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MULTIPLE RECURRENCE AND CONVERGENCE WITHOUT COMMUTATIVITY

NIKOS FRANTZIKINAKIS AND BERNARD HOST

ABSTRACT. We establish multiple recurrence and convergence results for pairs of zero entropy measure preserving transformations that do not satisfy any commutativity assumptions. Our results cover the case where the iterates of the two transformations are n and n^k respectively, where $k \geq 2$, and the case $k = 1$ remains an open problem. Our starting point is based on the observation that Furstenberg systems of sequences of the form $(f(T^{n^k}x))$ have very special structural properties when $k \geq 2$. We use these properties and some disjointness arguments in order to get characteristic factors with nilpotent structure for the corresponding ergodic averages, and then finish the proof using some equidistribution results on nilmanifolds.

1. INTRODUCTION AND MAIN RESULTS

A well known multiple recurrence result of Furstenberg and Katznelson [18] states that if T, S are commuting, measure preserving transformations, acting on a probability space (X, \mathcal{X}, μ) , then for every set $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that

$$(1) \quad \mu(A \cap T^{-n}A \cap S^{-n}A) > 0.$$

This result was extended by Bergelson and Leibman to cover iterates given by arbitrary integer polynomials with zero constant terms [4], and the case where T and S generate a nilpotent group [5]. For $f, g \in L^\infty(\mu)$, the existence in $L^2(\mu)$ of the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^n g$$

was established in the commutative case by Conze and Lesigne [10], and the extension to any number of transformations spanning a nilpotent group and to polynomial iterates was established by Walsh [29]. The goal of this article is to study similar recurrence phenomena and mean convergence results for pairs of measure preserving transformations that do not satisfy any commutativity assumptions.

We caution the reader at this point that a simple example of Furstenberg [17, Page 40] shows that if T and S do not satisfy any commutativity assumptions, then the recurrence property (1) may fail in general. Indeed, if T is the $(1/2, 1/2)$ -Bernoulli shift on the sequence space $X = \{0, 1\}^{\mathbb{Z}}$, and R is the transformation on X that fixes the 0th-coordinate and flips the 0's and the 1's in all other coordinates, then for

$$A := \{x \in \{0, 1\}^{\mathbb{Z}} : x(0) = 1\} \quad \text{and} \quad S := R^{-1}TR,$$

an easy computation shows that $\mu(T^{-n}A \cap S^{-n}A) = 0$ for every $n \in \mathbb{N}$. Other examples, which also cover non-convergence results and different iterates, are given in [2, Example 7.1], [6], and [15, Lemma 4.1]. Furthermore, Proposition 1.4 below shows that this situation is fairly general. So in the absence of additional assumptions, there is no hope for recurrence and convergence results that apply to pairs of general, not necessarily commuting, measure preserving transformations T, S .

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1.1. **Results.** On the positive side, it was recently established in [12] that if the transformation T has zero entropy (in the above mentioned counterexamples T and S have positive entropy), then for any positive non-integer a we have for all sets $A \in \mathcal{X}$ with $\mu(A) > 0$ that there exists $n \in \mathbb{N}$ such that

$$(2) \quad \mu(A \cap T^{-n}A \cap S^{-[n^a]}A) > 0.$$

It was also established in [12] that the corresponding multiple ergodic averages

$$(3) \quad \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{[n^a]}g$$

converge in $L^2(\mu)$ as $N \rightarrow \infty$ and a simple formula for the limit was obtained. The method of [12] makes essential use of the fact that the power of n in the second iterate is non-integral; this hypothesis gives access to properties of strongly stationary systems that the so called ‘‘Furstenberg systems’’ of sequences of the form $(g(S^{[n^a]}x))$ satisfy. Such information is lost when a is integral, and it was left as an open problem (see [12, Problem 2]) whether the recurrence property (2), or the mean convergence of the averages (3), still holds when $a \in \mathbb{N}$. The main objective of this article is to give a positive answer to these questions when $a \in \mathbb{N}$ is different than 1. The case where $a = 1$ remains open (see Section 1.4 for additional related problems).

Theorem 1.1. *Let T, S be measure preserving transformations acting on a probability space (X, \mathcal{X}, μ) such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$. Then for every $f, g \in L^\infty(\mu)$ the limit*

$$(4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{p(n)}g$$

exists in $L^2(\mu)$.

The rational Kronecker factor $\mathcal{K}_{\text{rat}}(T)$ is the factor spanned by eigenfunctions associated to rational eigenvalues of T . In the case where $T = S$ and $p(n) = n^2$, Furstenberg and Weiss [19] proved that this factor is characteristic, in the sense of [19], for mean convergence of the averages (4), a result that was later extended to the case of commuting transformations in [9]. In the context of Theorem 1.1, the methods used in these articles do not allow us to get similar characteristic factors for the averages (4). We use a different approach to establish the following result.

Theorem 1.2. *Under the hypothesis of Theorem 1.1, the limit (4) in $L^2(\mu)$ is equal to 0 if either $\mathbb{E}(f|\mathcal{K}_{\text{rat}}(T)) = 0$ or $\mathbb{E}(g|\mathcal{K}_{\text{rat}}(S)) = 0$.*

From this result it is an easy matter to deduce the following.

Theorem 1.3. *Let T, S be measure preserving transformations acting on a probability space (X, \mathcal{X}, μ) such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$ and $p(0) = 0$. Then for every $A \in \mathcal{X}$ and $\varepsilon > 0$, the set*

$$(5) \quad \{n \in \mathbb{N} : \mu(A \cap T^{-n}A \cap S^{-p(n)}A) \geq \mu(A)^3 - \varepsilon\}$$

has positive lower density.

Remarks. • Even proving that $\mu(A \cap T^{-n}A \cap S^{-p(n)}A) > 0$ for some $n \in \mathbb{N}$ is not much easier than the stronger statement mentioned above.

• It is possible to prove that the set (5) is syndetic but we do not address this problem here as this will .

When $T = S$ or T, S commute, this result was obtained without an entropy assumption in [14] and [9] respectively.

The convergence and recurrence results of Theorems 1.1-1.3 are non-trivial even in the case of distal or weakly mixing systems, the reason being that in the absence of any commutativity assumptions, the traditional methods for proving recurrence and mean convergence results do not seem to give us a useful starting point.

1.2. Methods. Our approach has some similarities with the one used in [12] to study the averages (3) for non-integral values of a . We proceed by first studying Furstenberg systems of sequences of the form $(f(T^{p(n)}x))$; the advantage being that this problem can be handled by analyzing multiple ergodic averages that involve only a single transformation, a problem that has been well studied and we can employ an arsenal of tools. A crucial difference with the approach taken in [12], is that unlike the case of iterates given by non-integral powers, systems arising from polynomial iterates are not in general strongly stationary. Nevertheless, when the polynomial p is non-linear the corresponding Furstenberg systems are expected to have very particular structure, and in fact, it was conjectured in [12] that they are direct products of Bernoulli systems and systems of algebraic structure. Although, we do not establish this structural result, we obtain some partial information in Proposition 3.2 that suffices for our purposes. It enables us to use a disjointness argument and reduce the study of the limiting behavior of the averages in (4) to the case where the transformation S is structured (see Proposition 4.1). We then proceed by using a variant of the main structural result in [22] and a Nilsequence Wiener-Wintner convergence result from [24], to conclude the proof of Theorem 1.1 (see Proposition 4.2). Theorem 1.2, as well as Theorem 1.3, require some additional work. After further reducing matters to the case where both transformations T and S have algebraic structure, we are left with verifying an equidistribution property for nilsystems (stated in Proposition 5.4). After this property is established, it is an easy matter to show that the rational Kronecker factor is a characteristic factor for mean convergence and from this we deduce the recurrence property of Theorem 1.3.

1.3. Is zero entropy necessary? It is natural to ask if the zero entropy assumption on T in the previous results is necessary. The answer is yes and in fact, by adapting known examples, we will show that if T has positive entropy, then for some measure preserving transformation S the conclusions of Theorems 1.1-1.3 fail. This is a direct consequence of the following result (which is a variant of [15, Lemma 4.1]).

Proposition 1.4. *Let (X, μ, T) be an ergodic system with positive entropy, $a, b: \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ be injective sequences that miss infinitely many integers, and F be an arbitrary subset of \mathbb{N} . Then there exist a system (X, μ, S) , with positive entropy, a measurable set A , and $c > 0$, such that*

$$\mu(T^{-a(n)}A \cap S^{-b(n)}A) = \begin{cases} 0 & \text{if } n \in F, \\ c & \text{if } n \notin F. \end{cases}$$

As a consequence, there exist a transformation S and a set A with $\mu(A) > 0$ so that $\mu(T^{-a(n)}A \cap S^{-b(n)}A) = 0$ for every $n \in \mathbb{N}$, and another transformation S and a set A so that the averages $\frac{1}{N} \sum_{n=1}^N \mu(T^{-a(n)}A \cap S^{-b(n)}A)$ do not converge as $N \rightarrow \infty$.

Remark. Theorems 1.1 and 1.3 show that if the transformation T has zero entropy, then such general constructions are not possible.

1.4. Further directions. Our arguments fail if in place of the iterates $n, p(n)$ in Theorems 1.1-1.3 we use the pair of iterates n, n or the pair n^2, n^3 . In the first case the problem is that Furstenberg systems of sequences of the form $(g(S^n x))$ no longer have special structure, and we therefore lose the crucial starting point provided by Proposition 3.2. In

the second case, the problem is that the zero entropy assumption on T does not pass to Furstenberg systems of the sequence $(f(T^{n^2}x))$ (which can be Bernoulli). In either case, we were not able to prove convergence or recurrence even when both T and S are weakly mixing or isomorphic to 2-step distal systems of the form $(x, y) \mapsto (x + \alpha, y + f(x))$ defined on \mathbb{T}^2 with the Lebesgue measure $m_{\mathbb{T}^2}$. This leads to the following problem.

Problem. *Does Theorem 1.1 hold in place of the iterates $n, p(n)$ we use the pair of iterates n, n or the pair n^2, n^3 ? Do we also have recurrence in these cases?*

One can ask a more general question for pairs of iterates given by arbitrary polynomials $p, q \in \mathbb{Z}[t]$ with $p(0) = q(0) = 0$. It is not clear which way the answer should go and we would not be surprised if the answer to these questions turns out to be negative (see though [1] for some positive results when the iterates are n, n).

Lastly, we expect that under the assumptions of Theorem 1.1 the averages (4) converge pointwise almost everywhere. In this regard, we have the following conditional result that lends credence to this conjecture.

Proposition 1.5. *Let T, S be measure preserving transformations acting on a probability space (X, \mathcal{X}, μ) such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$, and suppose that for all $\ell \in \mathbb{N}$ and functions $f_1, \dots, f_\ell \in L^\infty(\mu)$, the averages*

$$(6) \quad \frac{1}{N} \sum_{n=1}^N f_1(S^{p(n+1)}x) \cdot \dots \cdot f_\ell(S^{p(n+\ell)}x)$$

converge pointwise almost everywhere. Then for all functions $f, g \in L^\infty(\mu)$, the averages

$$\frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot g(S^{p(n)}x)$$

converge pointwise almost everywhere and they converge to 0 if either $\mathbb{E}(f|\mathcal{K}_{\text{rat}}(T)) = 0$ or $\mathbb{E}(g|\mathcal{K}_{\text{rat}}(S)) = 0$.

We remark that pointwise convergence of the averages (6) is expected to hold, but it is considered to be very hard to establish even when $\ell = 2, 3$ and $p(n) = n^2$.

1.5. Notation. For $N \in \mathbb{N}$ we let $[N] := \{1, \dots, N\}$. We usually denote sequences on \mathbb{N} or on \mathbb{Z} by $(a(n))$, instead of $(a(n))_{n \in \mathbb{N}}$ or $(a(n))_{n \in \mathbb{Z}}$; the domain of the sequence is going to be clear from the context. With \mathbb{T} we denote the one dimensional torus \mathbb{R}/\mathbb{Z} , and we often identify it with $[0, 1)$. We often denote elements of \mathbb{T} with real numbers and we are implicitly assuming that these real numbers are taken modulo 1. For $t \in \mathbb{R}$ or \mathbb{T} we let $e(t) := \exp(2\pi it)$.

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2. BACKGROUND AND TOOLS FROM ERGODIC THEORY

In this section we gather some basic facts needed in the sequel and establish some notation. In order to avoid unnecessary repetition, we refer the reader to [25] and [20] for some other standard notions from ergodic theory used in this article.

2.1. Measure preserving systems. Throughout, by a *measure preserving system*, or simply a *system*, we mean a Lebesgue probability space (X, \mathcal{X}, μ) together with an invertible, measurable, and measure preserving transformation $T: X \rightarrow X$. In general we omit the σ -algebra and write a system as (X, μ, T) and sometimes we abbreviate the notation and write X or T . The system is ergodic if the only T -invariant sets in \mathcal{X} have measure 0 or 1. If $f \in L^\infty(\mu)$ and $n \in \mathbb{Z}$, with $T^n f$ we denote the composition $f \circ T^n$, where for $n \in \mathbb{N}$, we let $T^n := T \circ \dots \circ T$ (n times), $T^{-n} = (T^{-1})^n$, and $T^0 = \text{id}$.

2.2. Factors and joinings. Let (X, \mathcal{X}, μ, T) be a system. By a *factor* of X we mean a T -invariant sub- σ -algebra of \mathcal{X} . A *factor map* from the system (X, \mathcal{X}, μ, T) to the system (Y, \mathcal{Y}, ν, S) is a measurable map $\pi: X \rightarrow Y$ with $\pi \circ T = S \circ \pi$ and such that ν is the image of the measure μ under π . In this case, the sub- σ -algebra $\pi^{-1}(\mathcal{Y})$ of \mathcal{X} is a factor of X and every factor of X can be obtained in this way. We say that a factor is *spanned* by a given collection of functions if it coincides with the smallest σ -algebra with respect to which all these functions are measurable.

A *joining* of two systems (X, μ, T) and (Y, ν, S) is a measure ρ on $X \times Y$, invariant under $T \times S$ and whose projections on X and Y are equal to μ and ν , respectively.

2.3. The rational Kronecker factor. Let (X, μ, T) be a system. For $r \in \mathbb{N}$ we let $\mathcal{I}(T^r)$ be the σ -algebra of T^r -invariant subsets of X . The *rational Kronecker factor* of the system is the σ -algebra $\bigvee_{r \in \mathbb{N}} \mathcal{I}(T^r)$. We denote it by $\mathcal{K}_{\text{rat}}(T)$ or $\mathcal{K}_{\text{rat}}(X, \mu, T)$. It follows from the definition that this factor is spanned by the functions $f \in L^2(\mu)$ with $Tf = e(s)f$ for some $s \in \mathbb{Q}$. It follows also that for $f \in L^2(\mu)$ we have $\mathbb{E}_\mu(f|\mathcal{K}_{\text{rat}}(T)) = 0$ if and only if $\mathbb{E}_\mu(f|\mathcal{I}(T^r)) = 0$ for every $r \in \mathbb{N}$, that is,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^{rn}x) = 0 \text{ for every } r \in \mathbb{N}, \mu\text{-a.e.}$$

Hence, if $\mu = \int \mu_x d\mu(x)$ is the ergodic decomposition of μ , then $\mathbb{E}_\mu(f|\mathcal{K}_{\text{rat}}(X, \mu, T)) = 0$ if and only if $\mathbb{E}_{\mu_x}(f|\mathcal{K}_{\text{rat}}(X, \mu_x, T)) = 0$ for μ -almost every $x \in X$.

2.4. The Pinsker factor. Every system (X, μ, T) admits a largest factor belonging to the class of systems of entropy zero. This factor is called the *Pinsker factor* of (X, μ, T) and is denoted by $\Pi(X, \mu, T)$, or $\Pi(T)$, if there is no danger of confusion. The next result follows from [28, Lemma 3] (see also [11] and [26, Proposition 2.2]).

Proposition 2.1. *Let ρ be a joining of a system (X, μ, T) of zero entropy and a system (Y, ν, S) . If $g \in L^2(\nu)$ is such that $\mathbb{E}_\nu(g|\Pi(S)) = 0$, then for every $f \in L^2(\mu)$ we have $\int f(x) \cdot g(y) d\rho(x, y) = 0$.*

We deduce from this the following fact (which is probably a folklore result but we were not able to find a textbook reference).

Corollary 2.2. *Let $\pi: (X, \mu, T) \rightarrow (Y, \nu, S)$ be a factor map and $g \in L^2(\nu)$. If g is measurable with respect to $\Pi(S)$, then $g \circ \pi$ is measurable with respect to $\Pi(T)$. If $\mathbb{E}_\nu(g|\Pi(S)) = 0$, then $\mathbb{E}_\mu(g \circ \pi|\Pi(T)) = 0$.*

Proof. The first assertion follows immediately from the maximality of $\Pi(T)$ in the family of factors of X of entropy 0. We prove the second assertion. If $g \in L^2(\nu)$ satisfies $\mathbb{E}_\nu(g|\Pi(S)) = 0$ and $f \in L^2(\mu)$ is measurable with respect to $\Pi(T)$, it suffices to prove that $\int f \cdot g \circ \pi d\mu = 0$. We denote the Pinsker factor of (X, μ, T) by $(\Pi(T), \lambda, R)$, with factor map $p: X \rightarrow \Pi(T)$. Then the function f can be written as $f = h \circ p$ for some function $h \in L^2(\lambda)$. Let ρ be the measure on $\Pi(T) \times Y$ that is the image of the measure μ under the map $p \times \pi$. Then ρ is a joining of $(\Pi(T), \lambda, R)$ and (Y, ν, S) and by Proposition 2.1 we have $\int h(z) \cdot g(y) d\rho(z, y) = 0$. By the definition of ρ this integral is equal to $\int f \cdot g \circ \pi d\mu$ and we are done. \square

The following simple fact will be crucial in the proof of Proposition 3.2 below.

Lemma 2.3. *Let (X, μ, T) be a system and $f \in L^\infty(\mu)$ be real valued. Suppose that for every $\ell \geq 0$ and all $n_1, \dots, n_\ell \in \mathbb{N}$, distinct or not, we have*

$$\int f \cdot \prod_{j=1}^{\ell} T^{n_j} f d\mu = 0.$$

Then $\mathbb{E}_\mu(f|\Pi(T)) = 0$.

(Here and below the empty product is equal to 1 by convention and thus the integral above is equal to $\int f d\mu$ if $\ell = 0$.)

Remark. If the function f is complex valued, then in order to get the same conclusion we have to consider products of iterates of f and \bar{f} on the assumption.

Proof. Let \mathcal{Y} be the σ -algebra spanned by $\{T^n f : n \in \mathbb{Z}\}$. Let also Y be the corresponding factor-system. By Corollary 2.2 it suffices to prove that f has zero conditional expectation on the Pinsker factor of Y . Therefore, after substituting Y for X , we can assume that $\mathcal{Y} = \mathcal{X}$.

Let \mathcal{F} be the σ -algebra spanned by $\{T^k f : k \geq 0\}$. For $n \geq 0$, $T^n \mathcal{F}$ is the σ -algebra spanned by $\{T^k f : k \geq -n\}$ and by our reduction we have $\mathcal{F}_\infty := \bigvee_{n=0}^\infty T^n \mathcal{F} = \mathcal{X}$. On the other hand, for $n \in \mathbb{N}$ we have that $T^{-n} \mathcal{F}$ is the σ -algebra spanned by $\{T^k f : k \geq n\}$. By [20, Lemma 18.7], the σ -algebra $\mathcal{F}_{-\infty} := \bigcap_{n=0}^\infty T^{-n} \mathcal{F}$ contains the Pinsker σ -algebra $\Pi(T)$ of X . By hypothesis, f has zero conditional expectation on $T^{-1} \mathcal{F}$ and thus on the smaller σ -algebra $\mathcal{F}_{-\infty}$. Since $\mathcal{F}_{-\infty}$ contains $\Pi(T)$ we are done. \square

2.5. Nilsystems and nilsequences. A *nilmanifold* is a homogeneous space $X = G/\Gamma$ where G is a nilpotent Lie group, and Γ is a discrete cocompact subgroup of G . If $G_{k+1} = \{e\}$, where G_k denotes the k -th commutator subgroup of G , we say that X is a *k-step nilmanifold*.

Let $X = G/\Gamma$ be a k -step nilmanifold. The group G acts on X by left translations and we write this action as $(g \cdot x) \mapsto g \cdot x$ for $g \in G$ and $x \in X$. The unique probability measure on X invariant under this action is called the *Haar measure of X* and is denoted by m_X . If $a \in G$, $x \in X$, and $f \in \mathcal{C}(X)$, following [3] we call the sequence $(f(a^n \cdot x))$ a *basic k-step nilsequence*. A *k-step nilsequence*, is a uniform limit of *basic k-step nilsequences*. If $a \in G$ and $T : X \rightarrow X$ is the map $x \mapsto a \cdot x$, then the system $(X = G/\Gamma, m_X, T)$ is called a *k-step nilsystem*. For classical properties of nilsystems see [25, Chapter 11].

2.6. Systems of finite order. Let (X, μ, T) be a system. For $k \in \mathbb{N}$, the definition of the seminorm $\|\cdot\|_k$ on $L^\infty(\mu)$, as well as related facts recorded in this subsection, can be found in Chapters 8 and 9 of [25]. We write $\|\cdot\|_{\mu,k}$ in case of ambiguity. We define also $\|f\|_\infty := \sup_{k \in \mathbb{N}} \|f\|_k$ and the properties of this seminorm follow immediately from the properties of the seminorms of finite index. For $k \in \mathbb{N} \cup \{\infty\}$ we say that (X, μ, T) is a *system of order k* if the seminorm $\|\cdot\|_{k+1}$ is a norm. For $k \in \mathbb{N} \cup \{\infty\}$, each system (X, μ, T) admits a maximal factor of order k , that we denote by $\mathcal{Z}_k(X, \mu, T)$ or $\mathcal{Z}_k(T)$ when there is danger of confusion. Each system of order ∞ is an inverse limit of systems of finite orders, and for every system we have $\mathcal{Z}_\infty(T) = \bigvee_{k \in \mathbb{N}} \mathcal{Z}_k(T)$.

Let (X, μ, T) be a system, $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ under T , and $k \in \mathbb{N} \cup \{\infty\}$. Then (X, μ, T) is a system of order k , if and only if (X, μ_x, T) is a system of order k for almost every $x \in X$. For $k \in \mathbb{N}$ this is [25, Chapter 9, Proposition 23] and the case $k = \infty$ follows.

For $k \in \mathbb{N}$, the structure theorem [25, Chapter 16, Theorem 1], originally proved in [22], states that ergodic systems of order k are exactly the inverse limits of ergodic k -step nilsystems.

Theorem 2.4 (Nilsequence Wiener-Wintner [24]). *Let (X, μ, T) be a system and $f \in L^\infty(\mu)$. Then there exists a subset X_0 of X with $\mu(X_0) = 1$ such that the limit*

$$(7) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \Psi(n) \cdot f(T^n x)$$

exists for every $x \in X_0$ and every nilsequence $(\Psi(n))$. Moreover, if $\mathbb{E}_\mu(f | \mathcal{Z}_\infty(T)) = 0$, then the limit (7) is equal to 0 for every $x \in X_0$.

Remarks. • It is important for our purposes that the set X_0 does not depend on the nilsequence Ψ .

• The first assertion is [24, Theorem 2.22] in the case of an ergodic system, and [25, Chapter 23, Theorem 14] in the general case. The second assertion is implicit in [24] and [25] and we give a short proof here.

Proof. Let $k \in \mathbb{N}$. We write \mathbf{N} for the sequence of intervals $([N])_{N \in \mathbb{N}}$. The proof uses the notion of the uniformity seminorms $\|\cdot\|_{\mathbf{N},k+1}$ for sequences in $\ell^\infty(\mathbb{Z})$ that was introduced in [24].

If $\mathbb{E}_\mu(f|\mathcal{Z}_\infty(T)) = 0$ we will show that there exists a subset X_0 of X with $\mu(X_0) = 1$ such that the limit (7) is equal to 0 for every $x \in X_0$ and for every k -step nilsequence Ψ .

Let $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ and for $x \in X$ let $\phi_x \in \ell^\infty(\mathbb{Z})$ be the sequence $(f(T^n x))$. Since $\mathbb{E}_\mu(f|\mathcal{Z}_\infty(T)) = 0$ we have for every $k \in \mathbb{N}$ that $\|f\|_{\mu,k+1} = 0$, and since $\int \|f\|_{\mu_x,k+1}^{2^{k+1}} d\mu(x) = \|f\|_{\mu,k+1}^{2^{k+1}} = 0$, it follows that $\|f\|_{\mu_x,k+1} = 0$ for μ -almost every $x \in X$. By [24, Corollary 3.10], for μ -almost every $x \in X$ and μ_x -almost every $x' \in X$ we have $\|\phi_{x'}\|_{\mathbf{N},k+1} = \|f\|_{\mu_x,k+1} = 0$. Therefore, $\|\phi_x\|_{\mathbf{N},k+1} = 0$ for a set X_0 of $x \in X$ that has full measure.

Furthermore, by [24, Corollary 2.14], for every $\delta > 0$ there exists a constant $C = C(\delta, \Psi) > 0$ such that for every bounded sequence ϕ we have

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \Psi(n) \cdot \phi(n) \right| \leq C \|\phi\|_{\mathbf{N},k+1} + \delta \|\phi\|_\infty.$$

Applying this with $\phi = \phi_x$ for $x \in X_0$ (recall that $\|\phi_x\|_{\mathbf{N},k+1} = 0$ for every $x \in X_0$) and letting $\delta \rightarrow 0$, gives the announced result. \square

2.7. Furstenberg systems of sequences. We reproduce here the notion of a Furstenberg system of a bounded sequence from [13] and record some basic related facts that will be used later. For notational convenience we restrict to real valued sequences.

Definition. Let $\mathbf{N} := ([N_k])_{k \in \mathbb{N}}$ be a sequence of intervals with $N_k \rightarrow \infty$ and I be a bounded closed interval in \mathbb{R} . We say that the sequence $z = (z(n))_{n \in \mathbb{Z}}$ with values in I admits correlations on \mathbf{N} , if the limits

$$(8) \quad \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \prod_{j=1}^s z(n + n_j)$$

exist for all $s \in \mathbb{N}$ and all $n_1, \dots, n_s \in \mathbb{Z}$ (not necessarily distinct).

Remarks. • Given $z: \mathbb{Z} \rightarrow I$, using a diagonal argument, we get that every sequence of intervals $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ has a subsequence $\mathbf{N}' = ([N'_k])_{k \in \mathbb{N}}$, such that the sequence z admits correlations on \mathbf{N}' .

• If we are given a one-sided sequence $(z(n))_{n \in \mathbb{N}}$ we extend it to \mathbb{Z} in an arbitrary way; then the existence and values of the correlations do not depend on the extension.

If a sequence $z: \mathbb{Z} \rightarrow I$ admits correlations on a given sequence of intervals, then we use a variant of the correspondence principle of Furstenberg [16, 17] in order to associate a measure preserving system that captures the statistical properties of this sequence. We briefly describe this process next.

Let $\Omega := I^{\mathbb{Z}}$. The elements of Ω are written $\omega := (\omega(n))_{n \in \mathbb{Z}}$ and the shift $\sigma: \Omega \rightarrow \Omega$ is defined by $(\sigma\omega)(n) := \omega(n+1)$, $n \in \mathbb{Z}$. We consider the sequence z as an element of Ω . Note that the algebra generated by functions of the form $\omega \mapsto \omega(k)$, $\omega \in \Omega$, for $k \in \mathbb{Z}$, separates points in Ω . We conclude that if the sequence $z: \mathbb{Z} \rightarrow I$ admits correlations on

\mathbf{N} , then for all $f \in \mathcal{C}(\Omega)$ the following limit exist

$$\lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} f(\sigma^n z).$$

Hence, the following weak*-limit exists

$$\nu := \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{\sigma^n z}$$

and we say that *the point z is generic for ν along \mathbf{N}* .

Definition. Let I be a compact interval and let $z: \mathbb{Z} \rightarrow I$ be a sequence that admits correlations on $\mathbf{N} := ([N_k])_{k \in \mathbb{N}}$, and Ω , σ , and ν as above.

- We call (Ω, ν, σ) the *Furstenberg system associated with z* on \mathbf{N} .
- We let $F_0 \in \mathcal{C}(\Omega)$ be defined by $F_0(\omega) := \omega(0)$, $\omega \in \Omega$, and call it the *0th-coordinate projection*. Note that $F_0(\sigma^n z) = z(n)$ for every $n \in \mathbb{Z}$ and

$$(9) \quad \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \prod_{j=1}^s z(n + n_j) = \int \prod_{j=1}^s \sigma^{n_j} F_0 d\nu$$

for all $s \in \mathbb{N}$, $n_1, \dots, n_s \in \mathbb{Z}$. This identity is often referred to as the *Furstenberg correspondence principle*.

- We say that the sequence z has a *unique Furstenberg system*, if z admits correlations on $([N])_{N \in \mathbb{N}}$, or equivalently, if z is generic for a measure along $([N])_{n \in \mathbb{N}}$.

Remark. A sequence $z: \mathbb{Z} \rightarrow I$ may have several non-isomorphic Furstenberg systems depending on which sequence of intervals \mathbf{N} we use in the evaluation of its correlations. We call any such system a *Furstenberg system of z* .

The following fact is an immediate consequence of the pointwise ergodic theorem.

Proposition 2.5. *Let (X, μ, T) be a system and suppose that $\mu = \int \mu_x d\mu(x)$ is the ergodic decomposition of μ . Then for every real valued $f \in L^\infty(\mu)$ and for almost every $x \in X$, the sequence $(f(T^n x))$ has a unique Furstenberg system that is ergodic and a factor of the system (X, μ_x, T) .*

(The factor map is the map $x \mapsto (f(T^n x))_{n \in \mathbb{Z}}$ from X to Ω .)

Remark. It follows that if (X, μ, T) has zero entropy, then for almost every $x \in X$, the Furstenberg system of the sequence $(f(T^n x))$ has zero entropy.

3. FURSTENBERG SYSTEMS OF POLYNOMIAL ITERATES

Let (X, μ, T) be a system, $f \in L^\infty(\mu)$, and $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$. In this section we are going to establish a crucial property that Furstenberg systems of sequences of the form $(f(T^{p(n)} x))$ satisfy. A key ingredient is the following result.

Theorem 3.1 ([23]). *Let $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$. Let also (X, μ, T) be an ergodic system, $\ell \in \mathbb{N}$, and $f_1, \dots, f_\ell \in L^\infty(\mu)$. Then the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{p(n+1)} f_1 \cdot \dots \cdot T^{p(n+\ell)} f_\ell$$

exists in $L^2(\mu)$ and it is zero if $\mathbb{E}(f_j | \mathcal{Z}_\infty(T)) = 0$ for some $j \in \{1, \dots, \ell\}$.

We will combine this result with Lemma 2.3 in order to deduce the following.

Proposition 3.2. *Let $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$, (X, μ, T) be a system, and $f \in L^\infty(\mu)$ be real valued. Then every strictly increasing sequence of positive integers (N_k) has a subsequence (N'_k) such that for almost every $x \in X$, the sequence $(f(T^{p(n)}x))$ admits correlations on $\mathbf{N}' := ([N'_k])_{k \in \mathbb{N}}$. Moreover, if $\mathbb{E}(f|\mathcal{Z}_\infty(T)) = 0$, then the 0th-coordinate projection $F_0: \Omega \rightarrow \mathbb{R}$ has zero conditional expectation on the Pinsker factor of the corresponding Furstenberg system (Ω, ν_x, σ) .*

Remarks. • By taking a suitable countable dense collection of functions in $L^2(\mu)$ and using a diagonal and an approximation argument, we can choose the subsequence (N'_k) independently of the function f . We cannot choose it independently of the system or the full measure set of $x \in X$ though.

• It seems likely that for almost every $x \in X$ the Furstenberg systems of $(f(T^{p(n)}x))$ along $\mathbf{N}' := ([N'_k])_{k \in \mathbb{N}}$ are ergodic and isomorphic to direct products of infinite-step nilsystems and Bernoulli systems (see [12, Problem 1]). Proving this seems non-trivial though.

Proof. By Theorem 3.1, for all $s \in \mathbb{N}$ and $h_1, \dots, h_s \in \mathbb{Z}$ (not necessarily distinct) the limit

$$(10) \quad \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \prod_{j=1}^s f(T^{p(n+h_j)}x)$$

exists in $L^2(\mu)$. Since mean convergence of a sequence of functions implies pointwise almost everywhere convergence along a subsequence, we can use a diagonal argument to find a subsequence (N'_k) of (N_k) such that for a subset X_0 of X with full measure and for $x \in X_0$ all the limits (10) exist. This means that for $x \in X_0$ the sequence $(f(T^{p(n)}x))$ admits correlations on the sequence of intervals $\mathbf{N}' = ([N'_k])_{k \in \mathbb{N}}$, which proves the first assertion.

For $x \in X_0$ let (Ω, ν_x, σ) be the associated Furstenberg system and $F_0: \Omega \rightarrow \mathbb{R}$ be the 0th-coordinate projection. Note also that the limit

$$(11) \quad \lim_{k \rightarrow \infty} \frac{1}{N'_k} \sum_{n=1}^{N'_k} f(T^{p(n)}x) \cdot \prod_{j=1}^{\ell} f(T^{p(n+n_j)}x)$$

exists for all $x \in X_0$, $\ell \geq 0$, and $n_1, \dots, n_\ell \in \mathbb{N}$ (not necessarily distinct). Suppose that $\mathbb{E}(f|\mathcal{Z}_\infty(T)) = 0$. Then by Theorem 3.1 we have that for every x in a full measure subset X_1 of X_0 the limit (11) is equal to 0. Using the correspondence principle (9) we get that

$$\int F_0 \cdot \prod_{j=1}^{\ell} \sigma^{n_j} F_0 d\nu_x = 0$$

for all $\ell \geq 0$ and $n_1, \dots, n_\ell \in \mathbb{N}$.

By Lemma 2.3 this implies that for $x \in X_1$, the function F_0 is orthogonal to the Pinsker factor of the Furstenberg system (Ω, ν_x, σ) . This completes the proof. \square

We remark that if f takes values in $\{-1, 1\}$, then the previous argument also gives that for almost every $x \in X$ the Furstenberg system of $(f(T^{p(n)}x))$ on $([N'_k])_{k \in \mathbb{N}}$ is Bernoulli. But this is no longer necessary if say f takes values in $\{0, \pm 1\}$. To see this, consider the ergodic system defined by the transformation $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by $T(x, y) := (x + \alpha, 2y)$, $x, y \in \mathbb{T}$, where α is irrational. We cut the unit square in four square pieces in the natural way, and let f be 1 on the first, -1 on the fourth, and 0 on the other two vertically adjacent squares. Then f is orthogonal to the $\mathcal{Z}_\infty(T)$ -factor. Suppose that the Furstenberg system of $(f(T^{n^2}x))$ was Bernoulli. Then the Furstenberg system of $(f^2(T^{n^2}x))$ would be a factor of this system. But f^2 is a non-trivial measurable function with respect to the Kronecker factor of the system (X, μ, T) . This easily implies

that the Furstenberg system of $(f^2(T^{n^2}x))$ is a non-trivial factor of the affine system $(x, y) \mapsto (x + \alpha, y + x)$, and any such factor cannot be a factor of a Bernoulli system (since it is not even weak mixing).

4. CONVERGENCE

The goal of this section is to prove Theorem 1.1 and to determine some convenient (but not optimal) characteristic factors for the mean convergence of the averages (4). We do this in Proposition 4.2 below.

4.1. A partial characteristic factor. Our first step is to combine Proposition 2.1 with Proposition 3.2 in order to establish the following result via a disjointness argument.

Proposition 4.1. *Let (X, \mathcal{X}, μ) be a probability space and $T, S: X \rightarrow X$ be measure preserving transformations such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$ and $f, g \in L^\infty(\mu)$ with $\mathbb{E}(g|\mathcal{Z}_\infty(S)) = 0$. Then*

$$(12) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{p(n)} g = 0$$

in $L^2(\mu)$.

(The case where $\mathbb{E}(f|\mathcal{Z}_\infty(T)) = 0$ will be treated in the next subsection.)

Proof. Note first that our assumption is satisfied for the real and imaginary part of g in place of g . Hence, we can restrict to the case where the functions f and g are real valued with range on a closed bounded interval I .

Arguing by contradiction, suppose that the conclusion fails. Then there exist $\varepsilon > 0$ and $N_k \rightarrow \infty$ such that

$$(13) \quad \left\| \frac{1}{N_k} \sum_{n=1}^{N_k} T^n f \cdot S^{p(n)} g \right\|_{L^2(\mu)} \geq \varepsilon$$

for every $k \in \mathbb{N}$. By Proposition 3.2 there exists a subsequence (N'_k) of (N_k) such that for μ -almost every $x \in X$ the sequence $(g(S^{p(n)}x))$ admits correlations on $\mathbf{N}' := ([N'_k])$. Let (Ω, ν_x, σ) be the corresponding Furstenberg system. Moreover, since $\mathbb{E}(g|\mathcal{Z}_\infty(S)) = 0$, by Proposition 3.2 again we have that for μ -almost every $x \in X$ the 0th-coordinate projection $F_0: \Omega \rightarrow \mathbb{R}$ satisfies $\mathbb{E}_{\nu_x}(F_0|\Pi(\Omega, \nu_x, \sigma)) = 0$.

Furthermore, by Proposition 2.5, we have that for μ -almost every $x \in X$ the sequence $(f(T^n x))$ admits correlations on the sequence of intervals \mathbf{N}' and the corresponding Furstenberg system (Ξ, ν'_x, σ) is ergodic and has zero entropy. We write $G_0: \Xi \rightarrow \mathbb{R}$ for the 0th-coordinate projection.

Let X_0 be a subset of X with $\mu(X_0) = 1$ and such that the previous properties hold for all $x \in X_0$. We claim that

$$(14) \quad \lim_{k \rightarrow \infty} \frac{1}{N'_k} \sum_{n=1}^{N'_k} f(T^n x) \cdot g(S^{p(n)} x) = 0 \quad \text{for almost every } x \in X_0.$$

If this is shown, then we get a contradiction from (13) using the bounded convergence theorem.

So suppose that (14) fails. Then there exists a subset X_1 of X_0 with $\mu(X_1) > 0$ such that for every $x \in X_1$ there exists a subsequence $(N'_{x,k})$ of (N'_k) such that

$$(15) \quad \lim_{k \rightarrow \infty} \frac{1}{N'_{x,k}} \sum_{n=1}^{N'_{x,k}} f(T^n x) \cdot g(S^{p(n)} x) \quad \text{exists and is non-zero.}$$

Let $x \in X_1$ be fixed for the moment. Let $w, z \in I^{\mathbb{Z}}$ be defined by $w(n) := f(T^n x)$ and $z(n) := g(S^{p(n)} x)$ for $n \in \mathbb{Z}$. The sequence $(N'_{x,k})$ admits a subsequence $(N''_{x,k})$ along which the point (w, z) of $I^{\mathbb{Z}} \times I^{\mathbb{Z}}$ is generic on $(I^{\mathbb{Z}} \times I^{\mathbb{Z}}, \sigma \times \sigma)$ for some measure ρ_x on this space. This means that for every continuous function Φ on $I^{\mathbb{Z}} \times I^{\mathbb{Z}}$ we have

$$\lim_{k \rightarrow \infty} \frac{1}{N''_{x,k}} \sum_{n=1}^{N''_{x,k}} \Phi(\sigma^n w, \sigma^n z) = \int \Phi d\rho_x.$$

In particular, since F_0 is continuous on Ω and G_0 is continuous on Ξ , and since $G_0(\sigma^n w) = f(T^n x)$ and $F_0(\sigma^n z) = g(S^{p(n)} x)$, $n \in \mathbb{N}$, we obtain

$$(16) \quad \lim_{k \rightarrow \infty} \frac{1}{N''_{x,k}} \sum_{n=1}^{N''_{x,k}} f(T^n x) \cdot g(S^{p(n)} x) = \int G_0 \otimes F_0 d\rho_x.$$

Note that the measure ρ_x is invariant under $\sigma \times \sigma$. Furthermore, since w is generic for the measure ν'_x on (Ξ, σ) along $([N])$ and since z is generic for the measure ν_x on (Ω, σ) along $([N'_k])$, and since $([N''_{x,k}])$ is a subsequence of $([N'_k])$, the two coordinate projections of ρ_x are ν'_x and ν_x respectively, and ρ_x is a joining of the systems (Ξ, ν'_x, σ) and (Ω, ν_x, σ) .

Recall that for μ -almost every $x \in X_1$ we have $\mathbb{E}_{\nu_x}(F_0 | \Pi(\Omega, \nu_x, \sigma)) = 0$ and the system (Ξ, ν'_x, σ) has entropy zero. By Proposition 2.1 we have

$$\int G_0 \otimes F_0 d\rho_x = 0$$

for μ -almost every $x \in X_1$. Taken together with identity (16), this gives a contradiction by (15) (recall that $(N''_{k,x})$ is a subsequence of $(N'_{k,x})$). This proves (14) and finishes the proof. \square

4.2. Convergence. We are now ready to prove Theorem 1.1. For convenience we restate it and also record some additional information that will be needed later.

Proposition 4.2. *Let T, S be measure preserving transformations acting on a probability space (X, \mathcal{X}, μ) such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$. Then for every $f, g \in L^\infty(\mu)$ the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{p(n)} g$$

exists in $L^2(\mu)$. Moreover, the limit is 0 if either $\mathbb{E}_\mu(f | \mathcal{Z}_\infty(T)) = 0$ or $\mathbb{E}_\mu(g | \mathcal{Z}_\infty(S)) = 0$.

Proof. We first prove mean convergence. By Proposition 4.1 we can restrict to the case where g is measurable with respect to $\mathcal{Z}_\infty(S)$ and using an $L^2(\mu)$ approximation argument we can assume that it is measurable with respect to the factor $\mathcal{Z}_k(S)$ for some $k \in \mathbb{N}$.

By [9, Proposition 3.1] (see also [25, Chapter 16, Theorem 10]), for every $\varepsilon > 0$ there exists $\tilde{g} \in L^\infty(\mu)$ such that

- (i) \tilde{g} is measurable with respect to $\mathcal{Z}_k(S)$ and $\|g - \tilde{g}\|_{L^1(\mu)} < \varepsilon$;
- (ii) for μ -almost every $x \in X$ the sequence $(\tilde{g}(S^n x))$ is a k -step nilsequence.

Therefore, it suffices to show the existence in $L^2(\mu)$ of the limit

$$(17) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \tilde{g}(S^{p(n)} x) \cdot f(T^n x),$$

where \tilde{g} satisfies (ii), and that this limit is 0 if $\mathbb{E}_\mu(f | \mathcal{Z}_\infty(T)) = 0$.

By [27, Proposition 3.14], for μ -almost every $x \in X$ the sequence $(\tilde{g}(S^{p(n)} x))$ is an ℓ -step nilsequence for some $\ell = \ell(k, p)$. Let X_0 be the full measure subset of X for which

this property holds. Let also X_1 be the subset of X associated with f by Theorem 2.4. Then for every $x \in X_0 \cap X_1$ the limit (17) exists. It follows that this limit exists in $L^2(\mu)$ and the first part of the proposition is proved. Furthermore, if $\mathbb{E}_\mu(f|\mathcal{Z}_\infty(T)) = 0$, then by the second part of Theorem 2.4 the limit is equal to 0 for every $x \in X_0 \cap X_1$. This completes the proof. \square

5. THE RATIONAL KRONECKER FACTOR IS CHARACTERISTIC

The goal of this section is to prove Theorem 1.2. Here is a brief sketch of our strategy. Using Proposition 4.2 we can reduce matters to the case where the functions f and g are measurable with respect to the factors $\mathcal{Z}_k(T)$ and $\mathcal{Z}_k(S)$ respectively, for some $k \in \mathbb{N}$. Using an ergodic decomposition argument and the Structure Theorem [22] for systems of order k , we can further reduce matters to proving the equidistribution property for nilsystems stated in Proposition 5.4. In order to establish this property we verify first that it holds for a particular class of nilsystems, namely, for ergodic unipotent affine transformations acting on some finite dimensional torus. We do this in Lemma 5.3 and our starting point is the simple observation of Lemma 5.1. We then combine Lemma 5.3 with the criterion in [27, Theorem 2.17], which enables us to reduce equidistribution properties of general nilsystems to the unipotent affine case, in order to finish the proof of Proposition 5.4 for general nilsystems.

5.1. Equidistribution for unipotent affine transformations. In what follows, we say that the real numbers $\alpha_1, \dots, \alpha_r$ are *rationaly independent*, if the equation $n_1\alpha_1 + \dots + n_r\alpha_r = 0 \pmod 1$ has no non-trivial integer solutions.

As usual, for $t \in \mathbb{R}$ we write $e(t) = \exp(2\pi it)$ and we use the same notation for $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $d \in \mathbb{N}$, $x \in \mathbb{T}^d$, and $k \in \mathbb{Z}^d$, we write $k \cdot x = k_1x_1 + \dots + k_dx_d$. Let $u = (u(n))$ be a sequence with values in \mathbb{T}^d . Then by the Weyl Equidistribution Theorem, this sequence is equidistributed in \mathbb{T}^d if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(k \cdot u(n)) = 0 \quad \text{for every non-zero } k \in \mathbb{Z}^d.$$

If the sequence u has polynomial coordinates this condition is equivalent to saying that, for every non-zero $k \in \mathbb{Z}^d$, the polynomial $n \mapsto k \cdot u(n)$ has at least one irrational non-constant coefficient.

Lemma 5.1. *Let $P \in \mathbb{R}[t]$ be a non-constant polynomial with rationaly independent coefficients, $Q \in \mathbb{R}[t]$ be arbitrary, and $p \in \mathbb{Z}[t]$ be such that $\deg(p) \geq 2$. Then the polynomial $P + Q \circ p$ has at least one irrational non-constant coefficient.*

Remark. Note that the conclusion is false if $\deg(p) = 1$.

Proof. If the non-constant coefficients of Q are rational, then the non-constant coefficients of the polynomial $Q \circ p$ are rational and thus the polynomial $P + Q \circ p$ has rationaly independent non-constant coefficients and we are done. So suppose that Q has at least one irrational non-constant coefficient. We write $Q(t) = t^{d+1}Q_1(t) + Q_2(t)$ where $d \in \mathbb{N}$, $Q_1 \in \mathbb{Q}[t]$, $\deg(Q_2) = d$, and Q_2 has an irrational leading coefficient. Then the polynomial $p(t)^{d+1}(Q_1 \circ p)(t)$ has rational coefficients. After replacing $P(t)$ with $P(t) + p(t)^{d+1}(Q_1 \circ p)(t)$ and Q with Q_2 , we are reduced to the case where Q has an irrational leading coefficient.

We distinguish two cases. If $\deg(Q) \geq \deg(P)$, then $\deg(Q \circ p) > \deg(Q) \geq \deg(P)$, because $\deg(p) \geq 2$ and Q is non-constant. Hence, the leading coefficient of $P + Q \circ p$ coincides with the leading coefficient of $Q \circ p$, which is irrational and we are done.

Suppose now that $\deg(Q) < \deg(P)$. Arguing by contradiction, suppose that all the non-constant coefficients of $P + Q \circ p$ are rational. Since the coefficients of P are rationaly independent, the linear span over \mathbb{Q} of the non-constant coefficients of the polynomial

$Q \circ p = -P + (P + Q \circ p)$ has dimension equal to $\deg(P)$ over \mathbb{Q} . Furthermore, the non-constant coefficients of $Q \circ p$ are integer combinations of the non-constant coefficients of Q . Hence, the dimension of the span of the non-constant coefficients of the polynomial $Q \circ p$ over \mathbb{Q} is at most $\deg(Q)$, which is strictly smaller than $\deg(P)$ by assumption, and we have a contradiction. This completes the proof. \square

Definition. A *unipotent affine transformation* is a transformation $T: \mathbb{T}^m \rightarrow \mathbb{T}^m$ of the form $Tx = Sx + b$, $x \in \mathbb{T}^m$, where $b \in \mathbb{T}^m$ and S is a unipotent automorphism of \mathbb{T}^m .

Lemma 5.2. *Let $m \in \mathbb{N}$ and $T: \mathbb{T}^m \rightarrow \mathbb{T}^m$ be an ergodic unipotent affine transformation. Then for almost every $x \in \mathbb{T}^m$, with respect to the Lebesgue measure in \mathbb{T}^m , the following holds: For every non-zero $k \in \mathbb{Z}^m$ the non-constant coefficients of the polynomial $n \mapsto k \cdot T^n x$ are rationally independent.*

Proof. We write $Tx = Sx + b$ where $b \in \mathbb{T}^m$ and S is a unipotent automorphism of \mathbb{T}^m . We are going to prove that the conclusion holds if the coordinates of $x \in \mathbb{T}^m$ are rationally independent of the coordinates of b , meaning, x satisfies the following property

$$(18) \quad \text{if } k \cdot b + \ell \cdot x = 0 \pmod{1} \text{ for some } k, \ell \in \mathbb{Z}^m, \text{ then } \ell = 0.$$

The Lebesgue measure of the set of points $x \in \mathbb{T}^m$ with this property is 1, and the result will follow.

We write S also for the $(m \times m)$ -matrix defining this automorphism and denote by S^t the transpose of S . Since S is unipotent we have $(S - \text{id})^m = 0$ and $(S^t - \text{id})^m = 0$. Let K be the subtorus $(S - \text{id})\mathbb{T}^m$ of \mathbb{T}^m . By [21, Theorem 4], the ergodicity of T is equivalent to the ergodicity of the rotation induced by b on the quotient \mathbb{T}^m/K , which in turn is equivalent to

$$(19) \quad \text{if } \ell \in \mathbb{Z}^m \text{ satisfies } (S^t - \text{id})\ell = 0 \text{ and } \ell \cdot b = 0 \pmod{1}, \text{ then } \ell = 0.$$

Let $k \in \mathbb{Z}^m$ be non-zero. Since $(S - \text{id})^m = 0$, using induction, we get that for every $n \in \mathbb{N}$ we have (by convention $\binom{n}{j} = 0$ if $j > n$)

$$(20) \quad \begin{aligned} k \cdot T^n x &= k \cdot x + \sum_{j=1}^m \binom{n}{j} (k \cdot (S - \text{id})^j x + k \cdot (S - \text{id})^{j-1} b) \\ &= k \cdot x + \sum_{j=1}^{m_0} \binom{n}{j} ((S^t - \text{id})^j k \cdot x + (S^t - \text{id})^{j-1} k \cdot b) \end{aligned}$$

where $m_0 := \min\{q \in \mathbb{N} : (S^t - \text{id})^q k = 0\}$.

Note that $m_0 \leq m$ and that the expression in (20) is a polynomial of degree at most m_0 . We will be done if we show that its non-constant coefficients are rationally independent. Arguing by contradiction, suppose that this is not the case. Note that the linear span over \mathbb{Q} of the polynomials $n \mapsto \binom{n}{1}, \dots, n \mapsto \binom{n}{m_0}$ is equal to the space of polynomials of degree at most m_0 without constant term, that is, to the linear span over \mathbb{Q} of the polynomials n, \dots, n^{m_0} . Therefore, the coefficients in (20) of the polynomials $\binom{n}{1}, \dots, \binom{n}{m_0}$ are rationally dependent and there exist integers $r_1, \dots, r_{m_0} \in \mathbb{Z}$, not all of them zero, such that

$$\sum_{j=1}^{m_0} r_j (S^t - \text{id})^j k \cdot x + \sum_{j=1}^{m_0} r_j (S^t - \text{id})^{j-1} k \cdot b = 0 \pmod{1}.$$

The first sum is a linear combination with integer coefficients of the coordinates of x , and the second sum is a linear combination with integer coefficients of the coordinates of b . By our hypothesis (18) we have

$$\sum_{j=1}^{m_0} r_j (S^t - \text{id})^j k = 0 \quad \text{and} \quad \sum_{j=1}^{m_0} r_j (S^t - \text{id})^{j-1} k \cdot b = 0 \pmod{1}.$$

We claim that

$$(21) \quad \sum_{j=1}^{m_0} r_j (S^t - \text{id})^{j-1} k = 0.$$

Indeed, writing ℓ for this sum we have $(S^t - \text{id})\ell = 0$ and $\ell \cdot b = 0 \pmod 1$ and thus $\ell = 0$ by (19), proving the claim. Let $j_0 := \min\{j : r_j \neq 0\}$. Applying $(S^t - \text{id})^{m_0-j_0}$ to both sides of (21) and recalling that $(S^t - \text{id})^{m_0} k = 0$, we obtain $(S^t - \text{id})^{m_0-1} k = 0$, contradicting the definition of m_0 . This completes the proof. \square

We will now combine the previous two lemmas in order to deduce an equidistribution property that will be crucial in the next subsection.

Lemma 5.3. *Let $m_1, m_2 \in \mathbb{N}$, and $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$. Let $T: \mathbb{T}^{m_1} \rightarrow \mathbb{T}^{m_1}$ be an ergodic unipotent affine transformation. Then for $m_{\mathbb{T}^{m_1}}$ -almost every $x \in \mathbb{T}^{m_1}$, the following holds: For every sequence $u: \mathbb{N} \rightarrow \mathbb{T}^{m_2}$ with polynomial coordinates that is equidistributed in \mathbb{T}^{m_2} , the sequence $(T^n x, (u \circ p)(n))$ is equidistributed in $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$.*

Proof. Let X_0 be the full measure subset of \mathbb{T}^{m_1} given by Lemma 5.2. We are going to prove that the conclusion holds if $x \in X_0$.

Note that the sequence $(T^n x, u(p(n)))$ with values in $\mathbb{T}^{m_1+m_2}$ has polynomial coordinates. By the Weyl Equidistribution Theorem it suffices to show that if $k_1 \in \mathbb{Z}^{m_1}$ and $k_2 \in \mathbb{Z}^{m_2}$ are not both zero, then the polynomial

$$(22) \quad n \mapsto k_1 \cdot T^n x + k_2 \cdot u(p(n))$$

has at least one non-constant irrational coefficient.

Suppose first that $k_1 = 0$, in which case we have $k_2 \neq 0$. Since the sequence $(u(n))$ is equidistributed in \mathbb{T}^{m_2} , the polynomial $k_2 \cdot u(t)$ has at least one irrational non-constant coefficient. We write this polynomial as $t^{d+1}R_1(t) + R_2(t)$ where $R_1(t)$ has rational coefficients, $d := \deg(R_2)$, and the leading coefficient of R_2 is irrational. Then

$$k_2 \cdot (u \circ p)(t) = p(t)^{d+1}(R_1 \circ p)(t) + (R_2 \circ p)(t).$$

The first of these polynomials has rational coefficients and the second one has an irrational leading coefficient. Therefore, the polynomial $k_2 \cdot u \circ p$ has at least one non-constant irrational coefficient and we are done.

Suppose now that $k_1 \neq 0$. Recall that for $x \in X_0$ the non-constant coefficients of the polynomial $n \mapsto k_1 \cdot T^n x$ are rationally independent. Using Lemma 5.1, we get that the polynomial in (22) has an irrational non-constant coefficient and we are done. This completes the proof. \square

5.2. A nil-equidistribution result. We proceed now to establish the key equidistribution property on nilmanifolds needed for our purposes.

Proposition 5.4. *Let $(X = G/\Gamma, m_X, T)$ be an ergodic nilsystem. Then there exists a subset X_0 of X with $m_X(X_0) = 1$ such that the following holds: For every $x \in X_0$, every nilsequence $(\Psi(n))$, polynomial $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$, and function $f \in \mathcal{C}(X)$ with $\mathbb{E}_\mu(f | \mathcal{K}_{\text{rat}}(T)) = 0$, we have*

$$(23) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot \Psi(p(n)) = 0.$$

Remark. It is important that the set of good $x \in X$ for which the conclusion holds is independent of the nilsequence $(\Psi(n))$.

Proof. The proof is similar to the proof of [9, Lemma 7.6]. It is based on an equidistribution result of Leibman [27, Theorem 2.17] that in some cases enables us to deduce an equidistribution result for general nilsystems from the special case of unipotent affine transformations on the torus. We give a rather sketchy account below, the reader will find more details about the notions and results used in [9, Section 7].

We first consider the case where X is connected. In this case the rational Kronecker factor \mathcal{K}_{rat} of X is trivial and $\int f dm_X = 0$.

Let G_0 be the connected component of G and $A_X := X/[G_0, G_0]$ be the affine torus of X . We denote by \tilde{T} the transformation induced by T on A_X . Then \tilde{T} is a unipotent affine transformation. For $r \in \mathbb{N}$ let B_r be the subset of full measure of A_X associated by Lemma 5.3 to the ergodic transformation \tilde{T}^r of A_X and let

$$X' := \bigcap_{r \in \mathbb{N}} \bigcap_{j=0}^{r-1} T^j \pi_X^{-1}(B_r)$$

where $\pi_X: X \rightarrow A_X$ is the natural projection. We claim that this subset of full measure of X satisfies the required property.

Note that it suffices to consider the case where $(\Psi(n))$ is a basic nilsequence. We write $\Psi(n) = \psi(b^n \cdot y)$ for some nilmanifold $Y = H/\Lambda$ with Haar measure m_Y , $b \in H$, $\psi \in C(Y)$, and $y \in Y$. We can assume that $\{b^n \cdot y: n \in \mathbb{N}\}$ is dense in Y , and thus that the translation by b on Y is ergodic.

For the moment we also assume that Y is connected. Let $A_Y := H/[H_0, H_0]$ be the affine torus of Y , $\pi_Y: Y \rightarrow A_Y$ the natural projection, and $u(n) := \pi_Y(b^n \cdot y)$ for $n \in \mathbb{N}$. Then the sequence $(u(n))$ is a polynomial sequence in A_Y and is equidistributed in A_Y . Let $x \in X'$ and $z := \pi_Y(x)$. By the definition of the set X' , the sequence $(\tilde{T}^n z, u(p(n)))$ is equidistributed in $A_X \times A_Y$. But $A_X \times A_Y$ is the affine torus of the connected nilmanifold $X \times Y$ and the sequence $(\tilde{T}^n z, u(p(n)))$ is the image under the quotient map $\pi_X \times \pi_Y$ of the polynomial sequence $(T^n x, b^{p(n)} \cdot y)$ in $X \times Y$. By [25, Chapter 14, Theorem 20] (originally established in [27, Theorem 2.17]), this sequence is equidistributed in $X \times Y$ and thus

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot \Psi(p(n)) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot \psi(b^{p(n)} y) = \int_X f dm_X \cdot \int_Y \psi dm_Y = 0$$

and we are done.

We continue to assume that X is connected but we no longer assume that Y is connected. Let Y_0 be the connected component of y in Y . Then there exists $r \in \mathbb{N}$ such that Y_0 is invariant and ergodic under the translation by b^r and the sets $b^j \cdot Y_0$, $j = 0, \dots, r-1$, form a partition of Y . The preceding case applied with x replaced by $T^j x$, T replaced by T^r , Y replaced by Y_0 , and the polynomial p replaced by $p(rn + j)$, gives

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^{rn+j} x) \cdot \Psi(p(rn + j)) = 0 \quad \text{for } j = 0, \dots, r-1 \text{ and } x \in X'.$$

Averaging for $j \in \{0, \dots, r-1\}$ gives the announced result.

We consider now the case where X is not connected. Then there exist an integer $r \in \mathbb{N}$ and a connected subnilmanifold X_0 of X , such that the sets $T^j X_0$, $j = 0, \dots, r-1$, form a partition of X , X_0 is invariant under T^r , and $(T^j X_0, T^r)$ is ergodic for $j = 0, \dots, r-1$. Writing m_{X_j} for the Haar measure of $T^j X_0$, we have $\int f dm_{X_j} = 0$ for $j = 0, \dots, r-1$ because $\mathbb{E}_\mu(f | \mathcal{K}_{\text{rat}}(T)) = 0$. For $j = 0, \dots, r-1$, the preceding step provides a subset of full measure X'_j of $T^j X_0$ and it is immediate that the union of these sets fulfills the required conditions. This completes the proof. \square

We deduce from the previous result and the main structural result in [22] the following statement that is more convenient for our purposes.

Corollary 5.5. *Let (X, μ, T) be an ergodic system and $f \in L^\infty(\mu)$ with $\mathbb{E}_\mu(f|\mathcal{K}_{\text{rat}}(T)) = 0$. Then there exists a subset X_0 of X with $\mu(X_0) = 1$ such that the following holds: For every nilsequence $(\Psi(n))$ and polynomial $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$ we have*

$$(24) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot \Psi(p(n)) = 0.$$

Proof. By the second part of Theorem 2.4 we can assume that (X, μ, T) is a system of infinite order and using an approximation argument we can assume that it is a system of finite order. By the Structure Theorem [25, Chapter 18, Theorem 1] (originally established in [22]), the system (X, μ, T) is an inverse limit of a sequence $((X_j, \mu_j, T))$ of ergodic k -step nilsystems, with factor maps $\pi_j: X \rightarrow X_j$, $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ there exists $f_j \in \mathcal{C}(X_j)$ such that $\|f - f_j \circ \pi_j\|_{L^1(\mu)} \rightarrow 0$ when $j \rightarrow \infty$. For $j \in \mathbb{N}$, substituting $\tilde{f}_j := f_j - \mathbb{E}_{\mu_j}(f_j|\mathcal{K}_{\text{rat}}(X_j, \mu_j, T))$ for f_j and noting that $\|f - \tilde{f}_j \circ \pi_j\|_{L^1(\mu)} \rightarrow 0$ since $\mathbb{E}_{\mu_j}(f|\mathcal{K}_{\text{rat}}(X_j, \mu_j, T)) = 0$, we can assume that $\mathbb{E}_{\mu_j}(f_j|\mathcal{K}_{\text{rat}}(X_j, \mu_j, T)) = 0$. By the ergodic theorem, for $j \in \mathbb{N}$ there exists a subset A_j of X of full μ -measure such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |f(T^n x) - f_j(\pi_j(T^n x))| = \|f - f_j \circ \pi_j\|_{L^1(\mu)} \quad \text{for every } x \in A_j.$$

For $j \in \mathbb{N}$, let B_j be the subset of full measure of X_j associated by Proposition 5.4 to the nilsystem (X_j, μ_j, T) . Then for every nilsequence Ψ and every polynomial $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_j(T^n x) \cdot \Psi(p(n)) = 0 \quad \text{for every } x \in B_j, j \in \mathbb{N}.$$

Let $X_0 := \bigcap_{j \in \mathbb{N}} (A_j \cap \pi_j^{-1}(B_j))$. Then $\mu(X_0) = 1$ and every $x \in X_0$ satisfies the announced property. This completes the proof. \square

5.3. The rational Kronecker is characteristic. We are now ready to verify Theorem 1.2. We restate it for convenience.

Theorem. *Let T, S be measure preserving transformations acting on a probability space (X, \mathcal{X}, μ) such that the system (X, μ, T) has zero entropy. Let also $p \in \mathbb{Z}[t]$ be a polynomial with $\deg(p) \geq 2$ and $f, g \in L^\infty(\mu)$ be such that $\mathbb{E}_\mu(f|\mathcal{K}_{\text{rat}}(T)) = 0$ or $\mathbb{E}_\mu(g|\mathcal{K}_{\text{rat}}(S)) = 0$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{p(n)} g = 0$$

in $L^2(\mu)$.

Proof. Suppose first that $\mathbb{E}_\mu(f|\mathcal{K}_{\text{rat}}(T)) = 0$. Using Proposition 4.2 and a standard approximation argument we can assume that there exists $k \in \mathbb{N}$ such that f is measurable with respect to $\mathcal{Z}_k(T)$ and g is measurable with respect to $\mathcal{Z}_k(S)$. Moreover, arguing as in the proof of Proposition 4.2 we can assume that there exists a subset X_0 of X with $\mu(X_0) = 1$ such that the sequence $(g(S^n x))$ is a k -step nilsequence for every $x \in X_0$.

Let $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ with respect to the transformation T . Then $\mu_x(X_0) = 1$ for μ -almost every $x \in X$, and as we remarked in Section 2.3, for μ -almost every $x \in X$ we have $\mathbb{E}_{\mu_x}(f|\mathcal{K}_{\text{rat}}(X, \mu_x, T)) = 0$. For $x \in X$, let X_x be the subset of X with $\mu_x(X_x) = 1$ associated by Corollary 5.5 to the system (X, μ_x, T) and

to the function f . Then for every $x' \in X_0 \cap X_x$ we have (here we use crucially that the set X_x does not depend on the nilsequence $(g(S^{p(n)}x'))$)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x') \cdot g(S^{p(n)} x') = 0.$$

We deduce using the bounded convergence theorem that for every $x \in X_0$ we have

$$\lim_{N \rightarrow \infty} \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x') \cdot g(S^{p(n)} x') \right|^2 d\mu_x(x') = 0.$$

Integrating with respect to the measure μ and using the bounded convergence theorem again, we obtain the required convergence.

Suppose now that $\mathbb{E}_\mu(g|\mathcal{K}_{\text{rat}}(S)) = 0$. By the first case we can assume that f is measurable with respect to $\mathcal{K}_{\text{rat}}(T)$. Then f can be approximated in $L^2(\mu)$ by a finite linear combination of eigenfunctions associated with rational eigenvalues. Therefore, we can assume that f is an eigenfunction of this type. We are thus reduced to showing that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(ns) \cdot g(S^{p(n)} x) = 0 \text{ in } L^2(\mu), \text{ for every } s \in \mathbb{Q}.$$

Since $\mathbb{E}_\mu(g|\mathcal{K}_{\text{rat}}(S)) = 0$, the spectral measure of g with respect to the system (X, μ, S) has no rational point masses. The asserted mean convergence to zero follows by combining this fact with the spectral theorem for unitary operators, Weyl's Equidistribution Theorem, and the bounded convergence theorem. This completes the proof. \square

6. RECURRENCE, POINTWISE CONVERGENCE, AND POSITIVE ENTROPY

The goal of this section is to complete the proof of Theorem 1.3, Proposition 1.4, and Proposition 1.5.

6.1. Proof Theorem 1.3 (Recurrence). Let $\varepsilon > 0$. It suffices to prove that there exists $r \in \mathbb{N}$ such that

$$(25) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-rn} A \cap S^{-p(rn)} A) \geq \mu(A)^3 - \varepsilon/2.$$

(The limit exists by Theorem 1.1 applied for the transformation T^r in place of T and polynomial $p(rn)$ in place of $p(n)$.)

Since $\mathcal{K}_{\text{rat}}(T) = \bigvee_{d \in \mathbb{N}} \mathcal{I}(T^{dl})$ and $\mathcal{K}_{\text{rat}}(S) = \bigvee_{d \in \mathbb{N}} \mathcal{I}(S^{dl})$, we get that there exists $r \in \mathbb{N}$ such that

$$\|\mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(T)) - \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(T^r))\|_{L^1(\mu)} < \varepsilon/4$$

and

$$\|\mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(S)) - \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(S^r))\|_{L^1(\mu)} < \varepsilon/4.$$

Note that for every $n \in \mathbb{N}$ the function $\mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(T^r))$ is invariant under T^{rn} and the function $\mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(S^r))$ is invariant under $S^{p(rn)}$ because $p(0) = 0$ and thus $p(rn)$ is divisible by r . Therefore, for every $n \in \mathbb{N}$ we have

$$\left| \int \mathbf{1}_A \cdot T^{rn} \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(T)) \cdot S^{p(rn)} \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(S)) d\mu - \int \mathbf{1}_A \cdot \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(T^r)) \cdot \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{I}(S^r)) d\mu \right| \leq \varepsilon/2.$$

By [8, Lemma 1.6] the last integral is greater than or equal to $\mu(A)^3$ and thus for every $n \in \mathbb{N}$ we have

$$\int \mathbf{1}_A \cdot T^{rn} \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(T)) \cdot S^{p(rn)} \mathbb{E}_\mu(\mathbf{1}_A|\mathcal{K}_{\text{rat}}(S)) d\mu \geq \mu(A)^3 - \varepsilon/2.$$

We apply Theorem 1.2 with T^r in place of T (note that $\mathcal{K}_{\text{rat}}(T^r) = \mathcal{K}_{\text{rat}}(T)$) and $p(rn)$ in place of $p(n)$. We get that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-rn} A \cap S^{-p(rn)} A) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \int \mathbf{1}_A \cdot T^{rn} \mathbb{E}_\mu(\mathbf{1}_A | \mathcal{K}_{\text{rat}}(T)) \cdot S^{p(rn)} \mathbb{E}_\mu(\mathbf{1}_A | \mathcal{K}_{\text{rat}}(S)) d\mu.$$

Combining the above we get the bound (25). This completes the proof. \square

6.2. Proof of Proposition 1.4 (Positive entropy). Let h be the entropy of the system (X, μ, T) . Let \tilde{T} be the shift on the sequence space $\tilde{X} = \{0, 1, 2\}^{\mathbb{Z}}$ and let $\tilde{\mu}$ be the $(s, s, 1 - 2s)$ -Bernoulli measure on \tilde{X} , where $s > 0$ is small enough so that the entropy of the Bernoulli shift $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is smaller than h . It follows from Sinai's factor theorem [20, Theorem 20.13] that $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a factor of (X, μ, T) . Let $\pi: X \rightarrow \tilde{X}$ be the factor map.

We claim that it suffices to construct a measure preserving transformation \tilde{S} of \tilde{X} and a subset \tilde{A} of \tilde{X} such that

$$\tilde{\mu}(\tilde{T}^{-a(n)} \tilde{A} \cap \tilde{S}^{-b(n)} \tilde{A}) = \begin{cases} 0 & \text{if } n \in F, \\ s^2 & \text{if } n \notin F. \end{cases}$$

Indeed, since (X, μ, T) is ergodic, it follows from Rohlin's skew-product theorem [20, Theorem 3.18] that there exist a probability space (Z, ν) and an isomorphism of probability spaces Φ from $(\tilde{X} \times Z, \tilde{\mu} \times \nu)$ to (X, μ) such that $\pi(\Phi(\tilde{x}, z)) = \tilde{x}$ for $(\tilde{\mu} \times \nu)$ -almost every (\tilde{x}, z) . We let $A := \pi^{-1}(\tilde{A})$ and $S := \Phi \circ (\tilde{S} \times \text{id}) \circ \Phi^{-1}$. Then S is a measure preserving transformation of (X, μ) , and the required properties hold.

Therefore, we can restrict to the case where (X, μ, T) is the $(s, s, 1 - 2s)$ -Bernoulli shift on $X = \{0, 1, 2\}^{\mathbb{Z}}$. We argue as in the proof of [15, Lemma 4.1], with small changes. Given a permutation π of \mathbb{Z} with $\pi(0) = 0$ we define the map $\psi_\pi: X \rightarrow X$ by

$$(\psi_\pi x)(n) := \begin{cases} x(0) & \text{if } n = 0, \\ 1 - x(\pi(n)) & \text{if } n \neq 0 \text{ and } x(\pi(n)) \in \{0, 1\}, \\ 2 & \text{if } n \neq 0 \text{ and } x(\pi(n)) = 2. \end{cases}$$

It is easy to verify that ψ_π is measure preserving, invertible, with $(\psi_\pi)^{-1} = \psi_{\pi^{-1}}$, and $(\psi_{\pi^{-1}} x)(0) = x(0)$. We define the measure preserving transformation $S: X \rightarrow X$ by

$$S := \psi_\pi^{-1} T \psi_\pi.$$

Then for $n \in \mathbb{N}$ we have

$$(S^n x)(0) = (T^n \psi_\pi x)(0) = (\psi_\pi x)(n) = \begin{cases} 1 - x(\pi(n)) & \text{if } x(\pi(n)) \in \{0, 1\}, \\ 2 & \text{if } x(\pi(n)) = 2. \end{cases}$$

Hence, if $A := \{x \in X: x(0) = 1\}$ we have

$$T^{-a(n)} A \cap S^{-b(n)} A = \{x \in X: x(a(n)) = 1, x(\pi(b(n))) = 0\}, \quad n \in \mathbb{N}.$$

We now choose the permutation π . Since the sequences $a, b: \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ are injective and miss infinitely many integers, we can choose π that fixes 0 such that $\pi(b(n)) = a(n)$ if $n \in F$ and $\pi(b(n)) \neq a(n)$ if $n \notin F$. Then

$$\mu(T^{-a(n)} A \cap S^{-b(n)} A) = \begin{cases} 0 & \text{if } n \in F, \\ s^2 & \text{if } n \notin F. \end{cases}$$

This completes the proof. \square

6.3. Proof of Proposition 1.5 (Pointwise convergence). We can assume that $f, g \in L^\infty(\mu)$ are real valued. We first establish existence μ -almost everywhere of the limit

$$(26) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \cdot g(S^{p(n)} x).$$

By Proposition 2.5, for almost every $x \in X$ the sequence $(f(T^n x))$ admits correlations on the sequence of intervals $([N])_{N \in \mathbb{N}}$ and the corresponding Furstenberg system has zero entropy. By our assumption, for almost every $x \in X$ the sequence $(g(S^{p(n)} x))$ admits correlations along the sequence of intervals $([N])_{N \in \mathbb{N}}$ and defines a (unique) Furstenberg system. Suppose that $\mathbb{E}(g|\mathcal{Z}_\infty(S)) = 0$. Combining our pointwise convergence assumption of the averages (6) with Theorem 3.1 and Lemma 2.3, we get that for almost every $x \in X$ the 0th-coordinate projection of the Furstenberg system of the sequence $(g(S^{p(n)} x))$ is orthogonal to the Pinsker factor of this system. Hence, using Proposition 2.1 and arguing as in Proposition 4.1, we deduce that the averages (26) converge pointwise almost everywhere to 0. Hence, in order to prove pointwise convergence of the averages (26) we can assume that g is measurable with respect to the factor $\mathcal{Z}_\infty(S)$.

By Bourgain’s maximal inequality for ergodic averages with polynomial iterates [7, Theorem 6], for $g \in L^2(\mu)$ we have

$$\left\| \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{n=1}^N S^{p(n)} |g| \right\|_{L^2(\mu)} \leq C \|g\|_{L^2(\mu)}$$

for some universal constant C . Thus for $f \in L^\infty(\mu)$ we have

$$\left\| \sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{n=1}^N |T^n f \cdot S^{p(n)} g| \right\|_{L^2(\mu)} \leq C \|f\|_{L^\infty(\mu)} \|g\|_{L^2(\mu)}.$$

By the argument used in the classical proof of the pointwise ergodic theorem, it follows that for f fixed, the family of functions g for which the convergence (26) holds almost everywhere is closed in $L^2(\mu)$. Therefore, in order to prove the existence almost everywhere of the limit (26) for g measurable with respect to $\mathcal{Z}_\infty(S)$, it suffices to restrict to the case where g is measurable with respect to $\mathcal{Z}_k(S)$ for some $k \in \mathbb{N}$.

If g is measurable with respect to $\mathcal{Z}_k(S)$, then the pointwise convergence of the averages (26) follows by repeating the argument used in the proof for Proposition 4.2. This finishes the convergence part of the proof.

Since the pointwise limit coincides with the $L^2(\mu)$ limit, the remaining part of the result follows from Theorem 1.2. This completes the proof. \square

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