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# A clean, robust 3D medial axis

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## Abstract

*Computing the medial axis of a 3D surface mesh is challenging. Points on the discrete medial axis can be defined as interior Voronoi vertices of the surface mesh, but the resulting medial structure rarely has clean connectivity and consistent geometry. In this paper, we provide a medial axis computation based on the Voronoi diagram able to generate manifold medial sheets with coherent topology and geometry, generating consistent geometric structures similar to those in the continuous setting. Because of the correspondences between the surface mesh and resulting medial mesh, we also provide an efficient method for separating the shape into coherent regions associated to medial structures. This correspondence allows for a medial-axis-based filtration of surface structures to generate a Hausdorff  $\epsilon$ -approximation of the surface points based on a simplified medial axis, thereby providing a robust medial representation with guaranteed surface approximation.*

**Keywords:** Skeleton, structure, mesh, delaunay, voronoi

## 1. Introduction

Medial representations are quite popular in computer graphics because they combine implicit and explicit shape representations, and offer a lower dimensional model of a 3D shape, the skeleton, that captures the geometric information from the boundary, while a radius or distance function describes the interior of the shape. The first such model, the Blum medial axis, was introduced in [Blu73] for 2D shapes. It consists of a skeleton and a radius function defined at each point of the skeleton that gives the distance of the skeleton point to the boundary. A useful discretization of the Blum medial axis relies on a shape's Delaunay triangulation and its dual, the Voronoi diagram.

As described in [TDS\*16], several approaches exist to compute a full skeleton. Voxel-based methods [HR08, ADBS11] which are a 3D version of classical 2D thinning methods, easily maintain the structure of the shape, and can be less subject to noise. In [JST15], the voxel grid is the support of a mass transport process, leading to a multiscale, simplified skeleton. In [YLJ18], the authors use a Delaunay tetrahedralization of the open voxels of a shape boundary to compute a voxel core that converges to the true medial axis as the voxelization of the shape becomes dense. A separate pruning step then removes noisy elements in the voxel core with a guarantee on the approximation and the homotopy equivalence. In these approaches, the shape must first be converted to a voxel grid, and its sampling density is directly linked to the skeleton accuracy, thereby introducing an additional parameter to the model. Also, in the voxel based approaches, no global and topologically coherent

structure of the skeleton is available. Other methods compute a skeleton from boundary samples [RAV\*19, MBC12, LBKP21]. Each boundary sample is characterized by a position and a normal, and no additional structure is needed. A set of tangent spheres can then be computed, filtering any non-descriptive ones. Methods based on deep learning have also been proposed. In [YYW\*20], a simplified skeleton is computed and accurately describes the shape (depending on the chosen precision) but they do not capture the shape structure, as the resulting skeleton is typically not a graph. In [CD23] the local structure of the skeleton is preserved in general, but there is no guarantee on the coherence of the connectivity, that is, a tetrahedron may belong to the resulting skeleton. In fact, by construction deep learning methods do not maintain a clear relationship to the shape boundary. In [RLS\*21], the shape and its skeleton are linked but the representation is implicit, giving neither connectivity nor structure.

Other approaches, including ours, are based on the computation of a Voronoi diagram. For simple shapes such as polyhedra, the Voronoi diagram can be explicitly computed [SPB96]. The power crust applies a power diagram to compute a mesh and a skeleton from a disorganized point cloud [ACK01].

Underlying many of these methods is an attempt to address the major drawback of the medial axis in applications: a lack of stability, in that a small change on the shape boundary can produce large changes in the medial axis. Moreover, edges linked to several medial axis surface sheet may appear in non singular regions, leading to connectivity and structural inconsistencies. Many approaches have been proposed to construct a robust medial axis for 3D shapes, with varying degrees of success. Classical approaches to robustness

are based on pruning [DZ03, AL01] prune the Voronoi diagram to obtain a sensible medial structure. [YSC\*16] compute a significance measure: the authors demonstrate that a global measure related to Erosion Thickness, generalized from 2D to 3D, is more efficient than a local measure. The well-known Scale Axis Transform [MGP10] has the advantage of taking into account multiple medial representations. They propose a multi-scale representation through the scale axis transform that takes into account a scaling of medial balls, but the approach results in some artifacts that require a separate pruning step. This method has generated multiple variants and improvements [LWS\*15, PWG\*19] that produce a simplification of a Voronoi skeleton. However, these methods do not maintain a robust link to the 3D mesh. In contrast, our goal in the current work is to be independent of a pruning step. We both maintain global connectivity consistency in the structure of the medial axis and geometric consistency with the surface mesh, in the sense that the Delaunay tetrahedralization of the output surface mesh is dual to the output Voronoi skeleton.

Many researchers compute a curve skeleton, which is a network of curves, instead of the full skeleton in 3D, [CSM07, SYJT13]. A curve skeleton is appealing because it addresses the previously mentioned instability while capturing most of the major structures of the shape. It can be estimated by mesh contraction [ATC\*08, LW18, ZLZ20], or through the computation of a 3D skeleton [DS06, TAOZ12], or directly from the mesh, using tools such as local separators [BR21]. Curve skeletons necessarily maintain only a weak relationship to the surface mesh, however, and there are some shape structures (funnel shapes, for example) that are not well-suited for description by a single curve.

Depending on particular goals and applications for a skeletal model, different properties of the skeleton may be desired, such as continuity, connectivity, geometric correspondence with the boundary, or a particular level of simplicity. These variations in goals have produced variations in skeletonization algorithms that make different trade-offs, as the work cited in the previous paragraphs demonstrates. In this article, we take as input a Delaunay surface mesh of a shape and compute a medial axis that has a full array of desirable properties. This approach is similar to a 2D skeletonization method by propagation [DCL\*19, DMC\*19], where the skeleton is computed step by step, and where a Hausdorff distance criterion identifies noise so that insignificant parts of the medial axis are not computed. The next section describes this simplified skeletonization.

### 1.1. Simplified skeletonization

In this paper, we present a method to compute a simplified, robust medial axis for a 3D shape from a Delaunay input mesh representing the shape surface. Our approach is based on a propagation of medial balls, or equivalently, Delaunay tetrahedra, which means we do not require a pruning step. A Delaunay tetrahedron is a tetrahedron such that its circumscribed sphere does not contain any other point from the surface mesh. Therefore, each skeletal point is associated to a tetrahedron, and the edges between skeletal points are associated to tetrahedra that share faces. Finally, a Delaunay mesh is a mesh for which each triangle is a triangle from a Delaunay tetrahedron. Local geometry is combined into a global structure on

the skeleton, which is then used for the simplification analysis. This global structure consists of sheets, which are surface portions of the skeleton, boundary paths, which are curve boundaries of a single sheet, and singular paths, which are curve portions separating different sheets (See Section 3 for details). As a result, our approach guarantees the following properties (*cf.* Figure 1):

- **The topological and geometric relationships between the surface and medial meshes are preserved.** Specifically, (a) the resulting medial mesh is homotopically equivalent to the output surface mesh by construction, and (b) the points on the medial mesh are Voronoi duals of Delaunay tetrahedra generated by the surface mesh. Each vertex, edge, face and sheet of the medial axis corresponds to a well-specified set of vertices, edges, faces, or region of the surface mesh. This means that we maintain correspondence between the surface and medial meshes.
- **The geometric structures of the medial mesh are clean and coherent.** The skeleton is divided into sub-structures of sheets with topologically consistent manifold interiors, which are separated and bounded by manifold paths. The skeleton mesh itself is connected for an input shape with a single component due to the homotopy equivalence.
- **The simplified medial axis provides a guaranteed  $\epsilon$ -approximation to the original surface points.** The simplified medial mesh generated for some choice of  $\epsilon > 0$  corresponds to a subset of the original surface points that are  $\epsilon$ -close in Hausdorff distance to the original surface points. All topological features at a scale larger than  $\epsilon$  are maintained, though features at a scale smaller than  $\epsilon$  may be deleted.

In what follows, we first describe the continuous and discrete constructions of the interior Blum medial axis (BMA) in Section 2. In Section 3, we describe the global structures of the discrete BMA and describe how these are related to local structures. Finally, we describe the algorithm for generating the simplified skeleton based on the  $\epsilon$  error parameter in Section 4, and present results in Section 5. We also include a brief appendix to explain some special cases that arise in this discrete setting, and how we handle them.

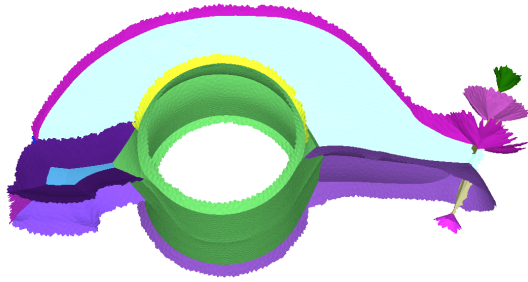
## 2. Geometry of the interior medial axis

The Blum medial axis is structured into parts that offers strong geometric properties in the smooth setting; we seek and provide analogs in the discrete setting. In this section, we describe these properties and the correspondences between smooth and discrete representations.

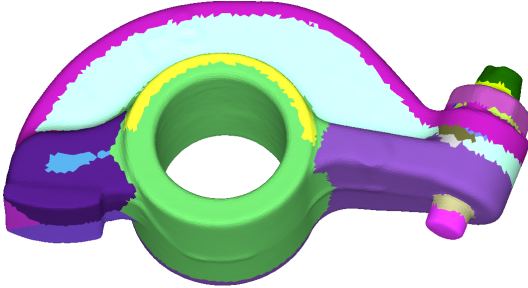
### 2.1. Smooth setting

In this section, we recall the structure of the medial axis in the smooth setting where the shape surface is differentiable, in order to provide an analogy with the discrete setting we are interested in.

Given a closed, non-self-intersecting surface  $S \subset \mathbb{R}^3$  with continuous curvature and a finite set of curvature extrema and ridge curves, the interior Blum medial axis can be defined as the locus of centers of maximally inscribed spheres within the surface, together with their radii. This produces a skeleton composed of sphere centers that is a Whitney stratified set  $M$  with a finite number of strata.



(a) Subdivision of a skeleton into sheets



(b) Surface mesh regions associated to each sheet of the skeleton

Figure 1: Global structures of the skeleton (a), and their consistency with the surface mesh (b). The color coding in the figures shows which global skeletal structures in (a) correspond to which global shape structures in (b).

165 The points of  $M$  are traced out by the centers of the inscribing  
 166 spheres resulting in a medial skeleton composed of smooth sur-  
 167 faces and curves joined along well-understood singularities. A ra-  
 168 dius function  $r$  is defined for each point of  $M$ , giving the associ-  
 169 ated radius of the inscribing sphere. Intuitively, a Whitney strati-  
 170 fied set is an arrangement of nicely behaved sets called strata, so  
 171 that any non-empty intersection of strata is a manifold of lower di-  
 172 mension, the intersection is transverse, and the tangent spaces of  
 173 the strata converge appropriately at these intersections. See, for ex-  
 174 ample, [Dam05] for details.

175 In the case of the medial axis for the surface of a 3D shape, such  
 176 as in Figure 2, each stratum is either a 2D medial sheet (in light blue  
 177 in the figure), a 1D medial curve (in red – generically, these will be  
 178 2D), an intersection point or curve where two higher dimensional  
 179 strata meet (in black), or a boundary point or curve bounding a  
 180 higher dimensional stratum (in blue). See [GK04] for an exhaustive  
 181 classification of medial axis points. Away from singularities and  
 182 sheet or curve boundaries, the degree of differentiability of  $M$  is  
 183 equal to the degree of differentiability of  $S$ .

184 As a stratified set, the points of  $M$  are generically one of three  
 185 types, see Figure 2:

- 186 1. **Regular points** in the interior of a top-dimensional stratum, in  
 187 light blue. For regular points, the medial spheres are bitangent to  
 188  $S$ , which means that each point of  $M$  corresponds to two points  
 189 of  $S$ , one on either side of  $M$  (“above” and “below”).

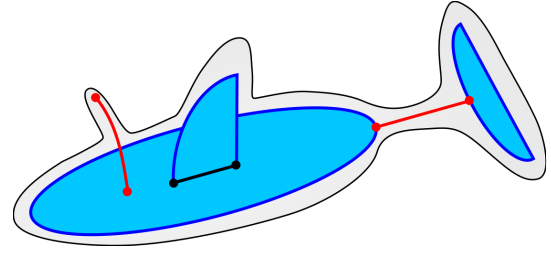


Figure 2: A skeleton structure in the continuous setting: a contin-  
 uous shape and its skeleton containing three medial sheets (and  
 corresponding strata). Regular skeleton points (in light blue) lie  
 in the interior of a skeleton sheet. They are bounded by singular  
 boundary points (in blue), and intersect at interior singular curves  
 (in black). The medial axis also contains two non-generic medial  
 strata of lower dimension, curves (in red).

- 190 2. **Singular interior points** where two strata intersect, in black.  
 191 For singular interior points, the medial spheres will be at least  
 192 tri-tangent to  $S$ , corresponding to the merging of bitangencies  
 193 on at least two strata meeting along the singularity.
- 194 3. **Singular boundary points** where the inscribing sphere oscu-  
 195 lates at an extremum of at least one principal curvature, in blue.  
 196 For singular boundary points, the medial sphere has only one  
 197 tangency, the point of extremal curvature, corresponding to the  
 198 merging of at least one bitangency where the points “above” and  
 199 “below” merge into a single point.

## 2.2. Discrete setting

200 A primary contribution of our paper is to construct well-defined dis-  
 201 crete versions of the medial structures described for differentiable  
 202 shape surfaces in the previous section. In applications, the smooth  
 203 surface  $S$  described above is typically replaced by a sampling of its  
 204 points and connectivity between those points, a surface mesh con-  
 205 sisting of vertices joined by edges and faces. Similarly, the smooth  
 206 strata of the medial skeleton are replaced by a medial mesh. In what  
 207 follows, we assume sampled points of a 3D shape surface are ver-  
 208 tices in a Delaunay surface mesh  $S$ , with an induced medial mesh  
 209  $M$ . Points on  $M$  can then be computed as interior Voronoi vertices  
 210 generated by the surface points. Because the Voronoi vertices will  
 211 be centers of the circumspheres of its dual, the Delaunay tetrahe-  
 212 dralization of the volume enclosed by  $S$ , the tetrahedra will provide  
 213 a discrete analogue of medial balls in the smooth case. Similarly,  
 214 distance from the Voronoi vertices to the surface points will pro-  
 215 vide a discrete version of the radius function  $r$ . As the sampling  
 216 on the surface becomes dense, converging to the smooth surface,  
 217 a subset of this discrete medial skeleton converges to the smooth  
 218 skeleton [DZ03].

220 With this characterization, much of the structure from the dif-  
 221 ferentiable setting carries over. The resulting discrete  $M$  will have  
 222 three types of points, see Figure 3 for illustration:

- 223 1. **Regular points**, in (3a). For regular points, each point of  $M$   
 224 will typically correspond to a tetrahedron that crosses the inter-  
 225 interior of  $S$ , with three points of  $S$  on one side of  $M$  and one  
 226 point on the other side (“above” and “below”), see left, but other

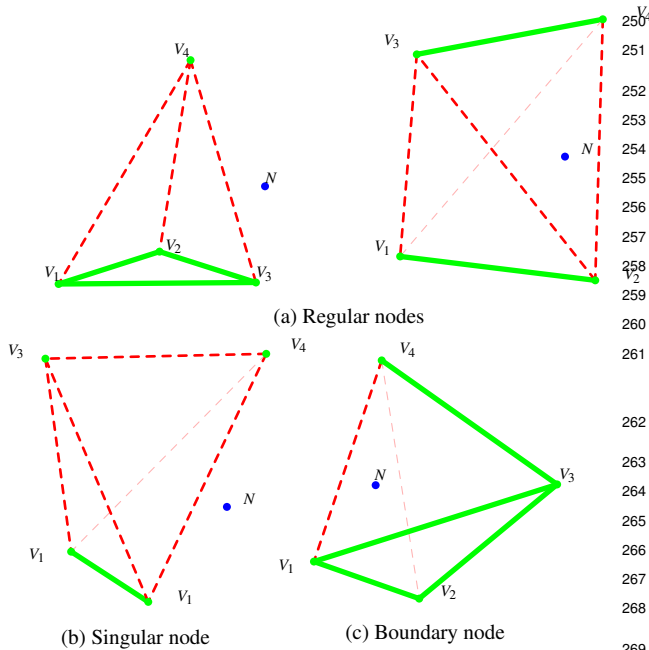


Figure 3: A sampling of some different (non-exhaustive) configurations for a discrete medial node  $N$ , in blue, corresponding to different point type classifications of  $N$ . The vertices  $(V_i)_{i=1..4}$  are its corresponding boundary points and tetrahedron. Among the tetrahedral edges, the green edges belong to the surface mesh and the red dashed ones traverse the volume inside the 3D shape. Note that the node  $N$  is not necessarily inside the tetrahedron.

227 configurations are possible such as two points above and two  
228 below [CDE\*00], see right.

- 229 2. **Singular interior points**, in (3b). For singular interior points,  
230 the tetrahedron will correspond to the convergence of tetrahedra  
231 of regular points on at least two strata meeting along the singu-  
232 larity. In this case, there will be at least two sets of points above  
233 and below corresponding to at least three disconnected regions  
234 of the surface.
- 235 3. **Singular boundary points**, but without a notion of tangency,  
236 in (3c). For singular boundary points, the points in  $S$  composing  
237 the tetrahedron will be adjacent and belong to a single connected  
238 component, with the two “sides” of  $M$  having converged.

239 This framework gives the foundation for the structures we use to  
240 generate the strata of the discrete skeleton. Unfortunately, guaran-  
241 tees of topological and geometric coherence from the smooth set-  
242 ting do not automatically carry over, as rounding errors and other  
243 issues often lead to medial skeletons with more complicated struc-  
244 tures [TDS\*16]. Our algorithm addresses these issues to produce a  
245 geometrically coherent skeleton.

### 246 3. Hierarchical structure of the discrete skeleton

247 In what follows, we distinguish between local medial structures  
248 (e.g., medial points and neighbors, corresponding surface points  
249 and neighbors) and global medial structures (e.g., medial sheets,

boundaries, singular sets). Our algorithm analyzes local structures  
250 to build coherent global structures.

251 The constructions presented in this section are necessary for un-  
252 derstanding the algorithm in Section 4. We define the global medial  
253 structures in Section 3.5, which allow us to segment the surface  
254 correspondingly in Section 3.2. In section 3.3, we present the local  
255 medial structures, derived from the Voronoi and Delaunay duality  
256 linking the surface mesh to the medial mesh, that provide the basis  
257 for the global structures and also for the simplification of the me-  
258 dial skeleton. Section 3.4 describes how the global structures are  
259 obtained from the local structures. Finally, section 3.5 describes  
260 how we compute the simplified 3D medial axis.

### 262 3.1. Global structures of the discrete skeleton

263 Medial sheets are typically two-dimensional for a 3D shape, but  
264 can also be one-dimensional. Sheet boundaries and singular paths  
265 are typically one-dimensional, but can also be zero-dimensional.  
266 For simplicity, we present our algorithm for two-dimensional case,  
267 but the process is analogous in the one-dimensional or zero-  
268 dimensional cases.

269 Sheets, the top-dimensional strata described in Section 2, are the  
270 fundamental building blocks of the skeleton. We define a discrete  
271 *sheet* of the medial axis to be a connected set of regular medial  
272 points for which the corresponding regions on the surface mesh  
273 have two connected components, together with any neighboring  
274 singular points (corresponding to more than two connected com-  
275 ponents) or boundary points (corresponding to one connected com-  
276 ponent) (see Figure 3). Each sheet corresponds to one or more con-  
277 nected portions of the input surface mesh. Furthermore, each sheet  
278 has two sides, and can be therefore viewed as two *oriented sheets*,  
279 where each oriented sheet corresponds to a different side of the sur-  
280 face mesh, “above” or “below”, see Figures 4a and 4b).

281 The non-regular medial points associated to a sheet are members  
282 of either a *boundary path*, Figure 4c, corresponding to the ridge  
283 curves on the shape surface where the two sides of the surface cor-  
284 respondence of a medial sheet have merged, or a *singular path*,  
285 Figure 4d, where two medial sheets intersect, or both. The bound-  
286 ary and singular paths are typically 1D strata. Boundary paths with  
287 no singular points belong to a single sheet, but all singular paths  
288 belong to multiple sheets. When boundary points are also singular,  
289 it is because a singular set has intersected a boundary path.

290 These global medial structures correspond to structures on the  
291 surface mesh  $S$ . Sheets correspond to coherent regions of the shape  
292 surface, such as protrusions. Boundary paths correspond to curva-  
293 ture extrema in the surface, such as ridges. Singular paths corre-  
294 spond to closed paths on the surface delineating connections be-  
295 tween coherent shape regions. Finally, we note that a path within a  
296 sheet, which means a path of regular points, generates two paths on  
297 the shape surface: one “above” and one “below”. This means that  
298 each surface path can be attributed to one of the two oriented sheets  
299 associated to a medial sheet.

### 300 3.2. Segmenting the surface using sheets and paths

Using these global structures, we can segment the shape surface  
301 into coherent regions. In Figure 4, we associate the singular path  
302

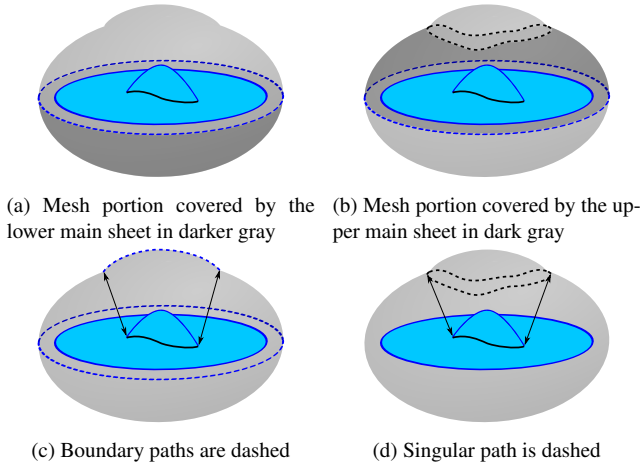


Figure 4: Correspondences between skeletal structures and surface regions. Solid line boundary paths (darker blue) and singular paths (black) represent paths on the mesh. Dashed paths represent the corresponding paths on the surface mesh. (a) The medial boundary path of the main (horizontal) sheet separates the surface into two regions, above and below, with the lower region shown in darker gray. (b) The upper surface region, however, is interrupted by region associated to the smaller medial sheet that is connected to one side of the main sheet. The singular path separating the two sheets corresponds to the black dashed path on the surface mesh and the resulting dark gray region corresponds to the surface that is above the main sheet. (c) If a medial boundary path is not a closed loop, the surface path will also not be a loop, as can be seen in the surface path generated by the medial boundary of the smaller sheet. If a medial boundary path is a loop, the corresponding path on the surface mesh will also be a loop, as can be seen in the surface path generated by boundary of the main medial sheet. (d) For medial singular paths, in contrast, a medial path that is an interval can correspond to a surface path that is a loop, as can be seen in the path on the surface mesh that corresponds to the closed black medial singular path.

sophically, the aggregation is straightforward – join connected sets of points that are regular, or singular, or boundary – but many local degeneracies arise in practice (some of which are collected in Appendix A) that make the process challenging. We now define the local structures that will allow us to generate global structures.

### 3.3.1. Voronoi and Delaunay duality between the skeleton and the surface mesh

Based on a Delaunay triangulation of a surface mesh  $S$ , we construct the Delaunay tetrahedralization  $T$  of the volume of the interior of  $S$ .  $T$  is the dual of the Voronoi diagram  $V$  of the vertices of  $S$ . Each of  $S$ ,  $M$ ,  $T$ , and  $V$  have vertices, edges, and faces, and  $T$  also has tetrahedra.

All vertices of  $T$  are vertices of the surface mesh. Note that while all edges and faces of  $S$  belong to the Delaunay tetrahedralization  $T$ , only a subset of edges and faces of  $T$  belong to  $S$ . Because of the number of types of collections of vertices, edges, and faces, we specify vocabulary and color-code mesh elements consistently in the text and figures to distinguish between them. All vertices, edges, faces (triangles in the general setting), and volumes (tetrahedra in the general setting) belonging to the **Delaunay tetrahedralization** are colored in **red**. We identify the edges and faces that cross the volume of the interior of  $S$  by appending the word *volumetric*. The faces of the **Delaunay tetrahedralization** that are also faces of the surface  $S$  are colored in **green** and carry the word *surface*.

Similarly for  $V$ , almost all vertices interior to  $S$  belong to the medial mesh  $M$  [DZ03], while only a subset of the edges and faces of  $V$  are contained in  $M$ . The elements in the **Voronoi diagram**  $V$  are colored in **dark blue** except for the subset of the elements that belong to the medial axis  $M$ , which are **lighter blue**. We refer to the faces of  $V$  as *Voronoi faces* to emphasize that they are not typically triangular. Identifying the faces, edges, and vertices of  $V$  that are completely inside the volume bounded by  $S$  defines the interior discrete medial axis that is our primary object of study. Note that there are also Voronoi vertices, edges, and faces that are completely outside the volume bounded by  $S$ . These define an exterior medial skeleton that we do not study here.

separating two sheets to a closed path on the shape surface that delineates the coherent surface region associated to that sheet. In Figure 4, consider the upper part of the object where there is a small medial sheet connected to the larger one. The small sheet corresponds to a small portion of the surface of the object that is surrounded by the dashed black loop induced by the black singular path at the base of the medial sheet. Repeating this analysis for each sheet in a medial axis produces a segmentation of the shape boundary where shape segments are bounded by surface loops induced by singular paths in the medial axis. This will be important in Section 4 where we determine whether or not to compute a sheet in the medial axis by evaluating the contribution of the corresponding shape segment to the overall shape.

### 3.3. Local structures of the discrete skeleton

The global structures described in the previous section can be formed by aggregation of the appropriate local structures. Philo-

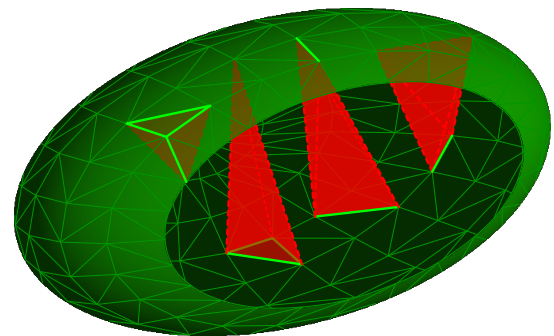


Figure 5: Difference between volumetric and surface tetrahedral components. Surface edges and faces (green) lie on the mesh of the object, while volumetric edges and faces (red) cross the volume of the object. The shapes of the drawn tetrahedra refer to Figure 3.

We illustrate the local constructions more precisely next, emphasizing the duality between them: the **Delaunay tetrahedralization**

357  $T$ , which is primal, creates connectivity between **surface vertices**  
 358 through the shape interior. In the typical case, this generates:

- 359 • **edges**, joining two **surface points**, that may be a **surface edge**  
 360 (Figures 5 and 3, plain green edges), or **volumetric edges** (Fig-  
 361 ures 5 and 3, dashed red edges);
- 362 • **triangles**, joining three **surface points**, that may be a **surface tri-**  
 363 **angle** in which case, its three edges are also **surface edges** (Fig-  
 364 ures 5 and 3, green triangles), or a **volumetric triangle** (Figures 5  
 365 and 3, red triangles);
- 366 • **tetrahedron**, joining four **surface points** that are volumetric, since  
 367 we consider only the interior **Delaunay tetrahedra** (Figures 5, red  
 368 tetrahedra).

369 These **primal elements** have corresponding **dual elements** in  $V$  :

- 370 • **tetrahedra** correspond to **Voronoi vertices** which are centers of  
 371 the circumspheres of the tetrahedra;
- 372 • **tetrahedral faces** correspond to **Voronoi edges**;
- 373 • **tetrahedral edges** correspond to **Voronoi faces**. Note that a *me-*  
 374 *dial face*, and more generally a **Voronoi face**, is a planar polygon  
 375 since it lies on the medial plane of the **dual edge** and can have an  
 376 arbitrary number of **nodes**.
- 377 • **medial vertices** are a subset of **Voronoi vertices** and we denote  
 378 them as **nodes**;
- 379 • **medial edges** are a subset of **Voronoi edges**;
- 380 • **medial faces** are a subset of **Voronoi faces**.

381 We construct our simplified medial axis by exploring the **Voronoi**  
 382 **diagram** to determine which elements to include in the **medial axis**,  
 383 and then computing the medial axis components according to their  
 384 importance to the overall shape as measured by the Hausdorff distance  
 385 between the original corresponding surface points and the  
 386 simplified version. The process requires both local and global con-  
 387 siderations of both primal and dual components.

388 Around a given **medial edge**, there can be as many **Voronoi faces**  
 389 as there are **Delaunay edges** (including **volumetric edges** and **sur-**  
 390 **face edges**) around the associated **Delaunay face**. This means that  
 391 **edges** are not necessarily surrounded by 3 **Voronoi faces**. A **Voronoi**  
 392 **face** dual to a **Delaunay edge** that belongs to the surface, that is, dual  
 393 to a **surface edge**, is called an *open Voronoi face*, cf. Figure 6. An  
 394 **open Voronoi face** contains a **node** that is outside the shape bound-  
 395 ary. This type of **Voronoi face** does not belong to the skeleton of  
 396 the medial axis, because it is not entirely contained in the interior  
 397  $S$ . This allows us to filter out **Voronoi face** that are not part of the  
 398 interior medial axis.

399 Using the same idea, an *open edge* is associated to a **Delaunay**  
 400 **face** that belongs to the surface mesh. For this type of **edge**, one  
 401 of its extremities is a **Voronoi nodes** on the inner medial axis, and  
 402 the other one is a **Voronoi vertex** on the outer medial axis. Like the  
 403 **open Voronoi face**, the **open edge** therefore crosses the surface  $S$ .  
 404 Naturally, an **open Voronoi face** contains at least one **open edge**.  
 405 These **edges** do not belong to the medial mesh.

### 406 3.3.2. Oriented medial edges and faces

407 As discussed in Section 3.2, medial sheets comprising the skele-  
 408 ton have two sides; each side corresponds to a region of the shape  
 409 surface that is either "above" or "below." This two-sidedness also

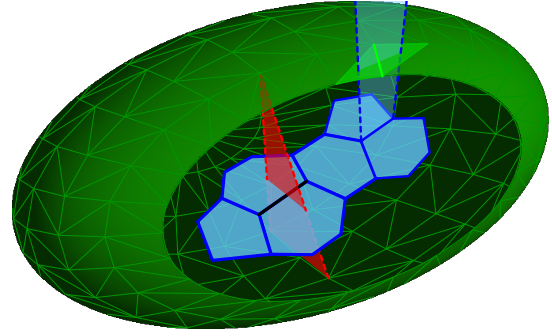


Figure 6: Illustration of the relationship between skeleton and sur-  
 face components. An open skeletal edge (dashed blue) is associated  
 to its dual, a surface mesh face (green triangle). An open face, con-  
 taining open edges is associated to its dual, a surface mesh edge  
 (green). This kind of face contains edges that lie on both the inter-  
 nal and the external skeletons. A Delaunay triangle (red) is asso-  
 ciated to its dual, a skeletal edge (black), and each of its crossing  
 edges (dashed red) is associated to its dual, a skeletal face.

410 applies to our local structures, producing two-sided edges and two-  
 411 sided face that we denote by **half edges** and **half facets**, as in  
 412 [DRJ15]. As shown in Figure 7, on a degree 3 edge, three **face**  
 413 are connected, which gives six **half facet**. In total, six **half edges** are  
 414 derived from a degree 3 edge, one for each **half facet**. For example,  
 415 viewing a singular path as composed of a succession of **half edges**  
 416 will result in a path that lies on a single side of a sheet.

417 We distinguish between the **edges** that are above and those be-  
 418 low by defining two **half edges** for each edge in the medial mesh.  
 419 These **half edges** will have orientations, moving in sequential order  
 420 around an **medial face**. The **half edges** are in counter-clockwise  
 421 order around the **medial face**, with respect to its normal vector. As  
 422 there are two possibilities for the normal orientation, there are two  
 423 sets of **half edges**, which are associated to the two sides of the **me-**  
 424 **dial face**. A **half medial face** consists of all the **half edges** that are on  
 425 the same side. Using the resulting graph structure to represent the  
 426 skeleton, we can easily navigate between the **medial faces**, **edges**  
 427 and **vertices** and their corresponding surface regions from the con-  
 428 nectivity information, and we can resolve ambiguities about neigh-  
 429 boring **medial faces**, since a neighbor must be on the same side of  
 430 the medial mesh.

### 431 3.4. Building global from local

432 We now aggregate the local structures described in Section 3.3 into  
 433 the global medial structures described in 3.1. In other words, we  
 434 determine medial sheets and identify which of its **nodes** and **edges**  
 435 are regular, singular, or belonging to the sheet boundary.

436 Each **sheet** is a subset of the **medial faces** of the skeleton. Two  
 437 **medial faces** are on the same sheet if there exists a path of **medial**  
 438 **faces** between them such that any two successive **medial faces** are  
 439 separated by a **regular edge**. Collecting all **medial faces** contained  
 440 in the same sheet produces the mesh for that sheet.

441 It remains to determine the regularity or singularity of the **nodes**

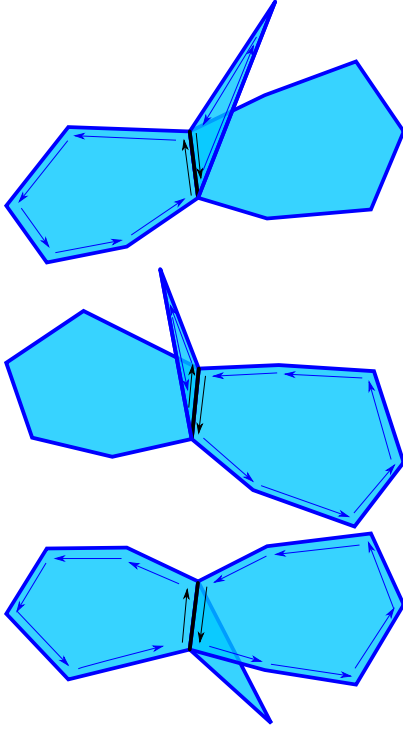


Figure 7: Representation of the half edges around a (black) singular medial edge, from different views (up left, up right, below). Around this degree 3 edge, there are 6 half edges represented by the black arrows. Each pair of (black arrows) half edges represents a different area on the surface mesh.

and edges within a (non-oriented) sheet. Recall that the degree of an edge is defined by the number of medial faces containing the edge. A singular edge is then an edge of degree 3 or more, a boundary edge is an edge of degree 1, and a regular edge is an edge of degree 2. These edges may belong to multiple sheets.

We now consider the half edges and how they compose paths on a sheet (cf. Figure 8). Since each half edge is associated to a half medial face, each of them will be associated to a unique sheet. Note that an open path of full medial edges corresponds to a closed path of half edges: one traversal of successive edges on one side of the sheet and another traversal of the same edges on the other side but with the opposite orientation. We say that two half edges are successive if they respect the following conditions:

- The last node of one half edge is the first node of the other,
- Their associated half medial faces are neighbors (or are the same).

A half singular path, or singular path (as every path in what follows will now be half edges) is defined by a set of successive half singular edges (degree 3 or more). As there are opposite half edges for each edge on a medial face, each singular path can be associated to an opposite singular path (cf. Figure 8).

Finally, a boundary path is a set of successive half boundary edges. As there are two half boundary edges for each boundary

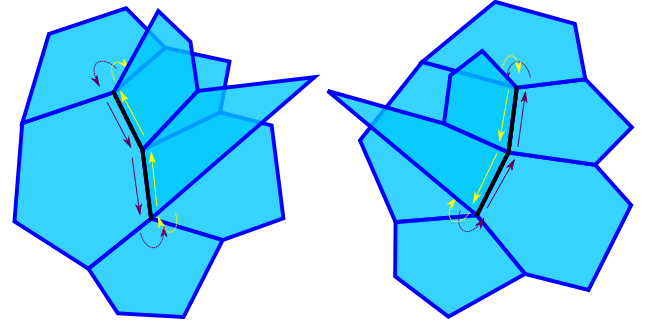


Figure 8: A path on the skeleton is a sequence of half edges. Here, we can see two sides of the same skeleton, where two paths are drawn (yellow and purple), representing the same (black) singular edges. By construction of half edges, these paths have opposite direction. Note that green and purple half paths do not lie on the same medial faces, and therefore are not on the same sheet.

edge on an medial face, a boundary path will be traced out twice by half paths, one on each side of the sheet.

In the following, we consider a specific subset of boundary and singular paths, such that all the half edges composing them are on the same sheet. In the case of the boundary path, it will help to simplify the boundary of a single sheet, and in the case of the singular path, it will separate a given sheet from other sheets.

### 3.4.1. Medial mesh and surface mesh correspondence

Given the global structures on the medial axis, we can recover the corresponding regions on the surface mesh (cf. Figure 9). Each sequence of half edges on the skeleton corresponds to a path on the surface mesh as follows: each half edge is associated to a half medial face on one side of the skeleton, and therefore determines one vertex of the surface mesh (the center of the unique Voronoi cell on which the half medial face lies). Therefore, the sequence of medial faces associated to the sequence of half edges gives us a sequence of vertices on the surface mesh. Indeed, if two medial faces are neighbors, the associated surface vertices are on the same Delaunay triangle. As they are on the same side by construction of the path, they necessarily are neighbors.

When this sequence of half edges corresponds to tracing out a half path loop along the entirety of a singular path, it generates a closed path on the surface mesh that distinguishes the surface region corresponding to the current sheet from the rest of the the shape surface.

While we primarily consider the paths on the surface mesh induced by singular skeletal paths, note that any skeletal path can be associated to a path on the surface mesh. For example, a boundary skeletal path, as it is two-sided, induces two closed paths on the surface mesh: each vertex and face between those paths is associated to a boundary skeletal node.

### 3.5. Evaluation of the significance of new sheets

Once a sheet has been computed, its singular edges are used to detect the singular paths that separate the current sheet from its



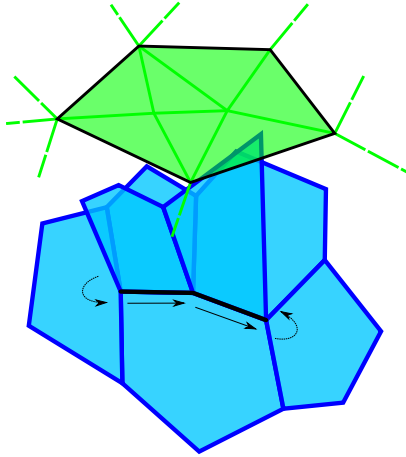


Figure 9: A (black) singular path on the skeleton associated to a (black) surface mesh path, whose vertices are the centers of the five Voronoi cells the path is crossing (as each medial face is separating two Voronoi cells). The computed surface path surrounds a surface portion for which we can evaluate an  $\epsilon$ -significance.

499 neighbor sheets. Suppose we are given a tolerance parameter  $\epsilon$ . We  
 500 only wish to compute those neighbor sheets that correspond to a  
 501 surface region, as described in 3.4.1, that has significance greater  
 502 than  $\epsilon$ , where significance is measured by Hausdorff distance to the  
 503 original surface.

504 Consider a singular path contained within a computed sheet. The  
 505 singular path corresponds both to a set of medial spheres centered  
 506 at its nodes, and also to the corresponding surface region for the  
 507 neighbor sheet whose significance we wish to determine. If the  
 508 Hausdorff distance between the vertices of the surface region and  
 509 the medial spheres is greater than  $\epsilon$ , the neighbor sheet is computed.  
 510 Otherwise, the sheet may be ignored: this means that the surface re-  
 511 gion is entirely covered by existing spheres of the skeleton, with a  
 512 radius increased by  $\epsilon$  (cf. Figure 10).

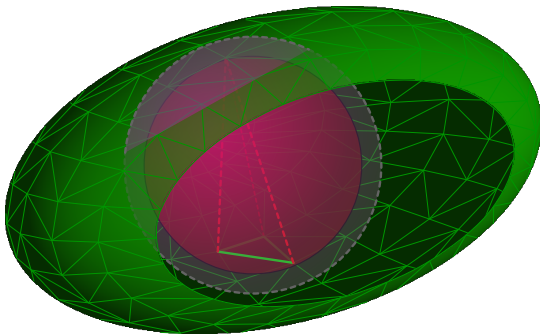


Figure 10: Illustration of the  $\epsilon$  criterion. A tetrahedron corresponds to a unique medial ball, passing through its four vertices. Adding  $\epsilon$  to the radius of the ball, we obtain a new (dashed) sphere, covering other vertices of the surface mesh.

#### 513 4. Computing a topologically coherent and simplified skeleton 514 representing an $\epsilon$ -approximation of the surface mesh

515 We now describe how to compute a simplified skeleton from an  
 516 input surface mesh. Using the structures developed in the previous  
 517 sections, we compute a provably, topologically coherent skeletal  
 518 structure sheet by sheet and check if each sheet is significant for  
 519 representing an  $\epsilon$  approximation of the original shape in terms of  
 520 the Hausdorff distance. As input, this method assumes a Delaunay  
 521 triangular mesh (cf. 4.1). The pseudocode of the algorithm is given  
 522 in Algorithm 1.

---

#### ALGORITHM 1: Skeletonization.

---

```

Input :  $\mathcal{M}$ : Delaunay mesh
Input :  $\epsilon$ : Hausdorff distance parameter
Output:  $\mathcal{S}$ : Simplified skeleton
Let  $\mathcal{M}_w$  be a working copy of  $\mathcal{M}$ 
Compute Delaunay tetrahedralization of  $\mathcal{M}_w$ 
Estimate a first medial face  $F$  of the skeleton
Compute full sheet  $\mathbf{S}$  containing  $F$ 
Let  $l_s$  be the list of degree 3 half edges on  $\mathbf{S}$ 
while  $l_s$  not empty do
   $E = \text{pop\_front}(l_s)$ 
   $E_{ext}$  = half edge neighboring  $E$ , on neighbor sheet
   $P$  = singular path containing  $E_{ext}$ , on neighbor sheet
   $C$  = set of faces on  $\mathcal{M}_w$  covered by neighbor sheet
   $d$  = maximal distance between skeletal spheres on  $P$  and mesh
    vertices on  $C$ 
  if  $d < \epsilon$  then
    Remove faces  $C$  from  $\mathcal{M}_w$ 
     $D$  = set of internal Delaunay faces closing the hole
    Add faces  $D$  to  $\mathcal{M}_w$ 
  else
    Compute neighbor sheet  $\mathbf{S}_i$ 
    Add list of degree 3 half edges on  $\mathbf{S}_i$  to  $l_s$ 
  end
end
Let  $l_b$  be the list of degree 1 half edges on skeleton
while  $l_b$  not empty do
   $E$  = steepest half edge of  $l_b$ 
   $F$  = set of medial faces containing  $E$  and  $E_n$ 
   $P$  = singular path separating  $F$  from the computed skeleton
   $C$  = set of faces on  $\mathcal{M}_w$  covered by the sheet beyond  $P$ 
   $d$  = distance between skeletal spheres on  $P$  and mesh vertices
    on  $C$ 
  if  $d < \epsilon$  then
    Remove faces  $C$  from  $\mathcal{M}_w$ 
     $D$  = set of internal Delaunay faces closing the hole
    Add faces  $D$  to  $\mathcal{M}_w$ 
  end
Let  $l_p$  be the list of problematic half edges on skeleton
while  $l_p$  not empty do
   $E = \text{pop\_front}(l_p)$ 
   $P$  = problematic path containing  $E$ 
   $F$  = set of medial faces separated by  $P$ 
  Set edges of  $P$  singular
  Assign new label to faces of  $F$ 
end

```

---

523 The algorithm proceeds through all sheets in the medial axis of  
 524 the input mesh, computing those with significance above the thresh-  
 525 old  $\epsilon$ . We summarize the pipeline here, then detail each step in the

526 paragraphs that follow. To begin, we compute a first sheet, starting  
 527 from a first **medial face** (cf. 4.2) and aggregating neighboring **me-**  
 528 **dial faces** into a complete sheet (cf. 4.3). Each singular **medial edge**  
 529 is attached to a new sheet. We compute the singular path connecting  
 530 this sheet (cf. 4.4) and determine if the significance if the sheet is  
 531 greater than  $\epsilon$  (cf. 3.5). If not, we discard the skeleton propagation  
 532 to the insignificant sheet. If the sheet is relevant, we allow the prop-  
 533 agation to compute the sheet. Repeating this operation over each  
 534 half singular path linking a sheet will produce the full skeleton of  
 535  $\epsilon$ -significant sheets. We then simplify the skeleton boundaries ac-  
 536 cording to the same  $\epsilon$  criterion, removing **medial faces** when possi-  
 537 ble (cf. 4.6). This step highly simplifies the resulting skeleton,  
 538 removing small peaks along sheet boundaries. Finally, we ensure  
 539 the sheet is manifold, removing problematic paths that create de-  
 540 generate geometry (cf. 4.7).

#### 541 4.1. Input mesh

542 The input of this algorithm is a Delaunay triangular mesh bounding  
 543 a 3D shape. Each triangle of such a mesh is on a Delaunay tetra-  
 544 hedron: thus a Delaunay tetrahedron is either inside or outside the  
 545 shape. As a consequence, we know which portion of the skeleton  
 546 is inside, and which is outside, offering a clear labeling of outside  
 547 and inside skeleton. Furthermore, because the mesh is Delaunay,  
 548 a direct connection exists between the connectivity of the skeleton  
 549 and the connectivity of the surface mesh, and between the topology  
 550 of the skeleton and the topology of the surface mesh.

551 A Delaunay triangular mesh can be built from a given manifold  
 552 surface mesh (cf. [DZM07]), using edges flipping and splitting. The  
 553 vertices of the original mesh are preserved in the Delaunay mesh,  
 554 and the skeletal structure we compute can still be applied to the  
 555 non-Delaunay version of a mesh, though the duality will not be  
 556 preserved. We also note that, since the atomic operations applied to  
 557 convert a mesh into a Delaunay mesh do not affect the topology of  
 558 the mesh, topology is preserved between original mesh and Delau-  
 559 nay mesh, and therefore maintains consistency with the skeleton.

#### 560 4.2. Computing the first medial face

561 The algorithm starts with an initial **medial face** of the skeleton. We  
 562 first choose a **node** of the skeleton. This **node** will likely belong to  
 563 multiple **medial faces**, and we select one.

564 To compute the first **node** on the skeleton, we need to find an  
 565 internal, or volumetric, **tetrahedron**. We select an arbitrary **triangle**  
 566 of the surface mesh. As the mesh itself is a subset of the Delaunay  
 567 tetrahedralization, this **triangle** is necessarily a face of a **tetrahe-**  
 568 **dron**. Once this **triangle** is chosen, two **tetrahedra** are candidates to  
 569 be our first **tetrahedron**: one interior and one exterior to the shape  
 570 volume. Using the orientation of the mesh surface, we can deter-  
 571 mine which one is interior, which gives us a first interior medial  
 572 **node**. Finally, to avoid choosing a first medial **node** that is not sig-  
 573 nificant, we detect the neighboring **node** with a maximal radius (as  
 574 each interior **tetrahedron** corresponds to a medial sphere), and it-  
 575 erate this operation so that that chosen initial node has a locally  
 576 maximal radius.

577 Finally, the first **medial face** is randomly chosen from the **medial**

578 **faces** surrounding the chosen **node**. More precisely, as an **medial**  
 579 **face** is the dual of a **volumetric edge**, we choose one **volumetric**  
 580 **edge** among the **edges** of the chosen interior **tetrahedron** which de-  
 581 termines an **medial face**.

#### 582 4.3. Single sheet propagation

583 We will assign a label to each skeletal sheet. To compute a sheet, we  
 584 start from a first **medial face**, which we label. We then add all the  
 585 neighbor **medial faces** sharing a **regular edge** in a propagation pro-  
 586 cess, and assign the same label to them. We repeat the propagation  
 587 for each added face, labeling the **medial faces** until no neighboring  
 588 unlabelled **medial faces** sharing a regular edge remain. This pro-  
 589 cess ends because the boundaries of sheets consist of **singular or**  
 590 **boundary edges** which are not regular edges, and a medial sheet for  
 591 a finite volume discrete surface mesh will have a finite number of  
 592 **medial faces**.

#### 593 4.4. Singular paths computation

594 For each **half singular edge** on the boundary of the processed me-  
 595 dial sheet, a **half singular edge** on a neighbor sheet can be identified.  
 596 From this neighbor **half singular edge**, a half singular path can be  
 597 estimated, which is associated to the neighbor sheet. (As described  
 598 in Section 3.4, a half singular path in a sheet is a set of successive  
 599 **half singular edges** belonging to the same sheet.) Recall that a full  
 600 singular path corresponds to a closed loop of half singular edges  
 601 that trace out the full path twice but in opposite directions (see e.g.  
 602 Figure 8).

We start from a **half singular edge** to form the candidate list  $l_s$   
 and identify a neighboring half singular edge (cf. Figure 11) using  
 Algorithm 2. We iterate until the loop of **half edges** closes, and  
 returns the complete singular path. We then remove every used **half**  
**edge** from the candidate list  $l_s$ .

---

**ALGORITHM 2:** Computing next half singular edge.

---

**Input** : Half singular edge  $pe$   
**Output:** Next half singular edge  $pe'$   
 $pe' = pe$   
**do**  
 Find  $pe_f$ , next half edge on same medial face than  $pe'$   
 Find  $pe_o$ , opposite half edge of  $pe_f$ , on the neighbor half  
 medial face  
 Find  $pe'$ , next half edge on same medial face than  $pe_o$   
**while**  $pe'$  is not singular;

---

608 The result of this operation is sequence of **half edges**, called a  
 609 singular path. A given singular path can be tested to identify in-  
 610 significant sheets, according to  $\epsilon$  parameter.

#### 611 4.5. Identifying an insignificant sheet beyond a singular path

612 If a singular path separates the current sheet from a sheet that has  
 613 been determined to be insignificant according to the  $\epsilon$  criterion,  
 614 we do not compute that sheet. In order to maintain the geometric  
 615 relationship between the skeleton and the mesh, we also update  
 616 the connectivity of the mesh when we do not compute a sheet (cf.  
 617 Figure 12).

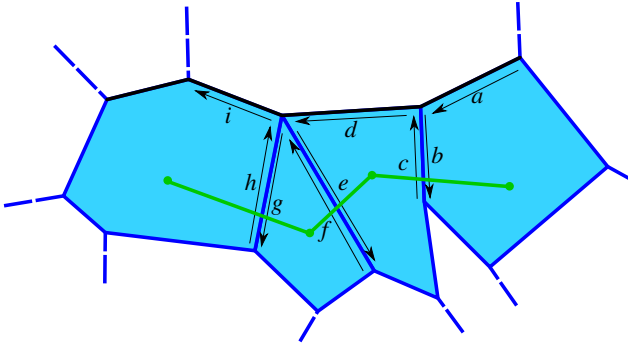


Figure 11: Identifying the next half edge to compute a singular path (here  $(a, d, i)$ ). The singular edges are in black. Two examples are shown here. From  $a$  to  $d$ , we first take the next edge on the same half medial face,  $b$ , then take the opposite edge to  $b$ ,  $c$ . Finally, we take the next edge to  $c$ , which is the singular edge  $d$ . More generally, to go from  $d$  to  $i$ , the same steps are repeated on each medial face until a singular half edge is found. Here, it is done twice: through half edges  $e$ ,  $f$  and  $g$ ,  $h$ . At this point, the singular path is composed by  $a$ ,  $d$  and  $i$ . During this process, each visited half medial face belongs to a unique Voronoi cell with associated dual (green) boundary point. Each new point is stored as the next vertex of the path. When the path closes, the result is a sequence of (green) vertices creating a closed path on the boundary.

618 The main step in not computing a sheet is removing any interior  
619 tetrahedra from the object volume. This implies removing some  
620 vertices, namely vertices strictly within the region bounded by the  
621 corresponding surface path, and remeshing this portion of the sur-  
622 face, creating a new set of edges and faces to fill the hole.

#### 623 4.6. Simplifying boundaries

624 Boundary paths for sheets are prone to noise. Using the same  $\epsilon$  sig-  
625 nificance measure, we simplify sheet boundaries by removing some  
626 medial faces. We define the notion of steepness of a consecutive  
627 pair of half boundary edges to be the angle between them. At each  
628 step of the simplification (cf. Algorithm 1), we check the steepest  
629 consecutive half edges along the boundaries. We then compute the  
630 singular path separating those medial faces from the skeleton, and  
631 use it to check whether the sheet beyond the path can be removed  
632 using the  $\epsilon$  criterion. If so, we remove the medial faces using same  
633 method as in 4.5.

#### 634 4.7. Handling problematic paths

635 A computed sheet may contain some non-manifold edges. This  
636 happens when skeletal edges are connected to three medial faces  
637 that are labeled as belonging to the same sheet, violating the ge-  
638 ometry of regular edges contained in a sheet interior. We first label  
639 these edges problematic edges, that is, any edge belonging to three  
640 different medial faces but labeled as members of the same sheet  
641 (see Figure 13). To resolve the problematic edges, we wish to create

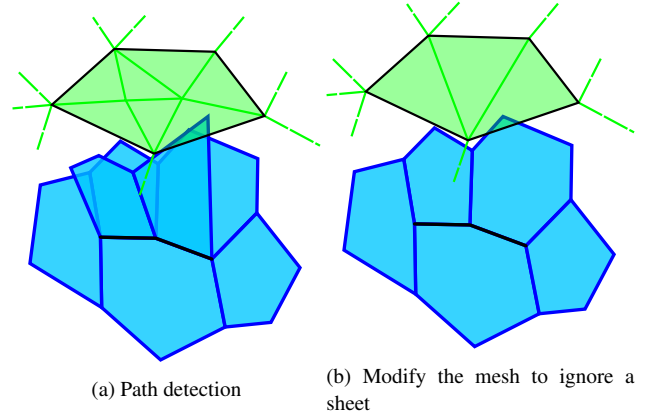


Figure 12: Singular path detection and ignoring a sheet. Here, to ignore the sheet, two edges, then three faces, are created on the mesh. The mesh vertices inside the region are removed from the mesh. To ignore the sheet, the mesh vertices within the closed boundary path are removed from the mesh and the corresponding region is re-triangulated. Note that that singular edges bordering the medial path have become regular.

642 a new sheet, distinct from the current sheet, as shown in Figure 13.  
643 To determine the boundaries of the new sheet, we resolve a prob-  
644 lematic path by joining problematic edges together with selected  
645 regular edges to border and be part of a new sheet's singular set.  
646 We then select the medial faces separated off by this path and cre-  
647 ate the new sheet by labeling them with a new label, as described  
648 in Algorithm 3. Splitting the new sheet from the original sheet re-  
649 solves the problematic edges.

---

#### ALGORITHM 3: Correcting problematic edges.

---

**Input** : Problematic half edge  $pe$   
**Output**: Set of medial faces with new label  
**Output**: Set of degree 2 edges set as singular  
 $P$  = Path of problematic half edges containing  $pe$   
 $N$  = Extremity of  $P$  non connected directly to a boundary edge  
 $S$  = Path of regular edges starting from  $N$  to boundary nodes of skeleton  
Set  $S$  edges singular  
 $F$  = Separated skeletal faces  
Assign new label to faces of  $F$

---

## 650 5. Results

651 To evaluate our methods, we apply them to 3D models from two  
652 popular databases using our implementation in Rust (<https://github.com/Ibujah/compact-skel-3d>):

- 654 • The McGill database [SZM\*08], which aggregates different  
655 classical databases, such as Princeton's, and the authors own  
656 models (<http://www.cim.mcgill.ca/~shape/benchMark/>).
- 657 • The Lavoué database [LCSL18] (<https://perso.liris.cnrs.fr/guillaume.lavoue/data/saliency/>).

659 We require manifold meshes, and so all non-manifold mesh are dis-  
660 carded. Moreover, our algorithm assumes that we have Delaunay

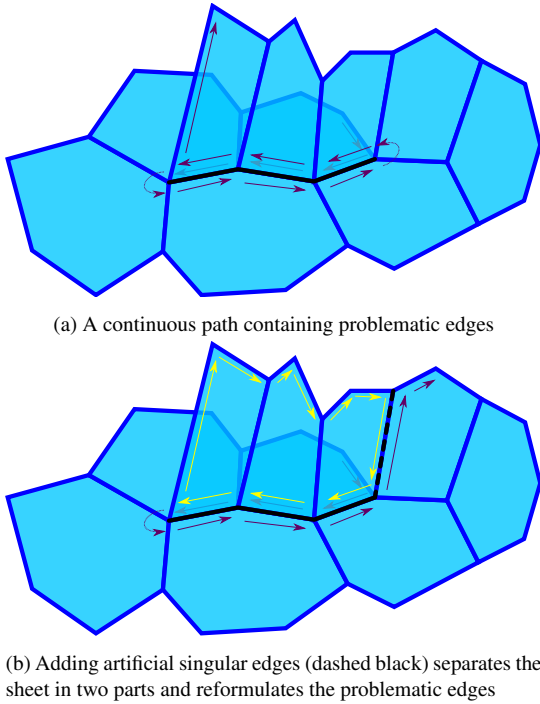


Figure 13: An example of problematic edges, and their resolution. A problematic edge belongs to three medial faces belonging to the same sheet. Note that in (a), all medial faces belong to the same sheet even though there are some faces connected by singular edges. In (a), the black edges are problematic because they are not cleanly separating two sheets. This situation cannot be characterized locally, but requires the full computation of the sheet to detect. Our solution is to relabel appropriate regular edges as singular edges (new edge in dashed black in the bottom figure) in order to split the affected medial faces into two separate sheets. In (b), the medial faces surrounded by the yellow edges now belong to a separate sheet. This means that singular edges (black edges in the bottom figure) now belong to at most two medial faces belonging to the same sheet.

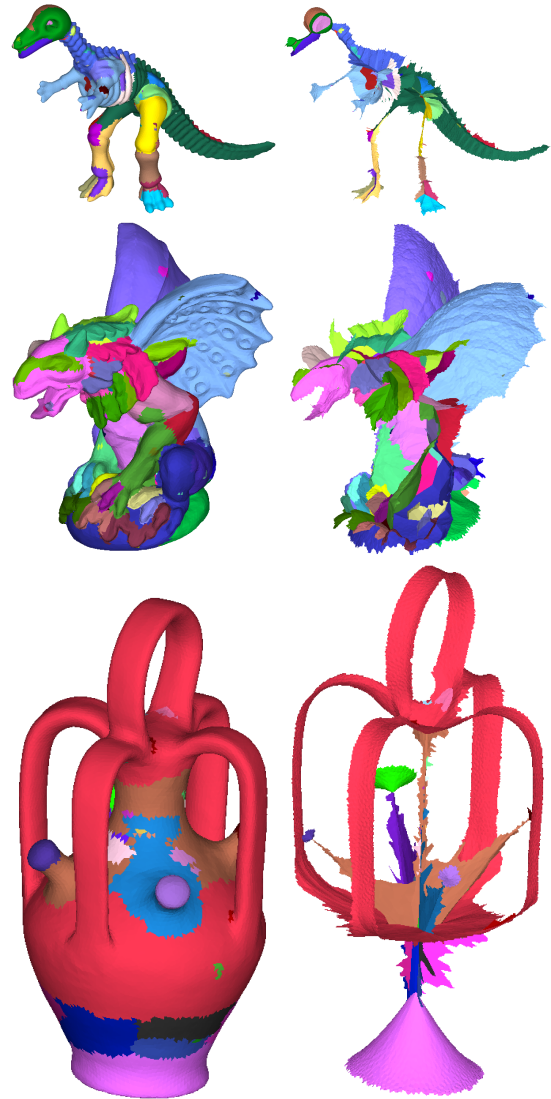


Figure 14: Skeletons computed with our algorithm, with  $\epsilon = 0.01$ . To highlight the link between skeletal sheets and mesh portions, we use the same color for a skeletal sheet (right) and its associated mesh portion (left).

661 meshes. Some meshes of these two databases are not Delaunay,  
 662 and so we apply Dyer’s algorithm [DZM07] to convert them to De-  
 663 launay.

664 In total, we have tested our algorithm on 411 models (in sup-  
 665 plementary material), with meshes with between 3000 and 30,000  
 666 vertices in the McGill database, and around 20,000 vertices in the  
 667 Lavoué database. Skeletonization takes in average less than one  
 668 minute per mesh, depending on the complexity of the given model.

669 Some examples are shown on Figure 14. For these examples,  
 670 we set  $\epsilon$  as 1% of the bounding box diagonal. Our algorithm can  
 671 accept Delaunay meshes of arbitrary genus. As shown in Figure  
 672 14, our algorithm separates the skeleton into different sheets which  
 673 are highlighted (right side of figure), and these sheets correspond  
 674 to surface mesh regions (left side of figure). The structure of the  
 675 skeleton is then easily accessible and usable.

676 In Figure 18, we compare our approach to Voxel Core [YLJ18]

677 and Scale Axis Transform (SAT) [MGP10]. Our method consis-  
 678 tently retains salient shape features across different types of shape  
 679 configurations, unlike the other two, while maintaining a similar  
 680 level of cleanliness. For example, SAT erodes important regions of  
 681 the body of the bunny. Similarly, Voxel Core erodes important re-  
 682 gions of the tool (bottom left) and the head and the wings of the  
 683 dragon, as well as legs of the chair. This is because our method  
 684 explicitly preserves  $\epsilon$ -fidelity with the input surface vertices while  
 685 simultaneously seeking geometric coherence. In other words, we  
 686 have the additional advantage that we maintain geometric consis-  
 687 tency with the surface mesh.

688 In Figure 15, we show skeletons of the same object computed  
 689 with different values of  $\epsilon$ . As  $\epsilon$  increases from 0.00, to 0.01, and

690 to 0.05, successive sheets are removed, but we can see a larger dif-  
 691 ference between 0.00 and 0.01 than between 0.01 and 0.05. This  
 692 may suggest that each input shape has an intrinsically appropriate  
 693  $\epsilon$  value.

694 Figure 16 shows skeleton computed with  $\epsilon = 0$ . These skeletons  
 695 have many spurious peaks.

696 Figure 17 shows effects of noise in the input surface mesh. We  
 697 observe that even if more skeletal peaks are present on the thin  
 698 parts, the global skeleton stays clean.

## 699 6. Conclusion

700 In this article, we present a 3D skeletonization approach to produce  
 701 a simplified and geometrically consistent medial axis. Our algo-  
 702 rithm identifies the primary structures in the medial skeleton, which  
 703 are medial sheets and curves, and their boundaries, which are singu-  
 704 lar or boundary structures. These structures are topologically valid,  
 705 manifold, and cleanly separated, thereby maintaining the Whitney  
 706 stratified structure of the Blum medial axis in the discrete set-  
 707 ting. Furthermore, we maintain the relationship between the sur-  
 708 face mesh and the medial axis by associating **nodes**, **edges**, and  
 709 **medial faces** of the skeleton with **tetrahedra**, **triangles** and **edges**  
 710 of the Delaunay structure generated by the surface mesh **points**.

711 Using these medial structures, we compute a simplified skeleton,  
 712 excluding insignificant sheets. Using the geometric links between  
 713 the mesh and the skeleton, we evaluate the relevance of a skeletal  
 714 sheet by evaluating the deviation from the surface mesh that would  
 715 result from removing the surface region associated to that sheet. As  
 716 a result, we do not compute insignificant sheets for which the Haus-  
 717 dorff distance of the points in the associated region of the surface  
 718 mesh to the remaining surface points is lower than an  $\epsilon$  threshold.

719 Future work might explore the relationship between the noise  
 720 level of the surface mesh and optimal  $\epsilon$  values for several distribu-  
 721 tions of noise. Additionally, exploring preprocessing approaches to  
 722 allow for more general formats for surface data beyond Delaunay  
 723 meshes would be useful.

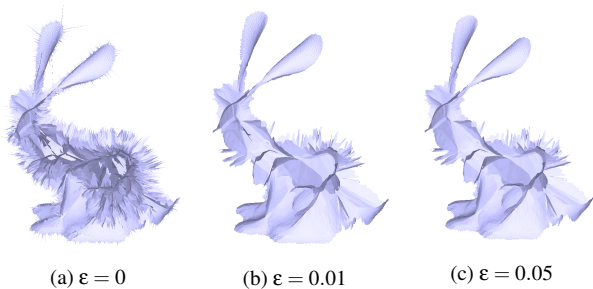


Figure 15: Comparison of the same skeleton with different  $\epsilon$  values. Between 15a and 15b, we can see that most of the noisy peaks were removed. Few sheets were removed between 15b and 15c.

## 724 Appendix A: Irregularities of the discrete skeleton

725 Generating the medial mesh from the Voronoi/Delaunay structures of  
 726 the surface mesh, even a nice one, can produce geometric irreg-

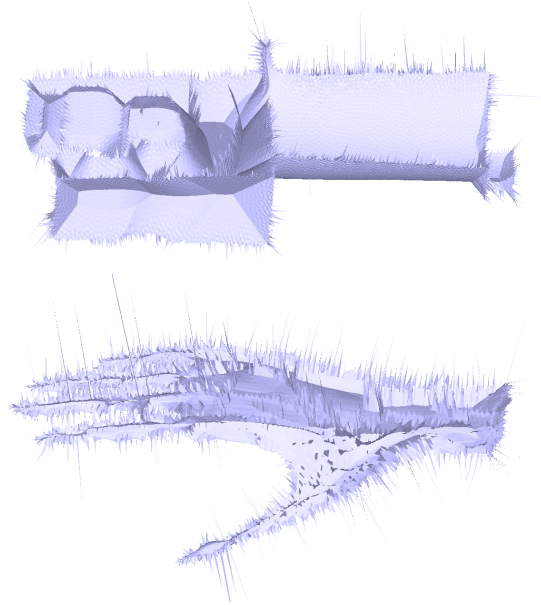


Figure 16: Computed skeleton without simplification ( $\epsilon = 0$ ). Unwanted peaks appear, including peaks going outside of the mesh.

727 ularities. These can be classified into two different categories: nu-  
 728 merical problems, which come from the fact we manipulate floating  
 729 point numbers instead of real numbers, and discrete skeleton prob-  
 730 lems, which are problems arising from particular configurations of  
 731 the surface points, potentially breaking the properties of continuous  
 732 skeletons.

- 733 • There can be more than 4 mesh points on a particular sphere, and  
 734 more than 3 points on a Delaunay face. Tetrahedra and triangles  
 735 are the most frequent type of Delaunay polyhedra, but not the  
 736 only one. The Delaunay library we use, Tetgen [Si15] computes  
 737 only standard tetrahedra, which avoids the problem, but this can  
 738 result in flat (very low volume) tetrahedra.
- 739 • More than three sheets can share the same singular set, which  
 740 means singular paths can lie on more than three sheets. There-  
 741 fore, when separating two sheets, several half singular paths can  
 742 be computed. When we test if a new sheet has to be computed,  
 743 we compute the half singular path which lies on the new sheet  
 744 (as explained in 4.4).
- 745 • Though it is rare, some “sheets” may be one-dimensional in cer-  
 746 tain places or in their entirety (for example, an arm that is a long  
 747 circular cylinder will have a curve as its skeleton). We handle  
 748 these cases separately: instead of propagating medial faces, we  
 749 compute successive degree 0 edges.
- 750 • Two singular paths can cross each other: this situation is not  
 751 generic behavior in the continuous setting, but can happen in the  
 752 discrete setting.
- 753 • Some medial nodes can be associated to a sphere outside the  
 754 mesh (*cf.* Figure 16). This is because, for some interior boundary  
 755 tetrahedra, the associated sphere center will be outside of the  
 756 mesh. This can happen due to overly coarse surface sampling or  
 757 to flat tetrahedra. Our algorithm detects and removes these nodes  
 758 by removing the associated tetrahedra.

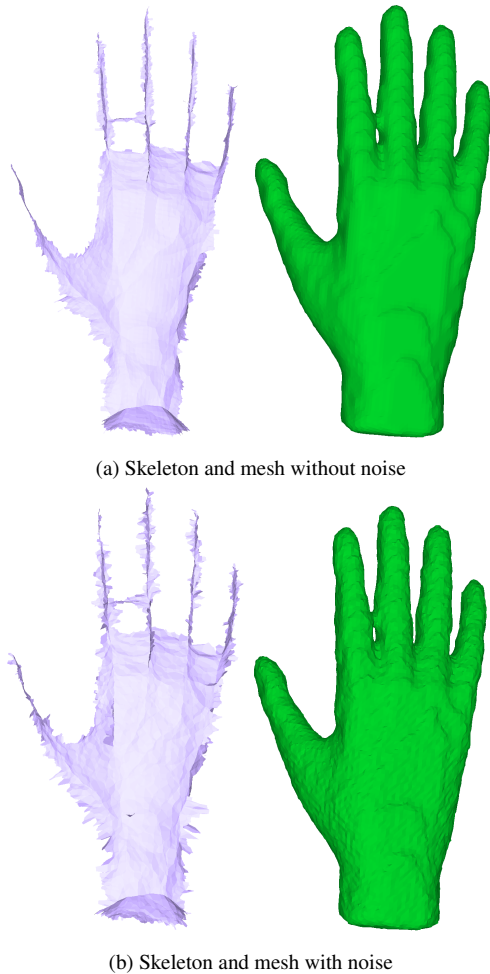


Figure 17: Effect of the mesh noise on the skeleton ( $\epsilon = 0.01$ ).

- As shown on Figure 13, some edges can be problematic, as they separate three medial faces from the same sheet. This problem, specific to the discrete case, is addressed in Section 4.7.

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Propagated skeleton	
Voxel core	
SAT	
Propagated skeleton	
Voxel core	
Scale-axis-transform	

Figure 18: Comparison between our approach ( $\epsilon = 0.01$ ), voxelcore ( $\lambda = 0.04$ ) and scale-axis-transform ( $s = 1.1$ ). Because of the resolution of the image it seems that the skeleton disconnects but the topology of the skeleton is kept in all approaches.



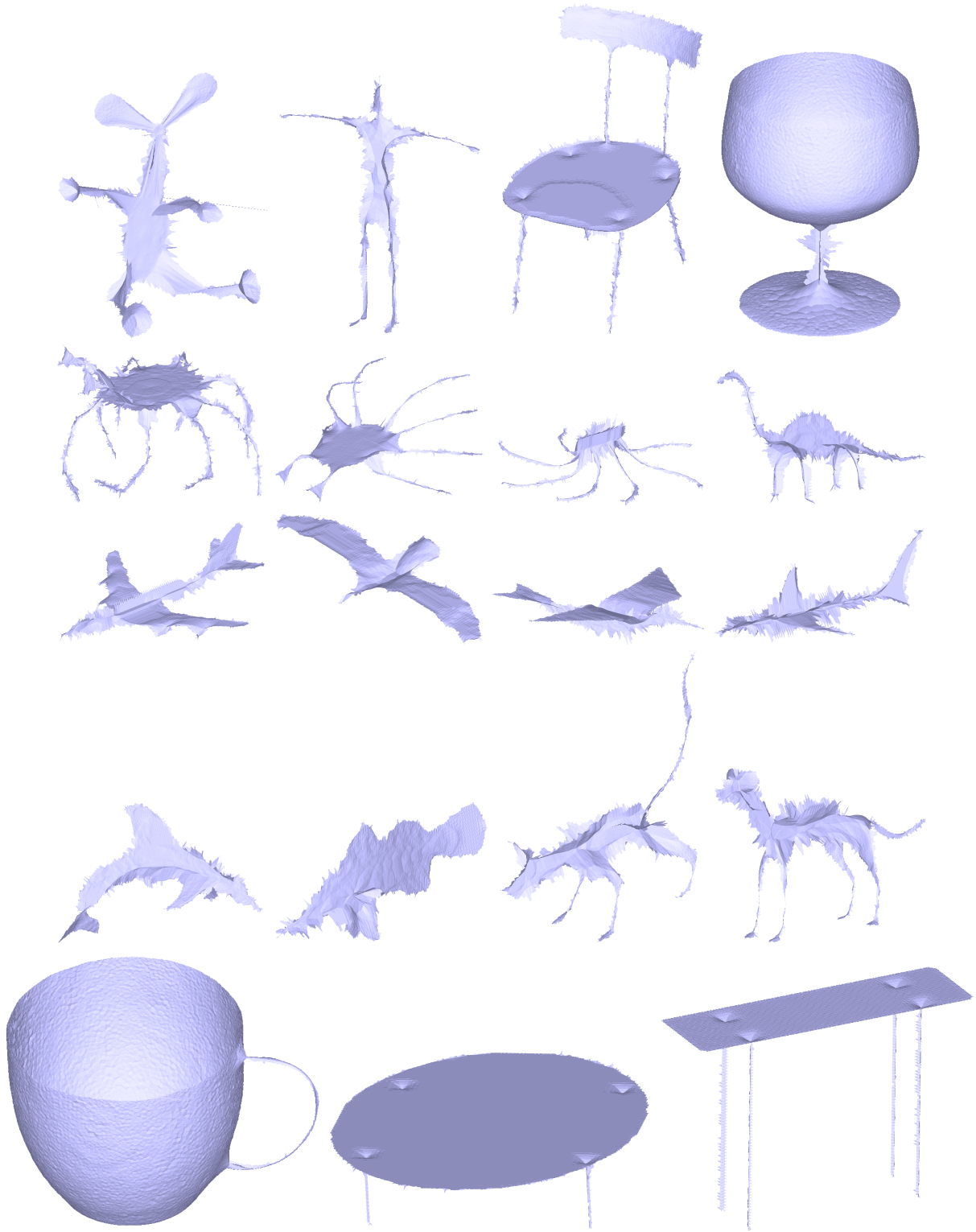


Figure 19: Some skeletons computed by our method ( $\epsilon = 0.01$ ).