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# TREND TO EQUILIBRIUM AND DIFFUSION LIMIT FOR THE INERTIAL KURAMOTO-SAKAGUCHI EQUATION

FRANCIS FILBET AND MYEONGJU KANG

ABSTRACT. In this paper, we study the inertial Kuramoto-Sakaguchi equation for interacting oscillatory systems. On the one hand, we prove the convergence toward corresponding phase-homogeneous stationary states in weighted Lebesgue norm sense when the coupling strength is small enough. In [10], it is proved that when the noise intensity is sufficiently large, equilibrium of the inertial Kuramoto-Sakaguchi equation is asymptotically stable. For generic initial data, every solutions converges to equilibrium in weighted Sobolev norm sense. We improve this previous result by showing the convergence for a larger class of functions and by providing a simpler proof. On the other hand, we investigate the diffusion limit when all oscillators are identical. In [19], authors studied the same problem using an energy estimate on renormalized solutions and a compactness method, through which error estimates could not be discussed. Here we provide error estimates for the diffusion limit with respect to the mass  $m \ll 1$  using a simple proof by imposing slightly more regularity on the solution.

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## 1. INTRODUCTION AND MAIN RESULTS

Synchronous behavior of a large but loosely organized group of agents is ubiquitously found in various social and biological phenomena, for example, flashing of fireflies, beating of cardiac cells, and hand clapping in opera, etc [1, 6, 15, 26, 28, 31, 30]. Recently, collective dynamics of an

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interacting oscillatory system has received more attention due to its diverse applications in research areas of control theory, physics, neuroscience [1].

Systematic studies on synchronization were invoked by the pioneer works of A. Winfree [31, 30] and Y. Kuramoto [23, 24, 22]. More precisely, these models consider a collection of  $N \in \mathbb{N}$  oscillators, represented by their phase-frequency pair  $(\theta^i, \omega^i) \in \mathbb{T} \times \mathbb{R}$  and by their natural frequency  $\nu^i$ . In the presence of inertia and stochastic noise effect, the dynamics of stochastic Kuramoto oscillators is governed by the following set of globally coupled ODEs [1, 22, 27]:

$$(1.1) \quad \begin{cases} d\theta_t^i = \omega_t^i dt, \\ m d\omega_t^i = \left( -\omega_t^i + \nu^i + \frac{\kappa}{N} \sum_{k=1}^N \sin(\theta_t^k - \theta_t^i) \right) dt + \sqrt{2\sigma} dB_t^i, \end{cases}$$

where nonnegative coefficients  $m$ ,  $\kappa$ , and  $\sigma$  represent mass, coupling strength, and noise intensity, respectively, and  $B_t^i$ 's are independent one-dimensional Brownian motions. We refer to [8, 11, 9, 17, 18] for the emergent behavior of the Kuramoto model (1.1) including zero inertia case ( $m = 0$ ) or noiseless case ( $\sigma = 0$ ).

The continuum approximation assumes that populations at the thermodynamic limit  $N \rightarrow +\infty$  are described by a continuous distribution  $f = f(t, \theta, \omega, \nu)$  at time  $t \in \mathbb{R}^+$ , phase  $\theta \in \mathbb{T}$ , frequency  $\omega \in \mathbb{R}$ , and natural frequency  $\nu \in \mathbb{R}$ . Under this assumption, the time evolution is governed by the following Vlasov-Fokker-Planck-type equation [1, 17]:

$$(1.2) \quad \begin{cases} \partial_t f + \omega \partial_\theta f + \partial_\omega (f \mathcal{T}[f]) = \frac{\sigma}{m^2} \partial_\omega^2 f, \\ \mathcal{T}[f](t, \mathbf{z}) = -\frac{\omega}{m} + \frac{\nu}{m} + \frac{\kappa}{m} \int_{\mathbb{T} \times \mathbb{R}^2} \sin(\theta_* - \theta) f(t, \mathbf{z}_*) d\mathbf{z}_*, \\ f(t=0) = f_{\text{in}} \geq 0, \end{cases}$$

where  $\mathbf{z} = (\theta, \omega, \nu) \in \mathbb{T} \times \mathbb{R}^2$ . Observe that since the  $\nu$  variable only appears as a parameter we have the following invariant

$$\int_{\mathbb{T} \times \mathbb{R}} f(t, \mathbf{z}) d\omega d\theta = \int_{\mathbb{T} \times \mathbb{R}} f_{\text{in}}(\mathbf{z}) d\omega d\theta =: g(\nu).$$

In order to highlight important parameters, we introduce rescaled coupling strength and noise intensity  $\tilde{\kappa}$  and  $\tilde{\sigma}$  as

$$\begin{cases} \tilde{\kappa} = \frac{\kappa}{m}, \\ \tilde{\sigma} = \frac{\sigma}{m} \end{cases}$$

and define the macroscopic quantity as

$$\rho(t, \theta) = \int_{\mathbb{R}^2} f(t, \mathbf{z}) d\omega d\nu.$$

Hence we simplify  $\mathcal{T}[f]$  as

$$\mathcal{T}[f](t, \mathbf{z}) = -\frac{\omega}{m} + \frac{\nu}{m} + \tilde{\kappa} (\sin * \rho(t))(\theta).$$

and system (1.2) can be written as

$$(1.3) \quad \begin{cases} \partial_t f + \omega \partial_\theta f + \tilde{\kappa} (\sin * \rho) \partial_\omega f = \mathcal{L}_{\text{FP}}[f], \\ f(t=0) = f_{\text{in}} \geq 0, \end{cases}$$

with

$$(1.4) \quad \mathcal{L}_{\text{FP}}[f] := \frac{1}{m} \partial_\omega \left( \tilde{\sigma} \partial_\omega f + (\omega - \nu) f \right).$$

For the existence and uniqueness theory on (1.2), we refer to [10, 19], in which global nonnegative weak solutions are constructed in weighted Sobolev spaces.

The goal of this paper is twofold. On the one hand, we study the long time behavior of the solution to (1.2), when the coupling strength  $\tilde{\kappa} > 0$  is small compared to the noise intensity  $\tilde{\sigma} > 0$ . In that case, the global equilibrium is characterized by a space homogeneous distribution and we prove that the solution of (1.2) converges exponentially fast to this equilibrium with an explicit rate of convergence. On the other hand, we consider the case of identical oscillators and investigate the limit when mass  $\varepsilon := m$  goes to zero. Hence, we prove that the local density  $\rho^\varepsilon$  converges to the solution of the drift-diffusion equation. Our methods provides directly error estimates with respect to  $\varepsilon$ .

In the next subsections, we describe more precisely our main results and the state of the art.

**1.1. Long time asymptotics.** First, note that nonnegative phase-homogeneous state  $f_{\text{eq}} = f_{\text{eq}}(\omega, \nu)$  becomes stationary solution of (1.3)-(1.4) if and only if

$$\mathcal{L}_{\text{FP}}[f_{\text{eq}}] = 0 \quad \text{and} \quad 2\pi \int_{\mathbb{T} \times \mathbb{R}} f_{\text{eq}}(\omega, \nu) d\omega = g(\nu).$$

Thus we denote by  $N_\infty(\nu) = \rho_\infty g(\nu)$  with  $\rho_\infty = \|f_{\text{in}}\|_{L^1}/(2\pi)$  and

$$\mathcal{M}(\omega, \nu) := \frac{1}{\sqrt{2\pi\tilde{\sigma}}} \exp\left(-\frac{(\omega - \nu)^2}{2\tilde{\sigma}}\right),$$

hence one can check that

$$f_\infty(\omega, \nu) := N_\infty(\nu) \mathcal{M}(\omega, \nu)$$

becomes phase-homogeneous stationary solution to (1.3)-(1.4).

Recently, the asymptotic stability of  $f_\infty$  has been studied in [10], where it has been shown that the solution  $f$  of (1.3)-(1.4) converges to  $f_\infty$  exponentially fast for sufficiently large  $\tilde{\sigma} > 0$ . More precisely, suppose there exists  $h$  such that  $f = f_\infty + \sqrt{f_\infty} h$  and

$$\int_{\mathbb{R}} \|h(t, \nu)\|_{H^s(\mathbb{T} \times \mathbb{R})}^2 d\nu < \infty, \quad \forall t \geq 0,$$

for  $s \geq 1$ , hence for  $\tilde{\sigma} > 0$  satisfying

$$\max\left(\frac{1}{m^2}, \tilde{\kappa}, m^2 \tilde{\kappa}^2\right) \ll \tilde{\sigma},$$

there exists  $C_1 > 0$  such that

$$\int_{\mathbb{R}} \|h(t, \nu)\|_{H^s}^2 d\nu \lesssim e^{-C_1 t} \int_{\mathbb{R}} \|h(0, \nu)\|_{H^s}^2 d\nu.$$

Here our aim is twofold. On the one hand, we give an explicit condition on the intensity of collision  $\tilde{\sigma}$  and the coupling strength  $\tilde{\kappa}$  to get the convergence to the homogeneous stationary state  $f_\infty$ . This condition requires that  $\tilde{\kappa}$  is sufficiently small compared to  $\tilde{\sigma}$ . On the other hand, we provide a simpler proof of convergence than the one presented in [10]. We apply the hypocoercivity method with a micro-macro decomposition, developed in [29, 20, 14], to get quantitative estimates on the convergence to equilibrium. The advantage of this approach is that it is simply based on a natural weighted  $L^2$  estimate. The key tool of our method is a modified energy functional  $\mathcal{E}[f]$ , whose square root is a norm equivalent to the weighted  $L^2$  norm, such that

$$\frac{d\mathcal{E}[f]}{dt} \leq -C \mathcal{E}[f],$$

for an explicitly computable positive constant  $C$ . It is worth to mention that this functional framework is well adapted to the development of structure preserving numerical schemes [5, 4] and will be the purpose of a forthcoming work [16]. Furthermore, this technique has also been applied for the asymptotic stability of related models [12, 13, 25].

Before we present our first main result, we introduce macroscopic quantities  $N$ ,  $J$ , and  $P$  defined as

$$(1.5) \quad \begin{cases} N(t, \theta, \nu) := \int_{\mathbb{R}} f(t, \mathbf{z}) d\omega & \text{(density),} \\ J(t, \theta, \nu) := \int_{\mathbb{R}} f(t, \mathbf{z}) (\omega - \nu) d\omega & \text{(first moment),} \\ P(t, \theta, \nu) := \int_{\mathbb{R}} f(t, \mathbf{z}) (\omega - \nu)^2 d\omega & \text{(second moment),} \end{cases}$$

which will be used throughout this paper. One can multiply the equation (1.3)-(1.4) by  $(1, \omega - \nu)$  and integrating in  $\omega \in \mathbb{R}$  to get following system of balance laws:

$$(1.6) \quad \begin{cases} \partial_t N + \partial_\theta (J + \nu N) = 0, \\ \partial_t J + \partial_\theta (P + \nu J) - \tilde{\kappa} (\sin * \rho) N = -\frac{J}{m}. \end{cases}$$

Next, we introduce a weighted Lebesgue space by considering a weight function  $\bar{\gamma} : \mathbb{R} \mapsto \mathbb{R}^+$  such that  $\bar{\gamma}(\nu) > 0$ , for all  $\nu \in \mathbb{R}$ ,

$$(1.7) \quad \int_{\mathbb{R}} \frac{d\nu}{\bar{\gamma}(\nu)} = 1 \quad \text{and} \quad \int_{\mathbb{R}} |g(\nu)|^2 \bar{\gamma}(\nu) d\nu < \infty.$$

Then we set

$$\gamma(\omega, \nu) := \bar{\gamma}(\nu) \mathcal{M}^{-1}(\omega, \nu),$$

and define the weighted  $L^2_\gamma(\mathbb{T} \times \mathbb{R}^2)$  norm as

$$\|h\|_{L^2_\gamma} = \left( \int_{\mathbb{T} \times \mathbb{R}^2} |h|^2 \gamma(\omega, \nu) d\mathbf{z} \right)^{1/2},$$

and the corresponding weighted  $L^2_{\bar{\gamma}}$  norm for the macroscopic quantity  $R : \mathbb{T} \times \mathbb{R} \mapsto \mathbb{R}$  as

$$\|R\|_{L^2_{\bar{\gamma}}} = \left( \int_{\mathbb{T} \times \mathbb{R}} |R(\theta, \nu)|^2 \bar{\gamma}(\nu) d\nu d\theta \right)^{1/2}.$$

Under these setting, we have exponential relaxation of the solution of (1.3)-(1.4) toward a phase-homogeneous stationary state.

**Theorem 1.1.** *Consider an initial data  $f_{\text{in}} \geq 0$  such that*

$$\iint_{\mathbb{T} \times \mathbb{R}^2} |f_{\text{in}}|^2 \gamma(\omega, \nu) d\mathbf{z} < +\infty.$$

*Then there exists a constant  $C_\infty > 0$ , only depending on  $\|N_\infty\|_{L^1}$  and  $\|N_\infty\|_{L^2_\gamma}$ , such that if the coupling strength  $\tilde{\kappa} > 0$  and the noise intensity  $\tilde{\sigma} > 0$  satisfy*

$$(1.8) \quad C_\infty \max \left( \sqrt{\frac{\tilde{\kappa}}{m}}, \tilde{\kappa}, m\tilde{\kappa} \right) \leq \tilde{\sigma},$$

*hence the solution  $f$  to (1.3)-(1.4) converges to a phase-homogeneous stationary state  $f_\infty$  exponentially fast*

$$\|f(t) - f_\infty\|_{L^2_\gamma} \leq 3 \|f(0) - f_\infty\|_{L^2_\gamma} e^{-Ct}, \quad \forall t \geq 0,$$

*where  $C > 0$  only depends on  $C_\infty$  and  $\max \left( \sqrt{\tilde{\kappa}/m}, \tilde{\kappa}, m\tilde{\kappa} \right)$ .*

The proof of this result is provided in Section 2. The key idea is to get advantage of the dissipation corresponding to the Fokker-Planck operator for the weighted  $L^2$  norm. Then the main difficulty consists in proving the convergence of the macroscopic quantity

$$N(t, \theta, \nu) := \int_{\mathbb{R}} f(t, z) d\omega,$$

toward the equilibrium  $N_\infty$ . We adapt the hypocoercivity method developed in [29, 14] to the present model (1.3)-(1.4). Instead of estimating directly the quantities of interest, we introduce modified energy functionals in order to recover dissipation and thus a convergence rate on  $N$ . Our approach is related to the one developed in [14] and [7, 21, 3, 2] for the Vlasov-Poisson-Fokker-Planck system. Even if the natural energy corresponding to the system (1.3)-(1.4) does not provide an estimate, the key point here is to exploit the regularity of the nonlocal term  $\sin * \rho$ .

Let us now make some comments how our results compares with the one presented in [10].

**Remark 1.1.** In [10], it is required that  $\tilde{\kappa} > 0$  and  $\tilde{\sigma} > 0$  satisfy

$$(1.9) \quad \tilde{C} \max \left( \frac{1}{m^2}, \tilde{\kappa}, m^2 \tilde{\kappa}^2 \right) \leq \tilde{\sigma},$$

for some sufficiently large  $\tilde{C} > 0$ . Note that as  $\kappa > 0$  goes to zero, left hand side of (1.9) does not converges to zero, whereas left hand side of (1.8) converges to zero. Moreover, as  $\tilde{\kappa} > 0$  goes to infinity, left hand side of (1.9) diverges to infinity with growth rate  $\mathcal{O}(m^2 \tilde{\kappa}^2)$ , whereas left hand side of (1.8) diverges to infinity with growth rate  $\mathcal{O}(m \tilde{\kappa})$ .

**1.2. Diffusion limit for identical oscillators.** We now present our second main result on the diffusion limit of (1.3)-(1.4) for identical oscillators, *i.e.*,  $g(\nu) = \delta_0(\nu)$ . We consider the following rescaling  $t \mapsto \varepsilon t$  and  $\varepsilon = m$  in (1.3)-(1.4), which yields :

$$(1.10) \quad \begin{cases} \varepsilon \partial_t f^\varepsilon + \omega \partial_\theta f^\varepsilon + \tilde{\kappa} (\sin * \rho^\varepsilon) \partial_\omega f^\varepsilon = \mathcal{L}_{\text{FP}}[f^\varepsilon], \\ f^\varepsilon(t=0) = f_{\text{in}}^\varepsilon \geq 0, \end{cases}$$

with

$$(1.11) \quad \mathcal{L}_{\text{FP}}[f^\varepsilon] := \frac{1}{\varepsilon} \partial_\omega \left( \tilde{\sigma} \partial_\omega f^\varepsilon + \omega f^\varepsilon \right)$$

and  $\rho^\varepsilon$  is given by

$$\rho^\varepsilon(t, \theta) = \int_{\mathbb{R}} f^\varepsilon(t, \theta, \omega) d\omega, \quad \forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{T}.$$

We remind (1.6) that  $\rho^\varepsilon$  and the first moment  $J^\varepsilon$  are given by

$$\begin{cases} \partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \partial_\theta J^\varepsilon = 0, \\ \varepsilon \partial_t J^\varepsilon + \partial_\theta P^\varepsilon - \tilde{\kappa} (\sin * \rho^\varepsilon) \rho^\varepsilon = -\frac{J^\varepsilon}{\varepsilon}, \end{cases}$$

hence differentiating the last equation with respect to  $\theta$  and combining it with the the former, it yields that

$$(1.12) \quad \partial_t (\rho^\varepsilon - \varepsilon \partial_\theta J^\varepsilon) - \partial_\theta \left( \partial_\theta P^\varepsilon - \tilde{\kappa} (\sin * \rho^\varepsilon) \rho^\varepsilon \right) = 0.$$

In the limit  $\varepsilon \rightarrow 0$ , it is expected that  $(f^\varepsilon)_{\varepsilon > 0}$  converges to  $\rho \mathcal{M}$ , that is,

$$P^\varepsilon = \int_{\mathbb{R}} f^\varepsilon \omega^2 d\omega \rightarrow \tilde{\sigma} \rho, \quad \text{as } \varepsilon \rightarrow 0.$$

Therefore, we formally get that the limit  $\rho$  is solution to the following drift-diffusion equation

$$(1.13) \quad \begin{cases} \partial_t \rho - \tilde{\sigma} \partial_\theta^2 \rho + \tilde{\kappa} \partial_\theta ((\sin * \rho) \rho) = 0, \\ \rho(0) = \rho_{\text{in}}. \end{cases}$$

Since the  $\nu$  variable now cancels, we consider the weighted  $L^2$  space where the weight is now given by invert of the centred Gaussian distribution  $\mathcal{M}$

$$\mathcal{M}(\omega) = \frac{1}{\sqrt{2\pi} \tilde{\sigma}} \exp\left(-\frac{\omega^2}{2\tilde{\sigma}}\right).$$

Thus, we prove the following result.

**Theorem 1.2.** *Suppose that the initial data  $(f_{\text{in}}^\varepsilon)_{\varepsilon>0}$  in (1.10)-(1.11), satisfy the following assumptions*

$$(1.14) \quad \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} < +\infty,$$

*uniformly with respect to  $\varepsilon$  and the initial datum  $\rho_{\text{in}}$  in (1.13) verifies*

$$\|\rho_{\text{in}}\|_{L^2} < +\infty.$$

*Moreover, we suppose that*

$$\|f_{\text{in}}^\varepsilon\|_{L^1} = \|\rho_{\text{in}}\|_{L^1} = M_0.$$

*Let  $f^\varepsilon$  be the solution to (1.10)-(1.11) and  $\rho$  be the solution to (1.13). Then the following statements hold true uniformly with respect to  $\varepsilon$*

$$\|f^\varepsilon(t) - \rho^\varepsilon(t) \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}} \leq \|f_{\text{in}}^\varepsilon - \rho_{\text{in}}^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}} e^{-\tilde{\sigma} t / (4\varepsilon^2)} + C \varepsilon \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) e^{Ct},$$

*and*

$$(1.15) \quad \|\rho^\varepsilon(t) - \rho(t)\|_{H^{-1}} \leq C \left( \|\rho_{\text{in}}^\varepsilon - \rho_{\text{in}}\|_{H^{-1}} + \varepsilon \left( \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) \right) e^{Ct},$$

*where  $C$  is a positive constant only depending on  $\tilde{\sigma}$ ,  $\tilde{\kappa}$  and  $M_0$ .*

**Remark 1.2.** *In [19], the authors also investigated the same problem using a compactness method and an energy estimate, they proved that*

$$\rho^\varepsilon \rightarrow \rho, \quad \text{in } L^1((0, T) \times \mathbb{T}), \quad \text{when } \varepsilon \rightarrow 0,$$

*considering the notion of renormalized solution. Our method is simpler than the former but requires more regularity. Moreover, it allows us to prove error estimates with respect to  $\varepsilon$  thanks to the propagation of regularity  $\theta$  uniformly in with respect to  $\varepsilon$ .*

*Recently, A. Blaustein provided error estimates for the diffusive limit of the Vlasov-Poisson-Fokker-Planck system by proving propagation of regularity in weighted  $L^p$  spaces [3].*

The rest of the paper is organized as follows. On the one hand, in the next section (Section 2), we establish some basic properties of (1.3)-(1.4), hence we study the propagation of the modified energy functional  $\mathcal{E}[f]$  and prove Theorem 1.1 on the exponential relaxation of  $f$  toward the phase homogeneous stationary state  $f_\infty$ . On the other hand, in Section 3, we consider the particular case when all oscillators are identical and study the diffusion limit to prove our second result (Theorem 1.2) on error estimates in the diffusion limit for identical oscillators. Finally, Section 4 is devoted to a brief summary and possible future works.

## 2. LONG TIME BEHAVIOR FOR SMALL COUPLING STRENGTH

In this section, we present *a priori* estimates which aim at describing the long time behavior of (1.3)-(1.4) when collisions dominate. Then, we focus on macroscopic quantities and provide a free energy estimate, which is the starting point of our analysis.

**2.1. Basic properties.** In this section, we study some basic properties of the inertial equation (1.3)-(1.4) showing the propagation of the weighted  $L_\gamma^2$  norm and estimate some macroscopic quantities for latter use. First we remind the estimate provided in [10, 19].

**Proposition 2.1.** *Let  $f = f(t, \theta, \omega, \nu)$  be a classical solution to (1.3)-(1.4) with a nonnegative initial data  $f_{\text{in}} \in L^1(\mathbb{T} \times \mathbb{R}^2)$ . Then, for all time  $t \geq 0$ , we have that  $f(t)$  is also nonnegative and*

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}^2} f(t, \mathbf{z}) \, d\mathbf{z} &= \int_{\mathbb{T} \times \mathbb{R}^2} f_{\text{in}}(\mathbf{z}) \, d\mathbf{z}, \\ \int_{\mathbb{T} \times \mathbb{R}} f(t, \mathbf{z}) \, d\omega \, d\theta &= \int_{\mathbb{T} \times \mathbb{R}} f_{\text{in}}(\mathbf{z}) \, d\omega \, d\theta = g(\nu). \end{aligned}$$

We aim to study the propagation of the weighted  $L_\gamma^2$  norm and first prove the following preliminary result.

**Lemma 2.1.** *Let  $f = f(t, \theta, \omega, \nu)$  be a classical solution to (1.3)-(1.4). Then for all time  $t \geq 0$ , we have*

$$\|f(t) - f_\infty\|_{L_\gamma^2}^2 \leq \mathcal{I}[f](t) + \|N(t) - N_\infty\|_{L_\gamma^2}^2,$$

where  $\mathcal{I}[f](t)$  corresponds to the dissipation of the Fokker-Planck operator and is defined as

$$(2.1) \quad \mathcal{I}[f](t) := \int_{\mathbb{T} \times \mathbb{R}^2} \left| \partial_\omega \left( \frac{f(t)}{\mathcal{M}} \right) \right|^2 \bar{\gamma}(\nu) \mathcal{M}(\omega, \nu) \, d\mathbf{z} \geq 0.$$

*Proof.* It follows

$$\begin{aligned} \|f - f_\infty\|_{L_\gamma^2}^2 &= \|f - N \mathcal{M}\|_{L_\gamma^2}^2 + \|(N - N_\infty) \mathcal{M}\|_{L_\gamma^2}^2, \\ &= \|f - N \mathcal{M}\|_{L_\gamma^2}^2 + \|N - N_\infty\|_{L_\gamma^2}^2. \end{aligned}$$

We use the Gaussian-Poincaré inequality with respect to probability measure  $\mathcal{M}d\omega$  to obtain

$$\begin{aligned} \|f - N \mathcal{M}\|_{L_\gamma^2}^2 &= \int_{\mathbb{T} \times \mathbb{R}} \left( \int_{\mathbb{R}} |f - N \mathcal{M}|^2 \mathcal{M}^{-1}(\omega, \nu) \, d\omega \right) \bar{\gamma}(\nu) \, d\nu \, d\theta, \\ &\leq \int_{\mathbb{T} \times \mathbb{R}} \left( \int_{\mathbb{R}} \left| \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \right|^2 \mathcal{M}(\omega, \nu) \, d\omega \right) \bar{\gamma}(\nu) \, d\nu \, d\theta, \end{aligned}$$

hence we have

$$\|f(t) - f_\infty\|_{L_\gamma^2}^2 \leq \mathcal{I}[f](t) + \|N(t) - N_\infty\|_{L_\gamma^2}^2.$$

□

From this latter lemma, we can prove the key estimate on the dissipation of the weighted  $L^2$  norm.

**Proposition 2.2.** *Let  $f = f(t, \theta, \omega, \nu)$  be a classical solution to (1.3)-(1.4). Then for all time  $t \geq 0$ , we have*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f(t) - f_\infty\|_{L_\gamma^2}^2 &\leq - \left[ \frac{\tilde{\sigma}}{m} - \frac{\tilde{\kappa}}{2} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L_\gamma^2} \right) \right] \mathcal{I}[f](t) \\ &\quad + \frac{\tilde{\kappa}}{2} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L_\gamma^2} \right) \|N(t) - N_\infty\|_{L_\gamma^2}^2, \end{aligned}$$

where the dissipation  $\mathcal{I}[f](t)$  is defined in (2.1).



*Proof.* We use conservation of mass (see Proposition 2.1) to have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T} \times \mathbb{R}^2} |f - f_\infty|^2 \gamma \, dz = \int_{\mathbb{T} \times \mathbb{R}^2} \partial_t f (f - f_\infty) \gamma \, dz.$$

From the definition of the weight  $\gamma = \mathcal{M}^{-1} \bar{\gamma}$ , it follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f - f_\infty\|_{L_\gamma^2}^2 &= \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} (-\omega \partial_\theta f - \tilde{\kappa} (\sin * \rho) \partial_\omega f + \mathcal{L}_{\text{FP}}[f]) \bar{\gamma} \, dz \\ &= - \int_{\mathbb{T} \times \mathbb{R}^2} \partial_\theta \left( \frac{\omega f^2}{2\mathcal{M}} \right) \bar{\gamma} \, dz - \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} (\sin * \rho) \partial_\omega f \bar{\gamma} \, dz \\ &\quad + \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} \mathcal{L}_{\text{FP}}[f] \bar{\gamma} \, dz \\ &= \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}^2} (\sin * \rho) f \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz + \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} \mathcal{L}_{\text{FP}}[f] \bar{\gamma} \, dz. \end{aligned}$$

We substitute

$$\partial_\omega \mathcal{M} = -\frac{\omega - \nu}{\tilde{\sigma}} \mathcal{M}$$

into  $\mathcal{L}_{\text{FP}}$  to observe

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} \mathcal{L}_{\text{FP}}[f] \bar{\gamma} \, dz &= \frac{\tilde{\sigma}}{m} \int_{\mathbb{T} \times \mathbb{R}^2} \frac{f}{\mathcal{M}} \partial_\omega \left( \partial_\omega f - \frac{\partial_\omega \mathcal{M}}{\mathcal{M}} f \right) \bar{\gamma} \, dz \\ &= -\frac{\tilde{\sigma}}{m} \int_{\mathbb{T} \times \mathbb{R}^2} \left( \partial_\omega \left( \frac{f}{\mathcal{M}} \mathcal{M} \right) - \frac{f}{\mathcal{M}} \partial_\omega \mathcal{M} \right) \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz \\ &= -\frac{\tilde{\sigma}}{m} \int_{\mathbb{T} \times \mathbb{R}^2} \left| \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \right|^2 \mathcal{M} \bar{\gamma} \, dz = -\frac{\tilde{\sigma}}{m} \mathcal{I}[f]. \end{aligned}$$

We combine the latter results to obtain

$$\frac{1}{2} \frac{d}{dt} \|f(t) - f_\infty\|_{L_\gamma^2}^2 = \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}^2} (\sin * \rho(t)) f(t) \partial_\omega \left( \frac{f(t)}{\mathcal{M}} \right) \bar{\gamma} \, dz - \frac{\tilde{\sigma}}{m} \mathcal{I}[f](t).$$

It is left to estimate the first term of the right hand side. Using that

$$\sin * \rho = \sin * (\rho - \rho_\infty),$$

and since  $f_\infty = N_\infty \mathcal{M}$ , we have

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}^2} (\sin * \rho) f \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz &= \int_{\mathbb{T} \times \mathbb{R}^2} \sin * \rho \frac{f}{\sqrt{\mathcal{M}}} \sqrt{\mathcal{M}} \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz \\ &= \int_{\mathbb{T} \times \mathbb{R}^2} \sin * \rho \frac{f - f_\infty}{\sqrt{\mathcal{M}}} \sqrt{\mathcal{M}} \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz \\ &\quad + \int_{\mathbb{T} \times \mathbb{R}^2} (\sin * (\rho - \rho_\infty)) N_\infty \sqrt{\mathcal{M}} \sqrt{\mathcal{M}} \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz. \end{aligned}$$

Then we get

$$\begin{aligned} &\left| \tilde{\kappa} \iint_{\mathbb{T} \times \mathbb{R}^2} (\sin * \rho) f \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} \, dz \right| \\ &\leq \tilde{\kappa} \left( \|\sin * \rho\|_{L^\infty} \|f - f_\infty\|_{L_\gamma^2} + \|\sin * (\rho - \rho_\infty)\|_{L^\infty} \|N_\infty\|_{L_\gamma^2} \right) \sqrt{\mathcal{I}[f]}(t). \end{aligned}$$

It follows from Young's convolution inequality that

$$\begin{cases} \|\sin * \rho\|_{L^\infty} \leq \|\rho\|_{L^1} = \|f\|_{L^1} = \|N_\infty\|_{L^1}, \\ \|\sin * (\rho - \rho_\infty)\|_{L^\infty} \leq \sqrt{\pi} \|\rho - \rho_\infty\|_{L^2}. \end{cases}$$

On the other hand, using (1.7) and

$$\begin{aligned} \rho - \rho_\infty &= \int_{\mathbb{R}} (N - N_\infty) d\nu = \int_{\mathbb{R}} \frac{1}{\bar{\gamma}^{1/2}} (N - N_\infty) \bar{\gamma}^{1/2} d\nu \\ &\leq \left( \int_{\mathbb{R}} (N - N_\infty)^2 \bar{\gamma} d\nu \right)^{1/2}, \end{aligned}$$

we get that

$$\|\rho - \rho_\infty\|_{L^2} \leq \|N - N_\infty\|_{L^2_{\bar{\gamma}}}.$$

Therefore, it yields the following estimate

$$\begin{aligned} &\left| \tilde{\kappa} \iint_{\mathbb{T} \times \mathbb{R}^2} \sin * \rho f \partial_\omega \left( \frac{f}{\mathcal{M}} \right) \bar{\gamma} dz \right| \\ &\leq \tilde{\kappa} \left( \|N_\infty\|_{L^1} \|f - f_\infty\|_{L^2_{\bar{\gamma}}} + \sqrt{\pi} \|N_\infty\|_{L^2_{\bar{\gamma}}} \|N - N_\infty\|_{L^2_{\bar{\gamma}}} \right) \sqrt{\mathcal{I}[f]}(t). \end{aligned}$$

Now using Lemma 2.1, we get that

$$\begin{cases} \|f(t) - f_\infty\|_{L^2_{\bar{\gamma}}} \leq \sqrt{\mathcal{I}[f]}(t) + \|N(t) - N_\infty\|_{L^2_{\bar{\gamma}}}, \\ \sqrt{\mathcal{I}[f]}(t) \|N(t) - N_\infty\|_{L^2_{\bar{\gamma}}} \leq \frac{1}{2} \left( \mathcal{I}[f](t) + \|N(t) - N_\infty\|_{L^2_{\bar{\gamma}}}^2 \right), \end{cases}$$

and gathering the latter results, it gives the desired inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f(t) - f_\infty\|_{L^2_{\bar{\gamma}}}^2 &\leq - \left[ \frac{\tilde{\sigma}}{m} - \frac{\tilde{\kappa}}{2} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\bar{\gamma}}} \right) \right] \mathcal{I}[f](t) \\ &\quad + \frac{\tilde{\kappa}}{2} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\bar{\gamma}}} \right) \|N(t) - N_\infty\|_{L^2_{\bar{\gamma}}}^2. \end{aligned}$$

□

Next, we remind the macroscopic quantities  $N$ ,  $J$ , and  $P$  defined in (1.6), which satisfy

$$\begin{cases} \partial_t N + \partial_\theta (J + \nu N) = 0, \\ \partial_t J + \partial_\theta (P + \nu J) - \tilde{\kappa} (\sin * \rho) N = -\frac{J}{m}. \end{cases}$$

This will be used later to define a modified energy functional  $\mathcal{E}[f]$  to prove convergence to equilibrium. Before to do that, we prove the following estimates on the macroscopic moments.

**Lemma 2.2** (Moments estimates). *Let  $J$  and  $P$  be the moments given by (1.5). Then it holds for all time  $t \geq 0$  that*

$$\begin{cases} \|J(t)\|_{L^2_{\bar{\gamma}}}^2 \leq \tilde{\sigma} \mathcal{I}[f](t), \\ \|P(t) - \tilde{\sigma} N(t)\|_{L^2_{\bar{\gamma}}}^2 \leq 3\tilde{\sigma}^2 \mathcal{I}[f](t). \end{cases}$$

*Proof.* For the first estimate, we observe that  $J$  reads as follows

$$\begin{aligned} J &= \int_{\mathbb{R}} (\omega - \nu) \mathcal{M}^{1/2} f \mathcal{M}^{-1/2} d\omega \\ &= \int_{\mathbb{R}} (\omega - \nu) \mathcal{M}^{1/2} (f - N \mathcal{M}) \mathcal{M}^{-1/2} d\omega. \end{aligned}$$

Hence, using that

$$\int_{\mathbb{R}} (\omega - \nu)^2 \mathcal{M} d\omega = \tilde{\sigma}$$

and applying the Cauchy-Schwarz inequality, we obtain

$$\|J(t)\|_{L^2_{\tilde{\sigma}}}^2 \leq \tilde{\sigma} \|f(t) - N(t) \mathcal{M}\|_{L^2_{\tilde{\sigma}}}^2, \quad \forall t \geq 0.$$

As in the proof of Lemma 2.1, from the Gaussian Poincaré inequality we have

$$\|J(t)\|_{L^2_{\tilde{\sigma}}}^2 \leq \tilde{\sigma} \mathcal{I}[f](t), \quad \forall t \geq 0.$$

Finally, we also have

$$\begin{aligned} P - \tilde{\sigma} N &= \int_{\mathbb{R}} (\omega - \nu)^2 f d\omega - \int_{\mathbb{R}} (\omega - \nu)^2 N \mathcal{M} d\omega, \\ &= \int_{\mathbb{R}} (\omega - \nu)^2 \mathcal{M}^{1/2} (f - N \mathcal{M}) \mathcal{M}^{-1/2} d\omega. \end{aligned}$$

We proceed as before using that

$$\int_{\mathbb{R}} (\omega - \nu)^4 \mathcal{M} d\omega = 3\tilde{\sigma}^2,$$

hence we get for any  $t \geq 0$ ,

$$\begin{aligned} \|P(t) - \tilde{\sigma} N(t)\|_{L^2_{\tilde{\sigma}}}^2 &\leq 3\tilde{\sigma}^2 \|f(t) - N(t) \mathcal{M}\|_{L^2_{\tilde{\sigma}}}^2 \\ &\leq 3\tilde{\sigma}^2 \mathcal{I}[f](t). \end{aligned}$$

□

Now, we study the asymptotic stability of  $f_{\infty}$  when  $\tilde{\sigma} \gg \tilde{\kappa}$ . The goal is first to modify  $\|f - f_{\infty}\|_{L^2_{\tilde{\sigma}}}$  to define a monotonically decreasing energy functional  $\mathcal{E}[f]$  which is equivalent to  $\|f - f_{\infty}\|_{L^2_{\tilde{\sigma}}}$ . Then we show exponential decaying directly on the new functional  $\mathcal{E}[f]$  to prove Theorem 1.1.

**2.2. Toward the modified energy  $\mathcal{E}[f]$ .** We aim to modify the functional  $\|f - f_{\infty}\|_{L^2_{\tilde{\sigma}}}$  in order to construct a monotonically decreasing energy functional  $\mathcal{E}[f]$ . The first step is to characterize the lack of coercivity on the estimate provided in Proposition 2.2. Indeed, thanks to the Fokker-Planck operator, we get a dissipation with respect to quantity  $\mathcal{I}(t)$ . Hence, the goal is now to modify the functional  $\|f(t) - f_{\infty}\|_{L^2_{\tilde{\sigma}}}$  to get a dissipation with respect to quantity  $\|N(t) - N_{\infty}\|_{L^2_{\tilde{\sigma}}}$ . To this aim, we begin with a preliminary result by considering the following elliptic equation for a given function  $S \in L^2(\mathbb{T} \times \mathbb{R})$  such that

$$(2.2) \quad \int_{\mathbb{T}} S(\theta, \nu) d\theta = 0.$$

We consider

$$(2.3) \quad \begin{cases} \partial_{\theta}^2 v = S, \\ \int_{\mathbb{T}} v d\theta = 0 \end{cases}$$

and provide some intermediate results on the solutions  $v$  to (2.3).

**Lemma 2.3.** *Consider any  $S \in L^2_{\bar{\gamma}}(\mathbb{T} \times \mathbb{R})$  which meets condition (2.2) and  $v$  the corresponding solution to (2.3). Then,  $v$  satisfies the following estimate*

$$(2.4) \quad \|\partial_{\theta} v\|_{L^2_{\bar{\gamma}}} \leq C_P \|S\|_{L^2_{\bar{\gamma}}},$$

and

$$(2.5) \quad \|\partial_{\theta}^2 v\|_{L^2_{\bar{\gamma}}} = \|S\|_{L^2_{\bar{\gamma}}},$$

where  $C_P$  is the Poincaré-Wirtinger constant in

$$\|v\|_{L^2_{\bar{\gamma}}} \leq C_P \|\partial_{\theta} v\|_{L^2_{\bar{\gamma}}}.$$

Moreover, considering now  $v$  the solution to (2.3) with source term  $S = N_{\infty} - N$ , where  $N$  is given by (1.5). Then it holds for all time  $t \geq 0$  that

$$(2.6) \quad \partial_t \partial_{\theta} v + \nu \partial_{\theta}^2 v = J - \frac{1}{2\pi} \int_{\mathbb{T}} J d\theta.$$

*Proof.* The first estimate (2.4) is obtained by testing the elliptic equation (2.3) against  $v$  and after an integration by part

$$\|\partial_{\theta} v\|_{L^2_{\bar{\gamma}}}^2 \leq \|S\|_{L^2_{\bar{\gamma}}} \|v\|_{L^2_{\bar{\gamma}}},$$

from which the Wirtinger-Poincaré inequality yields

$$\|\partial_{\theta} v\|_{L^2_{\bar{\gamma}}} \leq C_P \|S\|_{L^2_{\bar{\gamma}}}.$$

The second estimates (2.5) directly follows from the equation on  $v$  (2.3) and using that  $S \in L^2_{\bar{\gamma}}$ .

Now let us consider that  $N$  is given by (1.5) and  $S = N_{\infty} - N$ . We differentiate in time the elliptic equation (2.3) and use the equation (1.6) on  $N$  to get

$$\partial_{\theta}^2 \partial_t v = \partial_{\theta} (J + \nu N).$$

Using again (2.3) and since  $N_{\infty}$  does not depend on  $\theta$ , it follows that

$$\partial_{\theta} (\partial_t \partial_{\theta} v + \nu \partial_{\theta}^2 v) = \partial_{\theta} J,$$

hence, there exists  $C(t, \nu)$  such that

$$\partial_t \partial_{\theta} v + \nu \partial_{\theta}^2 v = J + C(t, \nu).$$

Integrating the latter equation in  $\theta \in \mathbb{T}$  and using periodic boundary condition, we obtain that

$$C(t, \nu) = -\frac{1}{2\pi} \int_{\mathbb{T}} J(t, \theta, \nu) d\theta,$$

and the result follows.  $\square$

Now, we are ready to modify the functional  $\|f(t) - f_{\infty}\|_{L^2_{\bar{\gamma}}}^2$  and define a new functional  $\mathcal{E}[f]$  as

$$\mathcal{E}[f](t) := \frac{1}{2} \|f(t) - f_{\infty}\|_{L^2_{\bar{\gamma}}}^2 + \alpha \mathcal{A}(t),$$

where  $\alpha$  is a small parameter to be determined later and  $\mathcal{A}$  is given by

$$\mathcal{A}(t) = \int_{\mathbb{T} \times \mathbb{R}} J(t) \partial_{\theta} v(t) \bar{\gamma} d\nu d\theta.$$

Here,  $v$  is the solution to (2.3) with source term  $S = N_{\infty} - N$ .

The last step consist in showing that this modified functional  $\mathcal{E}[f]$  is equivalent to  $\|f(t) - f_{\infty}\|_{L^2_{\bar{\gamma}}}^2$  as long as  $\alpha > 0$  is sufficiently small.

**Lemma 2.4.** *Suppose that  $\alpha > 0$  satisfies*

$$(2.7) \quad \alpha \sqrt{\tilde{\sigma}} C_P \leq \frac{1}{2}.$$

*Then we have*

$$\frac{1}{4} \|f(t) - f_\infty\|_{L^2_\gamma}^2 \leq \mathcal{E}[f](t) \leq \frac{3}{4} \|f(t) - f_\infty\|_{L^2_\gamma}^2.$$

*Proof.* We use the Cauchy-Schwarz inequality and Lemma 2.3 to have

$$|\alpha \mathcal{A}| \leq \alpha \|J\|_{L^2_\gamma} \|\partial_\theta v\|_{L^2_\gamma} \leq \alpha C_P \|J\|_{L^2_\gamma} \|N(t) - N_\infty\|_{L^2_\gamma},$$

and from the proof of Lemma 2.2 we obtain

$$\begin{aligned} |\alpha \mathcal{A}| &\leq \alpha \sqrt{\tilde{\sigma}} C_P \|f(t) - N(t) \mathcal{M}\|_{L^2_\gamma} \|N(t) - N_\infty\|_{L^2_\gamma} \\ &\leq \frac{\alpha}{2} \sqrt{\tilde{\sigma}} C_P \left( \|f(t) - N(t) \mathcal{M}\|_{L^2_\gamma}^2 + \|N(t) - N_\infty\|_{L^2_\gamma}^2 \right) \\ &= \frac{\alpha}{2} \sqrt{\tilde{\sigma}} C_P \|f(t) - f_\infty\|_{L^2_\gamma}^2. \end{aligned}$$

Hence when  $\alpha > 0$  satisfies (2.7), we get the expected estimate.  $\square$

**2.3. Proof of Theorem 1.1.** The goal is now to show that when  $\alpha$  is sufficiently small, energy functional  $\mathcal{E}[f]$  is dissipated, which will give the asymptotic behavior of  $\|f(t) - f_\infty\|_{L^2_\gamma}$ . We have

$$\frac{d\mathcal{A}}{dt}(t) = \mathcal{I}_1(t) + \mathcal{I}_2(t),$$

with

$$\begin{cases} \mathcal{I}_1(t) = \int_{\mathbb{T} \times \mathbb{R}} \partial_t J(t) \partial_\theta v(t) \bar{\gamma} \, d\nu d\theta, \\ \mathcal{I}_2(t) = \int_{\mathbb{T} \times \mathbb{R}} J(t) \partial_\theta \partial_t v(t) \bar{\gamma} \, d\nu d\theta. \end{cases}$$

We first compute the term  $\mathcal{I}_1$  using (1.6) which gives

$$\begin{aligned} \mathcal{I}_1 &= - \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta (P + \nu J) \partial_\theta v \bar{\gamma} \, d\nu d\theta \\ &\quad + \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}} (\sin * \rho) N \partial_\theta v \bar{\gamma} \, d\nu d\theta - \frac{1}{m} \int_{\mathbb{T} \times \mathbb{R}} J \partial_\theta v \bar{\gamma} \, d\nu d\theta. \end{aligned}$$

On the one hand, integrating by part and using the equation (2.3) on  $v$ , the first term on the right hand side can be written as

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta (P + \nu J) \partial_\theta v \bar{\gamma} \, d\nu d\theta &= \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta (P - \tilde{\sigma} N + \tilde{\sigma} (N - N_\infty) + \nu J) \partial_\theta v \bar{\gamma} \, d\nu d\theta \\ &= - \int_{\mathbb{T} \times \mathbb{R}} (P - \tilde{\sigma} N) (N_\infty - N) \bar{\gamma} \, d\nu d\theta \\ &\quad + \tilde{\sigma} \|N - N_\infty\|_{L^2_\gamma}^2 - \int_{\mathbb{T} \times \mathbb{R}} \nu J \partial_\theta^2 v \bar{\gamma} \, d\nu d\theta, \end{aligned}$$

where the desired dissipation in  $\|N - N_\infty\|_{L^2_\gamma}^2$  now appears. On the other hand, the second term is estimated as in the proof of Proposition 2.2, which yields

$$\begin{aligned} \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}} (\sin * \rho) N \partial_\theta v \bar{\gamma} \, d\nu \, d\theta &= \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}} (\sin * \rho) (N - N_\infty) \partial_\theta v \bar{\gamma} \, d\nu \, d\theta \\ &+ \tilde{\kappa} \int_{\mathbb{T} \times \mathbb{R}} (\sin * (\rho - \rho_\infty)) N_\infty \partial_\theta v \bar{\gamma} \, d\nu \, d\theta \\ &\leq C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_\gamma} \right) \|N - N_\infty\|_{L^2_\gamma}^2. \end{aligned}$$

Therefore, gathering the latter computations and applying Lemmas 2.2 and 2.3, we estimate the term  $\mathcal{I}_1$  as

$$\begin{aligned} \mathcal{I}_1 &\leq - \left[ \tilde{\sigma} - C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_\gamma} \right) \right] \|N - N_\infty\|_{L^2}^2 \\ &+ \sqrt{\tilde{\sigma}} \left[ \frac{C_P}{m} + \sqrt{3\tilde{\sigma}} \right] \|N - N_\infty\|_{L^2} \sqrt{\mathcal{I}[f]}(t) + \int_{\mathbb{T} \times \mathbb{R}} \nu J \partial_\theta^2 v \bar{\gamma} \, d\nu \, d\theta. \end{aligned}$$

Now we use the term  $\mathcal{I}_2$  to remove the last term on the right hand side of the latter equation. Indeed, using the equation (2.6) in Lemma 2.3, we have

$$\begin{aligned} \mathcal{I}_2 &= \int_{\mathbb{T} \times \mathbb{R}} J \left( J - \frac{1}{2\pi} \int_{\mathbb{T}} J(\theta') \, d\theta' - \nu \partial_\theta^2 v \right) \bar{\gamma} \, d\nu \, d\theta, \\ &\leq 2 \|J\|_{L^2_\gamma}^2 - \int_{\mathbb{T} \times \mathbb{R}} \nu J \partial_\theta^2 v \bar{\gamma} \, d\nu \, d\theta, \\ &\leq 2 \tilde{\sigma} \mathcal{I}[f] - \int_{\mathbb{T} \times \mathbb{R}} \nu J \partial_\theta^2 v \bar{\gamma} \, d\nu \, d\theta, \end{aligned}$$

Gathering the latter results, we obtain

$$\begin{aligned} \frac{d\mathcal{A}}{dt}(t) &\leq 2 \tilde{\sigma} \mathcal{I}[f](t) - \left[ \tilde{\sigma} - C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_\gamma} \right) \right] \|N(t) - N_\infty\|_{L^2}^2 \\ &+ \sqrt{\tilde{\sigma}} \left[ \frac{C_P}{m} + \sqrt{3\tilde{\sigma}} \right] \|N(t) - N_\infty\|_{L^2} \sqrt{\mathcal{I}[f]}(t). \end{aligned}$$

Applying the Young inequality to the last term with  $\eta > 0$  such that

$$\eta \left[ \frac{C_P}{m} + \sqrt{3\tilde{\sigma}} \right]^2 = 1,$$

or equivalently,

$$\frac{\eta}{2} \tilde{\sigma} \left[ \frac{C_P}{m} + \sqrt{3\tilde{\sigma}} \right]^2 = \frac{\tilde{\sigma}}{2},$$

it yields that

$$\begin{aligned} \frac{d\mathcal{A}}{dt}(t) &\leq \left[ 2 + \frac{1}{2} \left( \frac{C_P}{m\sqrt{\tilde{\sigma}}} + \sqrt{3} \right)^2 \right] \tilde{\sigma} \mathcal{I}[f](t) \\ &- \left[ \frac{\tilde{\sigma}}{2} - C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_\gamma} \right) \right] \|N(t) - N_\infty\|_{L^2}^2. \end{aligned}$$

Finally, from the definition of  $\mathcal{E}[f]$  and applying Proposition 2.2 with the latter inequality, we get that for any  $\alpha > 0$ ,

$$\begin{aligned} \frac{d\mathcal{E}[f]}{dt}(t) &\leq - \left[ \frac{\tilde{\sigma}}{m} - \frac{\tilde{\kappa}}{2} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \mathcal{I}[f](t) \\ &\quad - \alpha \left[ \frac{\tilde{\sigma}}{2} - C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \|N(t) - N_\infty\|_{L^2}^2 \\ &\quad + \frac{\tilde{\kappa}}{2} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \|N(t) - N_\infty\|_{L^2_{\tilde{\gamma}}}^2 \\ &\quad + \alpha \left[ 2 + \frac{1}{2} \left( \frac{C_P}{m\sqrt{\tilde{\sigma}}} + \sqrt{3} \right)^2 \right] \tilde{\sigma} \mathcal{I}[f](t). \end{aligned}$$

We choose  $\alpha > 0$  such that the last term in the previous inequality becomes sufficiently small to be absorbed by the first term, that is, taking  $\alpha > 0$  such that

$$\alpha m \left[ 2 + \frac{1}{2} \left( \frac{C_P}{m\sqrt{\tilde{\sigma}}} + \sqrt{3} \right)^2 \right] \leq \frac{1}{2},$$

for instance

$$(2.8) \quad \alpha = \min \left( \frac{1}{2C_P}, \frac{m\tilde{\sigma}}{2(5m^2\tilde{\sigma} + C_P^2)} \right),$$

we have

$$\begin{aligned} \frac{d\mathcal{E}[f]}{dt}(t) &\leq - \left[ \frac{\tilde{\sigma}}{2m} - \frac{\tilde{\kappa}}{2} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \mathcal{I}[f](t) \\ &\quad - \alpha \left[ \frac{\tilde{\sigma}}{2} - C_P \tilde{\kappa} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \|N(t) - N_\infty\|_{L^2}^2 \\ &\quad + \frac{\tilde{\kappa}}{2} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \|N(t) - N_\infty\|_{L^2_{\tilde{\gamma}}}^2 \\ &\leq - \left[ \frac{\tilde{\sigma}}{2m} - \frac{\tilde{\kappa}}{2} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \mathcal{I}[f](t) \\ &\quad - \alpha \left[ \frac{\tilde{\sigma}}{2} - \frac{\tilde{\kappa}}{\alpha} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \right] \|N(t) - N_\infty\|_{L^2}^2. \end{aligned}$$

Finally, when  $\tilde{\sigma} > 0$  is sufficiently large, that is,

$$(2.9) \quad \begin{cases} \frac{\tilde{\kappa}}{\alpha} \left( \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \leq \frac{\tilde{\sigma}}{4}, \\ \tilde{\kappa} \left( 3 \|N_\infty\|_{L^1} + \sqrt{\pi} \|N_\infty\|_{L^2_{\tilde{\gamma}}} \right) \leq \frac{\tilde{\sigma}}{2m}, \end{cases}$$

then the right hand side of the latter estimate corresponds to a dissipation of both  $\mathcal{I}[f](t)$  and  $\|N(t) - N_\infty\|_{L^2_{\tilde{\gamma}}}^2$ . With (2.8), diffusive regime (2.9) can be reformulated as follows: there exists a

constant  $C_\infty > 0$ , only depending on  $\|N_\infty\|_{L^1}$  and  $\|N_\infty\|_{L^2_\gamma}$ , such that  $\kappa > 0$  and  $\sigma > 0$  satisfy

$$C_\infty \max \left( \sqrt{\frac{\tilde{\kappa}}{m}}, \tilde{\kappa}, m\tilde{\kappa} \right) \leq \tilde{\sigma}.$$

Hence by applying Lemma 2.4, we have

$$\frac{d\mathcal{E}[f]}{dt}(t) \leq -C \mathcal{E}[f],$$

where  $C > 0$  only depends on  $C_\infty$  and  $\max \left( \sqrt{\tilde{\kappa}/m}, \tilde{\kappa}, m\tilde{\kappa} \right)$ . Again since  $\mathcal{E}[f]$  and  $\|f - f_\infty\|_{L^2_\gamma}$  are equivalent (Lemma 2.4), we get the expected results using the Gronwall's inequality, which completes our proof.

### 3. DIFFUSION LIMIT WHEN $g = \delta_0$

We now suppose that all oscillators are identical, that is, take  $g = \delta_0$ . It allows to remove the  $\nu$  variable in the previous system, which now becomes (1.10)-(1.11). Thus, we consider the diffusion limit to prove Theorem 1.2 by propagating some regularity in  $\theta$ .

**3.1. Basic estimates.** First, we aim to study the propagation of the weighted  $L^2_{\mathcal{M}^{-1}}$  norm for  $f^\varepsilon$  and  $\partial_\theta f^\varepsilon$  uniformly with respect to  $\varepsilon$ . We prove the following preliminary result.

**Proposition 3.1.** *Let  $f^\varepsilon = f^\varepsilon(t, \theta, \omega)$  be a classical solution to (1.10)-(1.11) such that the initial data satisfy (1.14). Then, we have*

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq \frac{(\tilde{\kappa} M_0)^2}{2\tilde{\sigma}} \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 - \frac{\tilde{\sigma}}{2\varepsilon^2} \mathcal{I}[f^\varepsilon](t).$$

Moreover, for all time  $t \geq 0$

$$(3.2) \quad \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}} + \|\partial_\theta f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}} \leq \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + 3 \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) e^{Ct},$$

where  $C = (\tilde{\kappa} M_0)^2 / \tilde{\sigma}$ .

*Proof.* We first proceed as in Proposition 3.1, but combine the terms in a different way to get uniform estimates with respect to  $\varepsilon$ . We consider the centred Gaussian distribution  $\mathcal{M}$  and multiply (1.10) by  $f^\varepsilon \mathcal{M}^{-1}$ , then we integrate with respect to  $\mathbf{z} := (\theta, \omega) \in \mathbb{T} \times \mathbb{R}$ , it yields

$$\frac{1}{2} \frac{d}{dt} \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 = \frac{\tilde{\kappa}}{\varepsilon} \int_{\mathbb{T} \times \mathbb{R}} (\sin * \rho^\varepsilon) f^\varepsilon(t) \partial_\omega \left( \frac{f^\varepsilon(t)}{\mathcal{M}} \right) d\mathbf{z} - \frac{\tilde{\sigma}}{\varepsilon^2} \mathcal{I}[f^\varepsilon](t),$$

where the dissipation  $\mathcal{I}[f^\varepsilon](t)$  is defined in (2.1).

It is left to estimate the first term of the right hand side as

$$\frac{\tilde{\kappa}}{\varepsilon} \int_{\mathbb{T} \times \mathbb{R}} (\sin * \rho^\varepsilon) f^\varepsilon \partial_\omega \left( \frac{f^\varepsilon}{\mathcal{M}} \right) d\mathbf{z} \leq \frac{\tilde{\kappa}^2}{2\eta} \|\sin * \rho^\varepsilon\|_{L^\infty}^2 \|f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 + \frac{\eta}{2\varepsilon^2} \mathcal{I}[f^\varepsilon],$$

where  $\eta > 0$  is a free parameter to be defined later. Using the Young's convolution inequality and the conservation of mass, we have

$$\|\sin * \rho^\varepsilon(t)\|_{L^\infty} \leq \|\rho^\varepsilon(t)\|_{L^1} = \|f^\varepsilon(t)\|_{L^1} = M_0$$

and choosing  $\eta = \tilde{\sigma}$ , it gives the first estimate (3.1)

$$\frac{1}{2} \frac{d}{dt} \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq \frac{(\tilde{\kappa} M_0)^2}{2\tilde{\sigma}} \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 - \frac{\tilde{\sigma}}{2\varepsilon^2} \mathcal{I}[f^\varepsilon](t).$$

Finally from the Gronwall's inequality, we get that

$$(3.3) \quad \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}} \leq \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} e^{C_0 t},$$



with  $C_0 = (\tilde{\kappa} M_0)^2 / (2\tilde{\sigma})$ .

Then we set  $h^\varepsilon = \partial_\theta f^\varepsilon$  and differentiate (1.10)-(1.11) with respect to  $\theta$ , it yields the following equation

$$\varepsilon \partial_t h^\varepsilon + \omega \partial_\theta h^\varepsilon + \tilde{\kappa} (\sin * \rho^\varepsilon) \partial_\omega h^\varepsilon = \mathcal{L}_{\text{FP}}[h^\varepsilon] - \tilde{\kappa} (\cos * \rho^\varepsilon) \partial_\omega f^\varepsilon,$$

which has the same structure as the equation on  $f^\varepsilon$  with the additional source term  $-\tilde{\kappa} (\cos * \rho^\varepsilon) \partial_\omega f^\varepsilon$ . Hence, proceeding as previously, we now obtain

$$\frac{1}{2} \frac{d}{dt} \|h^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq \frac{(\tilde{\kappa} M_0)^2}{\tilde{\sigma}} \left( \|h^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 + \|f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 \right) - \frac{\tilde{\sigma}}{2\varepsilon^2} \mathcal{I}[h^\varepsilon](t).$$

Then, using (3.3), we get the second estimate

$$(3.4) \quad \|h^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}} \leq \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + \sqrt{2} \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) e^{2C_0 t},$$

Gathering the latter estimates (3.3) and (3.4), we obtain (3.2).  $\square$

**3.2. Proof of Theorem 1.2.** From Proposition 3.1, we may now prove our second main result. On the one hand, we evaluate a kind of relative entropy in the weighted  $L^2$  space,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 &= \frac{1}{2} \frac{d}{dt} \|f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 - \frac{1}{2} \frac{d}{dt} \|\rho^\varepsilon\|_{L^2}^2 \\ &\leq -\frac{\tilde{\sigma}}{2\varepsilon^2} \mathcal{I}[f^\varepsilon] + \frac{(\tilde{\kappa} M_0)^2}{2\tilde{\sigma}} \|f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 + \frac{1}{\varepsilon} \int_{\mathbb{T}} \rho^\varepsilon \partial_\theta J^\varepsilon d\theta. \end{aligned}$$

Hence, after integrating by part and applying the Young inequality on the last term on the right hand side of the latter inequality, we apply Lemma 2.2, which yields

$$\frac{1}{2} \frac{d}{dt} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq -\frac{\tilde{\sigma}}{4\varepsilon^2} \mathcal{I}[f^\varepsilon] + \frac{(\tilde{\kappa} M_0)^2}{2\tilde{\sigma}} \|f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 + \|\partial_\theta \rho^\varepsilon\|_{L^2}^2.$$

Moreover, observing that

$$\|\partial_\theta \rho^\varepsilon\|_{L^2} \leq \|\partial_\theta f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}},$$

using again the Gaussian-Poincaré inequality with respect to probability measure  $\mathcal{M} d\omega$  to have

$$\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq \mathcal{I}[f^\varepsilon],$$

and the  $H^1$  estimate (3.2) of Proposition 3.1, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 &\leq -\frac{\tilde{\sigma}}{4\varepsilon^2} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 \\ &\quad + \max\left(\frac{C}{2}, 1\right) \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + 3 \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right)^2 e^{2Ct}, \end{aligned}$$

hence from the Gronwall's lemma, we get the first estimate of Theorem 1.2

$$(3.5) \quad \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}} \leq \|f_{\text{in}}^\varepsilon - \rho_{\text{in}}^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}} e^{-\tilde{\sigma}t/(4\varepsilon^2)} \\ + \sqrt{2}\varepsilon \max\left(\frac{\tilde{\kappa} M_0}{\tilde{\sigma}}, \sqrt{\frac{2}{\tilde{\sigma}}}\right) \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + 3 \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) e^{Ct}.$$

On the other hand to prove the convergence of  $\rho^\varepsilon$  to its limit  $\rho$  given by (1.13), we define  $\mathcal{A}(t)$  as

$$(3.6) \quad \mathcal{A}(t) = \frac{1}{2} \|\partial_\theta v^\varepsilon(t)\|_{L^2}^2,$$

where  $v$  is now solution to (2.3) with source term

$$S = \rho - \rho^\varepsilon + \varepsilon \partial_\theta J^\varepsilon.$$

First let us observe that  $v^\varepsilon$  is well defined since the compatibility condition (2.2) on  $S$  is well satisfied. Before proving the second estimate of Theorem 1.2, let us show that  $\mathcal{A}(t)$  gives a  $H^{-1}$  estimate on  $\rho^\varepsilon - \rho$ . Indeed, the following Lemma ensures that  $\mathcal{A}(t)$  is controlled by the squares of the weighted  $L^2_{\mathcal{M}^{-1}}$  norm of  $\partial_\theta f^\varepsilon$  and the  $H^{-1}$  norm of  $\rho^\varepsilon - \rho$ .

**Lemma 3.1.** *We consider  $\mathcal{A}(t)$  defined by (3.6). It holds uniformly with respect to  $\varepsilon$*

$$\mathcal{A}(t) \leq \|\rho^\varepsilon(t) - \rho(t)\|_{H^{-1}}^2 + \tilde{\sigma} C_P^2 \varepsilon^2 \|\partial_\theta f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2,$$

and

$$\frac{1}{4} \|\rho^\varepsilon(t) - \rho(t)\|_{H^{-1}}^2 - \tilde{\sigma} C_P^2 \frac{\varepsilon^2}{2} \|\partial_\theta f^\varepsilon(t)\|_{L^2_{\mathcal{M}^{-1}}}^2 \leq \mathcal{A}(t).$$

*Proof.* Defining  $w^\varepsilon$  and  $u^\varepsilon$  as the respective solutions to (2.3) with source term  $S = -\partial_\theta J^\varepsilon$  and  $\rho - \rho^\varepsilon$ , it holds

$$v^\varepsilon = u^\varepsilon - \varepsilon w^\varepsilon.$$

We apply operator  $\partial_\theta$  to the latter relation, take the  $L^2$  norm, and apply the triangular inequality, it yields

$$\sqrt{2\mathcal{A}} \leq \|\partial_\theta u^\varepsilon\|_{L^2} + \varepsilon \|\partial_\theta w^\varepsilon\|_{L^2},$$

and

$$\|\partial_\theta u^\varepsilon\|_{L^2} - \varepsilon \|\partial_\theta w^\varepsilon\|_{L^2} \leq \sqrt{2\mathcal{A}}.$$

We estimate  $\|\partial_\theta w^\varepsilon\|_{L^2}$  by applying (2.4) in Lemma 2.3 with source term  $S = -\partial_\theta J^\varepsilon$  and using that

$$\|\partial_\theta J^\varepsilon\|_{L^2} \leq \sqrt{\tilde{\sigma}} \|\partial_\theta f^\varepsilon\|_{L^2},$$

which yields

$$\sqrt{2\mathcal{A}} \leq \|\rho^\varepsilon - \rho\|_{H^{-1}} + \varepsilon C_P \sqrt{\tilde{\sigma}} \|\partial_\theta f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}},$$

and

$$\|\rho^\varepsilon - \rho\|_{H^{-1}} - \varepsilon C_P \sqrt{\tilde{\sigma}} \|\partial_\theta f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \leq \sqrt{2\mathcal{A}}.$$

We obtain the result taking the square of the latter inequalities and applying Young's inequality.  $\square$

Now let us evaluate  $\mathcal{A}(t)$  observing that

$$\frac{d\mathcal{A}}{dt}(t) = \langle \partial_t \partial_\theta v^\varepsilon(t), \partial_\theta v^\varepsilon(t) \rangle = \langle \partial_t (\rho^\varepsilon(t) - \varepsilon \partial_\theta J^\varepsilon(t) - \rho(t)), v^\varepsilon(t) \rangle.$$

Therefore, relying on equations (2.2) and (1.13), we deduce

$$(3.7) \quad \frac{d\mathcal{A}}{dt}(t) = -\tilde{\sigma} \|\rho^\varepsilon(t) - \varepsilon \partial_\theta J^\varepsilon(t) - \rho(t)\|_{L^2}^2 + \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

where

$$\begin{cases} \mathcal{E}_1(t) = -\langle P^\varepsilon(t) - \tilde{\sigma} (\rho^\varepsilon(t) - \varepsilon \partial_\theta J^\varepsilon(t)), \rho^\varepsilon(t) - \varepsilon \partial_\theta J^\varepsilon(t) - \rho(t) \rangle, \\ \mathcal{E}_2(t) = \tilde{\kappa} \langle \sin * \rho^\varepsilon(t) \rho^\varepsilon(t) - \sin * \rho(t) \rho(t), \partial_\theta v^\varepsilon(t) \rangle. \end{cases}$$

First observing that

$$\|P^\varepsilon - \tilde{\sigma} \rho^\varepsilon\|_{L^2} \leq \left( \int_{\mathbb{R}} |\omega|^4 \mathcal{M}(\omega) d\omega \right)^{1/2} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}} = \sqrt{3} \tilde{\sigma} \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}},$$

we have for any  $\eta_1 > 0$

$$\mathcal{E}_1(t) \leq \frac{3\tilde{\sigma}^2}{\eta_1} \|f^\varepsilon(t) - \rho^\varepsilon(t) \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 + \frac{\varepsilon^2 \tilde{\sigma}^2}{\eta_1} \|\partial_\theta J^\varepsilon(t)\|_{L^2}^2 + \frac{\eta_1}{2} \|\rho^\varepsilon(t) - \varepsilon \partial_\theta J^\varepsilon(t) - \rho(t)\|_{L^2}^2.$$

Then we evaluate the term  $\mathcal{E}_2$  as follows

$$\mathcal{E}_2(t) = \mathcal{E}_{21}(t) + \mathcal{E}_{22}(t),$$

where

$$\begin{cases} \mathcal{E}_{21}(t) = \tilde{\kappa} \langle \sin * \rho (\rho^\varepsilon - \rho)(t), \partial_\theta v^\varepsilon(t) \rangle, \\ \mathcal{E}_{22}(t) = \tilde{\kappa} \langle \sin * (\rho^\varepsilon - \rho)(t) \rho^\varepsilon(t), \partial_\theta v^\varepsilon(t) \rangle. \end{cases}$$

Again applying the Young's inequality, we have for any  $\eta_{21} > 0$ ,

$$\begin{aligned} \mathcal{E}_{21}(t) &\leq \tilde{\kappa} \|\rho_{\text{in}}\|_{L^1} \left( \|\rho^\varepsilon(t) - \rho(t) - \varepsilon \partial_\theta J^\varepsilon(t)\|_{L^2} + \varepsilon \|\partial_\theta J^\varepsilon(t)\|_{L^2} \right) \|\partial_\theta v^\varepsilon(t)\|_{L^2}, \\ &\leq \frac{\eta_{21}}{2} (\|\rho^\varepsilon(t) - \rho(t) - \varepsilon \partial_\theta J^\varepsilon(t)\|_{L^2}^2 + \varepsilon^2 \|\partial_\theta J^\varepsilon(t)\|_{L^2}^2) + \frac{2(\tilde{\kappa} M_0)^2}{\eta_{21}} \mathcal{A}(t), \end{aligned}$$

whereas the second term  $\mathcal{E}_{22}(t)$  is evaluated as

$$\begin{aligned} \mathcal{E}_{22}(t) &\leq \tilde{\kappa} \|\rho_{\text{in}}^\varepsilon\|_{L^1} \|\partial_\theta v^\varepsilon(t)\|_{L^\infty} \left( \|\sin * (\rho^\varepsilon - \rho - \varepsilon \partial_\theta J^\varepsilon)(t)\|_{L^\infty} + \varepsilon \|\sin * \partial_\theta J^\varepsilon(t)\|_{L^\infty} \right), \\ &\leq \tilde{\kappa} M_0 \|\partial_\theta v^\varepsilon(t)\|_{L^\infty} (\|\sin * (\rho^\varepsilon - \rho - \varepsilon \partial_\theta J^\varepsilon)(t)\|_{L^\infty} + \varepsilon \sqrt{\pi} \|\partial_\theta J^\varepsilon(t)\|_{L^2}). \end{aligned}$$

Hence, using that  $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ , we have

$$\|\partial_\theta v^\varepsilon\|_{L^\infty} \leq C_{S,1} \|\partial_\theta v^\varepsilon\|_{H^1} \leq C_{S,2} \|\rho^\varepsilon - \rho - \varepsilon \partial_\theta J^\varepsilon\|_{L^2},$$

where  $C_{S,j}$ , for  $j = 1, 2$  are two positive constants. Therefore, applying the Young's convolution inequality,

$$\|\sin * (\rho^\varepsilon - \rho - \varepsilon \partial_\theta J^\varepsilon)\|_{L^\infty} = \|\sin * \partial_\theta^2 v^\varepsilon\|_{L^\infty} = \|\cos * \partial_\theta v^\varepsilon\|_{L^\infty} \leq \sqrt{\pi} \|\partial_\theta v^\varepsilon\|_{L^2},$$

It yields that for any  $\eta_{22} > 0$ ,

$$\mathcal{E}_{22}(t) \leq \frac{\eta_{22}}{2} \|\rho^\varepsilon(t) - \rho(t) - \varepsilon \partial_\theta J^\varepsilon(t)\|_{L^2}^2 + \frac{\pi (\tilde{\kappa} M_0 C_{S,2})^2}{\eta_{22}} (2\mathcal{A}(t) + \varepsilon^2 \|\partial_\theta J^\varepsilon(t)\|_{L^2}^2).$$

Gathering the latter estimates on  $\mathcal{E}_{21}$  and  $\mathcal{E}_{22}$ , it gives

$$\begin{aligned} \mathcal{E}_2(t) &\leq \frac{\eta_{21} + \eta_{22}}{2} \|\rho^\varepsilon(t) - \rho(t) - \varepsilon \partial_\theta J^\varepsilon(t)\|_{L^2}^2 + \left( \frac{\eta_{21}}{2} + \frac{\pi (\tilde{\kappa} M_0 C_{S,2})^2}{\eta_{22}} \right) \varepsilon^2 \|\partial_\theta J^\varepsilon(t)\|_{L^2}^2 \\ &\quad + 2 (\tilde{\kappa} M_0)^2 \left( \frac{1}{\eta_{21}} + \frac{\pi C_{S,2}^2}{\eta_{22}} \right) \mathcal{A}(t). \end{aligned}$$

Choosing  $\eta_1 = \eta_{21} = \eta_{22} = \tilde{\sigma}/3$  on the estimates of  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , we get that there exists a constant  $C > 0$ , only depending on  $\tilde{\kappa}$ ,  $\tilde{\sigma}$  and  $M_0$ , such that

$$\begin{aligned} \mathcal{E}_1(t) + \mathcal{E}_2(t) &\leq + \frac{\tilde{\sigma}}{2} \|\rho^\varepsilon(t) - \rho(t) - \varepsilon \partial_\theta J^\varepsilon(t)\|_{L^2}^2 \\ &\quad + C \left( \mathcal{A}(t) + \varepsilon^2 \|\partial_\theta J^\varepsilon\|_{L^2}^2 + \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 \right). \end{aligned}$$

Substituting this latter estimate in (3.7) and using the estimates in (3.5) and

$$\|\partial_\theta J^\varepsilon\|_{L^2} \leq \sqrt{\tilde{\sigma}} \|\partial_\theta f^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}},$$

with (3.2), we deduce that there exists a constant  $C > 0$ , only depending on  $\tilde{\kappa}$ ,  $\tilde{\sigma}$  and  $M_0$ , such that,

$$\frac{d\mathcal{A}}{dt}(t) \leq C \left( \mathcal{A}(t) + \|f_{\text{in}}^\varepsilon - \rho_{\text{in}}^\varepsilon \mathcal{M}\|_{L^2_{\mathcal{M}^{-1}}}^2 e^{-\tilde{\sigma}t/(2\varepsilon^2)} + \varepsilon^2 \left( \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 + \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 \right) e^{Ct} \right).$$

Integrating this differential inequality, it yields that there exists a constant  $C > 0$ , only depending on  $\tilde{\kappa}$ ,  $\tilde{\sigma}$  and  $M_0$  such that,

$$\mathcal{A}(t) \leq \left( \mathcal{A}(0) + C \left( \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 + \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}}^2 \right) \varepsilon^2 \right) e^{Ct}.$$

Applying Lemma 3.1, we get the second estimate (1.15) of Theorem 1.2.

$$\|\rho^\varepsilon - \rho\|_{H^{-1}} \leq C \left( \|\rho_{\text{in}}^\varepsilon - \rho_{\text{in}}\|_{H^{-1}} + \varepsilon \left( \|f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} + \|\partial_\theta f_{\text{in}}^\varepsilon\|_{L^2_{\mathcal{M}^{-1}}} \right) \right) e^{Ct}.$$

#### 4. CONCLUSION

In this paper, we first studied the stability of a phase-homogeneous stationary state to the inertial Kuramoto-Sakaguchi equation. We showed that when the noise intensity is sufficiently and relatively larger than the coupling strength, the solutions of the inertial Kuramoto-Sakaguchi equation (1.3)-(1.4) converge to the corresponding phase-homogeneous stationary state exponentially fast in weighted  $L^2_\gamma$  norm sense. To achieve this, we employed an energy functional which is equivalent to the weighted  $L^2_\gamma$  norm and proved the exponential decaying of it. Note that there is no smallness assumption on the initial data. Furthermore, it is notable that we improved the existing results in [10]. Indeed, we proved the convergence for a larger class of functions. In addition, for the case of sufficiently small or large coupling strength, that is when coupling strength is near zero or infinity, we provided smaller lower bound for noise intensity. Finally when all oscillators are identical, we investigate a particular regime corresponding to the long time behavior and the mass  $m$  of the single oscillator converges to zero. This corresponds to the diffusive limit of the inertial Kuramoto-Sakaguchi equation for which we prove error estimate with respect to  $m$ .

It is worth to mention that the present contribution proposes a simple proof of two results already given in [10] and [19]. The advantage of our approach is to present a continuous framework which will be useful for the design and analysis of a fully discrete finite volume scheme for the inertial Kuramoto-Sakaguchi equation (1.3)-(1.4) written as an hyperbolic system using Hermite polynomials in velocity [5, 4]. This approach should allow to preserve the stationary solution and the weighted  $L^2$  relative energy.

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