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MIRNA CHARIF[1], AHMAD FINO[2], AND LECH ZIELINSKI[3]

Abstract. We prove that the spectrum of the asymmetric quantum Rabi model consists of two eigenvalue sequences $(E_m^+)_{m=0}^\infty$, $(E_m^-)_{m=0}^\infty$, satisfying a two-term asymptotic formula with error estimate of the form $O(m^{-1/4})$, when m tends to infinity.

Keywords: quantum Rabi model, unbounded self-adjoint operators, discrete spectrum, asymtotic distribution of eigenvalues

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1. General presentation of the paper

1.1. **Introduction.** The quantum Rabi model describes the simplest physical example of interactions between radiation and matter. It has its origin in the semi-classical model of interactions between a Two Level System (TLS) and light, due to I. I. Rabi (see [19], [20]) and its fully quantized version was considered in the famous paper of Jaynes and Cummings [14]. The Hamiltonian of the quantum Rabi model (QRM) is given in Definition 1.2(d) and it depends on two real parameters: g and Δ (the coupling constant and the level separation energy in the TLS). In this paper we consider the asymmetric quantum Rabi model (AQRM) given in Definition 1.2(c). It is usually referred to as the QRM with a bias and contains an additional parameter ϵ, called the bias of the model. The AQRM is a fundamental model in the quantum electrodynamics of superconducting circuits (see [23], [16]). The additional term appears due to the tunnelling between two current states (see [11]). We refer to [8] concerning the historical aspects and to [23] for a list of recent research works and experimental realizations of QRM and AQRM.

Let $\mathbf{H}_{\text{Rabi}}$ denote the AQRM Hamiltonian from Definition 1.2(c). Its spectrum is discrete and only in the case $\Delta = 0$, the spectrum of the corresponding Hamiltonian $\mathbf{H}_{0,\text{Rabi}}$, is explicitly known (see Theorem 1.3). Our purpose is to investigate the large n estimate

$$\lambda_n(\mathbf{H}_{\text{Rabi}}) - \lambda_n(\mathbf{H}_{0,\text{Rabi}}) = O(n^{-\rho}), \tag{1.1}$$

where $\rho > 0$ and $(\lambda_n(\mathbf{H}_{\text{Rabi}}))_{n=0}^\infty$ (respectively $(\lambda_n(\mathbf{H}_{0,\text{Rabi}}))_{n=0}^\infty$) is the non-decreasing sequence of eigenvalues of $\mathbf{H}_{\text{Rabi}}$ (respectively $\mathbf{H}_{0,\text{Rabi}}$), counting the multiplicities. Since $\lambda_n(\mathbf{H}_{0,\text{Rabi}})$ is explicitly known, the estimate (1.1) gives the asymptotic behaviour of $\lambda_n(\mathbf{H}_{\text{Rabi}})$ with error $\mathrm{O}(n^{-\rho})$.

The main result of this paper is Theorem 1.4, which states that the estimate (1.1) holds with $\rho = \frac{1}{4}$ and this result is new in the case $\varepsilon \neq 0$. Indeed, the asymptotic behaviour of $\lambda_n(\mathbf{H}_{\text{Rabi}})$ was investigated in the case $\varepsilon = 0$ only (see Section 1.2 for a presentation of known results). It appears that the exponent $\rho = \frac{1}{4}$ is optimal in the case $\varepsilon = 0$.

1.2. **Overview of earlier results.** The problem of the asymptotic behaviour of large eigenvalues of Rabi type models was mentioned e.g. in papers Feranchuk, Komarov, Ulyanenkov [9] and Tur [21]. However, concerning the Hamiltonian $\mathbf{H}_{\rm Rabi}$ given by (1.10), mathematical results have been obtained only in the case $\epsilon = 0$. In this case, $\mathbf{H}_{\rm Rabi}$ is unitarily equivalent to the direct sum $J_{\Delta/2}(g) \oplus J_{-\Delta/2}(g)$, where $J_s(g)$ is the self-adjoint operator defined in $\ell^2(\mathbb{N})$ by the Jacobi matrix

$$J_s(g) = \begin{pmatrix} s & g\sqrt{1} & 0 & 0 & 0 & \cdots \\ g\sqrt{1} & 1-s & g\sqrt{2} & 0 & 0 & \cdots \\ 0 & g\sqrt{2} & 2+s & g\sqrt{3} & 0 & \cdots \\ 0 & 0 & g\sqrt{3} & 3-s & g\sqrt{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.2}$$

The analysis of large eigenvalues of Jacobi matrices was initiated by J. Janas and S. Naboko in the paper [13], which contains fundamental ideas of the method of approximative diagonalizations and asymptotic estimates (see Theorem 5.7). These ideas were developed by M. Malejki in papers [17], [18], but they do not work for the matrix (1.2). In fact, the perturbation $J_s(g) - J_0(g) = \mathrm{diag}(s(-1)^n)_{n=0}^{\infty}$ is not compact and it is not clear whether large eigenvalues of $J_s(g)$ and $J_0(g)$ are close. For this reason, the results obtained by A. Boutet de Monvel, S. Naboko and L. O. Silva in [1], [2], [3], concern a simpler class of operators, called "modified Jaynes-Cummings models". For this class of operators, the oscillations $s(-1)^n$ do appear in the asymptotic formula for large eigenvalues.

The first proof of the estimate (1.1) was obtained by E. A. Yanovich (Tur) with $\rho = \frac{1}{16}$ in [24] (see also [22]). The estimate (1.1) with $\rho = \frac{1}{4}$ was proved in [5] (see also [4]). The paper [5] gives the three-term asymptotic formula for large eigenvalues of the matrix (1.2), which shows that the exponent $\rho = \frac{1}{4}$ cannot be improved. Moreover, this three-term asymptotic formula allows one to recover the values of parameters Δ, g, from the spectrum of the QRM (see [6]) and it appears that the corresponding approximation is the same as the famous GRWA (generalized rotating-wave approximation) introduced by Irish in [12] (and considered earlier in [9]).

We also mention the paper [7], where the asymptotic behaviour of large eigenvalues is investigated for the two-photon asymmetric quantum Rabi model. Our proof of Theorem 1.4 uses several ideas from [7], e.g. a method to overcome the difficulty of the case when $\mathbf{H}_{0,\rm Rabi}$ has double eigenvalues. However, the paper [7] uses an approximation of the two-photon Hamiltonian with $\Delta = 0$ by an operator similar to a first order operator on the circle so that its eigenvectors can be expressed explicitly. In the case of $\mathbf{H}_{0,\rm Rabi}$ given by (1.12), a similar idea gives a pseudo-differential operator of order 1/2 and one needs to construct an approximation of its eigenvectors (see Section 3 and 4).

1.3. **Basic definitions.**

Notation 1.1. (a) We denote by $\mathbb{Z}$ the set of integers and $\mathbb{N} := \{j \in \mathbb{Z} : j \geq 0\}$.

(b) If $\mathcal{J} \subset \mathbb{Z}$, then $\ell^2(\mathcal{J})$ denotes the complex Hilbert space of square-summable sequences $x : \mathcal{J} \to \mathbb{C}$ equipped with the scalar product

$$\langle x, y \rangle_{\ell^2(\mathcal{J})} = \sum_{j \in \mathcal{J}} \overline{x(j)} y(j) \tag{1.3}$$

and the norm $||x||_{\ell^2(\mathcal{J})} := \langle x, x\rangle^{1/2}_{\ell^2(\mathcal{J})}$. We write $\langle\cdot,\cdot\rangle_{\ell^2(\mathbb{N})} = \langle\cdot,\cdot\rangle$ and $||\cdot||_{\ell^2(\mathbb{N})} = ||\cdot||$ in the case $\mathcal{J} = \mathbb{N}$.

(c) $\mathcal{B}_+ = \{e_n\}_{n\in\mathbb{N}}$ denotes the canonical basis of $\ell^2(\mathbb{N})$ (i.e. $e_n(m) = \delta_{n,m}$).

(d) The annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$, are defined as closed linear operators in $\ell^2(\mathbb{N})$ satisfying

$$\hat{a}^\dagger\, e_n = \sqrt{n+1}\, e_{n+1} \text{ for } n \in \mathbb{N} \tag{1.4}$$

$$\hat{a}\, e_0 = 0 \quad \text{and} \quad \hat{a}\, e_n = \sqrt{n}\, e_{n-1} \text{ for } n \in \mathbb{N}\setminus\{0\}. \tag{1.5}$$

(e) Using $(1,0) \in \mathbb{C}^2$ and $(0,1) \in \mathbb{C}^2$ as the canonical basis of the Euclidean space $\mathbb{C}^2$, we denote by $\sigma_{\rm x}$, $\sigma_{\rm z}$, I_2, the linear operators in $\mathbb{C}^2$ defined by the matrices

$$\sigma_{\rm x} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\rm z} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.6}$$

Definition 1.2. (a) The Hamiltonian of the single-mode radiation, $H_{\rm rad}$, is the self-adjoint operator in $\ell^2(\mathbb{N})$ defined on

$$\ell^{2,1}(\mathbb{N}) := \{x \in \ell^2(\mathbb{N}) : \sum_{j\in\mathbb{N}} (1+j^2)|x(j)|^2 < \infty\} \tag{1.7}$$

by the formula

$$H_{\rm rad} e_n = \hat{a}^\dagger \hat{a}\, e_n = n e_n. \tag{1.8}$$

(b) The Two Level System (TLS) Hamiltonian is the linear operator in $\mathbb{C}^2$ defined by the matrix

$$H_{\rm TLS} = \tfrac{1}{2}\begin{pmatrix} \Delta & \epsilon \\ \epsilon & -\Delta \end{pmatrix} = \tfrac{1}{2}\big(\Delta\sigma_{\rm z} + \epsilon\sigma_{\rm x}\big) \tag{1.9}$$

where Δ and ϵ are real parameters.

(c) We define the AQRM Hamiltonian as the linear map $\mathbf{H}_{\rm Rabi} : \mathbb{C}^2\otimes\ell^{2,1}(\mathbb{N}) \to \mathbb{C}^2\otimes\ell^2(\mathbb{N})$ given by

$$\mathbf{H}_{\rm Rabi} = I_2 \otimes H_{\rm rad} + H_{\rm TLS} \otimes I_{\ell^2} + g\sigma_{\rm x} \otimes \big(\hat{a} + \hat{a}^\dagger\big), \tag{1.10}$$

where g is a real parameter. We can also write

$$\mathbf{H}_{\rm Rabi} = \mathbf{H}_{0,\rm Rabi} + \tfrac{1}{2}\Delta\, \sigma_{\rm z} \otimes I_{\ell^2} \tag{1.11}$$

with

$$\mathbf{H}_{0,\rm Rabi} := I_2 \otimes H_{\rm rad} + \sigma_{\rm x} \otimes \big(g\big(\hat{a} + \hat{a}^\dagger\big) + \tfrac{1}{2}\,\epsilon I_{\ell^2}\big). \tag{1.12}$$

(d) The QRM Hamiltonian is given by (1.11)-(1.12) with $\epsilon = 0$.

1.4. **Statement of the result.** We first state the following, well known result (see [11])

Theorem 1.3. *Let $\mathbf{H}_{0,\rm Rabi}$ be as in Definition 1.2. Then there is an orthonormal basis in $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ of the form $\{\mathbf{u}^-_{0,m}\}_{m\in\mathbb{N}} \cup \{\mathbf{u}^+_{0,m}\}_{m\in\mathbb{N}}$, such that*

$$\mathbf{H}_{0,\rm Rabi}\mathbf{u}^\pm_{0,m} = \big(m - g^2 \pm \tfrac{1}{2}\epsilon\big)\mathbf{u}^\pm_{0,m} \tag{1.13}$$

holds for every $m \in \mathbb{N}$.

Proof. See Section 2.2. □

The main result of this paper is the following

Theorem 1.4. *Let $\mathbf{H}_{\mathrm{Rabi}}$ be as in Definition 1.2. Then there is an orthonormal basis in $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ of the form $\{\mathbf{u}_m^-\}_{n\in\mathbb{N}} \cup \{\mathbf{u}_m^+\}_{n\in\mathbb{N}}$ such that*

$$\mathbf{H}_{\mathrm{Rabi}}\mathbf{u}_m^\pm = E_m^\pm \mathbf{u}_m^\pm \tag{1.14}$$

holds for every $m \in \mathbb{N}$ and the eigenvalue sequences $(E_m^-)_{m=0}^\infty$, $(E_m^+)_{m=0}^\infty$, satisfy the large m asymptotic formula

$$E_m^\pm = m - g^2 \pm \tfrac{1}{2}\epsilon + O(m^{-1/4}). \tag{1.15}$$

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.3 and the outline of the proof of Theorem 1.4. Our analysis is based on Proposition 2.6, which gives an approximation for the eigenvectors of $\mathbf{H}_{0,\mathrm{Rabi}}$. In Section 3 we introduce useful notations and prove auxiliary results. In Section 4 we prove Proposition 2.6 and in Section 5 we complete the proof of Theorem 1.4. In Section 6 we describe useful estimates of oscillatory integrals.

1.5. **General notations and conventions.** If $\mathcal{V}$ is a Banach space, then $\mathcal{B}(\mathcal{V})$ denotes the algebra of bounded operators $\mathcal{V} \to \mathcal{V}$ and $||\cdot||_{\mathcal{B}(\mathcal{V})}$ denotes its norm.

If $L : \mathcal{D}_L \to \mathcal{H}$ is a linear map defined on a dense subspace of the complex Hilbert space $\mathcal{H}$, then $\sigma(L)$ denotes the spectrum of L. If L has compact resolvent (i.e. there exists $\lambda_0 \in \mathbb{C} \setminus \sigma(L)$ such that $(L-\lambda_0)^{-1}$ is compact) and $\lambda \in \sigma(L)$, then λ is an isolated eigenvalue and has finite algebraic multiplicity $\mathrm{mult}(\lambda) := \mathrm{rank} P_\lambda(L)$, where $P_\lambda(L)$ denotes the associated Riesz projector (see Section III.5 in [15]).

We write $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and define $\mathrm{L}^2(\mathbb{T})$ as the Hilbert space of Lebesgue square integrable functions $]-\pi,\pi] \to \mathbb{C}$ equipped with the scalar product

$$\langle f, g\rangle_{\mathrm{L}^2(\mathbb{T})} := \int_{-\pi}^{\pi} \overline{f(\theta)} g(\theta)\, \frac{\mathrm{d}\theta}{2\pi}. \tag{1.16}$$

The Fourier transform $\mathcal{F}_{\mathbb{T}} : \mathrm{L}^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is defined by the formula

$$(\mathcal{F}_{\mathbb{T}} f)(j) = \int_{-\pi}^{\pi} f(\theta) \mathrm{e}^{-\mathrm{i}j\theta}\, \frac{\mathrm{d}\theta}{2\pi}. \tag{1.17}$$

For $s \in \mathbb{R}$ we write $\langle s\rangle := (1+s^2)^{1/2}$ and for $\mu \geq 0$ we define

$$\ell^{2,\mu}(\mathbb{Z}) := \{x \in \ell^2(\mathbb{Z}) :\ ||x||_{\ell^{2,\mu}(\mathbb{Z})} < \infty\} \tag{1.18}$$

with

$$||x||_{\ell^{2,\mu}(\mathbb{Z})} = \Big(\sum_{j\in\mathbb{Z}} \langle j\rangle^{2\mu} |x(j)|^2 \Big)^{1/2}. \tag{1.19}$$

We denote by $\mathcal{S}(\mathbb{Z})$ the set of fast decaying sequences $\mathbb{Z} \to \mathbb{C}$. By definition

$$\mathcal{S}(\mathbb{Z}) := \bigcap_{\mu\geq 0} \ell^{2,\mu}(\mathbb{Z}). \tag{1.20}$$

For $x \in \ell^2(\mathbb{Z})$ we denote $\mathrm{supp}\, x := \{j \in \mathbb{Z} :\ x(j) \neq 0\}$ and we define

$$\ell^2_{\mathrm{fin}}(\mathbb{Z}) := \{x \in \ell^2(\mathbb{Z}) : \mathrm{supp}\, x \text{ is finite}\}. \tag{1.21}$$

If $\mathcal{J} \subset \mathbb{Z}$ then we identify

$$\ell^2(\mathcal{J}) = \{x \in \ell^2(\mathbb{Z}) : \mathrm{supp}\, x \subset \mathcal{J}\}. \tag{1.22}$$

In the sequel $\{e_i\}_{i\in\mathbb{Z}}$ denotes the canonical basis of $\ell^2(\mathbb{Z})$ (i.e. $e_i(j)=\delta_{i,j}$ for $i,j\in\mathbb{Z}$). We remark that $\{e_i\}_{i\in\mathbb{N}}$ is a basis of $\{x\in\ell^2(\mathbb{Z}) : \operatorname{supp} x\subset\mathbb{N}\}=\ell^2(\mathbb{N})$, i.e. we can identify $\{e_i\}_{i\in\mathbb{N}}$ with $\mathcal{B}_+$, the canonical basis of $\ell^2(\mathbb{N})$ introduced in Notation 1.1(c).

If $\ell^{2,1}(\mathbb{N})$ is defined by (1.7), then $\ell^{2,1}(\mathbb{N})=\ell^{2,1}(\mathbb{Z})\cap\ell^2(\mathbb{N})$ is a Banach space with the norm induced by the norm of $||\cdot||_{\ell^{2,1}(\mathbb{Z})}$.

We also define $\mathcal{S}(\mathbb{N}):=\mathcal{S}(\mathbb{Z})\cap\ell^2(\mathbb{N})$, the set of fast decaying sequences $x:\mathbb{N}\to\mathbb{C}$.

2. Preliminaries

2.1. **Operators $\mathbf{H}_0$ and $\mathbf{H}$.** Our starting point consists in defining operators $\mathbf{H}_0$ and $\mathbf{H}$, which are unitarily similar to $\mathbf{H}_{0,\mathrm{Rabi}}$ and $\mathbf{H}_{\mathrm{Rabi}}$ respectively.

Notation 2.1. We define the linear maps $\mathbb{C}^2\otimes\ell^{2,1}(\mathbb{N})\to\mathbb{C}^2\otimes\ell^2(\mathbb{N})$ by

$$\mathbf{H}_0:=I_2\otimes\hat{a}^\dagger\hat{a}-\sigma_{\mathrm{z}}\otimes\left(g\big(\hat{a}+\hat{a}^\dagger\big)+\tfrac{1}{2}\epsilon I_{\ell^2(\mathbb{N})}\right), \tag{2.1}$$

$$\mathbf{V}:=\tfrac{1}{2}\Delta\sigma_{\mathrm{x}}\otimes I_{\ell^2(\mathbb{N})} \tag{2.2}$$

and

$$\mathbf{H}:=\mathbf{H}_0+\mathbf{V}. \tag{2.3}$$

Lemma 2.2. *$\mathbf{H}_0$ and $\mathbf{H}$ are unitarily similar to $\mathbf{H}_{0,\mathrm{Rabi}}$ and $\mathbf{H}_{\mathrm{Rabi}}$ respectively.*

Proof. Let $U_{\pi/4}=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ be the matrix of rotation by the angle $\pi/4$. Then

$$U_{\pi/4}\sigma_{\mathrm{x}}U_{\pi/4}^{-1}=-\sigma_{\mathrm{z}},\qquad U_{\pi/4}\sigma_{\mathrm{z}}U_{\pi/4}^{-1}=\sigma_{\mathrm{x}}.$$

We observe that the right hand side of (1.12) becomes the right hand side of (2.1) if σ_{x} is replaced by $-\sigma_{\mathrm{z}}$. If moreover σ_{z} is replaced by σ_{x}, then the right hand side of (1.11) becomes the right hand side of (2.3). □

We remark that $\{(1,0)\otimes e_j : j\in\mathbb{N}\}\cup\{(0,1)\otimes e_j : j\in\mathbb{N}\}$ is the canonical basis of $\mathbb{C}^2\otimes\ell^2(\mathbb{N})$ and we can identify $\mathbb{C}^2\otimes\ell^2(\mathbb{N})$ with $\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})$ writing

$$(1,0)\otimes e_j=(e_j,0)\in\ell^2(\mathbb{N})\times\ell^2(\mathbb{N}),\qquad (0,1)\otimes e_j=(0,e_j)\in\ell^2(\mathbb{N})\times\ell^2(\mathbb{N}).$$

This identification allows us to consider $\mathbf{H}$ as the linear map $\ell^{2,1}(\mathbb{N})\times\ell^{2,1}(\mathbb{N})\to\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})$ of the form

$$\mathbf{H}=\begin{pmatrix}H_{-1} & \frac{\Delta}{2}I_{\ell^2(\mathbb{N})}\\ \frac{\Delta}{2}I_{\ell^2(\mathbb{N})} & H_1\end{pmatrix} \tag{2.4}$$

where $H_\nu:\ell^{2,1}(\mathbb{N})\to\ell^2(\mathbb{N})$ is defined for $\nu=\pm1$ by

$$H_\nu:=\hat{a}^\dagger\hat{a}+\nu g\big(\hat{a}+\hat{a}^\dagger\big)+\tfrac{1}{2}\nu\epsilon. \tag{2.5}$$

We can also write

$$\mathbf{V}=\tfrac{1}{2}\Delta\begin{pmatrix}0_{\ell^2} & I_{\ell^2}\\ I_{\ell^2} & 0_{\ell^2}\end{pmatrix}, \tag{2.6}$$

where 0_{ℓ^2} denotes the null map in $\ell^2(\mathbb{N})$ and

$$\mathbf{H}_0=H_{-1}\oplus H_1.$$

2.2. **Proof of Theorem 1.3.**

Lemma 2.3. *Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{N})$ and let Q_+ denote the closure of the symmetric operator $\mathrm{i}(\hat{a}^\dagger - \hat{a})|_{\mathcal{S}(\mathbb{N})}$. Then*

(a) the operator Q_+ is self-adjoint in $\ell^2(\mathbb{N})$

(b) if $\nu = \pm 1$ and $\{v_{\nu,n}\}_{n\in\mathbb{N}}$ is the orthonormal basis given by

$$v_{\nu,n} := \mathrm{e}^{\mathrm{i}\nu g Q_+} e_n, \tag{2.7}$$

then $H_\nu v_{\nu,n} = (n - g^2 + \frac{1}{2}\nu\epsilon)v_{\nu,n}$ holds for every $n \in \mathbb{N}$.

Proof. The result is well known, but we indicate the details of its proof below.

(a) See Lemma 3.4 in [10].

(b) Due to Corollary 3.7 in [10], $\mathcal{S}(\mathbb{N})$ is an invariant subspace of $\mathrm{e}^{\mathrm{i}tQ_+}$ for every $t \in \mathbb{R}$ and $t \to \mathrm{e}^{\mathrm{i}tQ_+}x$ is a smooth function $\mathbb{R} \to \ell^{2,1}(\mathbb{N})$ if $x \in \mathcal{S}(\mathbb{N})$.

We claim that for every $t \in \mathbb{R}$ one has

$$\mathrm{e}^{-\mathrm{i}tQ_+}(\hat{a} + t)\,\mathrm{e}^{\mathrm{i}tQ_+} = \hat{a}, \qquad \mathrm{e}^{-\mathrm{i}tQ_+}(\hat{a}^\dagger + t)\,\mathrm{e}^{\mathrm{i}tQ_+} = \hat{a}^\dagger. \tag{2.8}$$

Indeed, if $G(t) := \mathrm{e}^{-\mathrm{i}tQ_+}(t + \hat{a})\,\mathrm{e}^{\mathrm{i}tQ_+}$ and $x \in \mathcal{S}(\mathbb{N})$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)x = \mathrm{e}^{-\mathrm{i}tQ_+}(I - [\mathrm{i}Q_+, \hat{a}])\,\mathrm{e}^{\mathrm{i}tQ_+}x = 0 \tag{2.9}$$

follows from $[\mathrm{i}Q_+, \hat{a}] = -[\hat{a}^\dagger, \hat{a}] = I$. It is clear that (2.9) implies $G(t) = G(0) = \hat{a}$ and the second equality (2.8) follows similarly. Using (2.8) with $t = \nu g$, we get

$$\hat{a}^\dagger\hat{a} = \mathrm{e}^{-\mathrm{i}\nu g Q_+}(\hat{a}^\dagger + \nu g)(\hat{a} + \nu g)\,\mathrm{e}^{\mathrm{i}\nu g Q_+} = \mathrm{e}^{-\mathrm{i}\nu g Q_+}(H_\nu + g^2 - \tfrac{1}{2}\nu\varepsilon)\,\mathrm{e}^{\mathrm{i}\nu g Q_+},$$

hence $(H_\nu + g^2 - \frac{1}{2}\nu\varepsilon)\,v_{\nu,n} = \mathrm{e}^{\mathrm{i}\nu g Q_+}\hat{a}^\dagger\hat{a}\,e_n = \mathrm{e}^{\mathrm{i}\nu g Q_+}\,ne_n = nv_{\nu,n}$. □

Proof of Theorem 1.3. Denote $\mathbf{v}^-_{0,m} := (v_{-1,m}, 0)$ and $\mathbf{v}^+_{0,m} := (0, v_{1,m})$. Then

$$(H_{-1} \oplus H_1)\mathbf{v}^\pm_{0,m} = \big(m - g^2 \pm \tfrac{1}{2}\epsilon\big)\mathbf{v}^\pm_{0,m} \tag{2.10}$$

holds for every $m \in \mathbb{N}$ and it is clear that the assertion of Theorem 1.3 follows from the fact that $H_{-1} \oplus H_1$ is unitarily similar to $\mathbf{H}_{0,\mathrm{Rabi}}$.

2.3. **Explicit expression of the n-th eigenvalue of $\mathbf{H}_0$.** Let $(\lambda_n(\mathbf{H}_0))_{n=0}^\infty$ denote the non-decreasing sequence of eigenvalues of $\mathbf{H}_0$, counting the multiplicities. The explicit expression for $\lambda_n(\mathbf{H}_0)$ is given in

Lemma 2.4. *Assume that $\epsilon \geq 0$. Denote $l_0 := \max\{i \in \mathbb{N} : i \leq \epsilon\}$ and define*

$$\mathbf{v}^0_n = (v_{-1,n}, 0) \text{ for } n = 0, \ldots, l_0, \tag{2.11}$$

$$\mathbf{v}^0_{l_0+2i} = (v_{-1,l_0+i}, 0) \text{ for } i \in \mathbb{N}, \tag{2.12}$$

$$\mathbf{v}^0_{l_0+1+2i} = (0, v_{1,i}) \text{ for } i \in \mathbb{N}. \tag{2.13}$$

Then $\{\mathbf{v}^0_n\}_{n\in\mathbb{N}}$ is an orthonormal basis such that $\mathbf{H}_0\mathbf{v}^0_n = d_n\mathbf{v}^0_n$ holds with

$$d_n = n - g^2 - \tfrac{1}{2}\epsilon \text{ for } n = 0, \ldots, l_0, \tag{2.14}$$

$$d_{l_0+2i} = l_0 + i - g^2 - \tfrac{1}{2}\epsilon \text{ for } i \in \mathbb{N}, \tag{2.15}$$

$$d_{l_0+1+2i} = i - g^2 + \tfrac{1}{2}\epsilon \text{ for } i \in \mathbb{N}. \tag{2.16}$$

Moreover $(d_n)_{n=0}^\infty$ is non-decreasing, i.e. one has $d_n = \lambda_n(\mathbf{H}_0)$ for all $n \in \mathbb{N}$.

Proof. We observe that $\{\mathbf{v}_n^0 : n \in \mathbb{N}\} = \{\mathbf{v}_{0,n}^- : n \in \mathbb{N}\} \cup \{\mathbf{v}_{0,n}^+ : n \in \mathbb{N}\}$ holds with $\mathbf{v}_{0,n}^{\pm}$ as in (2.10) and (2.10) ensures $\mathbf{H}_0\mathbf{v}_n^0 = d_n\mathbf{v}_n^0$ with $(d_n)_{n=0}^{\infty}$ given by (2.14)-(2.16). It remains to check that the sequence $(d_n)_{n=0}^{\infty}$ is non-decreasing. It is obvious that $(d_n)_{0\le n\le l_0}$, $(d_{l_0+2i})_{i\in\mathbb{N}}$ and $(d_{l_0+1+2i})_{i\in\mathbb{N}}$ are non-decreasing. Moreover,

$$d_{l_0+1+2i} - d_{l_0+2i} = \epsilon - l_0 \ge 0,$$
$$d_{l_0+1+2i} - d_{l_0+2+2i} = \epsilon - (l_0+1) < 0$$

hold by the choice of l_0. □

2.4. **Outline of the proof of Theorem 1.4.** Using the assertion of Theorem 1.3 and the fact that $\mathbf{H}_{\mathrm{Rabi}} - \mathbf{H}_{0,\mathrm{Rabi}}$ is bounded, we can conclude (see [15]) that $\mathbf{H}_{\mathrm{Rabi}}$ is a self-adjoint, bounded from below operator with compact resolvent. Due to the spectral theorem, there is an orthonormal basis $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that $\mathbf{H}_{\mathrm{Rabi}}\mathbf{u}_n = \lambda_n(\mathbf{H}_{\mathrm{Rabi}})\mathbf{u}_n$ and it remains to prove that (1.1) holds with $\rho = \frac{1}{4}$.

In the sequel we assume that $\epsilon \ge 0$. Then, due to Lemma 2.2 and 2.4, it remains to prove the large n estimate

$$\lambda_n(\mathbf{H}) = d_n + O(n^{-1/4}), \tag{2.17}$$

where $(d_n)_{n\in\mathbb{N}}$ is given by (2.14)-(2.16). The case of $\epsilon < 0$ can be treated similarly by exchanging the roles of H_{-1} and H_1.

Notation 2.5. We define the self-adjoint operators in $\ell^2(\mathbb{N})$ by

$$H = U\mathbf{H}U^{-1}, \tag{2.18}$$

where $U : \ell^2(\mathbb{N})\times\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is the isometric isomorphism satisfying $U\mathbf{v}_n^0 = e_n$ for every $n \in \mathbb{Z}$. Then we can write

$$H = D + V, \tag{2.19}$$

where $V := U\mathbf{V}U^{-1}$ and D is the diagonal operator satisfying

$$De_n = d_n e_n \text{ for } n \in \mathbb{N}. \tag{2.20}$$

Using this notation we rewrite (2.17) in the form

$$\lambda_n(D+V) = d_n + O(n^{-1/4}) \tag{2.21}$$

and in Section 6 we give the proof of (2.21) by showing that H is similar to a certain operator $H' = D + R$, where R is compact and satisfies

$$||R^* e_n|| = O(n^{-1/4}). \tag{2.22}$$

We deduce (2.21) using (2.22) and $\lambda_n(H) = \lambda_n(H')$ in Theorem 5.7.

The key estimate (2.22) follows from the analysis of the matrix

$$\big(\langle e_j, Ve_k\rangle\big)_{j,k\in\mathbb{N}} = \big(\langle \mathbf{v}_j^0, \mathbf{V}\mathbf{v}_k^0\rangle_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})}\big)_{j,k\in\mathbb{N}} \tag{2.23}$$

based on the approximation of $v_{\pm1,n}$ given in

Proposition 2.6. *For $t, \theta \in \mathbb{R}$, $k \in \mathbb{Z}$, we define*

$$\varphi(t;\ \theta, k) := \begin{cases} -2t\,k^{1/2}\sin\theta + t^2\sin\theta\cos\theta & \text{if } k \ge 1 \\ 0 & \text{if } k \le 0 \end{cases} \tag{2.24}$$

Then the following large n estimate

$$||v_{\pm1,n} - u_n(\pm g;\ \cdot)|| = O(n^{-1/2}) \tag{2.25}$$

holds with $u_n(t;\,\cdot) \in \ell^2(\mathbb{N})$ *given by the formula*

$$u_n(t;\,j) := \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\varphi(t;\,\theta,n)+\mathrm{i}(n-j)\theta}\,\frac{\mathrm{d}\theta}{2\pi}. \tag{2.26}$$

Proof. See Section 4. □

3. Auxiliary operators acting in $\ell^2(\mathbb{Z})$

3.1. **Matrices of Fourier type operators.** The proof of Proposition 2.6 is based on the Fourier transform. For this reason, before starting the proof of Proposition 2.6, we move from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{Z})$. At the beginning we introduce a class of matrices $\big(\mathrm{op}(p)(j,k)\big)_{j,k\in\mathbb{Z}}$, which define Fourier type operators acting in $\ell^2(\mathbb{Z})$. We recall the notation $\langle s\rangle := (1+s^2)^{1/2}$.

Notation 3.1. (a) We write $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and define $\mathrm{C}^\infty(\mathbb{T})$ as the set of all smooth 2π-periodic functions $\mathbb{R}\to\mathbb{C}$. We say that $\{f_\tau\}_{\tau\in\mathcal{T}}$ is bounded in $\mathrm{C}^\infty(\mathbb{T})$ if $f_\tau \in \mathrm{C}^\infty(\mathbb{T})$ for every $\tau\in\mathcal{T}$ and for every $\alpha\in\mathbb{N}$ one has

$$\sup_{\tau\in\mathcal{T}}\sup_{\theta\in\mathbb{R}}\Big|\frac{\mathrm{d}^\alpha}{\mathrm{d}\theta^\alpha} f_\tau(\theta)\Big| < \infty$$

(b) If $p(j,\cdot,k)\in C^\infty(\mathbb{T})$ for every $(j,k)\in\mathbb{Z}^2$, then we define the matrix $\big(\mathrm{op}(p)(j,k)\big)_{j,k\in\mathbb{Z}}$ by the formula

$$\mathrm{op}(p)(j,k) := \int_{-\pi}^{\pi} p(j,\theta,k)\mathrm{e}^{\mathrm{i}(k-j)\theta}\,\frac{\mathrm{d}\theta}{2\pi}. \tag{3.1}$$

3.2. **Non-stationary phase estimates.**

Lemma 3.2. *Assume that* $\{b(j,\cdot,k)\}_{j,k\in\mathbb{Z}}$ *and* $\{\phi(\cdot,k)\}_{k\in\mathbb{Z}}$ *are bounded in* $\mathrm{C}^\infty(\mathbb{T})$. *Assume moreover that* ϕ *is real valued and denote*

$$C_\phi := \sup_{(\theta,k)\in\mathbb{R}\times\mathbb{Z}} |\partial_\theta\phi(\theta,k)|, \tag{3.2}$$

$$Z_\phi := \{(j,k)\in\mathbb{Z}^2 :\, |k-j| > 2C_\phi\,\langle k\rangle^{1/2}\}. \tag{3.3}$$

If

$$p(j,\theta,k) = b(j,\theta,k)\mathrm{e}^{\mathrm{i}\phi(\theta,k)\langle k\rangle^{1/2}}, \tag{3.4}$$

then for every $N\in\mathbb{N}$ *there is* $C_N>0$ *such that the estimate*

$$|\mathrm{op}(p)(j,k)| \le C_N\langle j\rangle^{-N}\langle k\rangle^{-N} \tag{3.5}$$

holds for $(j,k)\in Z_\phi$.

Proof. For $(j,k)\in Z_\phi$ we can write

$$\mathrm{op}(p)(j,k) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-j)(\psi(j,\theta,k)+\theta)} b(j,\theta,k)\,\frac{\mathrm{d}\theta}{2\pi} \tag{3.6}$$

where

$$\psi(j,\theta,k) := \phi(j,\theta,k)\frac{\langle k\rangle^{1/2}}{k-j}. \tag{3.7}$$

However, $\{\psi(j,\cdot,k)\}_{(j,k)\in Z_\phi}$ is bounded in $\mathrm{C}^\infty(\mathbb{T})$ and

$$(j,k)\in Z_\phi \;\Rightarrow\; |\partial_\theta\psi(j,\theta,k)| \le \tfrac{1}{2} \;\Rightarrow\; \partial_\theta\left(\psi(j,\theta,k)+\theta\right) \ge \tfrac{1}{2}.$$

Thus the non-stationary phase estimates (see Lemma 6.1(b)) yield existence of $C_N > 0$ such that

$$|\mathrm{op}(p)(j,k)| \le C_N \langle k-j\rangle^{-5N} \tag{3.8}$$

holds for $(j,k) \in Z_\phi$. Using $\langle k-j\rangle^{-N} \le 2^N \langle j\rangle^{-N}\langle k\rangle^N$ and

$$(j,k) \in Z_\phi \;\Rightarrow\; \langle k-j\rangle^{-4N} \le 2^{-4N} C_\phi^{-4N} \langle k\rangle^{-2N},$$

we can estimate the right hand side of (3.8) by $C_N 2^{-3N} C_\phi^{-4N} \langle j\rangle^{-N}\langle k\rangle^{-N}$. □

Corollary 3.3. *If p satisfies the assumptions of Lemma 3.2, then one can define a linear operator $\mathrm{op}(p) : \ell^2_{\mathrm{fin}}(\mathbb{Z}) \to \mathcal{S}(\mathbb{Z})$ by the formula*

$$(\mathrm{op}(p)e_k)(j) = \langle e_j, \mathrm{op}(p)e_k\rangle = \mathrm{op}(p)(j,k), \tag{3.9}$$

where $\mathrm{op}(p)(j,k)$ is given by (3.1).

Proof. It suffices to observe that $\mathrm{op}(p)(\cdot,k) \in \mathcal{S}(\mathbb{Z})$ holds for every $k \in \mathbb{Z}$. Indeed, if $k \in \mathbb{Z}$ is fixed, then (3.5) holds for $j > k + 2C_\phi\langle k\rangle^{1/2}$ and for $j < k - 2C_\phi\langle k\rangle^{1/2}$. □

3.3. **Localisation of $u_n(t)$.**

Notation 3.4. (a) For t, $\theta \in \mathbb{R}$ and $j,k \in \mathbb{Z}$ we denote

$$\tilde\varphi(t;\; j,\theta,k) := \varphi(t;\; \theta,k), \tag{3.10}$$

where $\varphi(t;\; \theta,k)$ is given by (2.24).

(b) We define $u_n(t) \in \mathcal{S}(\mathbb{Z})$ by the formula

$$u_n(t) := \mathrm{op}(\mathrm{e}^{\mathrm{i}\tilde\varphi(t)})e_n. \tag{3.11}$$

It is clear that (3.11) defines an extension of $u_n(t;\; \cdot)$ introduced in (2.26).

Corollary 3.5. *Let $t_0 > 0$ be fixed.*

(a) Assume that $\widetilde{C}_{t_0} > 0$ is large enough and for $n \in \mathbb{N}^*$ denote

$$Z_n := \{j \in \mathbb{Z} :\; |j-n| \le \widetilde{C}_{t_0}\langle n\rangle^{1/2}\}. \tag{3.12}$$

Let Π_{Z_n} be the orthogonal projection on $\mathrm{span}\,\{e_k\}_{k\in Z_n}$. Then for every $N \in \mathbb{N}$ we can find $C_{N,t_0} > 0$ such that

$$\sup_{-t_0\le t\le t_0} ||(I-\Pi_{Z_n})u_n(t)||_{\ell^{2,N}(\mathbb{Z})} \le C_{N,t_0}\langle n\rangle^{-N}. \tag{3.13}$$

(b) For every N, $n \in \mathbb{N}$, the function $t \to u_n(t)$ is differentiable $[-t_0,t_0] \to \ell^{2,N}(\mathbb{Z})$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u_n(t) = \mathrm{op}(\mathrm{i}\partial_t\tilde\varphi(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)})e_n \tag{3.14}$$

and we can find $\tilde C_{N,t_0}$ such that

$$\sup_{-t_0\le t\le t_0} ||(I-\Pi_{Z_n})\tfrac{\mathrm{d}}{\mathrm{d}t}u_n(t)||_{\ell^{2,N}(\mathbb{Z})} \le \tilde C_{N,t_0}\langle n\rangle^{-N}. \tag{3.15}$$

Proof. (a) We fix $t_0 > 0$. Reasoning as in the proof of Lemma 3.2, for every $N \in \mathbb{N}$ we can find the constant C_{N,t_0} such that

$$j \in \mathbb{Z}\setminus Z_n \;\Rightarrow \sup_{-t_0\le t\le t_0} \big|\langle j\rangle^N\big(\mathrm{op}(\mathrm{e}^{\mathrm{i}\tilde\varphi(t)})e_n\big)(j)\big| \le C_{N,t_0}\langle n\rangle^{-N}\langle j\rangle^{-2}. \tag{3.16}$$

(b) We fix $t_0 > 0$, $n \in \mathbb{N}$ and denote

$$w_n(t,s) := (u_n(t+s)-u_n(t))/s, \qquad w_n(t) := \mathrm{op}(\mathrm{i}\partial_t\tilde\varphi(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)})e_n.$$

We claim that the dominated convergence theorem implies

$$||w_n(t,s) - w_n(t)||^2_{\ell^{2,N}(\mathbb{Z})} = \sum_{j\in\mathbb{Z}} \langle j\rangle^{2N} |w_n(t,s;\ j) - w_n(t;\ j)|^2 \xrightarrow[s\to 0]{} 0. \tag{3.17}$$

Indeed, $\lim_{s\to 0} w_n(t,s;\ j) = \frac{\mathrm{d}}{\mathrm{d}t} u_n(t;\ j) = w_n(t;\ j)$ for every $j \in \mathbb{N}$ and

$$j \in \mathbb{Z}\setminus Z_n \ \Rightarrow \sup_{-t_0\le s,t\le t_0} |w_n(t,s;\ j)| \le \sup_{-2t_0\le t\le 2t_0} |w_n(t;\ j)| \le C_{N,2t_0}\langle j\rangle^{-N-1}\langle n\rangle^{-N},$$

where we get the last estimate similarly as in the proof of Lemma 3.2. □

3.4. **An auxiliary result.**

Notation 3.6. For $\mu \in \mathbb{R}$ we define $\mathrm{S}^\mu_1(\mathbb{T}\times\mathbb{R})$ as the set of smooth functions $q : \mathbb{R}^2 \to \mathbb{C}$ satisfying $q(\theta + 2\pi, k) = q(\theta, k)$ for all $(\theta, k) \in \mathbb{R}^2$ and such that for every $\alpha_1, \alpha_2 \in \mathbb{N}$,

$$|\partial_\theta^{\alpha_1}\partial_k^{\alpha_2} q(\theta,k)| \le C_{\alpha_1,\alpha_2}\langle k\rangle^{\mu-\alpha_2} \tag{3.18}$$

holds with a certain constant C_{α_1,α_2} independent of $(\theta, k) \in \mathbb{R}^2$. We will use the following elementary properties of symbols:

$$q \in \mathrm{S}^\mu_1(\mathbb{T}\times\mathbb{R}) \ \Rightarrow\ \partial_\theta^{\alpha_1}\partial_k^{\alpha_2} q \in \mathrm{S}^{\mu-\alpha_2}_1(\mathbb{T}\times\mathbb{R})$$

$$q_1 \in \mathrm{S}^{\mu_1}_1(\mathbb{T}\times\mathbb{R}),\quad q_2 \in \mathrm{S}^{\mu_2}_1(\mathbb{T}\times\mathbb{R}) \ \Rightarrow\ q_1 q_2 \in \mathrm{S}^{\mu_1+\mu_2}_1(\mathbb{T}\times\mathbb{R}).$$

In Section 4.5 we will use the following

Lemma 3.7. *Assume that $q \in \mathrm{S}^{1/2}_1(\mathbb{T}\times\mathbb{R})$ and define*

$$\varphi(t;\ \theta,k) := tq(\theta,k) + \tfrac{1}{2}t^2\partial_k q(\theta,k)\partial_\theta q(\theta,k) \tag{3.19}$$

for t, θ, $k \in \mathbb{R}$. If

$$r_0(t;\ \theta,k) = q(\theta, k + \partial_\theta\varphi(t;\ \theta,k)) - \partial_t\varphi(t;\ \theta,k), \tag{3.20}$$

then there is $C > 0$ such that for all θ, $k \in \mathbb{R}$ and $t \in [-t_0, t_0]$ one has

$$|r_0(t;\ \theta,k)| \le C\langle k\rangle^{-1/2}. \tag{3.21}$$

Proof. Step 1. Let us write $\partial_1^{\alpha_1}\partial_2^{\alpha_2} q(\theta,k) := \partial_\theta^{\alpha_1}\partial_k^{\alpha_2} q(\theta,k)$ and denote

$$r_1(t;\ \theta,k) := q(\theta, k + \partial_\theta\varphi(t;\ \theta,k)) - q(\theta,k) - \partial_2 q(\theta,k)\partial_\theta\varphi(t;\ \theta,k). \tag{3.22}$$

We claim that there is a constant $C_1 > 0$ such that the estimate

$$|r_1(t;\ \theta,k)| \le C_1\langle k\rangle^{-1/2} \tag{3.23}$$

holds for all θ, $k \in \mathbb{R}$ and $t \in [-t_0, t_0]$. Indeed, since the Taylor expansion of order 2, gives

$$r_1(t;\ \theta,k) = \big(\partial_\theta\varphi(t;\ \theta,k)\big)^2 \int_0^1 \partial_2^2 q(\theta, k + s\,\partial_\theta\varphi(t;\ \theta,k))\,(1-s)\,\mathrm{d}s, \tag{3.24}$$

we deduce (3.23) from (3.24) due to $(\partial_\theta\varphi(t))^2 \in \mathrm{S}^1_1(\mathbb{T}\times\mathbb{R})$ and $\partial_2^2 q \in \mathrm{S}^{-3/2}_1(\mathbb{T}\times\mathbb{R})$.
Step 2. Using (3.19) in (3.20), we get the expression

$$r_0(t;\ \theta,k) = q(\theta, k + \partial_\theta\varphi(t;\ \theta,k)) - q(\theta,k) - t\partial_2 q(\theta,k)\partial_1 q(\theta,k), \tag{3.25}$$

and combining (3.25) with (3.22), we get

$$r_0(t) - r_1(t) = \partial_2 q\,\partial_\theta(\varphi(t) - tq). \tag{3.26}$$

Since $\partial_2 q \in \mathrm{S}^{-1/2}_1(\mathbb{T}\times\mathbb{R})$ and $\partial_\theta(\varphi(t)-tq) = \partial_\theta\big(\frac{1}{2}t^2\partial_2 q\,\partial_1 q\big)$ is bounded due to $\partial_2 q\,\partial_1 q \in \mathrm{S}^0_1(\mathbb{T}\times\mathbb{R})$, we can find $C_2>0$ such that for all θ, $k\in\mathbb{R}$, $t\in[-t_0,t_0]$, one has

$$|r_0(t)-r_1(t)| \leq C_2\langle k\rangle^{-1/2}. \tag{3.27}$$

Combining (3.23) and (3.27), we complete the proof of (3.21). □

4. Proof of Proposition 2.6

4.1. **Step 1.** We move to $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z}\setminus\mathbb{N})\oplus\ell^2(\mathbb{N})$ by introducing the operator

$$Q := 0_{\ell^2(\mathbb{Z}\setminus\mathbb{N})}\oplus Q_+, \tag{4.1}$$

where $0_{\ell^2(\mathbb{Z}\setminus\mathbb{N})}$ is the null map on $\ell^2(\mathbb{Z}\setminus\mathbb{N})$. Then Q is self-adjoint in $\ell^2(\mathbb{Z})$ and

$$\mathrm{e}^{\mathrm{i}tQ} = I_{\ell^2(\mathbb{Z}\setminus\mathbb{N})}\oplus \mathrm{e}^{\mathrm{i}tQ_+}.$$

If $\Pi_{\geq 0}\in\mathcal{B}(\ell^2(\mathbb{Z}))$ denotes the orthogonal projection on $\ell^2(\mathbb{N})$ and $u_n(t)$ is given by Notation 3.4(b), then the assertion of Proposition 2.6 says that the large n estimate

$$\big\|\Pi_{\geq 0}\big(u_n(t)-\mathrm{e}^{\mathrm{i}tQ}e_n\big)\big\|_{\ell^2(\mathbb{Z})} = O(n^{-1/2}) \tag{4.2}$$

holds for $t=\pm g$. In this section we will prove that for every $t_0>0$ one can find $C_{t_0}>0$ such that

$$\sup_{-t_0\leq t\leq t_0}\big\|u_n(t;\cdot)-\mathrm{e}^{\mathrm{i}tQ}e_n\big\|_{\ell^2(\mathbb{Z})} \leq C_{t_0}n^{-1/2} \tag{4.3}$$

holds for all $n\in\mathbb{N}^*$.

Notation 4.1. (a) In the sequel $\Lambda:\ell^{2,1}(\mathbb{Z})\to\ell^2(\mathbb{Z})$ is the self-adjoint operator in $\ell^2(\mathbb{Z})$ satisfying $\Lambda e_j = je_j$ for $j\in\mathbb{Z}$. Moreover, we denote $\langle\Lambda\rangle := (1+\Lambda^2)^{1/2}$.

(b) If $h:\mathbb{Z}\to\mathbb{C}$ then $h(\Lambda)=\mathrm{diag}(h(j))_{j\in\mathbb{Z}}$ is defined by the functional calculus, i.e. $h(\Lambda)$ is the closed linear operator satisfying

$$h(\Lambda)e_j = h(j)e_j \text{ for } j\in\mathbb{Z}. \tag{4.4}$$

(c) We define $S\in\mathcal{B}(\ell^2(\mathbb{Z}))$ as the shift satisfying $Se_j := e_{j+1}$ for $j\in\mathbb{Z}$.

Using these notations we can express the operator Q in the form

$$Q = \mathrm{i}\big(\Lambda_+^{1/2}S - S^{-1}\Lambda_+^{1/2}\big), \tag{4.5}$$

where $\Lambda_+^{1/2}=h(\Lambda)$ with $h(j)=j_+^{1/2}$.

4.2. **Step 2.** We fix $t_0>0$ and consider all estimates uniformly with respect to $t\in[-t_0,t_0]$. If $\mu\in\mathbb{R}$, then we write

$$w_n(t) = O(n^{-\mu}) \text{ for } t\in[-t_0,t_0]$$

if and only if there is $\widetilde{C}_{t_0}>0$ such that for all $n\in\mathbb{N}^*$ one has

$$\sup_{-t_0\leq t\leq t_0}\|w_n(t)\|_{\ell^2(\mathbb{Z})} \leq \widetilde{C}_{t_0}n^{-\mu}. \tag{4.6}$$

At the beginning we observe that Corollary 3.5 allows us to write

$$u_n(t)-\mathrm{e}^{\mathrm{i}tQ}\mathrm{e}_n = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{\mathrm{i}(t-s)Q}u_n(s)\big)\,\mathrm{d}s = -\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-s)Q}\Big(Q+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\Big)u_n(s)\,\mathrm{d}s,$$

hence it suffices to prove the estimate

$$\Big(Q+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big)u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0]. \tag{4.7}$$

We claim that (4.7) follows from the estimate

$$\Big(\widetilde{Q}+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big)u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0] \tag{4.8}$$

where

$$\widetilde{Q}:=\mathrm{i}\Lambda_+^{1/2}(S-S^{-1}). \tag{4.9}$$

To prove this claim, it suffices to show the estimate

$$(\widetilde{Q}-Q)u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0]. \tag{4.10}$$

We first observe that $(Q-\widetilde{Q})\langle\Lambda\rangle^{1/2}\in\mathcal{B}(\ell^2(\mathbb{Z}))$. Indeed, using $h(j):=j_+^{1/2}$ and $h_1(j):=h(j-1)$ we get $S^{-1}h(\Lambda)-h(\Lambda)S^{-1}=S^{-1}(h(\Lambda)-h(S\Lambda S^{-1}))=S^{-1}(h-h_1)(\Lambda)$ and $(h-h_1)(\Lambda)\langle\Lambda\rangle^{1/2}\in\mathcal{B}(\ell^2(\mathbb{Z}))$ follows from $(h-h_1)(j)=O(\langle j\rangle^{-1/2})$.

Due to $(Q-\widetilde{Q})\langle\Lambda\rangle^{1/2}\in\mathcal{B}(\ell^2(\mathbb{Z}))$, (4.10) holds if we know the estimate

$$\langle\Lambda\rangle^{-1/2}u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0]. \tag{4.11}$$

However, if Π_{Z_n} is as in Corollary 3.5, then $(I-\Pi_{Z_n})u_n(t)=O(n^{-N})$ for every $N\in\mathbb{N}$ and (4.11) follows from $\langle\Lambda\rangle^{-1/2}\Pi_{Z_n}=O(n^{-1/2})$.

4.3. **Step 3.** We claim that instead of (4.8), it suffices to prove the estimate

$$\Pi_{Z_n}\Big(\widetilde{Q}+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big)u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0] \tag{4.12}$$

where Π_{Z_n} is as in Corollary 3.5 and $\widetilde{Q}$ is given by (4.9).

Indeed, due to (3.15), it suffices to show that

$$(I-\Pi_{Z_n})\widetilde{Q}u_n(t)=O(n^{-1/2})\text{ for }t\in[-t_0,t_0] \tag{4.13}$$

By definition $((S-S^{-1})x)(j)=x(j-1)-x(j+1)$ and

$$x(j)=\mathrm{e}^{-\mathrm{i}j\theta}\;\Rightarrow\;(\mathrm{i}(S-S^{-1})x)(j)=\mathrm{i}(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta})x(j)=(-2\sin\theta)\,x(j),$$

hence

$$\mathrm{i}(S-S^{-1})u_n(t)=\mathrm{op}\big(p(t)\big)e_n\text{ with }p(t;j,\theta,k):=(-2\sin\theta)\,\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t;\,j,\theta,k)} \tag{4.14}$$

Therefore $\widetilde{Q}u_n(t)=\Lambda_+^{1/2}\mathrm{op}\big(p(t)\big)e_n$ and it remains to observe that reasoning as in the proof of Corollary 3.5, we find that for every $N\in\mathbb{N}$ one has

$$\langle\Lambda\rangle^{1/2}(I-\Pi_{Z_n})\mathrm{op}\big(p(t)\big)\mathrm{e}_n=O(n^{-N})\text{ for }t\in[-t_0,t_0] \tag{4.15}$$

4.4. **Step 4.**

Notation 4.2. (a) We define $q\in\mathrm{S}_1^{1/2}(\mathbb{T}\times\mathbb{R})$ by the formula

$$q(\theta,k)=-2\chi_0(k)|k|^{1/2}\sin\theta, \tag{4.16}$$

where $\chi_0\in\mathrm{C}^\infty(\mathbb{R})$ is a fixed function such that $\chi_0(k)=0$ if $k\le 0$, $\chi_0(k)=1$ if $k\ge 1$ and $0\le\chi_0(k)\le 1$ for $k\in[0,1]$.

(b) We define $\varphi:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ by

$$\varphi(t;\;\theta,k):=tq(\theta,k)+\tfrac{1}{2}\,t^2\,\partial_k q(\theta,k)\partial_\theta q(\theta,k) \tag{4.17}$$

with q given by (4.16). By direct calculation we check that for $k \geq 1$ one has

$$\varphi(t;\ \theta, k) = -2tk^{1/2} \sin\theta + t^2\ \sin\theta\cos\theta \tag{4.18}$$

and $\varphi(t;\ \theta, k) = 0$ if $k \leq 0$, i.e. φ satisfies (2.24) for $k \in \mathbb{Z}$.
(c) We define $\tilde{\varphi} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{\varphi}(t;\ j, \theta, k) := \varphi(t;\ \theta, k) \tag{4.19}$$

with φ given by (4.17). Obviously $u_n(t) = \mathrm{op}(\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n$ still holds.

Assume that r_0 is given by (3.20) with φ as in (4.17). Let

$$f_n(t;\ \theta) := r_0(t;\ \theta, n)\mathrm{e}^{\mathrm{i}\varphi(t;\ \theta, n)+\mathrm{i}n\theta}.$$

Then $\mathcal{F}_{\mathbb{T}} f_n(t) = \mathrm{op}(\tilde{r}_0(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n$ holds with

$$\tilde{r}_0(t;\ j, \theta, k) := r_0(t;\ \theta, k) \tag{4.20}$$

and

$$||\mathrm{op}(\tilde{r}_0(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n||_{\ell^2(\mathbb{Z})} = ||\mathcal{F}_{\mathbb{T}} f_n(t)||_{\ell^2(\mathbb{Z})} = ||f_n(t)||_{\mathrm{L}^2(\mathbb{T})} = O(n^{-1/2}), \tag{4.21}$$

where the last estimates results from (3.21).

We observe that using (3.14) and (4.14), we obtain the expression

$$\Big(\widetilde{Q} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big)u_n(t) = \mathrm{op}(\tilde{b}(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n \tag{4.22}$$

with

$$\tilde{b}(t;\ j, \theta, k) := q(\theta, j) - \partial_t\varphi(t;\ \theta, k), \tag{4.23}$$

hence the estimate (4.12) can be written in the form

$$\Pi_{Z_n}\mathrm{op}(\tilde{b}(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n = O(n^{-1/2}) \text{ for } t \in [-t_0, t_0]. \tag{4.24}$$

We introduce $b(t) := \tilde{b}(t) - \tilde{r}_0(t)$ and observe that due to (4.21), the estimate

$$\Pi_{Z_n}\mathrm{op}(b(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)})e_n = O(n^{-1/2}) \text{ for } t \in [-t_0, t_0] \tag{4.25}$$

is equivalent to (4.12). Moreover, using (4.23) and (3.20), we find

$$b(t;\ j, \theta, k) = (\tilde{b} - \tilde{r}_0)(t;\ j, \theta, k) = q(\theta, j) - q(\theta, \partial_\theta\varphi(t;\ \theta, k) + k). \tag{4.26}$$

4.5. **End of the proof of Proposition 2.6.** In remains to prove (4.25) with b given by (4.26). Using the Taylor's expansion of order 2, we can express

$$-b = b_1 + b_2 \tag{4.27}$$

with

$$b_1(t;\ j, \theta, k) := \partial_2 q(\theta, j)(\partial_\theta\varphi(t;\ \theta, k) + k - j) \tag{4.28}$$

$$b_2(t;\ j, \theta, k) := q_2(t;\ j, \theta, k)(\partial_\theta\varphi(t;\ \theta, k) + k - j)^2, \tag{4.29}$$

where

$$q_2(t;\ j, \theta, k) := \int_0^1 \partial_2^2 q(\theta, j + s(\partial_\theta\varphi(t;\ k, \theta) + k - j))\,(1 - s)\,\mathrm{d}s. \tag{4.30}$$

In order to prove

$$\mathrm{op}\big(b_1(t)\mathrm{e}^{\mathrm{i}\tilde{\varphi}(t)}\big)e_n = O(n^{-1/2}) \text{ for } t \in [-t_0, t_0] \tag{4.31}$$

we denote

$$q_1(t;\ j, \theta, k) := \partial_2 q(\theta, j) \tag{4.32}$$

$$\psi(t;\ j,\theta,k) := \varphi(t;\ \theta,k) + (k-j)\theta. \tag{4.33}$$

Using (4.32)-(4.33), we can write (4.28) in the form $b_1 = q_1\partial_\theta\psi$, hence

$$\mathrm{op}\big(b_1(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big)(j,k) = \int_{-\pi}^{\pi} \big(q_1\partial_\theta\psi\,\mathrm{e}^{\mathrm{i}\psi}\big)(t;\ j,\theta,k)\,\frac{\mathrm{d}\theta}{2\pi}. \tag{4.34}$$

Writing $\partial_\theta\psi\,\mathrm{e}^{\mathrm{i}\psi} = -\mathrm{i}\partial_\theta\big(\mathrm{e}^{\mathrm{i}\psi}\big)$ in the right hand side of (4.34), we can integrate by parts and express (4.34) in the form

$$\mathrm{i}\int_{-\pi}^{\pi} \big(\partial_\theta q_1\,\mathrm{e}^{\mathrm{i}\psi}\big)(t;\ j,\theta,k)\,\frac{\mathrm{d}\theta}{2\pi} = (\mathcal{F}_{\mathbb{T}} f_k(t))(j) \tag{4.35}$$

with $f_k(t;\ \theta) := \mathrm{i}\partial_1\partial_2 q(\theta,k)\mathrm{e}^{\mathrm{i}\varphi(t;\ \theta,k)+\mathrm{i}k\theta}$ and

$$||\mathrm{op}\big(b_1(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big)\mathrm{e}_n||_{\ell^2(\mathbb{Z})} = ||\mathcal{F}_{\mathbb{T}} f_n(t)||_{\ell^2(\mathbb{Z})} = ||f_n(t)||_{\mathrm{L}^2(\mathbb{T})} = O(n^{-1/2}), \tag{4.36}$$

where the last estimate follows from $\partial_1\partial_2 q \in \mathrm{S}_1^{-1/2}(\mathbb{T}\times\mathbb{R})$.

It remains to prove

$$\Pi_{Z_n}\mathrm{op}\big(b_2(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big)e_n = O(n^{-1/2}). \tag{4.37}$$

For this purpose we first observe that $b_2 = q_2(\partial_\theta\psi)^2$ and

$$\int_{-\pi}^{\pi} q_2\,(\partial_\theta\psi)^2\,\mathrm{e}^{\mathrm{i}\psi} = -\mathrm{i}\int_{-\pi}^{\pi} q_2\,\partial_\theta\psi\,\partial_\theta\big(\mathrm{e}^{\mathrm{i}\psi}\big) = \mathrm{i}\int_{-\pi}^{\pi} \partial_\theta(q_2\partial_\theta\psi)\,\mathrm{e}^{\mathrm{i}\psi}, \tag{4.38}$$

where the last equality follows from the integration by parts. However, (4.38) ensures

$$\mathrm{op}\,\big(\,b_2(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big) = \mathrm{op}\,\big(\,\mathrm{i}\,\partial_\theta(q_2\partial_\theta\psi)(t)\,\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big). \tag{4.39}$$

Using $\varphi(t;\cdot) \in \mathrm{S}_1^{1/2}(\mathbb{T}\times\mathbb{R})$ and $\partial_2^2 q \in \mathrm{S}_1^{-3/2}(\mathbb{T}\times\mathbb{R})$ in the expression of q_2, we obtain

$$\sup_{j\in Z_n} |(\partial_\theta(q_2\partial_\theta\psi)(t;\ j,\theta,n)| = O(n^{-1}). \tag{4.40}$$

To complete the proof we observe that (4.37) follows from (4.39) and

$$||\Pi_{Z_n}\mathrm{op}\ \big(\partial_\theta(q_2\partial_\theta\psi)(t)\mathrm{e}^{\mathrm{i}\tilde\varphi(t)}\big)e_n||_{\ell^2(\mathbb{Z})} \le \sum_{j\in Z_n} |(\partial_\theta(q_2\partial_\theta\psi)(t;\ j,\theta,n)| = O(n^{-1/2}),$$

where the last estimate is due to (4.40) and $\mathrm{card}\,Z_n = O(n^{1/2})$.

5. Proof of Theorem 1.4

5.1. Asymptotic orthogonality between eigenvectors of H_{-1} and H_1.

Lemma 5.1. *Let $\{v_{\pm1,j}\}_{j\in\mathbb{N}}$ be as in Lemma 2.3. If $c>0$ is fixed small enough, then*

$$|k-j| < c(j^{1/2}+k^{1/2}) \ \Rightarrow\ |\langle v_{-1,k}, v_{1,j}\rangle| \le C_0\langle j\rangle^{-1/4} \tag{5.1}$$

holds with a certain constant C_0, independent of $k,j\in\mathbb{N}$.

Proof. Due to Proposition 2.6,

$$\langle v_{-1,k}, v_{1,j}\rangle = \langle \mathrm{e}^{-\mathrm{i}gQ}e_k, \mathrm{e}^{\mathrm{i}gQ}e_j\rangle_{\ell^2(\mathbb{Z})} = \gamma_{k,j} + O(k^{-1/2}+j^{-1/2}) \tag{5.2}$$

holds with $\gamma_{k,j} := \langle u_k(-g), u_j(g)\rangle_{\ell^2(\mathbb{Z})}$. Let

$$f_n^{\pm}(\theta) := \mathrm{e}^{\mp 2\mathrm{i}gn^{1/2}\sin\theta + \mathrm{i}g^2\sin\theta\cos\theta + \mathrm{i}n\theta}. \tag{5.3}$$

Then $u_n(\pm g) = \mathcal{F}_{\mathbb{T}} f_n^{\pm}$ and

$$\gamma_{k,j} := \langle \mathcal{F}_{\mathbb{T}} f_k^-, \mathcal{F}_{\mathbb{T}} f_j^+ \rangle_{\ell^2(\mathbb{Z})} = \langle f_k^-, f_j^+ \rangle_{\mathrm{L}^2(\mathbb{T})} = \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(j^{1/2}+k^{1/2})\Psi(j,\theta,k)} \frac{\mathrm{d}\theta}{2\pi}$$

holds with

$$\Psi(j,\theta,k) := 2g\sin\theta + \frac{(k-j)\theta}{j^{1/2}+k^{1/2}}.$$

Denote $\mathcal{I}_0 := [-5\pi/6, -\pi/6] \cup [\pi/6, 5\pi/6]$. Then

$$\theta \in \mathcal{I}_0 \;\Rightarrow\; |\partial_\theta^2 \Psi(j,\theta,k)| = |2g\sin\theta| \geq g > 0$$

and van der Corput Lemma (see Section 6) ensures

$$\int_{\mathcal{I}_0} \mathrm{e}^{-\mathrm{i}(j^{1/2}+k^{1/2})\Psi(j,\theta,k)} \frac{\mathrm{d}\theta}{2\pi} = O((j^{1/2}+k^{1/2})^{-1/2}).$$

Denote $\tau_{j,k} := (k-j)/(j^{1/2}+k^{1/2})$ and assume that $|\tau_{j,k}| \leq g/2$. Then

$$\theta \in [-\pi,\pi] \setminus \mathcal{I}_0 \;\Rightarrow\; |\partial_\theta \Psi(j,\theta,k)| \geq |2g\cos\theta| - |\tau_{j,k}| \geq g/2 > 0$$

and the non-stationary phase estimate (see Lemma 6.1(a)) ensures

$$\int_{[-\pi,\pi]\setminus\mathcal{I}_0} \mathrm{e}^{-\mathrm{i}(j^{1/2}+k^{1/2})\Psi(j,\theta,k)} \frac{\mathrm{d}\theta}{2\pi} = O((j^{1/2}+k^{1/2})^{-1}).$$

□

Corollary 5.2. *For $j,\, k \in \mathbb{N}$ we denote $V(j,k) := \langle e_j, V e_k \rangle$. If $c > 0$ is small enough, then there exists $C > 0$ such that*

$$|m-n| < cn^{1/2} \;\Rightarrow\; |V(m,n)| \leq C\langle n \rangle^{-1/4}. \tag{5.4}$$

Proof. By definition of V, if $i, j \in \mathbb{N}$ then

$$V(l_0+2i, l_0+2j) = \tfrac{\Delta}{2}\langle (v_{-1,l_0+i}, 0), (0, v_{-1,l_0+j}) \rangle_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})} = 0 \tag{5.5}$$

$$V(l_0+2i+1, l_0+2j+1) = \tfrac{\Delta}{2}\langle (0, v_{1,i}), (v_{1,j}, 0) \rangle_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})} = 0 \tag{5.6}$$

$$V(l_0+2i, l_0+2j+1) = \tfrac{\Delta}{2}\langle (v_{-1,l_0+i}, 0), (v_{1,j}, 0) \rangle_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})} = \tfrac{\Delta}{2}\langle v_{-1,l_0+i}, v_{1,j} \rangle. \tag{5.7}$$

To complete the proof we fix $j_0 \in \mathbb{N}$ large enough and consider $k = l_0 + i$, $m = l_0 + 2i$, $n = l_0 + 2j + 1$ with $j \geq j_0$. Then one has

$$|m-n| < cn^{1/2} \Rightarrow |j-k| < c(j^{1/2}+k^{1/2}),$$

hence $|V(m,n)| = \left|\tfrac{\Delta}{2}\gamma_{j,k}\right| \leq C_0 \left|\tfrac{\Delta}{2}\right| \langle j \rangle^{-1/4} \leq C\langle n \rangle^{-1/4}$ if $n \geq l_0 + 2j_0 + 1$. □

5.2. **The auxiliary operator K.**

Notation 5.3. In the sequel $K \in \mathcal{B}(\ell^2(\mathbb{N}))$ is the operator defined by

$$\langle e_j, K e_k \rangle_{\ell^2} = K(j,k) = \begin{cases} \dfrac{V(j,k)}{d_j - d_k} & \text{if } d_j \neq d_k \\ 0 & \text{if } d_j = d_k \end{cases} \tag{5.8}$$

where $V(j,k) := \langle e_j, V e_k \rangle$ for $j,\, k \in \mathbb{N}$.

Lemma 5.4. *The operator K is compact and anti-Hermitian in $\ell^2(\mathbb{N})$. Moreover, there is $C > 0$ such that*

$$||K e_n|| \leq C\langle n \rangle^{-1/4} \text{ for all } n \in \mathbb{N}. \tag{5.9}$$

Proof. Step 1. We first observe that $K(k,j) = -\overline{K(j,k)}$, i.e. the matrix $(K(j,k))_{j,k\in\mathbb{Z}}$ is anti-Hermitian and (5.1) ensures

$$n \geq c^{-2}|k|^2 \quad \Rightarrow \quad |V(n+k,n)| \leq Cn^{-1/4}. \tag{5.10}$$

Let $c_0 > 0$ be fixed small enough. Then $d_j \neq d_k \Rightarrow |d_j - d_k| \geq c_0\langle j-k\rangle$ and

$$K(j,k) \neq 0 \quad \Rightarrow \quad d_j \neq d_k \quad \Rightarrow \quad |K(j,k)| \leq c_0^{-1}\frac{|V(j,k)|}{\langle j-k\rangle}. \tag{5.11}$$

We also observe that the Parseval's equality ensures

$$\sum_{j\in\mathbb{N}} |V(j,k)|^2 = \sum_{j\in\mathbb{N}} |\langle \mathbf{v}_j^0, \mathbf{V}\mathbf{v}_k^0\rangle_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})}|^2 = ||\mathbf{V}\mathbf{v}_k^0||^2_{\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})} \leq C_0. \tag{5.12}$$

Step 2. For $N \in \mathbb{N}$ we define $K_N \in \mathcal{B}(\ell^2(\mathbb{N}))$ by

$$K_N(j,k) = \langle e_j, K_N e_k\rangle_{\ell^2(\mathbb{N})} = \begin{cases} K(j,k) & \text{if } |j-k| < N \\ 0 & \text{if } |j-k| \geq N \end{cases} \tag{5.13}$$

Since K_N is a finite band matrix and each diagonal $K_N(k+n,n) \to 0$ when $n \to \infty$ due to (5.10), the operator K_N is compact. If $\widetilde{K}_N(m,n) := K(m,n) - K_N(m,n)$, then

$$\sum_{j\in\mathbb{N}} |\widetilde{K}_N(j,k)| \leq c_0^{-1} \sum_{\{j\in\mathbb{N}:\, |j-k|\geq N\}} \frac{|V(j,k)|}{\langle j-k\rangle} \tag{5.14}$$

and using the Cauchy-Schwarz inequality, we can estimate the right hand side of (5.14) by $c_0^{-1}C_0^{1/2}\delta_N^{1/2}$, where we have denoted $\delta_N := 2\sum_{m>N}\langle m\rangle^{-2}$ and C_0 is as in (5.12). Thus the Schur boundedness test ensures $K - K_N \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and

$$||K - K_N|| \leq c_0^{-1}C_0^{1/2}\delta_N^{1/2} \xrightarrow[N\to\infty]{} 0. \tag{5.15}$$

Since $K = K - K_0 \in \mathcal{B}(\ell^2(\mathbb{N}))$ and K_N are compact, (5.15) implies that K is compact.

Step 3. We can estimate

$$||Ke_n||^2 = \sum_{j\in\mathbb{N}} |K(j,n)|^2 \leq c_0^{-2}(M_n + M_n'),$$

where

$$M_n := \sum_{|j-n|\geq cn^{1/2}} \frac{|V(j,n)|^2}{\langle j-n\rangle^2} \leq \langle cn^{1/2}\rangle^{-2} \sum_{j\in\mathbb{N}} |V(j,n)|^2 \leq C_1 n^{-1}$$

due to (5.12) and

$$M_n' = \sum_{|j-n|<cn^{1/2}} \frac{|V(j,n)|^2}{\langle j-n\rangle^2} \leq C^2 n^{-1/2} \sum_{m\in\mathbb{N}} \langle m-n\rangle^{-2} \leq C_1' n^{-1/2}$$

due to (5.4). □

5.3. **Similarity transformation.**

Notation 5.5. In the sequel W denotes the operator defined in $\mathcal{B}(\ell^2(\mathbb{N}))$ by

$$\langle e_j, We_k\rangle = W(j,k) = (d_j - d_k)K(j,k) - V(j,k). \tag{5.16}$$

Lemma 5.6. *(a) If $\epsilon \notin \mathbb{N}$ then $W = 0$. If $\epsilon \in \mathbb{N}$ then W is compact, self-adjoint and there is $C > 0$ such that*

$$||We_n|| \leq Cn^{-1/4} \text{ for } n \in \mathbb{N}^*. \tag{5.17}$$

(b) The operator $I + K$ is invertible and $\ell^{2,1}(\mathbb{N})$ is an invariant subspace of $I + K$.

(c) The operator $R := (KV - W)(I + K)^{-1}$ is compact and one has

$$H = (I + K)^{-1}(D + R)(I + K). \tag{5.18}$$

Proof. (a) We first observe that $V(j,j) = 0$ implies $W(j,j) = 0$ and

$$d_j \neq d_k \implies W(j,k) = 0. \tag{5.19}$$

If $\epsilon \notin \mathbb{N}$ then $j \neq k$ implies $d_j \neq d_k$ and $W(j,k) = 0$ follows. If $\epsilon \in \mathbb{N}$ then $W(j,k) \neq 0$ implies $d_j = d_k$ and $|j - k| = 1$, hence all non zero entries of the matrix $(W(j,k))_{j,k\in\mathbb{Z}}$ have the form $W(j,j+1) = W(j+1,j) = -V(j+1,j)$ and W is compact because $V(j+1,j) \to 0$ when $j \to \infty$.

(b) Since the operator $\mathrm{i}K$ is self-adjoint, the operator $\mathrm{i} + \mathrm{i}K = \mathrm{i}(I + K)$ is invertible and

$$d_j(Ke_k)(j) = (d_j - d_k)(Ke_k)(j) + (KDe_k)(j) = \big((W + V + KD)e_k\big)(j) \tag{5.20}$$

implies that $d_j(Kx)(j) = \big((W + V + KD)x\big)(j)$ holds when $x \in \ell^{2,1}(\mathbb{N})$. Since $d_j \sim \frac{1}{2}j$ as $j \to \infty$, we can conclude that $x \in \ell^{2,1}(\mathbb{N}) \Rightarrow DKx \in \ell^2(\mathbb{N}) \Rightarrow Kx \in \ell^{2,1}(\mathbb{N})$ and

$$DK = W + V + KD \text{ on } \ell^{2,1}(\mathbb{N}). \tag{5.21}$$

(c) Due to (5.21) we can write $V + KD = DK - W$ on $\ell^{2,1}(\mathbb{N})$ and

$$(I + K)H = D + V + KD + KV = D + DK - W + KV =$$
$$= D(I + K) - W + KV = (D + R)(I + K).$$

Applying $(I + K)^{-1}$ from the left, we obtain (5.18). Moreover R is compact because K and W are compact. □

5.4. **End of the proof of** (1.15)**.** Let K and R be as in Lemma 5.6. We denote

$$H' := D + R. \tag{5.22}$$

We claim that $\sigma(H') = \sigma(H)$ and the corresponding eigenvalues have the same multiplicity. Indeed, (5.18) ensures

$$H - z = (I + K)^{-1}(H' - z)(I + K) \text{ for } z \in \mathbb{C} \tag{5.23}$$

and taking $z_0 \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H'))$ we find that

$$(I + K)|_{\ell^{2,1}(\mathbb{N})} = (H' - z_0)^{-1}(I + K)(H - z_0)$$

is invertible in $\mathcal{B}(\ell^{2,1}(\mathbb{N}))$. Thus $H - z$ is invertible $\ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ if and only if $H' - z$ is invertible $\ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$, i.e. $\sigma(H') = \sigma(H)$ and

$$(H - z)^{-1} = (I + K)^{-1}(H' - z)^{-1}(I + K) \text{ for } z \in \mathbb{C} \setminus \sigma(H). \tag{5.24}$$

Finally, the equality $\operatorname{rank} P_\lambda(H) = \operatorname{rank} P_\lambda(H')$ follows from the fact that (5.24) implies

$$P_\lambda(H) = (I + K)^{-1}P_\lambda(H')(I + K).$$

Let $(\lambda_n(H))_{n\in\mathbb{N}} = (\lambda_n(H'))_{n\in\mathbb{N}}$ be the non-decreasing sequence of eigenvalues of H (and of H'), counting the multiplicities. We need to prove the large n estimate

$$\lambda_n(H) = \lambda_n(H') = d_n + O(n^{-1/4}). \tag{5.25}$$

For this purpose we first observe that there exist constants C, C' such that

$$||R^*e_n|| \leq C(||K^*e_n|| + ||We_n||) \leq C'n^{-1/4} \tag{5.26}$$

holds for all $n \in \mathbb{N}^*$. It is easy to see that we obtain (5.25) using (5.26) and

Theorem 5.7. *Let $(d_n)_{n\in\mathbb{N}}$ be given by (2.14)-(2.16) and let $\widehat{\Pi}_n$ denote the orthogonal projection on the linear subspace generated by $\{e_m : d_m = d_n\}$. If R is as in Lemma 5.6 and $H' = D + R$, then one has the large n estimate*

$$\lambda_n(H') = d_n + O(||R^*\widehat{\Pi}_n||_{\mathcal{B}(\ell^2(\mathbb{N}))}). \tag{5.27}$$

Proof. If $\epsilon \notin \mathbb{N}$ then $||R^*\widehat{\Pi}_n|| = ||R^*e_n||$ and the estimate (5.27) was proved in Lemma 2.1 of the paper Janas and Naboko [13]. If $\epsilon \in \mathbb{N}$ then the eigenvalues are double for $n \geq l_0$ and the estimate (5.27) can be easily deduced from the result of Malejki [18] similarly as in Appendix of [7]. □

6. Estimates of oscillatory integrals

Lemma 6.1. *We us fix $c_0 > 0$ and θ_0, $\theta_1 \in \mathbb{R}$ satisfying $\theta_0 < \theta_1$. Let $\mathcal{T}$ be an arbitrary set of parameters. For every $\tau \in \mathcal{T}$ we consider smooth functions $h_\tau : [\theta_0, \theta_1] \to \mathbb{C}$, $\Psi_\tau : [\theta_0, \theta_1] \to \mathbb{R}$ and for $\lambda \in \mathbb{R}$ we denote*

$$M_\tau(\lambda) := \int_{\theta_0}^{\theta_1} \mathrm{e}^{\mathrm{i}\lambda\Psi_\tau(\theta)} h_\tau(\theta)\,\mathrm{d}\theta. \tag{6.1}$$

We assume that the derivative satisfies $|\Psi'_\tau(\theta)| \geq c_0$ for every $\theta \in [\theta_0, \theta_1]$ and $\tau \in \mathcal{T}$.
(a) If $\{h_\tau\}_{\tau\in\mathcal{T}}$ is bounded in $\mathrm{C}^1([\theta_0, \theta_1])$ and $\{\Psi_\tau\}_{\tau\in\mathcal{T}}$ is bounded in $\mathrm{C}^2([\theta_0, \theta_1])$, then there is a constant C_0 such that the estimate

$$|M_\tau(\lambda)| \leq C_0|\lambda|^{-1} \tag{6.2}$$

holds for all $\tau \in \mathcal{T}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.
(b) If $[\theta_0, \theta_1] = [-\pi, \pi]$, $\{h_\tau\}_{\tau\in\mathcal{T}}$ is bounded in $\mathrm{C}^\infty(\mathbb{T})$, $\{\Psi'_\tau\}_{\tau\in\mathcal{T}}$ is bounded in $\mathrm{C}^\infty(\mathbb{T})$ and $\Psi_\tau(\pi) - \Psi_\tau(-\pi) \in 2\pi\mathbb{Z}$ holds for every $\tau \in \mathcal{T}$, then for every $N \in \mathbb{N}$ one can find a constant $C_N > 0$ such that

$$|M_\tau(m)| \leq C_N m^{-N} \tag{6.3}$$

holds for all $\tau \in \mathcal{T}$ and $m \in \mathbb{N}^*$.

Proof. (a) The integration by parts gives

$$\mathrm{i}\lambda M_\tau(\lambda) = \int_{\theta_0}^{\theta_1} \left(\mathrm{e}^{\mathrm{i}\lambda\Psi_\tau}\right)' \frac{h_\tau}{\Psi'_\tau} = \int_{\theta_0}^{\theta_1} \mathrm{e}^{\mathrm{i}\lambda\Psi_\tau} \mathcal{L}_\tau(h_\tau) + \left[\mathrm{e}^{\mathrm{i}\lambda\Psi_\tau} \frac{h_\tau}{\Psi'_\tau}\right]_{\theta_0}^{\theta_1} \tag{6.4}$$

where $\mathcal{L}_\tau$ is the differential operator defined by $\mathcal{L}_\tau(f) := -(f/\Psi'_\tau)'$. To complete the proof of (6.2), it suffices to observe that the right hand side of (6.4) is bounded uniformly with respect to $\tau \in \mathcal{T}$ and $\lambda \in \mathbb{R}^*$.

(b) If $[\theta_0, \theta_1] = [-\pi, \pi]$ and $\lambda = m \in \mathbb{N}$, then the last term in the right hand side of (6.4) disappears due to the assumptions of 2π-periodicity and we obtain

$$(\mathrm{i}m)^N M_\tau(\lambda) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}m\Psi_\tau} \mathcal{L}_\tau^N(h_\tau) \tag{6.5}$$

after N integrations by parts. To complete the proof of (6.3), we observe that the right hand side of (6.5) is bounded uniformly with respect to $\tau \in \mathcal{T}$ and $m \in \mathbb{N}^*$. □

Lemma 6.2. (J. van der Corput) *Assume that $c_0 > 0$. If $\Psi\colon [\theta_0, \theta_1] \to \mathbb{R}$ is smooth and its second derivative satisfies $|\Psi''| \geq c_0$, then there is a constant C_0 depending only on c_0, such that for all $\lambda \in \mathbb{R}^*$ one has*

$$\Big| \int_{\theta_0}^{\theta_1} \mathrm{e}^{\mathrm{i}\lambda\Psi(\theta)}\, \mathrm{d}\theta \Big| \leq C_0 |\lambda|^{-1/2}. \tag{6.6}$$

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