x_2^*, v\rangle). \end{array}$$

Observe that Θ is onto. Indeed, assume that $\mathrm{Im}(\Theta) \subsetneq \mathbb{C}^2$. In particular, there is $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $(\lambda, \mu) \perp \mathrm{Im}(\Theta)$. That means that, for every $v \in X$, we have $\overline{\lambda}\langle x_1^*, v\rangle + \overline{\mu}\langle x_2^*, v\rangle = 0$. In other words, we get $\overline{\lambda}x_1^* + \overline{\mu}x_2^* = 0$, but, since x_1^* and x_2^* are independent, we deduce that $\lambda = \mu = 0$, which is absurd. Hence Θ is onto.

Assume now that T is cyclic and let $x \in X$ be a cyclic vector for T. Consider the following subset of $\mathbb{C}^2$ defined by

$$A = \{(\langle x_1^*, p(T)x\rangle, \langle x_2^*, p(T)x\rangle) : p \text{ is a polynomial}\}.$$

Since x is a cyclic vector and Θ is onto, the set A should be dense in $\mathbb{C}^2$. Indeed, let $(\lambda, \mu) \in \mathbb{C}^2$. Using that Θ is onto, we get $u \in X$ such that $\lambda = \langle x_1^*, u\rangle$ and $\mu = \langle x_2^*, u\rangle$. Using now that x is a cyclic vector, given $\varepsilon > 0$, we can find a polynomial p such that $\|p(T)x - u\|_X \leq \varepsilon$. Hence

$$|\langle x_1^*, p(T)x\rangle - \lambda| = |\langle x_1^*, p(T)x - u\rangle| \leq \|x_1^*\|\varepsilon$$

and similarly

$$|\langle x_2^*, p(T)x\rangle - \mu| = |\langle x_2^*, p(T)x - u\rangle| \leq \|x_2^*\|\varepsilon.$$

These two inequalities imply that (λ, μ) is in the closure of A, and thus A is dense in $\mathbb{C}^2$. It remains to prove that this is absurd. First observe that

$$\begin{aligned} A &= \{(\langle p(T^*)x_1^*, x\rangle, \langle p(T^*)x_2^*, x\rangle) : p \text{ is a polynomial}\} \\ &= \{p(\lambda)(\langle x_1^*, x\rangle, \langle x_2^*, x\rangle) : p \text{ is a polynomial}\} \\ &= \{p(\lambda)\Theta(x) : p \text{ is a polynomial}\}. \end{aligned}$$

If $\langle x_1^*, x\rangle = 0$ or $\langle x_2^*, x\rangle = 0$, then either $A \subset \{0\} \times \mathbb{C}$ or $A \subset \mathbb{C} \times \{0\}$, and of course A cannot be dense in $\mathbb{C}^2$. Now assume that $\langle x_1^*, x\rangle \neq 0$ and $\langle x_2^*, x\rangle \neq 0$, and take

$$z = \frac{\langle x_2^*, x\rangle}{2\langle x_1^*, x\rangle}.$$

Since A is dense in $\mathbb{C}^2$, the point $(1, z)$ belongs to the closure of A. In particular, there exists a sequence of polynomials $(p_n)_n$ such that $p_n(\lambda)\Theta(x) \to (1, z)$, as $n \to \infty$. That means that

$$p_n(\lambda)\langle x_1^*, x\rangle \to 1, \quad \text{as } n \to \infty, \tag{5.1}$$

and

$$p_n(\lambda)\langle x_2^*, x\rangle \to z, \quad \text{as } n \to \infty. \tag{5.2}$$

In particular, $zp_n(\lambda)\langle x_1^*, x\rangle - p_n(\lambda)\langle x_2^*, x\rangle \to 0$, as $n \to \infty$. But observe that

$$\begin{aligned} zp_n(\lambda)\langle x_1^*, x\rangle - p_n(\lambda)\langle x_2^*, x\rangle &= p_n(\lambda)\left(\frac{\langle x_2^*, x\rangle}{2} - \langle x_2^*, x\rangle\right) \\ &= -p_n(\lambda)\frac{\langle x_2^*, x\rangle}{2}, \end{aligned}$$

which implies that $p_n(\lambda) \to 0$, as $n \to \infty$. That contradicts (5.1). Hence A cannot be dense, proving that indeed T is not cyclic.

We are now ready to prove the following.

THEOREM 5.11. *Let T be defined as in Question 5.9, and let $\omega_j := \alpha_1 \dots \alpha_j$, $j \geq 1$. Then T is cyclic if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{|\omega_j|^2} = \infty.$$

PROOF. First assume that $\displaystyle\sum_{j=1}^{\infty} \frac{1}{|\omega_j|^2} < \infty$ and let us prove that T is not cyclic. Let $a = (a_0, a_1, a_2, \dots) \in \ell^2$. Observe that $T^*a = a$ if and only if

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \overline{\alpha_1} & 0 & 0 & \dots \\ 0 & 0 & 0 & \overline{\alpha_2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix},$$

which is equivalent to

$$\overline{\alpha_j} a_{j+1} = a_j, \qquad j \geq 1.$$

Thus $a \in \ker(T^* - I)$ if and only if $a_0, a_1 \in \mathbb{C}$ and $a_j = \frac{1}{\overline{\omega_{j-1}}} a_1$, $j \geq 2$. In particular, we get that $\dim(\ker(T^* - I)) = 2$. Lemma 5.10 implies now that T is not cyclic.

Conversely, assume that $\displaystyle\sum_{j=1}^{\infty} \frac{1}{|\omega_j|^2} = \infty$ and let us prove that $f = (1, 1, 0, 0, \dots)$ is cyclic for T. Argue by absurdity and assume that f is not cyclic for T. Then there exists a non zero element $b = (b_0, b_1, b_2, \dots) \in \ell^2$ such that $b \perp T^k f$, for every $k \geq 0$. We get the following system

$$\begin{cases} b_0 + b_1 = 0 \\ b_0 + \alpha_1 b_2 = 0 \\ b_0 + \alpha_1 \alpha_2 b_3 = 0 \\ \vdots \end{cases}$$

This is equivalent to

$$\begin{cases} b_0 + b_1 = 0 \\ b_0 + \omega_j b_{j+1} = 0, \ j \geq 1. \end{cases}$$

Clearly if $b_0 = 0$, then $b_j = 0$ for all j, hence $b = 0$, which is a contradiction. If $b_0 \neq 0$, then $b_1 = -b_0$ and $b_{j+1} = -\frac{1}{\omega_j} b_0$. But since by hypothesis we have $\sum_{j=1}^{\infty} \frac{1}{|\omega_j|^2} = \infty$, we deduce that $b = (b_j)_{j \geq 0}$ does not belong to ℓ^2, which is a contradiction. We are done.

We immediately deduce from Theorem 5.11 a solution of the remaining case for cyclicity of C_{az+b}.

COROLLARY 5.12. *Let $\beta_* \neq \pm\infty$, $a > 1$, and suppose (λ_n) has precisely one zero and one non-zero initial point with respect to a. Let $\omega_j = \prod_{k=2}^{j+1} r_k(a, b)$, $j \geq 2$. Then C_{az+b} is cyclic if and only if*

$$\sum_{j=2}^{\infty} \frac{1}{|\omega_j|^2} = \infty.$$

This characterization leads to a condition which is much easier to check.

COROLLARY 5.13. *Let $\beta_* \neq \pm\infty$, $a > 1$, and suppose (λ_n) has precisely one zero and one non-zero initial point with respect to a.*

(i) *If $\Re(b) > (a-1)\beta_*$, then C_{az+b} is cyclic.*
(ii) *If $\Re(b) = (a-1)\beta_*$, then both cyclic and non-cyclic behaviors can be exhibited.*

Before we begin the proof, we remark that the above corollary completes the characterization of cyclicity for this case. It is easy to check that for such a (λ_n), we have $L = 0$. Since C_{az+b} is necessarily bounded, by Theorem 3.12 we must have $\Re(b) \geq (1-a)\left(\frac{L}{2} - \beta_*\right) = (a-1)\beta_*$.

PROOF OF COROLLARY 5.13. A direct computation shows that

$$\begin{aligned} |\omega_j|^{-1} = \prod_{k=2}^{j+1} |r_k(a,b)|^{-1} &= \frac{\beta_2}{\beta_{j+2}} \exp\left(\Re(b) \sum_{k=2}^{j+1} \lambda_k \right) \\ &= \frac{\beta_2}{\beta_{j+2}} \exp\left(\lambda_2 \Re(b) \frac{a^j - 1}{a - 1} \right). \end{aligned}$$

(i) Assume first that $\Re(b) > (a-1)\beta_*$. Since $\beta_* = \liminf_{n\to\infty} \dfrac{\log \beta_n}{\lambda_n}$ is finite, there exists $\varepsilon > 0$ and a strictly increasing sequence of natural numbers $(n_p)_p$ such that, for all p,

$$\log \beta_{n_p} \leq \frac{\Re(b) - \varepsilon}{a-1} \lambda_{n_p}.$$

In other words,

$$\log \beta_{n_p} \leq \frac{\Re(b) - \varepsilon}{a-1} a^{n_p - 2} \lambda_2.$$

Set $j_p = n_p - 2$ and observe that

$$\begin{aligned} |\omega_{j_p}|^{-1} &\geq \beta_2 \exp\left(\lambda_2 \Re(b) \frac{a^{j_p} - 1}{a-1} - \lambda_2 \frac{\Re(b) - \varepsilon}{a-1} a^{j_p} \right) \\ &= \beta_2 \exp\left(\lambda_2 \frac{a^{j_p}}{a-1} (\varepsilon + o(1)) \right). \end{aligned}$$

In particular, $|\omega_{j_p}|^{-1} \to \infty$ as $p \to \infty$, so that $(\omega_j)^{-1} \notin \ell^2$. By Corollary 5.12, we deduce that C_{az+b} is cyclic.

(ii) First take $\beta_n = \exp\left(\frac{\lambda_2 a^{n-2}}{a-1}\right)$ and $\Re(b) = 1$. We shall verify such a choice of C_{az+b} is valid for $\mathcal{H}(\beta, \Lambda)$. The sequence $(r_n(a, \Re(b)))$ is bounded, by Theorem 3.12. Indeed, for $n \geq 2$,

$$r_n(a, \Re(b)) = e^{-\lambda_n \Re(b)} \frac{\beta_{n+1}}{\beta_n} = \exp\left(-\lambda_2 a^{n-2} + \frac{\lambda_2 a^{n-1}}{a-1} - \frac{\lambda_2 a^{n-2}}{a-1} \right) = e^0 = 1.$$

Furthermore,

$$\beta_* = \liminf_{n\to\infty} \frac{\lambda_2 a^{n-2}}{(a-1)\lambda_2 a^{n-2}} = \frac{1}{a-1},$$

so that $\Re(b) = (a-1)\beta_* = 1$. By Theorem 3.12, this choice is valid. Finally,

$$|\omega_j|^{-1} = \frac{\beta_2}{\beta_{j+2}} \exp\left(\lambda_2 \frac{a^j - 1}{a-1} \right) = \beta_2 \exp\left(-\frac{\lambda_2}{a-1} \right).$$

Hence, $(|\omega_j|^{-1})_j \notin \ell^2$ and then C_{az+b} is cyclic.

On the other hand, if we take $\beta_n = n^2 \exp\left(\frac{\lambda_2 a^{n-2}}{a-1}\right)$ and $\Re(b) = 1$, we can verify in the same manner that $r_n(a, 1) \to 1$ as $n \to \infty$, and have again

$$\beta_* = \liminf_{n\to\infty} \frac{2\log n + \lambda_2 a^{n-2}}{(a-1)\lambda_2 a^{n-2}} = \frac{1}{a-1},$$

and

$$|\omega_j|^{-1} = \frac{\beta_2}{\beta_{j+2}} \exp\left(\lambda_2 \frac{a^j - 1}{a-1} \right) = \frac{\beta_2 \exp\left(-\frac{\lambda_2}{a-1}\right)}{(j+2)^2}.$$

Thus $(|\omega_j|^{-1})_j \in \ell^2$ and then C_{az+b} is not cyclic. This completes the proof.

We end this section with a result which completes Proposition 5.5 on supercyclicity. As the proof methodology is similar to arguments in Proposition 5.6, we omit the full details and provide a sketch.

PROPOSITION 5.14. *Let $\beta_* = \pm\infty$, $a > 1$, and suppose (λ_n) has precisely one zero and one non-zero initial point with respect to a. Then, C_{az+b} is not supercyclic.*

PROOF (OUTLINE). The idea is for every $f \in \mathcal{H}(\beta, \Lambda)$, we can find a function g and a constant $B > 0$ such that $\operatorname{dist}(g, \{\mu y : y \in \operatorname{Orb}(C_{az+b}, f), \mu \in \mathbb{C}\}) \geq B$. The proof is split into four cases.

- Case 1: $f(z) \equiv a_1$ $(a_1 \in \mathbb{C})$. The result is obvious as $\operatorname{Orb}(C_{az+b}, f)$ has only one element, so $\{\mu y : y \in \operatorname{Orb}(C_{az+b}, f), \mu \in \mathbb{C}\}$ has dimension 1.

- Case 2: $f(z) = a_2 e^{-\lambda_2 z}, a_2 \neq 0$. Pick $g(z) \equiv b > 0$. Since all $h \in \operatorname{Orb}(C_{az+b}, f)$ have zero constant part, it can be verified that taking $B = |b|\beta_1$ works.

- Case 3: $f(z) = a_1 + a_2 e^{-\lambda_2 z}$ $(a_1, a_2 \neq 0)$. Pick $g(z) = a_1 + (a_2 + 1)e^{-\lambda_2 z}$. Since λ_2 is a non-zero initial point, following the proof of Proposition 5.6 (Case 2) yields $\|\mu C_{az+b}^k f - g\|^2 \geq |a_2 + 1|^2 \beta_2^2$ for all $k \geq 1$. Furthermore the computation for $k = 0$ gives

$$\|\mu f - g\|^2 \geq |\mu - 1|^2 |a_1|^2 \beta_1^2 + |(\mu - 1)a_2 + 1|^2 \beta_2^2,$$

which is minimized in a similar manner as in Proposition 5.7.

- Case 4: $\exists k \geq 3$ such that $a_k \neq 0$. Pick $\kappa \neq 0, -a_2$ and $g(z) = \kappa e^{-\lambda_2 z}$. This is again similar to Proposition 5.7. We are done.

6. Complex symmetry

In this section we investigate the complex symmetry property of C_φ. In this section we adopt the convention that

$$e_n(z) = \beta_n q_n(z) = e^{-\lambda_n z}. \tag{6.1}$$

6.1. *Composition conjugations*

DEFINITION 6.1. Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space. A map $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ satisfying the conditions of

(1) isometry: $\|\mathcal{C}x\| = \|x\|, \forall x \in \mathcal{H}$,
(2) involutivity: $\mathcal{C}\mathcal{C} = I$, the identity map, and
(3) anti-linearity: $\mathcal{C}(\lambda x + \mu y) = \overline{\lambda}\mathcal{C}x + \overline{\mu}\mathcal{C}y, \forall x, y \in \mathcal{H}, \forall \lambda, \mu \in \mathbb{C}$,

is called a conjugation on $\mathcal{H}$.

DEFINITION 6.2. Let $\xi : \mathbb{C}_{L/2-\beta_*} \to \mathbb{C}_{L/2-\beta_*}$ be an analytic function. Define

$$J_\xi f(z) = \overline{f(\overline{\xi(z)})}, \qquad \forall f \in \mathcal{H}(\beta, \Lambda).$$

If J_ξ satisfies the definition of a conjugation, then it is called a composition conjugation.

We will investigate composition conjugations induced by polynomial functions ξ.

THEOREM 6.3. *Assume that ξ is a polynomial. The following statements are equivalent.*

(1) *J_ξ is a conjugation on $\mathcal{H}(\beta,\Lambda)$.*
(2) *$\xi(z)=z+ci, c\in\mathbb{R}$.*

PROOF. (1) $\Longrightarrow$ (2): first let us note that since J_ξ is a conjugation on $\mathcal{H}(\beta,\Lambda)$, then ξ induces a bounded composition operator on $\mathcal{H}(\beta,\Lambda)$. Indeed, let $f(z)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\in\mathcal{H}(\beta,\Lambda)$. Then the function

$$\widetilde{f}(z)=\sum_{n=1}^{\infty}\overline{a_n}e^{-\lambda_n z}$$

also belongs to $\mathcal{H}(\beta,\Lambda)$, and so $J_\xi\widetilde{f}\in\mathcal{H}(\beta,\Lambda)$. Note now that

$$J_\xi\widetilde{f}(z)=\overline{\widetilde{f}(\overline{\xi(z)})}=\sum_{n=1}^{\infty}a_n e^{-\lambda_n\xi(z)}=(C_\xi f)(z).$$

Therefore, we get $C_\xi f\in\mathcal{H}(\beta,\Lambda)$ as required. In particular, according to Theorem 3.12, ξ should be of the form $\xi(z)=az+b$, where $a=0$ and $\Re(b)>\frac{L}{2}-\beta_*$ or $a\geq 1$ and $\Re(b)\geq(1-a)(\frac{L}{2}-\beta_*)$.

First, we will show that $a\neq 0$. Argue by absurdity and assume that $a=0$. Then $\xi(z)=b$ and $(J_\xi f)(z)=\overline{f(\overline{b})}$. Apply that equation to the vector of the orthonormal basis $q_k(z)=\frac{1}{\beta_k}e^{-\lambda_k z}$, which gives

$$(J_\xi q_k)(z)=\frac{1}{\beta_k}e^{-\lambda_k z}.$$

Since J_ξ is an isometry, we get

$$\frac{\beta_1}{\beta_k}e^{-\lambda_k\Re(b)}=1,$$

that is

$$-\Re(b)=\frac{\log(\beta_k)}{\lambda_k}-\frac{\log(\beta_1)}{\lambda_k},\qquad k\geq 2.$$

Taking the limit when $k\to\infty$, we get $\Re(b)=-\beta_*$. In particular, $\beta_*\neq\pm\infty$ and the condition $\Re(b)>\frac{L}{2}-\beta_*$ gives the contradiction. Hence $a\neq 0$.

Let us show now that $a = 1$ and $b \in i\mathbb{R}$. Remind that $e_n(z) = e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$. Since J_ξ is conjugation on $\mathcal{H}(\beta, \Lambda)$, we have $e_n = J_\xi^2 e_n$, which gives

$$\begin{aligned} e^{-\lambda_n z} =& J_\xi(\overline{(e_n)(\overline{\xi(z)})}) \\ =& J_\xi(e^{-\overline{\lambda_n}\xi(z)}) \\ =& e^{-\lambda_n \overline{\xi(\overline{\xi(z)})}}. \end{aligned}$$

By analyticity, we get that for every $n \geq 1$, there exists $k_n \in \mathbb{Z}$ such that

$$-\lambda_n \overline{\xi(\overline{\xi(z)})} = -\lambda_n z + 2i\pi k_n.$$

In particular, since $\lambda_2 > \lambda_1 \geq 0$, we deduce that

$$\overline{\xi(\overline{\xi(z)})} = z - \frac{2i\pi k_2}{\lambda_2} = z + ic_1,$$

where $c_1 = -2\pi \frac{k_2}{\lambda_2} \in \mathbb{R}$. Recall now that $\xi(z) = az + b$. An easy computation shows that $\overline{\xi(\overline{\xi(z)})} = a^2 z + \bar{a}b + \bar{b}$. Hence, we see that $a^2 = 1$, that $a = 1$ (because $a \geq 0$) and $b + \bar{b} = ic_1$, that is $2\Re(b) = ic_1$. This implies that $\Re(b) = c_1 = 0$. Finally, we get that $\xi(z) = z + i\Im(b) = z + ic$, with $c \in \mathbb{R}$.

(2) $\Longrightarrow$ (1): assume that $\xi(z) = z + ic$, $c \in \mathbb{R}$. Since $\Re(\xi(z)) = \Re(z)$, it is clear that ξ maps $\mathbb{C}_{\frac{L}{2} - \beta_*}$ into itself. Then J_ξ is well defined. Now, if $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(\beta, \Lambda)$, then

$$(J_\xi f)(z) = \sum_{n=1}^{\infty} \overline{a_n} e^{-\lambda_n \xi(z)} = \sum_{n=1}^{\infty} \overline{a_n} e^{-ic\lambda_n} e^{-\lambda_n z}.$$

Since $\left|\overline{a_n} e^{-ic\lambda_n}\right| = |a_n|$, we see that $J_\xi f \in \mathcal{H}(\beta, \Lambda)$. Moreover,

$$\|J_\xi f\|^2 = \sum_{n=1}^{\infty} \beta_n^2 |a_n|^2 = \|f\|^2.$$

Hence, J_ξ is an isometry from $\mathcal{H}(\beta, \Lambda)$ into itself. It is of course antilinear. It remains to check that J_ξ is involutive. But, note that

$$(J_\xi e_n)(z) = e^{-\lambda_n \xi(z)} = e^{-ic\lambda_n} e^{-\lambda_n z} = e^{-ic\lambda_n} e_n(z),$$

and so

$$J_\xi^2(e_n) = J_\xi(e^{-ic\lambda_n} e_n) = e^{ic\lambda_n} J_\xi(e_n) = e^{ic\lambda_n} e^{-ic\lambda_n} e_n = e_n$$

By linearity and continuity of J_ξ^2, and density of $\bigvee(e_n : n \geq 1)$ in $\mathcal{H}(\beta, \Lambda)$, we get that $J_\xi^2 = I$. This proves that J_ξ is a conjugation on $\mathcal{H}(\beta, \Lambda)$ and completes the proof.

6.2. *Complex symmetry*
Having proven the form of composition conjugations J_ξ on $\mathcal{H}(\beta,\Lambda)$, we can now consider the complex symmetry property of bounded composition operators on $\mathcal{H}(\beta,\Lambda)$ with respect to conjugations J_ξ.

Definition 6.4. Let T be a continuous linear operator mapping a Hilbert space $\mathcal{H}$ to itself. Given that $\mathcal{C}$ is a conjugation, we say that T is *$\mathcal{C}$-symmetric*, or *complex symmetric with respect to $\mathcal{C}$*, if

$$\mathcal{C}T\mathcal{C} = T^*.$$

If such a $\mathcal{C}$ exists, we say that T is *complex symmetric*.

Recall that Proposition 4.7 provides explicitly the adjoint functions C^*_{az+b}. The following two results follow easily (cf. [9]).

Proposition 6.5. *The following are true.*
(1) $C^*_{z+b} = C_{z+\overline{b}}$.
(2) *If $a > 1$, then $\ker C^*_{az+b}$ is non-trivial.*

Proof. (1) follows directly from Proposition 4.7 (note that since $a = 1$ we have $m_n = n$). Let us now prove (2). If $\lambda_1 \neq 0$, then $\lambda_{m_1} = a\lambda_1 > \lambda_1$, and so $m_1 > 1$. It follows from (4.2) that since $j \mapsto m_j$ is strictly increasing, we have $C^*_{az+b}e_1 = 0$, that is $e_1 \in \ker C^*_{az+b}$. If $\lambda_1 = 0$, then $m_1 = 1$ and arguing as before, we show that $e_2 \in \ker C^*_{az+b}$. This completes the proof.

As already mentioned, composition operators associated to non-constant analytic symbols are injective. Thus we immediately get the following.

Corollary 6.6. *Suppose $\beta_* \neq \pm\infty$. Let $a > 1$ and suppose C_{az+b} is a bounded composition operator on $\mathcal{H}(\beta,\Lambda)$. Then, no symbol ψ exists such that $C^*_{az+b} = C_\psi$ (here, ψ need not be a polynomial).*

Proof. First note that since $a > 1$, then there cannot exists a constant symbol ψ such that $C^*_{az+b} = C_\psi$. Then it remains to apply Proposition 6.5 (2) to get the result. Done.

Proposition 6.7. *Let $\xi(z) = z + ci, c \in \mathbb{R}$. Let $b \in \mathbb{C}$ and suppose that $\Re(b) \geq 0$. Then C_{z+b} is J_ξ-symmetric.*

Proof. Remind that $e_n(z) = e^{-\lambda_n z}$, $n \geq 1$. Since $a = 1$, $m_n = n$, $n \geq 1$, and so, on one hand, we have

$$(C_{z+b}J_\xi e_n)(z) = C_{z+b}\left(e^{-\lambda_n ci}e^{-\lambda_n z}\right) = e^{-\lambda_n(b+ci)}e^{-\lambda_n z}.$$

On the other hand, by (4.2), we have

$$(J_\xi C^*_{z+b}e_n)(z) = J_\xi\left(e^{-\lambda_n \overline{b}}e^{-\lambda_n z}\right) = e^{-\lambda_n(b+ci)}e^{-\lambda_n z}.$$

Thus $C_{z+b}J_\xi$ and $J_\xi C^*_{z+b}$ coincide on an orthogonal basis, whence $C_{z+b}J_\xi = J_\xi C^*_{z+b}$. Hence C_{z+b} is J_ξ-symmetric. This completes the proof.

The above shows that all bounded composition operators with symbol C_{z+b} is complex symmetric. To determine the complex symmetry property of C_{az+b} when $a > 1$, we recall the following result.

LEMMA 6.8 ([10]). *Let T be a complex symmetric operator. Then,* $\dim\ker T^* = \dim\ker T$.

Now we may prove the following result.

PROPOSITION 6.9. *Let $\beta_* \neq \pm\infty$ and $a > 1$. Then, C_{az+b} is not complex symmetric on $\mathcal{H}(\beta, \Lambda)$.*

PROOF. It follows immediately from Lemma 6.8 and Proposition 6.5 and the fact that C_{az+b} is one to one.

Finally we consider the constant case.

LEMMA 6.10 ([11]). *Any rank one operator is complex symmetric.*

We deduce immediately the following.

COROLLARY 6.11. *Let $\lambda_1 = 0$ and $\beta_* \neq \pm\infty$. Then, C_b is complex symmetric on $\mathcal{H}(\beta, \Lambda)$.*

We can summarize all the previous results to obtain the following characterization of complex symmetric composition operators.

THEOREM 6.12. *Let $\beta_* \neq \pm\infty$ and C_φ be a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$. Then:*

1. *If $\varphi(z) = z + b$, then C_φ is complex symmetric. More precisely, C_φ is J_ξ-symmetric for all composition conjugations J_ξ where $\xi(z) = z + ci, c \in \mathbb{R}$.*
2. *If $\varphi(z) = az + b$ where $a \in \mathcal{R}_1(\lambda_n)$, then C_φ is never complex symmetric.*
3. *If $\lambda_1 = 0$, then C_b is complex symmetric.*

7. On similar results when $\beta_* = \infty$

In [8], a characterization for boundedness of C_φ on $\mathcal{H}(\beta, \Lambda)$ in the case of $\beta_* = \infty$ was proven. Specifically, the authors prove the following analogue to Theorem 3.12:

THEOREM 7.1 ([8]). *Let $\beta_* = \infty$. Let φ be an entire function. The following are true.*

(1) *If $\lambda_1 > 0$, then C_φ is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$ if and only if $\varphi(z) = z + b$, $\Re(b) \geq 0$.*
(2) *If $\lambda_1 = 0$, then C_φ is a bounded composition operator on $\mathcal{H}(\beta, \Lambda)$ if and only if $\varphi(z) = z + b$, $\Re(b) \geq 0$, or if φ is constant.*

The authors also estimate $\|C_b\|_{\text{op}}$ and compute $\|C_{z+b}\|_{\text{op}}$. They conjectured that results on certain properties of bounded composition operators on spaces $\{\mathcal{H}(E, \beta_S)\}$, a proper subset of the set of spaces $A := \{\mathcal{H}(\beta, \Lambda) : \beta_* = \infty\}$, hold for bounded composition operators on all spaces in A.

We remark that up to minor modifications in the proof, the following results in our paper hold for all spaces in A. **In the following list, the condition $\boldsymbol{\beta_* = \infty}$ is assumed.** We will use the following observation:

$$r_n(1, \Re(b)) = e^{-\lambda_n \Re(b)}. \tag{7.1}$$

- Operator norms: Propositions 3.5 and 3.11. In the latter case the operator norm simplifies via (7.1) to $\|C_{z+b}\|_{\text{op}} = e^{-\lambda_1 \Re(b)}$ (cf. [20]).
- Essential norms and compactness: Proposition 4.1 and Theorem 4.3 (cf. [19, 20]).
- Schatten class and adjoint: Propositions 4.6 (cf. [27]) and 4.7.
- Compact differences: Theorem 4.15 and Corollary 4.16. In the former, condition (2)(i) simplifies to $\Re(b) = \Re(b') = 0$ (c.f. [19]).
- Closed range: Section 5.1 preamble (on C_b) and Proposition 5.1. The latter simplifies via (7.1) to the statement that $R(C_{z+b})$ is closed if and only if $\Re(b) = 0$.
- Cyclicity: Section 5.2 preamble (on C_b) and Proposition 5.5.
- Conjugations: Theorem 6.3 (cf. [9]). It is worth noting that due to Theorem 7.1, we can weaken the assumption on ξ to simply assuming ξ is entire.
- Complex symmetry: Theorem 6.12, leaving out statement (2) (cf. [9]).

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