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The minimal monomial lifting of cluster algebras I: branching problems

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Abstract

Let Y be a complex scheme with cluster structure, T be a complex torus and \mathfrak{X} be a suitable partial compactification of $T \times Y$. We produce, through a technique called minimal monomial lifting, a canonically graded upper cluster algebra $\overline{\mathcal{A}}$ inside $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, which is, in a precise sense, the best candidate to give a cluster structure on \mathfrak{X} compatible with the one on Y . We develop some geometric criteria to prove the equality between $\overline{\mathcal{A}}$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, which doesn't always hold and has some remarkable consequences. As a special case, through this construction, we recover in a purely geometric and combinatorial method some cluster algebras already appearing in the literature.

If \widehat{G} is a complex reductive subgroup of the complex, semisimple, simply connected algebraic group G , we set up a geometric framework to study the branching problem in representation theory, for this pair of groups, through the minimal monomial lifting. We study in detail the case of a Levi subgroup and the tensor product case, for which we have the best possible outcome. Finally, we identify a cluster structure on the base affine space associated to G and discuss some consequences of these results.

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1 Introduction

This is the first step in a project which aims to study branching problems in representation theory, through the use of cluster algebras. The aims for this project are multiple, among them we have:

1. Provide a possibly uniform framework to study these problems.
2. Reveal hidden features of this classical question through the lens of cluster theory.
3. Try to build polyhedral models for multiplicities using cluster algebra theory.

The branching problem

Let \widehat{G} be a complex, connected, reductive algebraic subgroup of the connected reductive group G . The branching problem in representation theory asks to understand how, irreducible representations of G , decompose under the natural \widehat{G} -action.

Fix maximal tori $\widehat{T} \subseteq T$ and Borel subgroups $B \supseteq T$ and $\widehat{B} \supseteq \widehat{T}$ of G and \widehat{G} respectively. Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. For $\lambda \in X(T)^+$, $V(\lambda)$ denotes the irreducible representation of highest weight λ . Similarly, we use the notation $X(\widehat{T})$, $X(\widehat{T})^+$, $V(\widehat{\lambda})$ relatively to \widehat{G} . For any $\lambda \in X(T)^+$, there is a natural \widehat{G} -equivariant isomorphism:

$$\bigoplus_{\widehat{\lambda} \in X(\widehat{T})^+} \text{Hom}(V(\widehat{\lambda}), V(\lambda))^{\widehat{G}} \otimes V(\widehat{\lambda}) \longrightarrow V(\lambda)$$

$$f \otimes v \longmapsto f(v).$$

Hence the *multiplicity* of $V(\widehat{\lambda})$ in $V(\lambda)$ is $c_{\lambda}^{\widehat{\lambda}} = \dim \text{Hom}(V(\widehat{\lambda}), V(\lambda))^G$. A result one would like to achieve, in solving the branching problem is the following.

Objective: for the pair (G, \widehat{G}) , construct a positive combinatorial model for the multiplicities.

Example of such models are given by the Littlewood-Ricardson rule, Gelfand-Tsetlin's patterns, Littelmann's paths [Lit95] or Sundaram's dominos [Sun90].

Though, for some specific pairs (G, \widehat{G}) such a model exists, no general technique for constructing those models, uniformly for some families of pairs, is known. To the author best knowledge, the only remarkable exceptions, for which a positive model for multiplicities can be constructed in families, are the following.

1. Weight spaces: \widehat{G} is a maximal torus of G .
2. The Levi case: \widehat{G} is a Levi subgroup of G .
3. The tensor product case: \widehat{G} is diagonally embedded in $G = \widehat{G} \times \widehat{G}$, for which the branching problem amounts to decompose the tensor product of irreducible representations of \widehat{G} .

Note that the first family is a subfamily of the second one. A breakthrough has been realised by Berenstein and Zelevinsky [BZ01], who constructed, under the assumption that G is semisimple and simply connected, (many) polyhedral models for multiplicities, for any branching problem belonging to the previous list. Their proof is actually quite involved and uses deep results and objects, such as: the canonical basis and the Lusztig's parametrisations [Lus10], its relation to total positivity and double Bruhat cells [Lus94] [FZ99] and tropicalisation. We refer to [Zel02] for a beautiful survey on the proof. One key ingredient to link all these objects, are the determinantal identities [FZ99][Theorem 1.16,1.17], which also stand at the very base of all the known cluster algebra structures on the ring of varieties related to G , such as double Bruhat cells [BFZ05] and T -stable subgroups of the unipotent radical of B [GLS11], [GY21]. From this perspective, it's no surprise that cluster algebras have been defined by Fomin and Zelevinsky [FZ02] to study the combinatorial properties of the dual canonical basis.

A more evident link between cluster algebras and branching problems has been recently realised by Gross, Hacking, Keel, Kontsevich [GHKK18][Corollary 0.20, 0.21]: they constructed (many) polyhedral models for the weight space decomposition and for the tensor product decomposition of SL_n , using cluster algebras, and recovered some well known models. Fei obtained the same results [Fei21][Theorem 8.1, 9.2] for any simple, simply laced and simply connected algebraic group.

Why cluster algebras?

Starting from a combinatorial data called seed, usually denoted by t , and an iterative procedure called mutation, one can define the *cluster algebra* $\mathcal{A}(t)$ and the *upper cluster algebra* $\overline{\mathcal{A}}(t)$. In general one has $\mathcal{A}(t) \subseteq \overline{\mathcal{A}}(t)$. The cluster algebra is more combinatorial in nature, while the upper cluster algebra is more geometric. In particular, though $\overline{\mathcal{A}}(t)$ may not be noetherian [GHK15], [Spe13], it may be interpreted as the ring of global sections of a locally of finite type smooth scheme. Thanks to a huge effort due to many authors, there's nowadays a quite well developed structural theory of cluster algebras.

To explain some features of this theory, we recall that part of the defining data of a seed t is the vertex set I , which has a partition $I = I_{uf} \sqcup I_f$ into un-frozen (or mutable) and frozen vertices. Each seed also possesses a distinguished set of variables $x = (x_i)_{i \in I}$, indexed by the vertex set, which are elements of the cluster algebra $\mathcal{A}(t)$. Defining the degree of the cluster variables, under some constraints arising from the mutation process, defines a global graduation on both $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$ to which we refer as a *cluster graduation*. There's a well developed theory for constructing special bases, called *good bases*, of $\overline{\mathcal{A}}(t)$ [Dup11], [QIN17], [GHKK18], [Qin22]. If we allow some frozen variables not to be invertible in $\overline{\mathcal{A}}(t)$, some results are still conjectural. Nevertheless, under some technical assumption on the seed t good bases exist and have many parametrisations by rational points into polyhedral cones.

We say that a scheme \mathfrak{X} has cluster structure if $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) = \overline{\mathcal{A}}(t)$ for a certain seed t . Our strategy to study branching problems through cluster algebras is the following.

The strategy

Step 1: Define a $T \times \widehat{T}$ -scheme $\mathfrak{X} = \mathfrak{X}(G, \widehat{G})$ such that for any $(\lambda, \widehat{\lambda}) \in X(T) \times X(\widehat{T})$

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})_{\lambda, \widehat{\lambda}} = \begin{cases} \text{Hom}(V(\widehat{\lambda}), V(\lambda))^G & \text{if } (\lambda, \widehat{\lambda}) \in X(T)^+ \times X(\widehat{T})^+ \\ 0 & \text{otherwise} \end{cases}$$

Here $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})_{\lambda, \widehat{\lambda}}$ denotes the homogeneous component, of weight $(\lambda, \widehat{\lambda})$, for the $T \times \widehat{T}$ action.

Step 2: Identify a cluster structure on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.

Step 3: Interpret the $X(T) \times X(\widehat{T})$ graduation on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ as a cluster graduation.

Step 4: Prove the existence of good bases, for the upper cluster algebra $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, and derive polyhedral models for multiplicities.

Step 1 is rather easy and exploits the algebraic Peter-Weyl theorem. To achieve Step 2 and Step 3, we want to use a new construction called minimal monomial lifting and the results of [GLS11], [GY21]. For step 4, we want to use the results of [Qin22] and some ideas of [GHKK18].

In the present paper, we want to describe the minimal monomial lifting technique, perform step 1 and explain how one could perform step 2 for a given pair (G, \widehat{G}) . Moreover, we perform step 2 and half of step 3 for the three families of branching problems described above.

For some of these families, the second half of step 3 and step 4 are achieved. This is part of an ongoing work.

Outline of the paper and main results

Section 2 contains the preliminaries on cluster algebras. We formalise two notions already existing in the literature: the difference between *highly-frozen* and *semi-frozen* vertices and define the cluster valuation associated to a frozen vertex. The cluster variable associated to a highly-frozen vertex is not invertible, while the one associated to a semi-frozen one is. Then, we recall the notion of graded seed and degree configuration, which is the combinatorial datum defining a cluster graduation.

In Section 3, we introduce the fundamental notion of *lifting configuration*. Let t be a seed, with vertex set I , and D be a finite set. By little abuse, a lifting configuration ν , on t , is a matrix $\nu \in \mathbb{Z}^{D \times I}$. Given a lifting configuration ν , on t , we introduce a seed called *monomial lifting* of t associated to ν , which is denoted by $\uparrow t$. The set D is a subset of the semi-frozen vertex set of $\uparrow t$. For every vertex $d \in D$, the associated cluster variable is denoted by $\uparrow x_d$. We introduce the notion of *mutation of lifting configurations* and we prove the following properties.

Theorem 1.0.1. *Let $\nu \in \mathbb{Z}^{D \times I}$ be a lifting configuration on t .*

1. *Lifting a seed commutes with mutation.*
2. *For every seed t' mutation equivalent to t , through the mutation process, we can associate a well defined lifting configuration ν' on t' .*
3. $\mathcal{A}(\uparrow t) = \mathcal{A}(t)[\uparrow x_d^{\pm 1}]_d$.
4. *If t is of maximal rank, $\overline{\mathcal{A}}(\uparrow t) = \overline{\mathcal{A}}(t)[\uparrow x_d^{\pm 1}]_d$.*

Moreover, if the seed t is graded by a degree configuration σ , we produce a degree configuration $\uparrow \sigma$, on $\uparrow t$, called the *lifting* of the degree configuration σ . Since every seed t is trivially graded, the cluster algebra $\overline{\mathcal{A}}(\uparrow t)$ is canonically \mathbb{Z}^D -graded by the lifting of the trivial degree configuration on t , which is denoted by $\uparrow 0$. By Theorem 1.0.1, if Y is a scheme with cluster structure, under mild assumptions on Y , any lifting configuration on any seed of the cluster structure of Y allows to produce a cluster structure of $(\mathbb{C}^*)^D \times Y$. Section 3.1 is devoted to understand geometrically the canonical \mathbb{Z}^D -graduation on $\overline{\mathcal{A}}(\uparrow t)$ and the kind of graduations we could hope to understand, using the minimal monomial lifting technique.

About the minimal monomial lifting. In Section 4 we give the definition of schemes \mathfrak{X} suitable for lifting. Such a scheme contains an open subset of the form $T \times Y$ and functions on this open subset can be "homogenised" to global regular functions on \mathfrak{X} . If $\mathcal{O}_Y(Y) = \overline{\mathcal{A}}(t)$, then, we can identify geometrically a special lifting configuration ν , on t , called the *minimal lifting configuration*. The seed $\uparrow t^D$, obtained from $\uparrow t$ by changing the nature of the vertices of D from semi-frozen to highly-frozen, is called *minimal monomial lifting* of t , with respect to \mathfrak{X} . The seed $\uparrow t^D$ is the best possible candidate, to give \mathfrak{X} a cluster structure compatible with the one on Y , in the sense of Theorem 4.0.10.

Theorem 1.0.2. *We have that*

$$\overline{\mathcal{A}}(\uparrow t^D) \subseteq \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \quad \text{and} \quad \overline{\mathcal{A}}(\uparrow t^D)_{\prod_d \uparrow x_d} = \overline{\mathcal{A}}(\uparrow t) = \mathcal{O}_{\mathfrak{X}}(T \times Y).$$

Moreover, if a seed \tilde{t} is compatible with t in the sense of Theorem 4.0.10, and $\overline{\mathcal{A}}(\tilde{t}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, then $\tilde{t} = \uparrow t^D$.

In the last part of Section 4, we develop some elementary geometric criteria to study whether $\overline{\mathcal{A}}(\uparrow t^D)$ equals $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, since the inclusion between these two algebras may be strict. Ultimately, the difference between $\overline{\mathcal{A}}(\uparrow t^D)$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, depends on the behaviour of $\uparrow t^D$ along the divisors of \mathfrak{X} in the complement of $T \times Y$. We deduce some consequences of such an equality, that suggest that it is reasonable to construct a cluster structure on a suitable for lifting scheme \mathfrak{X} , through the minimal monomial lifting, if the T -action on $T \times Y$ extends to \mathfrak{X} . This is a necessary condition, for

the equality between $\overline{\mathcal{A}}(1t^D)$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ to hold, if \mathfrak{X} is affine. Moreover, under the existence of such an action, the algebra $\overline{\mathcal{A}}(1t^D)$, which is canonically graded via the lifting of the trivial degree configuration 10 , is a graded subalgebra of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, which is $X(T)$ -graded by the T -action.

Section 5 is devoted to some preliminaries on algebraic groups. We study in detail some technical properties of generalised minors, that are crucial for the proof of Theorem 1.0.4. The reader can skip this part at first. In Section 6 we consider an example of minimal monomial lifting where the inclusion $\overline{\mathcal{A}}(1t^D) \subseteq \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is strict. We report this toy example since we consider it instructive.

In Section 7, we define the branching scheme $\mathfrak{X}(G, \widehat{G})$, for the pair $\widehat{G} \subseteq G$, where G is semisimple and \widehat{G} is reductive. The scheme $\mathfrak{X}(G, \widehat{G})$ realizes step 1 of our strategy and we prove that it is suitable for lifting. As a consequence of the results of Section 3.1, we derive the following geometric realization of the multiplicities spaces of the branching from G to \widehat{G} .

Proposition 1.0.3. *For any $(\lambda, \widehat{\lambda}) \in X(T)^+ \times X(\widehat{T})^+$, we have an isomorphism*

$$\mathrm{Hom}(V(\lambda), V(\widehat{\lambda}))^{\widehat{G}} \longrightarrow \mathbb{C}[U]_{\lambda, \widehat{\lambda} - \rho(\lambda)}^{\widehat{U}}$$

where

$$\mathbb{C}[U]_{\lambda, \widehat{\lambda} - \rho(\lambda)}^{\widehat{U}} = \{f \in \mathbb{C}[U]^{\widehat{U}} : \begin{array}{ll} f(\widehat{h}^{-1}u\widehat{h}) = (\widehat{\lambda} - \lambda)(\widehat{h})f(u) & \text{for } \widehat{h} \in \widehat{T}, u \in U, \text{ and} \\ \mathcal{V}_{\alpha}(1 \otimes f) \geq -\langle \lambda, \alpha^{\vee} \rangle & \text{for } \alpha \in D \end{array}\}.$$

Actually, the previous proposition can be interpreted as a generalisation of [Zel99][Proposition 1.4], which is the starting point for the interest in perfect bases. Perfect bases, which include the classical limit of the dual canonical basis, have been arguably one of the most used items to study tensor product decomposition. See [Kam22] for a beautiful survey on perfect bases. Finally, we prove the main result of the paper.

Theorem 1.0.4. *In the following two cases:*

1. G is simple, simply connected and \widehat{G} is a Levi subgroup of G .
2. \widehat{G} is semisimple, simply connected and diagonally embedded in $G = \widehat{G} \times \widehat{G}$.

Then, if $\mathfrak{X} = \mathfrak{X}(G, \widehat{G})$, there exists a seed t constructed in [GY21], and an isomorphism $\mathbb{Z}^D \simeq X(T)$, such that $\overline{\mathcal{A}}(1t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ as graded algebras.

The construction of cluster structures for branching to a Levi subgroup is, to my best knowledge, new. For the tensor product decomposition it is mostly new, except for the following few exceptions. In [GHKK18], the authors construct a cluster structure on a variation of the branching scheme for tensor product decomposition of $\mathrm{SL}_n(\mathbb{C})$. In [Fei21], the author constructs a cluster structure on the branching scheme for tensor product decomposition assuming that the group is simple, simply laced and simply connected. In Section 8.2 we prove that Fei's structures are obtained through minimal monomial lifting. It's probable that Fei's structures agree with the ones of the present text, but I'm not capable of doing this comparison in detail. Finally, Example 8.2.14 and Theorem 1.0.4 partially answer to [Fei21][Conjecture 8.2] in the G_2 -case.

In Theorem 1.0.4, $\overline{\mathcal{A}}(1t^D)$ is \mathbb{Z}^D -graded via the lifting of the trivial degree configuration on t , while $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is $X(T)$ -graded by restriction of the $X(T) \times X(\widehat{T})$ -graduation. That is, if $\lambda \in X(T)$,

$\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})_{\lambda, \bullet} = \bigoplus_{\widehat{\lambda} \in X(\widehat{T})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})_{\lambda, \widehat{\lambda}}$. Roughly speaking, the canonical cluster graduation on $\overline{\mathcal{A}}(1t^D)$ detects, for a single irreducible representation of G , into how many \widehat{G} -irreducible representations it decomposes. Still, it doesn't distinguish different \widehat{G} -modules. Incorporating this information into a cluster graduation becomes a purely combinatorial problem, that could be achieved lifting a certain conjectural degree configuration on the seeds constructed in [GY21]. See Question 5.3.4. As a final remark, we are also interested in Questions 7.4.1 .

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2 Preliminaries on cluster algebras

2.1 Cluster algebras

We give a definition of cluster algebras and upper cluster algebras of geometric type in the spirit of [GLS13]. The only differences with the standard setting are encoded in the notions of *semi-frozen* (invertible) and *highly-frozen* (not invertible) variables. The type of frozen variables affects the definition of the coefficient ring of the cluster algebra. This choice provides a natural framework for studying functions over some partial compactifications of cluster varieties. Note that our cluster algebras are very special cases of the ones defined in [BMS19] and of *generalized cluster algebras* [GSV18].

Definition 2.1.1 (seed). Let \mathbb{K} be a field extension of \mathbb{C} . A seed t of \mathbb{K} is a collection

$$(I_{uf}, I_{sf}, I_{hf}, B, x)$$

consisting of:

- Three disjoint finite sets I_{uf}, I_{sf}, I_{hf} . An element of $I := I_{uf} \sqcup I_{sf} \sqcup I_{hf}$ (resp. $I_f := I_{hf} \sqcup I_{sf}$) is called *vertex* (resp. *frozen vertex*). A vertex is respectively called *unfrozen* (or *mutable*), *semi-frozen*, *highly frozen* if it belongs to I_{uf}, I_{sf}, I_{hf} .
- The *extended exchange matrix* $B \in \mathbb{Z}^{I \times I_{uf}}$, which is an integer matrix, such that its *principal part* $B^\circ := B|_{I_{uf} \times I_{uf}}$ is *skew-symmetrizable*, meaning that there exists $d_i \in \mathbb{N}_{>0}$, for $i \in I_{uf}$, such that for any $i, j \in I_{uf}$, $d_i b_{i,j} = -d_j b_{j,i}$.
- An *extended cluster* $x \in (\mathbb{K}^*)^I$, consisting of a transcendence basis of \mathbb{K} over \mathbb{C} , whose elements are called *cluster variables*. A variable is said to be *unfrozen* (or *mutable*) (resp. *semi-frozen*, resp. *highly-frozen*, resp. *frozen*) if its corresponding vertex is unfrozen (resp. semi-frozen, resp. highly frozen, resp. frozen).

If a seed is denoted by t , we implicitly assume that its defining data are denoted as in the previous definition. If the seed is called t^\bullet (resp. t_\bullet), where \bullet is any superscript (resp. subscript), then we add a \bullet superscript (resp. subscript) to all the notation. For example, we write $t' = (I'_{uf}, I'_{sf}, I'_{hf}, B', x')$ or $t_i = (I_{i,uf}, \dots, x_i)$. If needed, we add the dependence on t writing $B(t)$, $x(t)$ and so on.

Graphical notation for seeds: given a seed t , the only data required to identify the isomorphism class of the associated cluster and upper cluster algebra is the vertex set and the extended exchange matrix. This is encoded graphically in a *valued quiver* Q (or $Q(t)$ if the dependence on t is needed) as follows.

- The vertex set of Q is I . For $i \in I$, the corresponding vertex of Q is pictured by a symbol \circ (resp. \square , resp. \blacksquare) if it is unfrozen (resp. semi-frozen, resp. highly frozen) and labelled by i .

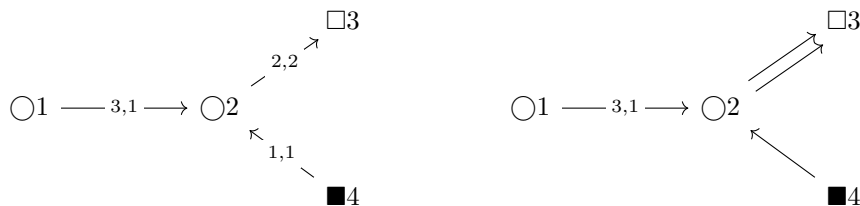
Given two vertices i and j of Q , we use the convention that $b_{i,j} = 0$ if i, j are both frozen and that $b_{i,j} = -b_{j,i}$ if exactly one between i and j is frozen. Note that, in the last case, exactly one between $b_{i,j}$ and $b_{j,i}$ is defined as a coefficient of the generalised exchange matrix, accordingly to which vertex is unfrozen. Then

- There is an arrow between i and j , pointing towards j , if and only if $b_{i,j} > 0$. In this case, it is labelled by: " $b_{i,j}, -b_{j,i}$ ". Moreover, if $b_{i,j} = -b_{j,i}$, for low values of $b_{i,j}$, we may write $b_{i,j}$ distinct arrows from i to j instead of a labelled arrow.

Example 2.1.2. Let $t = (\{1, 2\}, \{3\}, \{4\}, B, x)$ be a seed whose generalised exchange matrix is

$$B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \\ 0 & -2 \\ 0 & 1 \end{pmatrix}$$

where the rows and columns of B are labelled in the obvious way. The following two pictures are both a graphical representation of the seed t



General notation: If $b \in \mathbb{R}$ then

$$b^+ := \max\{b, 0\} \quad b^- := \max\{-b, 0\} \quad \text{so that} \quad b = b^+ - b^-.$$

If J, K are finite sets and $M \in \mathbb{R}^{J \times K}$, then $M^\pm \in \mathbb{R}_{\geq 0}^{J \times K}$ is the matrix obtained applying component-wise to M the corresponding operation. If $k \in K$, the k -th column of M is $M_{\bullet, k} = M_{|J \times \{k\}} \in \mathbb{R}^J$. Similarly, if $M \in \mathbb{R}^{J \times K}$ and $J_1 \subseteq J, K_1 \subseteq K$, then $M_{J_1 \times K_1} = M_{|J_1 \times K_1}$ and $M_{\bullet, K_1} = M_{J \times K_1}$.

If S is any set and we consider S -valued matrices of size $J \times K$, that is elements of $S^{J \times K}$, then we use analogue notations for restrictions.

Let H be an abelian group and $h \in H^J$. If $m \in \mathbb{Z}^J$ and $M \in \mathbb{Z}^{J \times K}$, then

$$h^m := \sum m_j h_j \quad \text{while} \quad h^M = (h^{M_{\bullet, k}})_{k \in K} \in (H)^K.$$

If $H = \mathbb{K}^*$, then h^m is a monomial in the h_j . Note that, if I is a third finite set and $N \in \mathbb{Z}^{K \times I}$ then

$$h^{MN} = (h^M)^N$$

where the product $\mathbb{Z}^{J \times K} \times \mathbb{Z}^{K \times I} \longrightarrow \mathbb{Z}^{J \times I}$ corresponds to composition of morphisms in the canonical identification between $\mathbb{Z}^{J \times K}$ and $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^K, \mathbb{Z}^J)$. Finally, the elements of the canonical basis of \mathbb{Z}^K are denoted by e_k , for $k \in K$.

Given a seed t and a mutable vertex $k \in I_{uf}$, we have an operation called *seed mutation at k* , denoted by μ_k , that produces a new seed $\mu_k(t) = t'$, of \mathbb{K} , defined as follows.

- The vertex sets are the same of t , that is $I'_{uf} = I_{uf}$, $I'_{sf} = I_{sf}$ and $I'_{hf} = I_{hf}$.
- The extended exchange matrix B' satisfies:

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k \\ b_{i,j} + b_{i,k}^+ b_{k,j}^+ - b_{i,k}^- b_{k,j}^- & \text{otherwise.} \end{cases} \quad (1)$$

- The extended cluster x' is defined by

$$x'_i = \begin{cases} x_i & \text{if } i \neq k \\ x^{B_{\bullet,k}^+ - e_k} + x^{B_{\bullet,k}^- - e_k} & \text{if } i = k. \end{cases} \quad (2)$$

The identity

$$x_k x'_k = x^{B_{\bullet,k}^+} + x^{B_{\bullet,k}^-} \quad (3)$$

is called *exchange relation*. We will often refer to the two monomials on the right hand side of the exchange relation as M_k^+ and M_k^- , in the obvious way.

Since the mutation is an involution, that is $\mu_k \circ \mu_k(t) = t$, we can define an equivalence relation \sim on the set of seeds of \mathbb{K} with the same set of vertices: we say that $t \sim t'$ if there exists a sequence of mutations that transforms t in t' . That is, if there exists $i_1, \dots, i_l \in I_{uf}$ such that

$$\mu_{i_l} \circ \dots \circ \mu_{i_1}(t) = t'.$$

We call Δ the set of seeds of \mathbb{K} equivalent to t .

From now on, we fix an equivalence class of seeds Δ of \mathbb{K} . Since any seed in Δ has the same set of frozen variables, say x_i for $i \in I_f$, we can define the *coefficient ring*

$$\text{Cf}[\Delta] := \mathbb{C}[x_i]_{i \in I_{hf}}[x_i^{\pm 1}]_{i \in I_{sf}}. \quad (4)$$

Definition 2.1.3 (cluster algebra). The *cluster algebra* $\mathcal{A}(\Delta)$ is the $\text{Cf}[\Delta]$ -algebra generated by all the cluster variables of all the seeds in Δ .

If $t \in \Delta$, we define the ring of *Laurent Polynomials*

$$\mathcal{L}(t) := \text{Cf}[\Delta][x_i^{\pm 1}]_{i \in I_{uf}} \quad (5)$$

Note that the unfrozen and semi-frozen variable of t are invertible in $\mathcal{L}(t)$, while the highly frozen are not.

Definition 2.1.4 (upper cluster algebra). The *upper cluster algebra* $\overline{\mathcal{A}}(\Delta)$ is

$$\overline{\mathcal{A}}(\Delta) := \bigcap_{t \in \Delta} \mathcal{L}(t).$$

Both $\mathcal{A}(\Delta)$ and $\overline{\mathcal{A}}(\Delta)$ are domains, moreover $\overline{\mathcal{A}}(\Delta)$ is normal since it is defined as the intersection of normal rings. Still, both the cluster algebra and the upper cluster algebra are not noetherian in general. Note that, the choice of the ambient field \mathbb{K} (as the choice of the variables x) is immaterial for the isomorphism class of the cluster algebra and the upper cluster algebra. It becomes important when we try to identify cluster algebra structures on a given ring. So, when we only deal with (upper) cluster algebras, we often omit to specify the ambient field \mathbb{K} . The following theorem can be found in [GHKK18][Corollary 0.4] or in [LS15] for skew-symmetric seeds.

Theorem 2.1.5 (Positivity of the Laurent phenomenon). *For any $t, t' \in \Delta$ and $i \in I$,*

$$x'_i \in \mathbb{N}[x_j]_{j \in I_f} [x_k^{\pm 1}]_{k \in I_{uf}}$$

In particular $\mathcal{A}(\Delta) \subseteq \overline{\mathcal{A}}(\Delta)$.

The weaker statement that $x'_i \in \mathbb{Z}[x_j]_{j \in I_f} [x_k^{\pm 1}]_{k \in I_{uf}}$, usually called *Laurent phenomenon*, is much more elementary and also sufficient for most applications. It can be found in [FZ03, Proposition 11.2]. (See also [FZ02, Theorem 3.1]).

Clearly, if $t \in \Delta$, we also use the notation $\mathcal{A}(t)$ (resp. $\overline{\mathcal{A}}(t)$) to denote $\mathcal{A}(\Delta)$ (resp. $\overline{\mathcal{A}}(\Delta)$).

Definition 2.1.6. Let $t \in \Delta$. For $i \in I_{uf}$, let $t_i := \mu_i(t)$ and $t_0 := t$. The *upper bound* $\mathcal{U}(t)$ at t is

$$\mathcal{U}(t) := \bigcap_{i \in I_{uf} \cup \{0\}} \mathcal{L}(t_i).$$

If the matrix B is of maximal rank, we say that t is of *maximal rank*. Note that the rank of the generalised exchange matrix is invariant under mutation by [BFZ05, Lemma 3.2]. The following theorem is a very special case of [GSV18, Theorem 3.11]. The case with no highly frozen vertices had already been proved in [BFZ05, Corollary 1.7].

Theorem 2.1.7. *If t is of maximal rank, $\overline{\mathcal{A}}(\Delta) = \mathcal{U}(t)$.*

We say that no mutable vertex of t is completely disconnected if, for any $k \in I_{uf}$, there exists $i \in I$ such that $b_{i,k} \neq 0$. It's easy to see that this property is invariant under mutation, hence if Δ is an equivalence class of seeds, we say that no mutable vertex of Δ is completely disconnected if the same property holds for one, hence for any, of its seeds. Note that completely disconnected mutable vertices are the ones that produce trivial exchange relations.

Theorem 2.1.8 ([GLS13]). *Suppose that no mutable vertex of Δ is completely disconnected. Let $t, t' \in \Delta$ and $i, j \in I_{uf} \sqcup I_{hf}$, then*

1. *The variable x_i is irreducible in $\overline{\mathcal{A}}(\Delta)$.*
2. *The ideals (x_i) and (x'_j) are equal if and only if $x_i = x'_j$.*
3. *The invertible elements of $\overline{\mathcal{A}}(\Delta)$, up to scalar, are the monomials in the semi-frozen variables.*

The above theorem is proved in [GLS13, Theorem 1.3, Corollary 2.3] for the ordinary cluster algebra under the assumption that the seed t is connected (see [GLS13, Section 1.2]). The exact same proof also works for the upper cluster algebra and the upper bound under the weaker assumption that no mutable vertex is completely disconnected.

2.2 Disjoint union of seeds

This section is only needed for an application in Section 6.

We introduce a notation. If $M \in \mathbb{R}^{(J_1 \sqcup J_2) \times (K_1 \times K_2)}$, then

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \quad (6)$$

means that $M_{J_a \times K_b} = M_{a,b}$, for $a, b \in \{1, 2\}$.

Definition 2.2.1. Let t and t' be two seeds. The *disjoint union* of t and t' is the seed $t|t' = (J_{uf}, J_{sf}, J_{hf}, C, z)$ of $\mathbb{C}(x_i, x'_{i'})_{i \in I, i' \in I'}$ defined by

- The set of vertices

$$J_{uf} = I_{uf} \sqcup I'_{uf} \quad J_{sf} = I_{sf} \sqcup I'_{sf} \quad J_{hf} = I_{hf} \sqcup I'_{hf}.$$

- The generalised exchange matrix

$$C = \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix}.$$

- The cluster z , which is defined by

$$z_I = x \quad z_{I'} = x'.$$

We identify $\mathbb{C}(x_i, x'_{i'})_{i \in I, i' \in I'}$ with the fraction field of $\mathcal{L}(t) \otimes \mathcal{L}(t')$ in the natural way. It's clear that $\mathcal{L}(t|t') = \mathcal{L}(t) \otimes \mathcal{L}(t')$. The following property is immediate to verify:

Lemma 2.2.2. *Let t, t' as above, $i \in I_{uf}, i' \in I'_{uf}$, then*

$$\mu_i(t|t') = \mu_i(t)|t' \quad \text{and} \quad \mu_{i'}(t|t') = t|\mu_{i'}(t').$$

Lemma 2.2.3. *If t and t' are of maximal rank, then $\overline{\mathcal{A}}(t|t') = \overline{\mathcal{A}}(t) \otimes \overline{\mathcal{A}}(t')$.*

Proof. Clearly $t|t'$ is of maximal rank. The lemma follows easily from repeated application of Theorem 2.1.7, the previous lemma and the fact that the tensor product commutes with finite intersections. \square

2.3 Cluster valuations induced by frozen variables

We introduce the notion of *cluster valuation* at a frozen vertex. This is something which implicitly appears in the existing literature, but for which we didn't find an appropriate reference, or at least not at the level of detail we need.

If $t \in \Delta$, we denote by $\mathbb{C}[t]$ (resp. $\mathbb{C}(t)$) the polynomial ring (resp. the field of fractions) in the cluster variables x_i . By the Laurent phenomenon (Theorem 2.1.5) we have that $\mathbb{C}[t] \subseteq \overline{\mathcal{A}}(\Delta)$ and the fraction field of $\overline{\mathcal{A}}(\Delta)$ is $\mathbb{C}(t)$, which from now on will be denoted by $\mathbb{C}(\Delta)$.

Any cluster variable x_i generates a prime ideal in $\mathbb{C}[t]$, so we can consider the induced discrete valuation

$$\mathcal{V}_i^t : \mathbb{C}(\Delta) \longrightarrow \mathbb{Z} \cup \{\infty\}.$$

For an element $p \in \mathbb{C}[t] \subseteq \mathbb{C}(\Delta)$, the exponent of x_i in the factorisation of p into irreducible factors, in the ring $\mathbb{C}[t]$, is $\mathcal{V}_i^t(p)$.

Lemma 2.3.1. *If $i \in I_f$ and $t \sim t'$, then $\mathcal{V}_i^t = \mathcal{V}_i^{t'}$. In particular, we refer to any such valuation as the cluster valuation at the frozen vertex i and denote it by \mathcal{CV}_i .*

Proof. It's sufficient to prove the case of $t' = \mu_k(t)$, where $k \in I_{uf}$. Let

$$\mathbb{C}[I \setminus \{k\}] := \mathbb{C}[x_j]_{j \in I \setminus \{k\}} = \mathbb{C}[x'_j]_{j \in I \setminus \{k\}}$$

and $\mathbb{C}(I \setminus \{k\})$ the fraction field of $\mathbb{C}[I \setminus \{k\}]$. Since the fraction field of $\mathbb{C}(I \setminus \{k\})[x_k^{\pm 1}]$ is equal to $\mathbb{C}(\Delta)$, it's also sufficient to prove that $\mathcal{V}_i^t(f) = \mathcal{V}_i^{t'}(f)$ for any $f \in \mathbb{C}(I \setminus \{k\})[x_k^{\pm 1}]$.

It's easy to see that, if $\mathcal{V}_i^{I \setminus \{k\}} : \mathbb{C}(I \setminus \{k\}) \longrightarrow \mathbb{Z} \cup \{\infty\}$ is the valuation induced by the prime ideal (x_i) of $\mathbb{C}[I \setminus \{k\}]$, and if

$$f = \sum_{n=-N}^N f_n x_k^n \tag{7}$$

with $f_n \in \mathbb{C}(I \setminus \{k\})$, then

$$\mathcal{V}_i^t(f) = \min\{\mathcal{V}_i^{I \setminus \{k\}}(f_n) : -N \leq n \leq N\}. \tag{8}$$

To prove (8), it's sufficient to notice that if $f_n \neq 0$, then

$$\mathcal{V}_i^{I \setminus \{k\}}(f_n) = v_n \iff f_n = x_i^{v_n} \frac{p_n}{q_n} \tag{9}$$

with $p_n, q_n \in \mathbb{C}[I \setminus \{k\}]$ not divisible by x_i . Then, we can compute $\mathcal{V}_i^t(f)$ plugging the right hand side of (9) into (7) and we easily obtain (8).

Using the exchange relation (3), we compute that

$$f = \sum_{n=-N}^N f_n (M_k^+ + M_k^-)^n (x'_k)^{-n} \in \mathbb{C}(I \setminus \{k\})[(x'_k)^{\pm 1}]$$

But $M_k^+ + M_k^-$ is not divisible by x_i , hence

$$\mathcal{V}_i^{I \setminus \{k\}}(f_n) = \mathcal{V}_i^{I \setminus \{k\}}(f_n (M_k^+ + M_k^-)^n).$$

The statement follows using (8) and the similar formula relative to the seed t' . □

At the level of cluster varieties, the previous lemma says that the gluing maps between cluster tori preserve the valuation defined by the frozen variables. We can strengthen Theorem 2.1.8 in the case of highly frozen variables as follows.

Corollary 2.3.2. *If $i \in I_{hf}$, then x_i generates a prime ideal of $\overline{\mathcal{A}}(\Delta)$. Moreover, $\overline{\mathcal{A}}(\Delta)_{(x_i)}$ is a discrete valuation ring whose induced valuation on $\mathbb{C}(\Delta)$ equals the cluster valuation \mathcal{CV}_i .*

Proof. Fix $t \in \Delta$. Take $a, b \in \overline{\mathcal{A}}(\Delta)$ such that $ab \in (x_i)$. Since x_i generates a prime ideal in $\mathcal{L}(t)$, we can assume that $\frac{a}{x_i} \in \mathcal{L}(t)$. Note that $\mathcal{L}(t)$ is a localisation of $\mathbb{C}[t]$, hence the restriction of the valuation \mathcal{V}_i^t on $\mathcal{L}(t)$ agrees with the valuation defined by the prime ideal of $\mathcal{L}(t)$ generated by x_i . In particular, $\mathcal{V}_i^t(\frac{a}{x_i}) \geq 0$. By the definition of $\overline{\mathcal{A}}(\Delta)$, it's clear that for any $t' \in \Delta$, $\frac{a}{x_i} \in \frac{1}{x_i}\mathcal{L}(t') \subseteq \mathcal{L}(t')_{x_i}$. Lemma 2.3.1 implies that $\mathcal{V}_i^{t'}(\frac{a}{x_i}) \geq 0$, hence $\frac{a}{x_i} \in \mathcal{L}(t')$. It follows that $\frac{a}{x_i} \in \overline{\mathcal{A}}(\Delta)$ and the first part of the statement follows. For the second statement, notice that we have inclusions

$$\mathbb{C}[t] \subseteq \overline{\mathcal{A}}(\Delta) \subseteq \mathcal{L}(t).$$

In particular, by the definition of $\mathcal{L}(t)$, localizing the above inclusions at $\prod_{j \in I_{uf} \sqcup I_{sf}} x_j$ gives a sequence of isomorphisms. Thus $\mathbb{C}[t]_{(x_i)} = \overline{\mathcal{A}}(\Delta)_{(x_i)} = \mathcal{L}(t)_{(x_i)}$. \square

Remark 2.3.3. One can easily prove that the cluster valuation is a tropical valuation on $\overline{\mathcal{A}}(\Delta)$ in the sense of [BFZ05, Definition 7.1], in particular, it agrees with one of the valuation constructed in [BFZ05, Lemma 7.3].

2.4 Highly-freezing and semi-freezing

We introduce two operations on seeds, that change the nature of some frozen vertices from highly-frozen to semi-frozen, or the other way around. The notation is designed to be coherent with the fact that, passing from t to t_F corresponds to a localisation at the level of upper cluster algebras, while passing from t to t^F is the inverse operation, as described in Lemma 2.4.2.

Definition 2.4.1. Let t be a seed.

- If $F \subseteq I_{hf}$, we define the seed $t_F = (I_{F,uf}, I_{F,sf}, I_{F,hf}, B_F, x_F)$ by

$$I_{F,uf} = I_{uf} \quad I_{F,sf} = I_{sf} \sqcup F \quad I_{F,hf} = I_{hf} \setminus F \quad B_F = B \quad x_F = x.$$

We say that t_F is obtained from t by *semi-freezing* the vertices in F . If $\Delta = \Delta(t)$, then we denote $\Delta_F = \Delta(t_F)$.

- If $F \subseteq I_{sf}$, we define the seed $t^F = (I_{uf}^F, I_{sf}^F, I_{hf}^F, B^F, x^F)$ by

$$I_{uf}^F = I_{uf} \quad I_{sf}^F = I_{sf} \setminus F \quad I_{hf}^F = I_{hf} \sqcup F \quad B^F = B \quad x^F = x.$$

We say that t^F is obtained from t by *highly-freezing* the vertices in F . If $\Delta = \Delta(t)$, then we denote $\Delta^F = \Delta(t^F)$.

Lemma 2.4.2. *Let t be a seed, $\Delta = \Delta(t)$, $F \subseteq I_{hf}$, $G \subseteq I_{sf}$, k (resp. j) a mutable (resp. frozen) vertex of t .*

1. $(t_F)^F = t$.
2. $(t^G)_G = t$.
3. $\mathbb{C}(\Delta) = \mathbb{C}(\Delta_F) = \mathbb{C}(\Delta^G)$.
4. $\mathcal{V}_j^{t^G} = \mathcal{V}_j^t = \mathcal{V}_j^{t_F}$.
5. $\mu_k(t_F) = \mu_k(t)_F$ and $\mu_k(t^G) = \mu_k(t)^G$.
6. $\Delta_F = \{t'_F : t' \in \Delta\}$ and $\Delta^G = \{(t')^G : t' \in \Delta\}$.
7. If $z := \prod_{i \in F} (x_i)$, the natural inclusion $\overline{\mathcal{A}}(\Delta) \subseteq \overline{\mathcal{A}}(\Delta_F)$ induces an equality

$$\overline{\mathcal{A}}(\Delta)_z = \overline{\mathcal{A}}(\Delta_F).$$

8. $\overline{\mathcal{A}}(\Delta^G) = (\bigcap_{i \in G} \mathcal{C}\mathcal{V}_i^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})) \cap \overline{\mathcal{A}}(\Delta)$.

Proof. The first five statements are obvious. Statement 6 follows from 5 and the fact that semi-freezing or highly-freezing a seed doesn't change the set of mutable vertices. Then it's clear that $\mathcal{L}(t^G) \subseteq \mathcal{L}(t) \subseteq \mathcal{L}(t_F)$. Using 6, we get that

$$\overline{\mathcal{A}}(\Delta^G) \subseteq \overline{\mathcal{A}}(\Delta) \subseteq \overline{\mathcal{A}}(\Delta_F).$$

Moreover, z is invertible in $\overline{\mathcal{A}}(\Delta_F)$, hence we have an inclusion $\overline{\mathcal{A}}(\Delta)_z \subseteq \overline{\mathcal{A}}(\Delta_F)$. If $f \in \overline{\mathcal{A}}(\Delta_F)$, by Lemma 2.3.1 and the definition of upper cluster algebra, we get that

$$\left(\prod_{i \in F} x_i^{-\mathcal{V}_i^t(f)} \right) f \in \overline{\mathcal{A}}(\Delta).$$

This proves 7. Statement 8 also follows immediately from Lemma 2.3.1 and the definition of upper cluster algebra. □

2.5 Graded seeds

Cluster algebras and upper cluster algebras can be graded almost as if they were polynomial rings. In particular, defining the degree of the cluster variables of an initial seed and using the mutation process induces, under some hypothesis, global graduations. This is formalised by the notion of graded seed. We refer to [Gra15] for more details.

Let t be a seed.

Definition 2.5.1. Let H be an abelian group and $\sigma \in H^I$. We say that σ is an H -degree configuration on t if

$$\sigma^{B^+} = \sigma^{B^-}. \tag{10}$$

A seed with an H -degree configuration is called H -graded seed.

We drop the dependence on H , in the notation, if the group is clear from the context or meaningless. If we denote a graded seed by t (resp. t^\bullet , resp. t_\bullet for a certain symbol \bullet), we implicitly assume that its degree configuration is σ (resp. σ^\bullet , resp. σ_\bullet) and that the grading group is H .

From now on, t is an H -graded seed. We have a notion of mutation of degree configuration: if $k \in I_{uf}$ and $t' = \mu_k(t)$, then $\sigma' = \mu_k(\sigma)$ is the H -degree configuration, on t' , defined by the following formula:

$$\sigma'_i = \begin{cases} \sigma_i & \text{if } i \neq k \\ \sigma^{B_{\bullet,k}^+} - \sigma_k & \text{if } i = k. \end{cases} \quad (11)$$

We say that σ' is the *mutation at k* of the degree configuration σ . Any sequence of mutations defines a degree configuration on the resulting seed, which actually only depends on the seed and not on the sequence of mutations. Hence, any $t' \in \Delta(t)$ is in a canonical way a graded seed.

Note that the algebra $\mathcal{L}(t)$ is canonically H -graded by setting $\deg(x_i) = \sigma_i$. The set of Laurent monomials is an homogeneous base of $\mathcal{L}(t)$.

Proposition 2.5.2. *Let $t^* \in \Delta(t)$. Then $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$ are graded subalgebras of $\mathcal{L}(t^*)$. The graduation defined on $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$, by this inclusion, is independent on t^* . In particular, $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$ are canonically graded. Moreover, cluster variables are homogeneous.*

Remark 2.5.3. Since any cluster variable is homogeneous, the construction above is compatible with the operation of highly freezing or semi-freezing a set of vertices.

Note that all the basis constructed in [Qin22] consist of homogeneous elements. Finally, if t is a not necessarily graded seed, and we identify B as an element of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{I_{uf}}, \mathbb{Z}^I)$, then we have an universal $\text{coker}(B)$ -graduation $\bar{\sigma}$ on t defined by $\bar{\sigma}_i = \bar{e}_i$, where \bar{e}_i is the class of e_i in $\text{coker}(B)$. In particular we see that, roughly speaking, having many frozen vertices is good for constructing fine graduation on $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$.

3 Monomial liftings

From now on, D denotes a finite set. We define the notion of D -lifting configuration on a seed t . For any such configuration, we construct a seed $\uparrow t$, to which we refer as a *monomial lifting* of t . The name is explained by the fact that, under mild assumptions, if Y is a variety carrying a cluster structure, then any monomial lifting of t gives to $(\mathbb{C}^*)^D \times Y$ a cluster structure, compatible with the one on Y . Conversely, any cluster structure on $(\mathbb{C}^*)^D \times Y$, which is compatible with the one on Y in the sense of Proposition 3.0.9, is defined by a monomial lifting. We start with some definitions.

From now on, t is a seed of the field \mathbb{K} and $\Delta = \Delta(t)$. We assume that $I \cap D = \emptyset$.

Definition 3.0.1. A D -pointed field extension of \mathbb{K} is a field extension \mathbb{L}/\mathbb{K} , with a D -uple $(x_d)_{d \in D} \in (\mathbb{L}^*)^D$ of \mathbb{K} -algebraically independent elements, such that $\mathbb{L} = \mathbb{K}(x_d)_{d \in D}$.

If we denote a D -pointed field extension of \mathbb{K} by \mathbb{L} , we implicitly assume that its defining D -uple of elements is denoted by $(x_d)_d$. Whenever a seed t of \mathbb{K} and a D -pointed field extension \mathbb{L} of \mathbb{K} are given, we denote by

$$\widehat{x} = (x_j)_{j \in I \sqcup D} \in (\mathbb{L}^*)^{I \sqcup D}. \quad (12)$$

Definition 3.0.2. A D -seed extension of t is a seed t^* , of a field extension \mathbb{L}/\mathbb{K} , such that

- $I_{uf} = I_{uf}^*$ $I_{sf} \subseteq I_{sf}^*$ $I_{hf} \subseteq I_{hf}^*$ and $I_f^* \setminus I_f = D$.
- $B_{I \times I_{uf}}^* = B$.

If moreover \mathbb{L} is D -pointed and for any $d \in D$ we have $x_d^* = x_d$, then we say that t^* is a D -pointed seed extension of t .

In the next definition, we repeatedly use the notation introduced in (6).

Definition 3.0.3 (monomial lifting). A D -lifting configuration on t is a pair (\mathbb{L}, ν) consisting of a D -pointed extension \mathbb{L} of \mathbb{K} and a lifting matrix $\nu \in \mathbb{Z}^{D \times I}$. The monomial lifting of t , defined by (\mathbb{L}, ν) , is the seed $\uparrow t = (\uparrow I_{uf}, \uparrow I_{sf}, \uparrow I_{hf}, \uparrow B, \uparrow x)$ defined by:

- The set of vertices $\uparrow I_{uf} = I_{uf}$, $\uparrow I_{sf} = I_{sf} \sqcup D$, $\uparrow I_{hf} = I_{hf}$.
- The matrix

$$\uparrow B = \begin{pmatrix} B \\ -\nu B \end{pmatrix} \in \mathbb{Z}^{(I \sqcup D) \times I_{uf}}. \quad (13)$$

The coefficients of $\uparrow B$ are denoted by $\uparrow b_{i,j}$ for $i \in \uparrow I, j \in \uparrow I_{uf}$.

- The variables are

$$\uparrow x = \widehat{x}^{\widehat{\nu}} \quad \text{where} \quad \widehat{\nu} = \begin{pmatrix} Id & 0 \\ \nu & Id \end{pmatrix} \in \mathbb{Z}^{(I \sqcup D) \times (I \sqcup D)}. \quad (14)$$

In this case we say that ν lifts t to $\uparrow t$ and write: $\uparrow t \xleftarrow{\nu} t$.

Any monomial lifting $\uparrow t$ is a D -pointed extension of t . Note that, according to our notation,

$$\widehat{x}_D = (x_d)_d \in (\mathbb{L}^*)^D.$$

Formula (14) implies that for any $i \in I$

$$\uparrow x_i = x_i \widehat{x}_D^{\nu \bullet, i}.$$

Hence, we can think about ν as a collection of monomials, indexed by I , in the variables \widehat{x}_D . The cluster variables $\uparrow x_i$, for $i \in I$, are the product of x_i and the corresponding monomial.

Example 3.0.4. Let t be a seed of the field \mathbb{K} , which graphically is described by the following quiver

$$\bigcirc 1 \longleftarrow \bigcirc 2 \longleftarrow \blacksquare 3 .$$

That is: $I_{uf} = \{1, 2\}$, $I_{sf} = \emptyset$, $I_{hf} = \{3\}$ and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $D = \{d\}$, here d is considered as a symbol, $\mathbb{L} = \mathbb{K}(x_d)$ where x_d is a variable and $\nu = (1, 2, 3)$. Then, a simple matrix multiplication implies that $\uparrow t$ corresponds graphically to the quiver

$$\begin{array}{c}
 \begin{array}{ccc}
 & & d \\
 & \nearrow & \uparrow \\
 \circ 1 & \longleftarrow & \circ 2 \longleftarrow \blacksquare 3
 \end{array}
 \end{array}
 \quad \text{and} \quad
 \uparrow B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{pmatrix}.$$

Moreover, from the definition, we have that

$$\uparrow x_1 = x_1 x_d \quad \uparrow x_2 = x_2 x_d^2 \quad \uparrow x_3 = x_3 x_d^3 \quad \uparrow x_d = x_d.$$

Monomial liftings can be performed in steps. In fact, suppose that $D = D_1 \sqcup D_2$ and \mathbb{L} is a D -field extension of \mathbb{K} . Then the field $\mathbb{L}_1 = \mathbb{K}(x_{d_1})_{d_1 \in D_1}$ is a D_1 -pointed extension of \mathbb{K} and \mathbb{L} is a D_2 -pointed extension of \mathbb{L}_1 . The following lemma is obvious from the definitions.

Lemma 3.0.5. *Let (\mathbb{L}, ν) be a D -lifting data on t . If $D = D_1 \sqcup D_2$, and $\nu_j = \nu_{D_j, \bullet}$, for $j = 1, 2$, then the following diagram is commutative.*

$$\begin{array}{ccc}
 & \bullet & \\
 (\nu_2, 0) \swarrow & & \nwarrow \nu_1 \\
 \bullet & \xleftarrow{\nu} & t
 \end{array}$$

We introduce the following notion of mutation of lifting configuration.

Definition 3.0.6 (mutation of lifting configuration). Let (\mathbb{L}, ν) be a D -lifting configuration on t , $k \in I_{uf}$ and $t' := \mu_k(t)$. Consider the D -lifting matrix ν' on t' defined by

$$\nu'_{\bullet, i} = \begin{cases} \nu_{\bullet, i} & \text{if } i \neq k \\ \max\{\nu B_{\bullet, k}^+, \nu B_{\bullet, k}^-\} - \nu_{\bullet, k} & \text{if } i = k \end{cases} \quad (15)$$

where the max in (15) is taken component-wise. We say that $\nu' = \mu_k(\nu)$ is the mutation at k of the lifting matrix ν and that the lifting configuration $(\mathbb{L}, \nu') = \mu_k(\mathbb{L}, \nu)$, on t' , is the mutation at k of the lifting configuration (\mathbb{L}, ν) .

Remark 3.0.7. For any $a, b \in \mathbb{R}$ we have that $\max\{a, b\} = (a - b)^+ + b$. Using this formula and the fact that $b = b^+ - b^-$, we get two alternative forms of formula (15), namely:

$$\nu'_{\bullet, k} = (\nu B_{\bullet, k})^+ + \nu B_{\bullet, k}^- - \nu_{\bullet, k} \quad (16)$$

and

$$\nu'_{\bullet, k} = (-\nu B_{\bullet, k})^+ + \nu B_{\bullet, k}^+ - \nu_{\bullet, k}. \quad (17)$$

If the D -pointed field \mathbb{L} is clear, we identify a lifting configuration with its lifting matrix. From now on, we fix \mathbb{L} and suppose that any lifting configuration has \mathbb{L} as defining field. The following lemma is crucial.

Lemma 3.0.8. *Lifting seeds commutes with mutation. That is, for any lifting configuration ν on t and any $k \in I_{uf}$, if we denote by $t' = \mu_k(t)$ and $\nu' = \mu_k(\nu)$, then the following diagram is commutative*

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\uparrow & \xrightarrow{\mu_k} & \uparrow \\
t & & t' \\
\downarrow & & \downarrow \\
t & \xrightarrow{\mu_k} & t'
\end{array}$$

Proof. Call $t^* = \mu_k(\uparrow t)$. We have to prove that $t^* = \uparrow t'$. It's clear that the two seeds have the same vertices sets. To avoid possible confusion, we stress that $\uparrow B^\pm$ denotes $(\uparrow B)^\pm$ and that $\uparrow b_{i,j}^\pm$ are the coefficients of $\uparrow B^\pm$.

First we prove that $x^* = \uparrow x'$. Clearly, if $d \in D$, then $x_d^* = \uparrow x'_d = x_d$. If $i \in I$ and $i \neq k$, then we have $x_i^* = \uparrow x_i = \uparrow x'_i$ because of (15) and (2). Finally, we can compute that:

$$\begin{aligned}
x_k^* &= \uparrow x^{\uparrow B_{\bullet,k}^+ - e_k} + \uparrow x^{\uparrow B_{\bullet,k}^- - e_k} \\
&= \widehat{x}^{\widehat{\nu} \cdot \uparrow B_{\bullet,k}^+ - \widehat{\nu}_{\bullet,k}} + \widehat{x}^{\widehat{\nu} \cdot \uparrow B_{\bullet,k}^- - \widehat{\nu}_{\bullet,k}} \\
&= x^{\uparrow B_{\bullet,k}^+ - e_k} \widehat{x}_D^{\nu \cdot \uparrow B_{\bullet,k}^+ + (-\nu \cdot B_{\bullet,k})^+ - \nu_{\bullet,k}} + x^{\uparrow B_{\bullet,k}^- - e_k} \widehat{x}_D^{\nu \cdot \uparrow B_{\bullet,k}^- + (-\nu \cdot B_{\bullet,k})^- - \nu_{\bullet,k}}
\end{aligned} \tag{18}$$

The first equality is the mutation rule (2), the second is obtained from the definition of $\uparrow x$ (14) and the third follows from the following expressions:

$$\begin{aligned}
\widehat{\nu} \cdot \uparrow B^+ &= \begin{pmatrix} Id & 0 \\ \nu & Id \end{pmatrix} \begin{pmatrix} B^+ \\ (-\nu B)^+ \end{pmatrix} = \begin{pmatrix} B^+ \\ \nu B^+ + (-\nu B)^+ \end{pmatrix} \\
\widehat{\nu} \cdot \uparrow B^- &= \begin{pmatrix} Id & 0 \\ \nu & Id \end{pmatrix} \begin{pmatrix} B^- \\ (-\nu B)^- \end{pmatrix} = \begin{pmatrix} B^- \\ \nu B^- + (-\nu B)^- \end{pmatrix}
\end{aligned} \tag{19}$$

Recall that $\uparrow B^+ - \uparrow B^- = \uparrow B$, in particular

$$\widehat{\nu} \cdot \uparrow B^+ - \widehat{\nu} \cdot \uparrow B^- = \begin{pmatrix} Id & 0 \\ \nu & Id \end{pmatrix} \begin{pmatrix} B \\ -\nu B \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{20}$$

From (19) and (20) we deduce that the exponents of \widehat{x}_D in the two monomials of expression (18) are the same. Hence, using (2), we deduce from (18) that

$$x_k^* = x'_k \widehat{x}_D^{\nu \cdot \uparrow B_{\bullet,k}^+ + (-\nu \cdot B_{\bullet,k})^+ - \nu_{\bullet,k}}.$$

Comparing this formula to the expression of $\uparrow x'_k$, we see that it's sufficient to prove that

$$\nu B_{\bullet,k}^+ + (-\nu B_{\bullet,k})^+ - \nu_{\bullet,k} = \nu'_{\bullet,k}.$$

But this is Formula (17).

Next, we consider the exchange matrices, that is we prove that $B^* = \uparrow B'$. It's clear from (1) and (13) that for any $j \in I$ and $i \in I_{uf}$, $b_{j,i}^* = b'_{j,i} = \uparrow b'_{j,i}$. For $d \in D$, then

$$b_{d,k}^* = -\uparrow b_{d,k} = \nu_{d,\bullet} B_{\bullet,k}.$$

and

$$\uparrow b'_{d,k} = -\nu'_{d,\bullet} B'_{\bullet,k} = \nu'_{d,\bullet} B_{\bullet,k}.$$

The equality $b_{d,k}^* = \uparrow b'_{d,k}$ follows from the fact that $b_{k,k} = 0$ and the definition of ν' (15). Next, if $i \in I_{uf}$, $i \neq k$, using (1) and (13) we get that

$$\begin{aligned} b_{d,i}^* &= \uparrow b_{d,i} + \uparrow b_{d,k}^+ b_{k,i}^+ - \uparrow b_{d,k}^- b_{k,i}^- \\ &= \uparrow b_{d,i} + (-\nu_{d,\bullet} B_{\bullet,k})^+ b_{k,i}^+ - (-\nu_{d,\bullet} B_{\bullet,k})^- b_{k,i}^-. \end{aligned}$$

Similarly we have that

$$\begin{aligned} \uparrow b'_{d,i} &= -\nu'_{d,\bullet} B'_{\bullet,i} \\ &= -\nu'_{d,k} b'_{k,i} + \sum_{j \neq k} -\nu_{d,j} b'_{j,i} \\ &= -\nu_{d,k} b_{k,i} + \max\{\nu_{d,\bullet} B_{\bullet,k}^+, \nu_{d,\bullet} B_{\bullet,k}^-\} b_{k,i} - \sum_{j \in I \setminus \{k\}} \nu_{d,j} (b_{j,i} + b_{j,k}^+ b_{k,i}^+ - b_{j,k}^- b_{k,i}^-) \\ &= -\nu_{d,\bullet} B_{\bullet,i} + \max\{\nu_{d,\bullet} B_{\bullet,k}^+, \nu_{d,\bullet} B_{\bullet,k}^-\} b_{k,i} - \nu_{d,\bullet} B_{\bullet,k}^+ b_{k,i}^+ + \nu_{d,\bullet} B_{\bullet,k}^- b_{k,i}^- \\ &= \uparrow b_{d,i} + (-\nu_{d,\bullet} B_{\bullet,k})^+ b_{k,i} + \nu_{d,\bullet} B_{\bullet,k}^+ b_{k,i} - \nu_{d,\bullet} B_{\bullet,k}^+ b_{k,i}^+ + \nu_{d,\bullet} B_{\bullet,k}^- b_{k,i}^- \end{aligned} \quad (21)$$

The first equality follows from formula (13), the second, third and fourth are obtained from formulas (1), (15) and some elementary manipulations. The last equality is a consequence of formulas (13) and (17). Finally, using that $b_{k,i} = b_{k,i}^+ - b_{k,i}^-$ in Formula (21), we obtain immediately that $\uparrow b'_{d,i} = b_{d,i}^*$. \square

The following characterisation of monomial liftings turns out to be useful.

Proposition 3.0.9. *Let \tilde{t} be a seed of \mathbb{L} . Suppose that there exist $\nu \in \mathbb{Z}^{D \times I}$, and $\lambda \in \mathbb{Z}^{D \times I_{uf}}$ such that*

1. \tilde{t} is a D -pointed field extension of t with $D \subset \tilde{I}_{sf}$.
2. For any $i \in I$, $\tilde{x}_i = x_i \hat{x}_D^{\nu_{\bullet,i}}$.
3. For any $k \in I_{uf}$, $\mu_k(\tilde{x}_k) = \mu_k(x_k) \hat{x}_D^{\lambda_{\bullet,k}}$.

Then $\tilde{t} \xleftarrow{\nu} t$ and for any $k \in I_{uf}$, $\lambda_{\bullet,k} = \mu_k(\nu)_{\bullet,k}$.

Proof. Denote by $\uparrow t$ the monomial lifting of t determined by (\mathbb{L}, ν) . We have to prove that $\tilde{t} = \uparrow t$. The first assumption implies that \tilde{t} and $\uparrow t$ have the same vertices set. Moreover, assumption 1 and 2 imply that $\tilde{x} = \uparrow x$. By assumption 1, $\tilde{B}_{I,\bullet} = B = \uparrow B_{I,\bullet}$. So we just have to prove that $\tilde{B}_{D,\bullet} = -\nu B$. Fix $k \in I_{uf}$. Call $t' = \mu_k(t)$ and $t^* = \mu_k(\tilde{t})$. By assumption 3 and (3), we have that

$$\begin{aligned} x_k^* \tilde{x}_k &= x'_k x_k \hat{x}_D^{\nu_{\bullet,k} + \lambda_{\bullet,k}} \\ &= x^{B_{\bullet,k}^+} \hat{x}_D^{\nu_{\bullet,k} + \lambda_{\bullet,k}} + x^{B_{\bullet,k}^-} \hat{x}_D^{\nu_{\bullet,k} + \lambda_{\bullet,k}}. \end{aligned}$$

Always using (3), we can compute that

$$\begin{aligned}
x_k^* \tilde{x}_k &= \tilde{x}^{\tilde{B}^+_{\bullet,k}} + \tilde{x}^{\tilde{B}^-_{\bullet,k}} \\
&= \hat{x}^{\hat{\nu} \tilde{B}^+_{\bullet,k}} + \hat{x}^{\hat{\nu} \tilde{B}^-_{\bullet,k}} \\
&= x^{B^+_{\bullet,k} \hat{x}^{\nu B^+_{\bullet,k} + \tilde{B}^+_{D \times k}} + x^{B^-_{\bullet,k} \hat{x}^{\nu B^-_{\bullet,k} + \tilde{B}^-_{D \times k}}.
\end{aligned}$$

We deduce that

$$\nu B^+_{\bullet,k} + \tilde{B}^+_{D \times k} = \nu_{\bullet,k} + \lambda_{\bullet,k} = \nu B^-_{\bullet,k} + \tilde{B}^-_{D \times k}.$$

In particular

$$\tilde{B}^-_{D \times k} = \tilde{B}^+_{D \times k} - \tilde{B}^-_{D \times k} = \nu B^-_{\bullet,k} - \nu B^+_{\bullet,k} = -\nu B_{\bullet,k}.$$

Finally, the statement about λ can be easily deduced from the two above identities and (17). \square

We introduce some notation. For any finite sets J and K , then $\mathbb{Z}^{J \times K}$ and $(\mathbb{Z}^J)^K$ are canonically identified by the map that assigns to a matrix the collection of its columns. From now on, we make no difference between these two objects. Let $M \in \mathbb{Z}^{J \times K} = (\mathbb{Z}^J)^K$. Note that if I is a third finite set and $N \in \mathbb{Z}^{K \times I}$, then $MN = M^N$. We write MN when we think about M as a matrix and M^N when we think about M as the collection of its columns.

If H, K are groups and $h \in H^J, k \in K^J$, then the notation $h \times k$ stands for the element of $(H \times K)^J$ defined by $(h \times k)_j = (h_j, k_j)$.

From now on, fix a lifting matrix $\nu \in \mathbb{Z}^{D \times I}$. In particular (\mathbb{L}, ν) is a fixed D -lifting configuration on t .

If the seed t is a graded seed, then the degree configuration of t can be lifted along with t .

Definition 3.0.10 (lifting of degree configuration). Let $\uparrow 0 := (\nu, \text{Id}) \in (\mathbb{Z}^D)^{I \sqcup D}$. If $\sigma \in H^I$ is a degree configuration on t , then we define

$$\uparrow \sigma = \uparrow 0 \times (\sigma, 0) \in (\mathbb{Z}^D \times H)^{I \sqcup D}. \quad (22)$$

In this case we say that ν lifts σ to $\uparrow \sigma$ and write: $\uparrow \sigma \xleftarrow{\nu} \sigma$.

Lemma 3.0.11. *For any degree configuration σ on t , $\uparrow \sigma$ is a degree configuration on $\uparrow t$.*

Proof. Note that $\uparrow \sigma^{1B} = \uparrow 0^{1B} \times (\sigma, 0)^{1B}$. Moreover $(\sigma, 0)^{1B} = \sigma^B = 0$ and

$$\uparrow 0^{1B} = (\nu, \text{Id}) \begin{pmatrix} B \\ -\nu B \end{pmatrix} = 0.$$

\square

Remark 3.0.12. Note that $\uparrow t$ has a canonical \mathbb{Z}^D -degree configuration given by $\uparrow 0 = (\nu, \text{Id})$, which is the lifting of the trivial degree configuration on t . The data of this degree configuration is equivalent to the one of ν . As a warning, the degree configuration $\uparrow 0$ depends both on t and on ν . To avoid ambiguity, we use the convention that $\uparrow 0$ is always considered as a degree configuration on a seed which is denoted by $\uparrow t$ and whose corresponding lifting matrix ν is clear from the context. If a seed is called t^\bullet (resp. t_\bullet), for a certain symbol \bullet , and a lifting $\uparrow t^\bullet$ (resp. $\uparrow t_\bullet$) of t^\bullet (resp. t_\bullet) is defined, then we use the notation $\uparrow 0^\bullet$ (resp. $\uparrow 0_\bullet$).

Lemma 3.0.13. *Lifting a degree configuration commutes with mutation. In particular, for any degree configuration σ on t and any $k \in I_{uf}$, if $\sigma' = \mu_k(\sigma)$, then we have a commutative diagram*

$$\begin{array}{ccc} \uparrow\sigma & \xrightarrow{\mu_k} & \uparrow\sigma' \\ \uparrow\nu & & \uparrow\nu' \\ \sigma & \xrightarrow{\mu_k} & \sigma' \end{array}$$

Proof. Let $t' = \mu_k(t)$. From the mutation rule (11) it's clear that $\mu_k(\uparrow\sigma) = \mu_k(\uparrow 0) \times (\sigma', 0)$. Hence it's sufficient to prove that $\mu_k(\uparrow 0) = \uparrow 0'$. Call $0^* = \mu_k(\uparrow 0)$. For $i \in I \setminus \{k\}$ and $d \in D$, the identities

$$0_i^* = \nu_i = \nu'_i = \uparrow 0'_i \quad \text{and} \quad 0_d^* = e_d = \uparrow 0'_d$$

follows at once from (11) and (15). From the same formulas we compute that

$$\begin{aligned} 0_k^* &= \uparrow 0^{1B_{\bullet,k}^+} - \nu_{\bullet,k} \\ &= \nu B_{\bullet,k}^+ + (-\nu B_{\bullet,k})^+ - \nu_{\bullet,k}. \end{aligned}$$

Then the identity $0_k^* = \uparrow 0'_k$ follows from (17). \square

Corollary 3.0.14. *For any $k_1, \dots, k_r \in I_{uf}$ and $j_1, \dots, j_s \in I_{uf}$ such that $\mu_{k_r} \circ \dots \circ \mu_{k_1}(t) = \mu_{j_s} \circ \dots \circ \mu_{j_1}(t)$, then $\mu_{k_r} \circ \dots \circ \mu_{k_1}(\nu) = \mu_{j_s} \circ \dots \circ \mu_{j_1}(\nu)$. In particular, any $t' \in \Delta(t)$ has a canonical lifting configuration ν' defined by the composition of (15) along any sequence of mutations from t to t' . Moreover, the map sending t' to $\uparrow t'$ is a bijection between $\Delta(t)$ and $\Delta(\uparrow t)$.*

Proof. One can recover ν from $\uparrow 0$. We know that $\mu_{k_r} \circ \dots \circ \mu_{k_1}(\uparrow 0) = \mu_{j_s} \circ \dots \circ \mu_{j_1}(\uparrow 0)$, then Lemma 3.0.13 implies that $\mu_{k_r} \circ \dots \circ \mu_{k_1}(\nu) = \mu_{j_s} \circ \dots \circ \mu_{j_1}(\nu)$. Alternatively, the same result can be obtained in the same way as a direct consequence of Lemma 3.0.8. By Lemma 3.0.8, the map $\Delta(t) \rightarrow \Delta(\uparrow t)$ sending t' to $\uparrow t'$ is well defined. It is surjective since t and $\uparrow t$ have the same set of mutable vertices. Moreover, it's clear from Definition 3.0.3 that if $\uparrow t' \xleftarrow{\nu'} t'$, then $\uparrow t'$ completely determines ν' and t' . In particular, the map is injective. \square

Note that, with little abuse of notation, we can consider the x_d as abstract independent variables over \mathbb{K} . Any $t' \in \Delta$ gives an isomorphism

$$\begin{aligned} \mathbb{K}[x_d^{\pm 1}]_d &\longrightarrow \mathbb{C}[x_d^{\pm 1}]_d \otimes \mathbb{K} \\ \uparrow x'_i &\longmapsto \begin{cases} x_d \otimes 1 & \text{if } i = d \in D \\ (x_d)_d^{\nu'_{\bullet,i}} \otimes x'_i & \text{if } i \in I \end{cases} \end{aligned} \quad (23)$$

where ν' is the lifting matrix on t' determined by ν . Because of Lemma 3.0.8, the above isomorphism doesn't depend on t' but only on the chosen initial lifting matrix ν on t . So, whenever ν is clear, we identify the above two rings by the previous isomorphism. In particular, we can think about $\uparrow t$ as a seed of the fraction field of $\mathbb{C}[x_d^{\pm 1}]_d \otimes \mathbb{K}$. With this identification, it's immediate that $\mathcal{L}(\uparrow t) = \mathbb{C}[x_d^{\pm 1}]_d \otimes \mathcal{L}(t)$.

Theorem 3.0.15. *We have equality*

$$\mathcal{A}(\uparrow t) = \mathcal{A}(t)[x_d^{\pm 1}]_d.$$

Moreover, if t is of maximal rank, then

$$\overline{\mathcal{A}}(\uparrow t) = \overline{\mathcal{A}}(t)[x_d^{\pm 1}]_d.$$

Proof. By Corollary 3.0.14, for any $t^* \in \Delta(\uparrow t)$, there exists a unique $t' \in \Delta(t)$, with a well defined lifting configuration ν' , such that $t^* = \uparrow t'$. The equality involving $\mathcal{A}(\uparrow t)$ is then obvious from the definition of $\uparrow x'$ and the fact that, for any $d \in D$, then $x_d^{\pm 1} \in \mathcal{A}(\uparrow t)$ since $D \subseteq \uparrow I_{sf}$.

In the following, we think about $\uparrow t$ as a seed of the fraction field of $\mathbb{C}[x_d^{\pm 1}]_d \otimes \mathbb{K}$. If t is of maximal rank, so is $\uparrow t$. Then using Theorem 2.1.7 and the convention that $\mu_0(t) = t$, we deduce that (intersections run over $I_{uf} \cup \{0\}$):

$$\begin{aligned} \overline{\mathcal{A}}(\uparrow t) &= \bigcap \left[\mathbb{C}[x_d^{\pm 1}]_d \otimes \mathcal{L}(\mu_k(t)) \right] \\ &= \mathbb{C}[x_d^{\pm 1}]_d \otimes \left[\bigcap \mathcal{L}(\mu_k(t)) \right] \\ &= \overline{\mathcal{A}}(t)[x_d^{\pm 1}]_d \end{aligned}$$

where central equality follows because $\mathbb{C}[x_d^{\pm 1}]_d \otimes -$ commutes with finite intersections. \square

We expect the answer to the following question to be negative in general.

Question 3.0.16. *Does the equality between $\overline{\mathcal{A}}(\uparrow t)$ and $\overline{\mathcal{A}}(t)[x_d^{\pm 1}]_d$ hold without the assumption that t is of maximal rank?*

Remark 3.0.17. In the special case of t_{prin} (the seed obtained from t by adding principal coefficients), the second statement of the previous theorem is a consequence of [GHKK18, Proposition 8.27]. We stress the fact that, the variables $(x_d)_d$ being algebraically independent over \mathbb{K} , $\mathcal{A}(t)[x_d^{\pm 1}]_d$ and $\overline{\mathcal{A}}(t)[x_d^{\pm 1}]_d$ are Laurent polynomial rings over $\mathcal{A}(t)$ and $\overline{\mathcal{A}}(t)$ respectively.

Corollary 3.0.18. *The deletion map $s : \overline{\mathcal{A}}(\uparrow t) \rightarrow \overline{\mathcal{A}}(t)$, that is the morphism defined by $s(\uparrow x_i) = x_i$ for $i \in I$ and $s(\uparrow x_d) = 1$ for $d \in D$, is surjective. The kernel is the ideal generated by $\uparrow x_d - 1$, for $d \in D$.*

Proof. This is obvious from Theorem 3.0.15. \square

Lemma 3.0.19. *Let \tilde{t} be a D -seed extension of t such that $D \subseteq \tilde{I}_{sf}$. If $B^t : \mathbb{Z}^I \rightarrow \mathbb{Z}^{\uparrow I_{sf}}$ is surjective, there exist a lifting matrix $\nu \in \mathbb{Z}^{D \times I}$ such that $\uparrow B = \tilde{B}$.*

Proof. Since B^t is surjective, we can find ν such that $\tilde{B}_{D, \bullet} = -\nu B$. \square

To the author best knowledge, the next statement is new.

Corollary 3.0.20. *Let \tilde{t} be a D -seed extension of t such that $D \subseteq \tilde{I}_{sf}$. If $B^t : \mathbb{Z}^I \rightarrow \mathbb{Z}^{\uparrow I_{sf}}$ is surjective, then the deletion map $s : \overline{\mathcal{A}}(\tilde{t}) \rightarrow \overline{\mathcal{A}}(t)$ is surjective and its kernel is the ideal generated by $\tilde{x}_d - 1$, for $d \in D$.*

Proof. Note that the isomorphism class of an upper cluster algebra only depends on the B -matrix and on the vertices sets of one of its seeds, and so does the desired property of s . The statement follows from the previous lemma and Corollary 3.0.18. \square

If \tilde{t} is a seed extension of t , the deletion map $s : \overline{\mathcal{A}}(\tilde{t}) \rightarrow \overline{\mathcal{A}}(t)$ is not always surjective,. An example is given in [MM15][Section 7.1, Corollary 7.1.2]. The deletion map is always surjective at the level of cluster algebras.

Remark 3.0.21. Let t be a seed and set $D := \{i' : i \in I\}$. We identify \mathbb{Z}^D with \mathbb{Z}^I in the obvious way. If we consider $\nu = \text{Id} \in \mathbb{Z}^{I \times I}$, then

$${}_{1}B = \begin{pmatrix} B \\ -B \end{pmatrix}.$$

Consider the degree transformation ψ defined in [Qin22, Definition 3.3.1]. By [Qin22, Remark 3.3.4], for any $k \in I_{uf}$,

$$e_k - \psi_{t, \mu_k(t)} \psi_{\mu_k(t), t}(e_k) = {}_{1}B_{D \times k}.$$

In particular, the monomial lifting for this special ν encodes the non-trivial monodromy of the maps ψ after one mutation. See [Qin22, Remark 3.3.4] for more details. It turns out that, under some hypothesis, the lifting matrix ν has an interpretation in terms of the Cox ring of a certain partial compactification of the cluster manifold. This is part of an ongoing work.

The results of this section should also work, with the same proofs, for cluster and upper cluster algebras defined over \mathbb{Z} instead then over \mathbb{C} .

Remark 3.0.22. If $\nu, \mu \in \mathbb{Z}^{D \times I}$ are distinct lifting matrices and ${}_{1}t_\nu \xleftarrow{\nu} t, {}_{1}t_\mu \xleftarrow{\mu}$ are the respective monomial liftings, clearly ${}_{1}t_\nu$ and ${}_{1}t_\mu$ are distinct. Nevertheless, they are quasi-equivalent in the sense of [Fra16] by [Fra16, Corollary 4.5].

3.1 Pole filtration and lifting graduation

It turns out that the graduation induced by ${}_{1}0$ on $\overline{\mathcal{A}}({}_{1}t^D)$ has a nice geometric interpretation. To explain this fact, we switch for a moment to a more general setting. This allows us to apply the same ideas in the context of schemes which are suitable for lifting, which are introduced in Section 4.

In this section, D is a finite set and T is a torus of rank $|D|$. We denote by $X(T)$ the character group of T .

Definition 3.1.1. Let A be a \mathbb{C} -algebra. An almost polynomial ring over A is a quadruple $(R, \mathcal{V}, X, \phi^*)$ where

1. R is a normal (non necessarily noetherian) \mathbb{C} -algebra which is a domain.
2. \mathcal{V} is a collection of discrete valuations $\mathcal{V}_d : R \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$, indexed by $d \in D$.
3. $X \subseteq R^D$ is a collection of elements such that $\mathcal{V}_{d_1}(X_{d_2}) = \delta_{d_1, d_2}$ and

$$R = \bigcap_{d \in D} \mathcal{V}_d^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\}) \bigcap R_{\prod X_d}.$$

4. $\phi^* : R_{\prod X_d} \rightarrow \mathbb{C}[T] \otimes A$ is an isomorphism such that $\phi^*(X_d) = x_d \otimes 1$, where the collection $(x_d)_d$ is a basis of $X(T)$.

When \mathcal{V}, X and ϕ^* are clear, we just write that R is an almost polynomial ring over A . If an almost polynomial ring is called R , we implicitly assume that the rest of the data is denoted by \mathcal{V}, X, ϕ^* and that R is almost polynomial over A . Moreover, $R_{\prod X_d}$ is canonically identified with $\mathbb{C}[T] \otimes A$ via ϕ^* . In particular, we write $X_d = x_d \otimes 1$.

Example 3.1.2. Let t be a seed of maximal rank, ν a D -lifting matrix and $\uparrow t \xleftarrow{\nu} t$. Then we have an almost polynomial ring over $\overline{\mathcal{A}}(t)$:

$$(\overline{\mathcal{A}}(\uparrow t^D), \mathcal{CV}, \uparrow x_D, \phi^*).$$

Recall that $\uparrow x_d = x_d$ and $\overline{\mathcal{A}}(\uparrow t^D)_{\prod_d x_d} = \overline{\mathcal{A}}(\uparrow t)$ because of Lemma 2.4.2. Moreover, the map $\phi^* : \overline{\mathcal{A}}(\uparrow t) \rightarrow \mathbb{C}[x_d^{\pm 1}]_d \otimes \overline{\mathcal{A}}(t)$ defined by formula (23) is an isomorphism because of Theorem 3.0.15. The third axiom of Definition 3.1.1 holds by statement 8 of Lemma 2.4.2. The other axioms are trivial to check.

We refer to this as the *standard quasi polynomial structure* of $\overline{\mathcal{A}}(\uparrow t^D)$. Of course, $\text{Spec}(\mathbb{C}[x_d^{\pm 1}]_d)$ is a torus with $(x_d)_d$ as base of characters.

Form now on, we fix an almost polynomial ring R over A . Let $s : R \rightarrow A$ be the morphism induced by the map $\mathbb{C}[T] \otimes A \rightarrow A$ sending $p \otimes a$ to $p(1)a$. Note that, if $R = \overline{\mathcal{A}}(\uparrow t^D)$, considered with its standard quasi-polynomial structure, then s is the deletion map of Corollary 3.0.18. We consider \mathbb{Z}^D ordered by $\lambda \leq \mu$ if the inequality holds component-wise.

Definition 3.1.3 (Pole filtration). The *pole filtration* on the algebra A is the (\mathbb{Z}^D, \leq) filtration defined by

$$A_\lambda := \{a \in A : \forall d \in D, \mathcal{V}_d(1 \otimes a) \geq -\lambda(d)\}.$$

If H is an abelian group and $A = \bigoplus_{h \in H} A_h$ is a graduation, we set $A_{\lambda, h} = A_\lambda \cap A_h$. We say that the pole filtration is H -graded, if for any $\lambda \in \mathbb{Z}^d$, $A_\lambda = \bigoplus_h A_{\lambda, h}$.

It's easy to verify that, if $\mu \leq \lambda$, then $A_\mu \subseteq A_\lambda$, moreover for any λ, μ we have $A_\lambda \cdot A_\mu \subseteq A_{\lambda+\mu}$ and finally $A_\lambda + A_\mu \subseteq A_{\max\{\lambda, \mu\}}$. We express this last inclusion saying that the pole filtration is a *tropical filtration*.

We introduce a notation. Let G and H be abelian groups and $B = \bigoplus_{(g, h) \in G \times H} B_{g, h}$ be a $G \times H$ -graded \mathbb{C} -algebra. The G -graduation induced by the $G \times H$ one, on B , is defined as follows:

$$B = \bigoplus_{g \in G} B_{g, \bullet} \quad \text{where, for } g \in G \quad B_{g, \bullet} = \bigoplus_{h \in H} B_{g, h}.$$

Similarly, we have an induced H -graduation, for which we use an analogue notation.

Definition 3.1.4 (lifting graduation). A \mathbb{Z}^D -graduation $R = \bigoplus_{\lambda \in \mathbb{Z}^D} R_\lambda$ is called *lifting graduation*, of the pole filtration, if the following conditions hold

1. For any $d \in D$, $X_d \in R_{e_d}$.
2. The isomorphism ϕ^* identifies $1 \otimes A$ with the 0-degree component of $R_{\prod X_D}$.
3. For any $\lambda \in \mathbb{Z}^D$, the restriction of s to R_λ is an isomorphism with A_λ .

If the pole filtration is H -graded, we say that a $\mathbb{Z}^D \times H$ graduation $R = \bigoplus_{\lambda, h} R_{\lambda, h}$ is an H -graded lifting graduation if

1. $R = \bigoplus_{\lambda} R_{\lambda, \bullet}$ is a lifting graduation.
2. For any $(\lambda, h) \in \mathbb{Z}^D \times H$, the restriction of s to $R_{\lambda, h}$ is an isomorphism with $A_{\lambda, h}$.

It often happens, in representation theory, that we study the components of the pole filtration on some algebra, because they are the shadows of the homogeneous components of the lifting graduation, which is the real object of interest. For example, some classical expressions for the generalised Littlewood-Richardson coefficients, such as [Zel02, Proposition 1.4], can be interpreted in this setting. We explain this in section 7.1.

The following lemma is obvious but useful.

Lemma 3.1.5. *A $\mathbb{Z}^D \times H$ graduation $R = \bigoplus_{\lambda, h} R_{\lambda, h}$ is an H -graded lifting graduation if and only if*

1. $R = \bigoplus_{\lambda} R_{\lambda, \bullet}$ is a lifting graduation.
2. The map s is H -equivariant.

Recall that if \tilde{T} is a torus and \tilde{R} is a \mathbb{C} -algebra, there is a bijection between left actions of \tilde{T} on $\text{Spec}(\tilde{R})$ and (multiplicative) gradations $\tilde{R} = \bigoplus_{\lambda \in X(\tilde{T})} \tilde{R}_{\lambda}$. Explicitly, if $\alpha : \tilde{T} \times \text{Spec}(\tilde{R}) \rightarrow \text{Spec}(\tilde{R})$ is an action, then \tilde{R}_{λ} is the set of $r \in \tilde{R}$ such that $\alpha^*(r) = \lambda \otimes r$. We adopt the convention that, for any \mathbb{C} -scheme Z , $\tilde{T} \times Z$ is canonically acted on by \tilde{T} by multiplication on the \tilde{T} component. In particular, when $Z = \text{Spec}(\tilde{A})$, for a \mathbb{C} -algebra \tilde{A} , then $(\mathbb{C}[\tilde{T}] \otimes \tilde{A})_{\lambda} = \lambda \otimes \tilde{A}$. When \tilde{T} has a fixed base of characters $\varpi \subseteq X(\tilde{T})^J$, then we identify \mathbb{Z}^J with $X(\tilde{T})$ through the map $n \mapsto \varpi^n$.

In our context, the torus T has a fixed base of characters determined by R : $(x_d)_d$.

Proposition 3.1.6. *Let $R = \bigoplus_{\lambda \in \mathbb{Z}^D} R_{\lambda}$ be a graduation and consider the associated T action on $\text{Spec}(R)$. The graduation on R is a lifting graduation if and only if the morphism $\phi : T \times \text{Spec}(A) \rightarrow \text{Spec}(R)$, induced by ϕ^* , is T -equivariant. In particular, if the lifting graduation exists, it is unique.*

Proof. We identify R with a subring of $\mathbb{C}[T] \otimes A$, using ϕ^* . Suppose that $R = \bigoplus_{\lambda \in \mathbb{Z}^D} R_{\lambda}$ is a lifting graduation. We need to prove that $R_{\lambda} \subseteq x^{\lambda} \otimes A$. Let $r \in R_{\lambda}$, by condition 3 in Definition 3.1.4, $s(r) \in A_{\lambda}$. Recall that $X_d = x_d \otimes 1$. Since $R = \bigcap_{d \in D} \mathcal{V}_d(\mathbb{Z}_{\geq 0} \cup \{\infty\}) \cap R_{\prod X_d}$ and $\mathcal{V}_{d_1}(X_{d_2}) = \delta_{d_1, d_2}$, we have that $x^{\lambda} \otimes s(r) \in R$. Moreover, since $X_d \in R_{e_d}$ and $1 \otimes s(r)$ is of degree zero, then $x^{\lambda} \otimes s(r) \in R_{\lambda}$. But the restriction of s to R_{λ} is injective. Hence $r = x^{\lambda} \otimes s(r)$.

Conversely, suppose that ϕ is equivariant. Since $X_d = x_d \otimes 1$, we clearly have that $X_d \in R_{e_d}$. Similarly, condition 2 of Definition 3.1.4 is trivial to check. We want to verify condition 3. Take $r \in R_{\lambda}$, then $r \in x^{\lambda} \otimes A$, in particular $r = x^{\lambda} \otimes s(r)$. Again, since $R = \bigcap_{d \in D} \mathcal{V}_d^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\}) \cap R_{\prod X_D}$ and $\mathcal{V}_{d_1}(X_{d_2}) = \delta_{d_1, d_2}$, we deduce that $s(r) \in A_{\lambda}$. Moreover, if $s(r) = 0$, then clearly $r = 0$. This proves that the restriction of s to R_{λ} is injective with image in A_{λ} . For the surjectivity, notice that if $a \in A_{\lambda}$, then $x^{\lambda} \otimes a \in R_{\lambda}$ and $s(x^{\lambda} \otimes a) = a$.

For the unicity part, it's sufficient to prove that if a torus action as in the statement of the proposition exists, then it is unique. But this is clear since R is a domain, hence $\text{Spec}(R_{\prod X_D})$ is schematically dense in $\text{Spec}(R)$. \square

The following corollary is a useful reformulation of the above proposition.

Corollary 3.1.7. *A graduation $R = \bigoplus_{\lambda \in \mathbb{Z}^D} R_\lambda$ is the lifting graduation if and only if, for any $r \in R_\lambda$, then $\phi^*(r) = x^\lambda \otimes s(r)$.*

Proposition 3.1.8. *If the lifting graduation $R = \bigoplus_{\lambda \in \mathbb{Z}^D} R_\lambda$ exists, then $\ker(s) = (X_d - 1 : d \in D)$.*

Proof. Let $J = (X_d - 1 : d \in D)$. A standard calculation allows see that if $\lambda \geq 0$, then $X^\lambda - 1$ is in J .

We claim that, for any $r \in R$ such that $s(r) \neq 0$ and for any sufficiently large ν , $r - x^\nu \otimes s(r)$ is in J . Let's see that the claim allows to conclude. Indeed, take $y \in \ker(s)$ and any $r_\lambda \in R_\lambda$ such that $s(r_\lambda) \neq 0$. In particular, $r_\lambda = x^\lambda \otimes s(r_\lambda)$. Then for any ν we can write

$$y = (y + r_\lambda) - r_\lambda = (y + r_\lambda - x^\nu \otimes s(r_\lambda)) + r_\lambda(x^{\nu-\lambda} \otimes 1 - 1 \otimes 1).$$

Observe that $s(y + r_\lambda) = s(r_\lambda) \neq 0$. In particular, if ν is sufficiently large, the claim implies that $(y + r_\lambda - x^\nu \otimes s(r_\lambda))$ is in J , but if $\nu \geq \lambda$, we also have that $r_\lambda(x^{\nu-\lambda} \otimes 1 - 1 \otimes 1) = r_\lambda(X^{\nu-\lambda} - 1)$ is in J .

Let's prove the claim. Write $r = \sum r_\mu$, where $r_\mu \in R_\mu$ and the sums runs over a finite set M , such that $r_\mu \neq 0$ if $\mu \in M$. By the definition of lifting graduation we have that, for any $d \in D$, $\mathcal{V}_d(1 \otimes s(r_\mu)) \geq -\mu(d)$. Then, if $\nu \geq \max\{\mu : \mu \in M\}$, we have that for any $\mu \in M$, $x^\nu \otimes s(r_\mu) \in R_\nu$, hence $x^\nu \otimes s(r) \in R_\nu$. Then, for any such ν ,

$$r - x^\nu \otimes s(r) = \sum_{\mu} (x^\mu \otimes s(r_\mu))(1 \otimes 1 - x^{\nu-\mu} \otimes 1) = \sum_{\mu} r_\mu(1 - X^{\nu-\mu}).$$

Since for any μ in the above sum $\nu - \mu \geq 0$, this proves the claim. \square

Proposition 3.1.9. *In the notation of Example 3.1.2, the \mathbb{Z}^D graduation on $\overline{\mathcal{A}}(1t^D)$ induced by $\uparrow 0$ is the lifting graduation. If t is a graded seed, then the pole filtration is H -graded and the graduation induced by $\uparrow \sigma$, on $\overline{\mathcal{A}}(1t^D)$, is the H -graded lifting graduation.*

Proof. For the first statement, we want to apply Corollary 3.1.7. Recall that $\uparrow 0$ defines a graduation on $\mathcal{L}(1t^D)$, for which the cluster monomials form a basis of homogeneous elements. Moreover, $\overline{\mathcal{A}}(1t^D)$ is a graded subring of $\mathcal{L}(1t^D)$. In particular, it's sufficient to prove that for any $i \in \uparrow I$, $\phi^*(\uparrow x_i) = \widehat{x}_D^{\uparrow 0_i} \otimes s(\uparrow x_i)$. But this is obvious from the definitions.

For the second statement, notice that the same argument as before proves that $s : \overline{\mathcal{A}}(1t^D) \rightarrow \overline{\mathcal{A}}(t)$ is H -equivariant. Using the first statement of the proposition, we have that for any $\lambda \in \mathbb{Z}^D$, $s : \overline{\mathcal{A}}(1t^D)_{\lambda, \bullet} \rightarrow \overline{\mathcal{A}}(t)_\lambda$ is an isomorphism. It follows at once that the pole filtration is H -graded. Then the second statement follows from Lemma 3.1.5. \square

Corollary 3.1.10. *The deletion map $s : \overline{\mathcal{A}}(1t^D) \rightarrow \overline{\mathcal{A}}(t)$ is surjective with kernel the ideal generated by $\uparrow x_d - 1$, for $d \in D$.*

Proof. Apply Propositions 3.1.9 and 3.1.8. \square

To the author it's not clear if Corollary 3.1.10 can be deduced from Corollary 3.0.18.

Corollary 3.1.11. *Suppose that $D = D_1 \sqcup D_2$. Denote by $\nu_1 = \nu_{D_1, \bullet}$, $\uparrow t \xleftarrow{\nu} t$ and $\tilde{\uparrow} t \xleftarrow{\nu_1} t$. Then $\overline{\mathcal{A}}(\tilde{\uparrow} t^{D_1})$ is the quotient of $\overline{\mathcal{A}}(\uparrow t^D)$ by the ideal $(\uparrow x_d - 1 : d \in D_2)$.*

Proof. By Lemma 3.0.5, we have that if $\nu_2 = \nu_{D_2, \bullet}$, then $\uparrow t^{D_1} \xleftarrow{(\nu_2, 0)} \tilde{\uparrow} t^{D_1}$. The statement follows from Corollary 3.1.10. \square

4 The minimal monomial lifting

We use the following notation. If Z is a \mathbb{C} -scheme and $f \in \mathcal{O}_Z(Z)$, we denote with $V(f)$ the zero locus of f in Z , which is a closed subspace of the topological space underlying Z . We write $Z^{(1)}$ for the set of codimension one points of Z . If Z is normal and integral, for $p \in Z^{(1)}$, we denote with $\mathcal{V}_p : \mathbb{C}(Z) \rightarrow \mathbb{Z} \cup \{\infty\}$ the valuation induced, on the fraction field of Z , by the discrete valuation ring $\mathcal{O}_{Z,p}$. If $D \subseteq Z$ is closed, irreducible and of codimension one, it has a unique generic point p . Then we set $\mathcal{V}_D = \mathcal{V}_p$ and we say that \mathcal{V}_p is the valuation associated to D . We say that $f, g \in \mathcal{O}_Z(Z)$ are *coprime* on Z if $V(f) \cap V(g) \cap Z^{(1)} = \emptyset$. Finally, we say that an open subset $\Omega \subseteq Z$ is *big* if $\Omega^{(1)} = Z^{(1)}$.

From now on, D is a finite set and T is torus of rank $|D|$.

Definition 4.0.1. A *suitable for D -lifting* scheme is a triple (\mathfrak{X}, ϕ, X) such that:

1. \mathfrak{X} is a noetherian, normal, integral \mathbb{C} -scheme.
2. $X \subseteq \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})^D$ is a collection of global sections such that, for each $d \in D$, $V(X_d)$ is irreducible and if \mathcal{V}_d is the associated valuation, then $\mathcal{V}_{d_1}(X_{d_2}) = \delta_{d_1, d_2}$.
3. $\phi : T \times Y \rightarrow \mathfrak{X} \setminus \cup_d V(X_d)$ is an open embedding such that the image is big and $\phi^*(X_d) = (x_d \otimes 1)$, where the collection of x_d is a base of $X(T)$. Here Y is an irreducible \mathbb{C} -scheme.

When the data of ϕ and X is clear, we just say that \mathfrak{X} is suitable for D -lifting. If a scheme suitable for D -lifting is called \mathfrak{X} , we implicitly assume that the rest of the data defining it is denoted by ϕ, Y and X and that \mathcal{V}_d is the valuation associated to $V(X_d)$.

We use the following convention. If \mathfrak{X} is suitable for D -lifting, then we identify $T \times Y$ with an open subset of \mathfrak{X} using ϕ . Hence, we write $\mathbb{C}(\mathfrak{X}) = \mathbb{C}(T \times Y)$ and we identify $\mathbb{C}(Y)$ with $1 \otimes \mathbb{C}(Y) \subseteq \mathbb{C}(\mathfrak{X})$. Then, $\mathbb{C}(\mathfrak{X})$ is canonically a D -field extension of $\mathbb{C}(Y)$: its D -uple is X . We stress that, with these identifications, if t is a seed of $\mathbb{C}(Y)$ such that $\overline{\mathcal{A}}(t) = \mathcal{O}_Y(Y)$, then $\widehat{x} \subseteq (\mathbb{C}(\mathfrak{X})^*)^{I \sqcup D}$ is defined by

$$\widehat{x}_i = \begin{cases} 1 \otimes x_i & \text{if } i \in I \\ X_d & \text{if } i = d \in D \end{cases}$$

We can construct affine schemes which are suitable for lifting as follows

Example 4.0.2. Let R be a finite type \mathbb{C} -algebra which is a normal domain. Consider a collection of elements $X \subseteq R^D$ such that the ideals (X_d) are prime and pairwise different. Suppose that there is an isomorphism $\psi : R_{\prod_d X_d} \rightarrow \mathbb{C}[T] \otimes A$ such that $\psi(X_d) = x_d \otimes 1$, where the x_d form a base of $X(T)$. Here A is a \mathbb{C} -algebra. Then $(\text{Spec}(R), \psi^*, X)$ is suitable for D -lifting.

We now state the main theorem of this section.

Theorem 4.0.3. *Let \mathfrak{X} be a suitable for D -lifting scheme and t be a maximal rank seed of $\mathbb{C}(Y)$ such that $\overline{\mathcal{A}}(t) = \mathcal{O}_Y(Y)$. Suppose that, for any non-equal $i, j \in I_{uf}$, x_i and x_j are coprime on Y and that, for any $k \in I_{uf}$, x_k and $x'_k = \mu_k(x_k)$ are coprime on Y .*

Consider $(\mathbb{C}(\mathfrak{X}), \nu)$: the D -lifting configuration on t defined by

$$\nu_{d,i} := -\mathcal{V}_d(1 \otimes x_i)$$

Then

1. $\overline{\mathcal{A}}(\uparrow t) = \mathcal{O}_{\mathfrak{X}}(T \times Y)$.
2. $\overline{\mathcal{A}}(\uparrow t^D) \subseteq \mathcal{O}_{\mathfrak{X}}(X)$.

Definition 4.0.4. In the setting of Theorem 4.0.3, ν and $\uparrow t^D$ are called respectively the *minimal lifting matrix* and the *minimal monomial lifting* of t , associated to \mathfrak{X} .

The proof is based on the well known algebraic Hartogs' lemma

Lemma 4.0.5 (Hartogs'). *Let Z be a locally noetherian normal integral scheme, then $\mathcal{O}_Z(Z) = \bigcap_{p \in Z^{(1)}} \mathcal{O}_{Z,p}$, where the intersection takes place in $\mathbb{C}(Z)$.*

Note that $\mathcal{O}_{Z,p}$ consists of the rational functions with non-negative valuation on p .

For the proof, we need the following, which is usually called starfish lemma. Notice that the proof is exactly the same of the affine case [FP16, Proposition 3.6].

Lemma 4.0.6 (starfish lemma). *Let Z be a locally noetherian normal integral scheme over \mathbb{C} and t be a seed of $\mathbb{C}(Z)$ such that*

1. *For any $i \in I$, $x_i \in \mathcal{O}_Z(Z)$ and if $i \in I_{sf}$, $x_i \in \mathcal{O}_Z(Z)^*$.*
2. *For any $k \in I_{uf}$, $x'_k := \mu_k(x_k) \in \mathcal{O}_Z(Z)$.*
3. *For any non equal $i, j \in I_{uf}$, x_i and x_j are coprime on Z and for any $i \in I_{uf}$, x_i and x'_i are coprime on Z .*

Then $\overline{\mathcal{A}}(t) \subseteq \mathcal{O}_Z(Z)$.

Proof. Let $f \in \overline{\mathcal{A}}(t)$ and $p \in Z^{(1)}$, since $f \in \mathcal{L}(t)$ and for any $i \in I_{sf}$ $x_i \in \mathcal{O}_Z(Z)^*$, if there is no $k \in I_{uf}$ such that $x_k(p) = 0$, then $f \in \mathcal{O}_{Z,p}$. If it's not the case, take $k \in I_{uf}$ such that $x_k(p) = 0$. Then, for any $j \in I_{uf} \setminus \{k\}$, $x_j(p) \neq 0$ and $x'_k(p) \neq 0$. But $f \in \mathcal{L}(\mu_k(t))$, hence we deduce that $f \in \mathcal{O}_{Z,p}$. We conclude using Lemma 4.0.5. Actually, this proves the stronger statement that the upper bound $\mathcal{U}(t)$ is included in $\mathcal{O}_Z(Z)$. \square

Proof of Theorem 4.0.3. First, if $e \in T(\mathbb{C})$ is the identity, identifying Y with $\{e\} \times Y$ gives an immersion (neither closed nor open) $\iota : Y \rightarrow \mathfrak{X}$. Since ι is a monomorphism and \mathfrak{X} is quasi-separated, being noetherian, it follows that Y is quasi-separated. Moreover, \mathfrak{X} is locally noetherian and ι is an immersion, hence ι is quasi-compact. In particular, Y is quasi-compact since \mathfrak{X} is. Then, we can cover Y with finitely many affine open subsets Y_i . Since Y is quasi-separated, we can cover

each intersection $Y_i \cap Y_j$ with finitely many affine open subsets $Y_{i,j,k}$. Consider the natural exact sequence

$$0 \longrightarrow \mathcal{O}_Y(Y) \longrightarrow \prod_i \mathcal{O}_Y(Y_i) \longrightarrow \prod_{i,j,k} \mathcal{O}_Y(Y_{i,j,k})$$

Note that the two products in the exact sequence are finite. Since $\mathbb{C}[T] \otimes (-)$ is exact, commutes with finite products and the $Y_i, Y_{i,j,k}$ are affine, tensoring with $\mathbb{C}[T]$ yields the exact sequence

$$0 \longrightarrow \mathbb{C}[T] \otimes \mathcal{O}_Y(Y) \longrightarrow \prod_i \mathcal{O}_{T \times Y}(T \times Y_i) \longrightarrow \prod_{i,j,k} \mathcal{O}_{T \times Y}(T \times Y_{i,j,k})$$

But since the $T \times Y_i$ cover $T \times Y$, and the $T \times Y_{i,j,k}$ cover $(T \times Y_i) \cap (T \times Y_j)$, we deduce from the above exact sequence that $\mathbb{C}[T] \otimes \mathcal{O}_Y(Y) \simeq \mathcal{O}_{T \times Y}(T \times Y)$. Finally, we deduce the first statement from Theorem 3.0.15 and the fact that ϕ is an open embedding.

For the second statement, we want to apply the starfish lemma. Let $p \in \mathfrak{X}^{(1)}$. Since \mathfrak{X} is suitable for lifting, $p \in (T \times Y)^{(1)}$ or $\bar{p} = V(X_d)$ for a certain $d \in D$. In the last case, d is unique. Note that for any $m \in \mathbb{Z}^D$, then

$$\mathcal{V}_p(\widehat{x}_D^m) = \begin{cases} 0 & \text{if } p \in (T \times Y)^{(1)} \\ m_d & \text{if } \bar{p} = V(X_d). \end{cases}$$

Then we easily compute that, for any $i \in I$,

$$\mathcal{V}_p(\upharpoonright x_i) = \begin{cases} \mathcal{V}_p(1 \otimes x_i) & \text{if } p \in (T \times Y)^{(1)} \\ 0 & \text{if } \bar{p} = V(X_d). \end{cases} \quad (24)$$

It follows from the Hartogs' lemma that $\upharpoonright x_i \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. Moreover, if $i \in]I_{sf} = I_{sf}$ then $\upharpoonright x_i \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})^*$ and clearly $\upharpoonright x_d = X_d \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. Now fix $i, j \in I_{uf} =]I_{uf}^D$ with $i \neq j$, we want to prove that $\upharpoonright x_i$ and $\upharpoonright x_j$ are coprime on X . By the previous calculation, it's sufficient to prove that they're coprime on $T \times Y$, which is equivalent to the comprimality of $1 \otimes x_i$ and $1 \otimes x_j$ because, for any $d \in D$, $X_d \in \mathcal{O}_{\mathfrak{X}}(T \times Y)^*$. Take a point $p \in T \times Y$ such that $1 \otimes x_i(p) = 0 = 1 \otimes x_j(p)$. Take an affine open subset $U \simeq \text{Spec} A$, of Y , such that $p \in T \times U$. Note that A is a normal domain. Then we can identify p with a prime of $\mathbb{C}[T] \otimes A$ which contains $1 \otimes x_i$ and $1 \otimes x_j$. If we look at the natural map $A \rightarrow \mathbb{C}[T] \otimes A$ that sends a to $1 \otimes a$, we have that p^c is a prime ideal of A that contains x_i and x_j ($(-)^c$ denotes the contraction of ideals). Since x_i and x_j are coprime on Y , there exists a non-zero prime q of A such that q is strictly contained in p^c . Since \mathbb{C} is algebraically closed and $\mathbb{C}[T]$ is a finitely generated \mathbb{C} -algebra which is a domain, then for any \mathbb{C} -algebra B which is a domain, $\mathbb{C}[T] \otimes B$ is again a domain. In particular, tensoring with $\mathbb{C}[T]$ sends prime ideals of A to prime ideals of $\mathbb{C}[T] \otimes A$ and it preserves strict inclusion because of exactness. It follows that $0 \subsetneq \mathbb{C}[T] \otimes q \subsetneq \mathbb{C}[T] \otimes p^c \subseteq p$. Hence p has height at least 2, so $1 \otimes x_i$ and $1 \otimes x_j$ are coprime on Y .

Finally, observe that for any $k \in I_{uf}$, $x_k^* := \mu_k(\upharpoonright x_k)$ is regular on $T \times Y$ because of Lemma 3.0.8. Moreover, from the exchange relation (3) and the fact that, for any $d \in D$, $\mathcal{V}_d(\upharpoonright x_k) = 0$, it follows that $\mathcal{V}_d(x_k^*) \geq 0$. Hence $x_k^* \in \mathcal{O}_X(X)$ by Hartogs' Lemma 4.0.5. Finally, call $t' = \mu_k(t)$ and $\nu' = \mu_k(\nu)$. Since $x_k^* = (1 \otimes x'_k) \widehat{x}^{\nu', k}$ by Lemma 3.0.8, the previous argument also proves that x_k^* and $\upharpoonright x_k$ are coprime on X . Then, $\overline{\mathcal{A}}(\upharpoonright t^D) \subseteq \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ follows from Lemma 4.0.6. \square

Corollary 4.0.7 (Of the proof). *If \mathfrak{X} is suitable for D -lifting, then $(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}), \mathcal{V}, X, \phi^*)$ is an almost polynomial ring over $\mathcal{O}_Y(Y)$.*

The following lemma allows to produce a nice class of suitable for D -lifting schemes.

Lemma 4.0.8. *Let (\mathfrak{X}, ϕ, X) be a triple that satisfies the first two conditions of Definition 4.0.1 and such that $\phi : T \times Y \rightarrow \mathfrak{X} \setminus \cup_d V(X_d)$ is an open embedding with big image. If moreover Y is an irreducible \mathbb{C} -scheme such that $\mathcal{O}_Y(Y)^* = \mathbb{C}^*$, and there exist a point $y \in Y$ such that, for any $d \in D$ $\phi^*(X_d)(e, y) = 1$, then (\mathfrak{X}, ϕ, X) is suitable for D -lifting.*

Proof. We just need to prove that, for any $d \in D$, $\phi^*(X_d) = x_d \otimes 1$ and the collection of x_d is a base of $X(T)$.

The argument at the beginning of the proof of Theorem 4.0.3 implies that $\mathcal{O}_{\mathfrak{X}}(T \times Y) = \mathbb{C}[T] \otimes \mathcal{O}_Y(Y)$. Since $\mathcal{O}_Y(Y)$ is a domain and $\mathcal{O}_Y(Y)^* = \mathbb{C}^*$, it follows that

$$\mathcal{O}_{\mathfrak{X}}(T \times Y)^* = \{\lambda \otimes c : \lambda \in X(T), c \in \mathbb{C}^*\}.$$

But $\phi^*(X_d) \in \mathcal{O}_{\mathfrak{X}}(T \times Y)^*$ by Hartogs' lemma, moreover $\phi^*(X_d)(e, y) = 1$. Hence there exist $x_d \in X(T)$ such that $\phi^*(X_d) = x_d \otimes 1$. Let $\lambda \in X(T)^D$ be a base of $X(T)$ and $\text{Div} \in \mathbb{Z}^{D \times D}$ be defined by

$$\text{Div}_{d_1, d_2} = \mathcal{V}_{d_1}(\lambda_{d_2} \otimes 1).$$

Since λ is a base of $X(T)$, there exists $A \in \mathbb{Z}^{D \times D}$ such that $\lambda^A = (x_d)_d$. Hence $\text{Div} \cdot A = \text{Id}$. This implies that $(x_d)_d$ is a base of $X(T)$. □

From now on we suppose to be in the setting of Theorem 4.0.3. In particular, \mathfrak{X} , t are fixed, ν , $\uparrow t^D$ denote respectively the minimal lifting matrix and the minimal monomial lifting of t associated to \mathfrak{X} .

The hypothesis of coprimality of Theorem 4.0.3 often holds. For example:

Lemma 4.0.9. *If Y is affine and $\mathcal{O}_Y(Y)$ is factorial then, for any $i, j \in I_{uf}$ with $i \neq j$, x_i and x_j are coprime on Y . Moreover, for any $k \in I_{uf}$, x_k and $x'_k := \mu_k(x_k)$ are coprime on Y .*

Proof. No mutable vertex of t is completely disconnected because t is of maximal rank. Then, by Theorem 2.1.8, any non semi-frozen cluster variable z is irreducible in $\mathcal{O}_Y(Y)$, hence (z) is prime. In particular, two cluster variables z and z' are coprime if and only if $(z) \neq (z')$. The lemma follows from the second part of Theorem 2.1.8. □

To the author's best knowledge, no general coprimality criteria for cluster variables in an upper cluster algebra is known. In [CKQ22], the authors develop some usefull criteria to study factoriality in upper cluster algebras of maximal rank.

The minimal monomial lifting is the only possible candidate to give a cluster structure on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ compatible with the one on $\mathcal{O}_Y(Y)$ in the following sense.

Theorem 4.0.10. *Let \tilde{t} be a seed of $\mathbb{C}(\mathfrak{X})$, $\mu \in \mathbb{Z}^{D \times I}$ and $\lambda \in \mathbb{Z}^{D \times I_{uf}}$ such that:*

1. \tilde{t} is a D -pointed field field extension of t .

2. For any $i \in I$, $\tilde{x}_i = x_i \widehat{x}_D^{\mu_{\bullet,i}}$.

3. For any $k \in I_{uf}$, $\mu_k(\tilde{x}_k) = \mu_k(x_k) \widehat{x}_D^{\lambda_{\bullet,k}}$.

If $\overline{\mathcal{A}}(\tilde{t}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, then $\mu = \nu$ and $\tilde{t} = \uparrow t^D$.

Proof. For any $d \in D$, $X_d \notin \mathcal{O}_X(X)^*$. It follows that $D \subseteq \tilde{I}_{hf}$. We claim that, for any $d \in D$ and $i \in I$, $\mathcal{V}_d(\tilde{x}_i) = 0$. This implies that $\mu = \nu$ because

$$\mathcal{V}_d(\tilde{x}_i) = \mathcal{V}_d(x_i \otimes 1) + \mu_{d,i} = -\nu_{d,i} + \mu_{d,i}.$$

If $i \in I_{sf}$, since $I_{sf} \subseteq \tilde{I}_{sf}$, then $\tilde{x}_i \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})^*$. Then the claim follows from the Hartogs' lemma. Similarly, if $i \notin I_{sf}$, hence $i \notin \tilde{I}_{sf}$, then \tilde{x}_i is irreducible in $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. Then the claim follows from Hartogs' lemma and the second hypothesis in Definition 4.0.1. For the final statement, it's sufficient to apply Proposition 3.0.9 to \tilde{t}_D . \square

Corollary 4.0.11. *Let $k \in I_{uf}$, $t' = \mu_k(t)$ and $\nu' = \mu_k(\nu)$. If $\overline{\mathcal{A}}(\uparrow t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, then ν' and $\mu_k((\uparrow t')^D)$ are respectively the minimal lifting matrix and the minimal monomial lifting of t' associated to \mathfrak{X} .*

Proof. Because of statement 5 of Lemma 2.4.2 and Lemma 3.0.8, we can apply the previous theorem with $\tilde{t} = \mu_k(\uparrow t^D)$ (considered as an extension of t'), $\mu = \nu'$ and $\lambda_{\bullet,i} = \mu_i(\nu')_{\bullet,i}$. \square

We expect the answer to the following question to be negative, in general, without the assumption that $\overline{\mathcal{A}}(\uparrow t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.

Question 4.0.12. *If $k \in I_{uf}$, is $\mu_k(\nu)$ the minimal lifting matrix of $\mu_k(t)$ associated to \mathfrak{X} ?*

It's interesting to notice that, reinterpreting the results of [Fei21] in this light, as it is done in Section 8.2, one can construct an example where $\mathcal{A}(t) = \overline{\mathcal{A}}(t)$, $\overline{\mathcal{A}}(\uparrow t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ but $\mathcal{A}(\uparrow t^D) \neq \overline{\mathcal{A}}(\uparrow t^D)$. See [Fei21][Example 8.3].

4.1 Equality conditions

In this section we suppose to be in the setting of Theorem 4.0.3. In particular, \mathfrak{X} , t are fixed, ν and $\uparrow t^D$ denote the minimal lifting matrix and the minimal monomial lifting of t , associated to \mathfrak{X} . It's easy to see that we don't always have equality between $\overline{\mathcal{A}}(\uparrow t^D)$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. Because of Theorem 4.0.3, the lack of equality should be caused by some bad behaviour along the divisors $V(X_d)$.

Example 4.1.1. We make an affine example in the spirit of Example 4.0.2. Consider $R = \mathbb{C}[y, z, X, \frac{y+z}{X}]$ where y, z and X are abstract independent variables. Notice that R is a polynomial ring in the variables $y, X, \frac{y+z}{X}$ and we have an isomorphism $R_X \simeq \mathbb{C}[y, z] \otimes \mathbb{C}[X^{\pm 1}]$. The ring $\mathbb{C}[y, z]$ is the upper cluster algebra of the seed t which has only highly-frozen vertices and has y, z as cluster variables. Here D consists of one element associated to the prime ideal generated by X . Applying the minimal monomial lifting we get $\overline{\mathcal{A}}(\uparrow t^D) = \mathbb{C}[y, z, X]$ which is not equal to R . In this very simple example we clearly see the problem: the cluster valuation of $y + z$ along the new frozen vertex is zero. If we had equality between $\overline{\mathcal{A}}(\uparrow t^D)$ and R , it should be one.

Lemma 4.1.2. *For any $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $d \in D$, we have $\mathcal{V}_d(f) \geq C\mathcal{V}_d(f)$. Here $C\mathcal{V}_d$ is the cluster valuation defined in 2.3.1.*

Proof. It's clear from (24) that, for any $i \in \mathbb{1}I$ and $d \in D$, $\mathcal{V}_d(\mathbb{1}x_i) = \mathcal{CV}_d(\mathbb{1}x_i)$. Since $f \in \overline{\mathcal{A}}(\mathbb{1}t)$ by Theorem 4.0.3, then we can write $f = \frac{P}{M}$ where P is a polynomial in the cluster variables of $\mathbb{1}x$, which is not divisible by any unfrozen or semi-frozen variable of $\mathbb{1}t$, and

$$M = \prod_{i \in I_{u,f} \sqcup I_{s,f} \sqcup D} \mathbb{1}x_i^{m_i}$$

is a Laurent monomial. Note that, for any $d \in D$, we have $\mathcal{CV}_d(P) = 0$. Moreover, since $P \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ because of Theorem 4.0.3 and the Laurent phenomenon, then $\mathcal{V}_d(P) \geq 0$. Then

$$\mathcal{CV}_d(f) = \mathcal{CV}_d(P) - \mathcal{CV}_d(M) = -m_d = -\mathcal{V}_d(M) \leq \mathcal{V}_d(P) - \mathcal{V}_d(M) = \mathcal{V}_d(f).$$

□

Proposition 4.1.3. *Suppose that $t^* \in \Delta(\mathbb{1}t)$. The following are equivalent*

1. $\overline{\mathcal{A}}(\mathbb{1}t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.
2. For any $d \in D$, $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \subseteq \mathcal{L}((t^*)^{\{d\}})$.
3. For any $d \in D$, $\mathcal{CV}_d(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})) \subseteq \mathbb{N} \cup \{\infty\}$.
4. For any $d \in D$, $\mathcal{CV}_d \geq \mathcal{V}_d$ over $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.
5. For any $d \in D$, $\mathcal{CV}_d = \mathcal{V}_d$ over $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.

Proof. Condition 1 implies 2 and 2 implies 3 are obvious. For 3 implies 4, suppose there exists $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $d \in D$ such that $n_d = \mathcal{V}_d(f) > \mathcal{CV}_d(f)$. Then $\frac{f}{X_d^{n_d}} \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and $\mathcal{CV}_d(f) < 0$, which is a contradiction. Then 4 implies 5 because of Lemma 4.1.2. Finally, 5 implies that $\mathcal{CV}_d = \mathcal{V}_d$ holds over the fraction field of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, that we denote by \mathbb{F} . By Theorem 4.0.3, \mathbb{F} is also the fraction field of $\overline{\mathcal{A}}(\mathbb{1}t)$. Then Hartogs' lemma and Theorem 4.0.3 imply that $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) = \bigcap_{d \in D} \mathcal{V}_d^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\}) \cap \overline{\mathcal{A}}(\mathbb{1}t)$. Hence, using part 8 of Lemma 2.4.2, we deduce that 5 implies 1. □

Corollary 4.1.4 (of the proof). *Conditions 2 to 5 of Proposition 4.1.3 are equivalent at d fixed, for any $d \in D$.*

Proposition 4.1.5. *Let $D' \subseteq D$, then $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \subseteq \overline{\mathcal{A}}(\mathbb{1}t^{D'})$ if and only if $D' \subseteq \{d : \mathcal{V}_d = \mathcal{CV}_d\}$.*

Proof. Note that $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \subseteq \overline{\mathcal{A}}(\mathbb{1}t^{D'})$ if and only if for any $t^* \in \Delta(\mathbb{1}t)$ and $d \in D'$, $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) \subseteq \mathcal{L}((t^*)^{\{d\}})$. Then the statement follows from Corollary 4.1.4. □

Observe that $\mathcal{O}_Y(Y)$ has two pole filtrations: one coming from $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and the other one coming from the standard almost polynomial structure of $\overline{\mathcal{A}}(\mathbb{1}t^D)$. The one coming from $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ may not admit a lifting graduation.

Corollary 4.1.6. *If $\overline{\mathcal{A}}(\mathbb{1}t^D) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$, then the two pole filtrations on $\mathcal{O}_Y(Y)$ coincide. In particular there exists an action of T on $\text{Spec}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}))$ such that the map $T \times \text{Spec}(\mathcal{O}_Y(Y)) \rightarrow \text{Spec}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}))$ induced by ϕ^* is equivariant.*

Proof. Because of Proposition 4.1.3, $\mathcal{V} = \mathcal{CV}$. Hence the two pole filtrations on $\mathcal{O}_Y(Y)$ coincide. The rest of the corollary is a direct consequence of Propositions 3.1.9 and 3.1.6. □

Thus, when \mathfrak{X} is affine, the existence of a torus action on \mathfrak{X} extending the one on $T \times Y$ is a necessary condition for having equality between $\overline{\mathcal{A}}(\mathbb{1}^D)$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. In general, we have the following lemma.

Lemma 4.1.7. *Suppose that there is an action of T on \mathfrak{X} which extends the one on $T \times Y$. Then $\overline{\mathcal{A}}(\mathbb{1}^D)$, with the graduation induced by $\mathbb{1}0$, is a graded subalgebra of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ which is \mathbb{Z}^D -graded by the action of T .*

Proof. Consider the induced T -action on $\text{Spec}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}))$. By Theorem 4.0.3 we have an open embedding $T \times \text{Spec}(\mathcal{O}_Y(Y)) \rightarrow \text{Spec}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}))$ induced by ϕ^* , which is clearly T -equivariant. By Lemma 3.1.6, the graduation on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is a lifting graduation. We denote by $\deg(f)$ the degree of an homogeneous element $f \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$. From the definition of lifting graduation we have that, for $d \in D$, $\deg(\mathbb{1}x_d) = \deg(X_d) = e_d$. Moreover, for $i \in I$,

$$\deg(\mathbb{1}x_i) = \deg(\mathbb{1} \otimes x_i) + \deg(X^{\nu_{\bullet, i}}) = \nu_{\bullet, i}.$$

Hence, for any $i \in I$, the degree of $\mathbb{1}x_i$ as an element of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is the same as its degree as an element of $\overline{\mathcal{A}}(\mathbb{1}^D)$. In particular, the two graduations coincide on $\mathcal{L}(\mathbb{1}t)$. The lemma follows because both $\overline{\mathcal{A}}(\mathbb{1}^D)$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ are graded subalgebras of $\mathcal{L}(\mathbb{1}t)$. \square

Definition 4.1.8. A suitable for D -lifting scheme is said to be *homogeneously suitable for D -lifting* if the T -action of T on $T \times Y$ extends to an action on \mathfrak{X} .

Corollary 4.1.6 and Lemma 4.1.7 suggest that homogeneous suitable for D -lifting schemes, should provide a good environment where the minimal monomial lifting technique could give important informations.

Finally, we improve Proposition 4.1.3 in the affine case. That is (\mathfrak{X}, ϕ, X) is constructed as in Example 4.0.2 from a given (R, ψ, X) . Let $t^* \in \Delta(\mathbb{1}t)$ and recall that $\mathbb{C}[t^*]$ denotes the polynomial ring in the cluster variables of the seed t^* . By Theorem 4.0.3, we have inclusions $\mathbb{C}[t^*] \subseteq R \subseteq \mathcal{L}(t^*)$, which corresponds to maps

$$\text{Spec}(\mathcal{L}(t^*)) \xrightarrow{\iota_{t^*}} \mathfrak{X} \xrightarrow{\pi_{t^*}} \text{Spec}(\mathbb{C}[t^*])$$

where ι_{t^*} is an open embedding whose image is a principal open subset.

Proposition 4.1.9. *The following are equivalent*

1. $\overline{\mathcal{A}}(\mathbb{1}^D) = R$.
2. For any $d \in D$ there exists a map $\iota_{t^*}^d : \text{Spec}(\mathcal{L}((t^*)^{\{d\}})) \rightarrow \mathfrak{X}$ extending ι_{t^*} . In this case $\iota_{t^*}^d$ is an open embedding.
3. If p_d (resp. q_d^*) is the point of \mathfrak{X} (resp. $\text{Spec}(\mathbb{C}[t^*])$) corresponding to the ideal generated by X_d , then $\pi_{t^*}(p_d) = q_d^*$. That is π_{t^*} does not contract the divisor $\overline{\{p_d\}}$.

Proof. Condition 1 implies 2 is a reformulation of the same implication in Proposition 4.1.3. The fact that, if $\iota_{t^*}^d$ exists, then it is an open embedding, is because if $R \subseteq \mathcal{L}((t^*)^{\{d\}})$, then the localisation of R at all the unfrozen and semi-frozen variables of $(t^*)^{\{d\}}$ is $\mathcal{L}((t^*)^{\{d\}})$ since $\mathbb{C}[t^*] \subseteq R$. If 2 holds, then $\text{Spec}(\mathcal{L}((t^*)^{\{d\}}))$ can be identified with an open subscheme of \mathfrak{X} containing p_d . Since

$\pi_{t^*} \circ \iota_{t^*}^d$ corresponds to the natural inclusion $\mathbb{C}[t^*] \subseteq \mathcal{L}((t^*)^{\{d\}})$, it clearly sends p_d to q_d^* . Finally, if condition 3 holds, then we have an inclusion $\mathcal{O}_{\text{Spec}(\mathbb{C}[t^*]), q_d^*} \subseteq \mathcal{O}_{\mathfrak{X}, p_d}$. But these two rings are discrete valuation domains with the same fraction field, hence they have to be equal. In particular $\mathcal{CV}_d = \mathcal{V}_d$ on $\mathbb{C}(\mathfrak{X})$, hence condition 3 implies 1 by Proposition 4.1.3. \square

5 Preliminaries on algebraic groups

5.1 Some classical facts and notation

In this section we introduce the notation and recall some classical facts about algebraic groups, we mostly follow [FZ99]. From now on, unless explicitly stated, G is a complex, semisimple, simply connected algebraic group of rank r and B, B^- are opposite Borel subgroups with unipotent radicals U and U^- respectively. Let $T = B \cap B^-$, which is a maximal torus of G , and $W = N_G(T)/T$ be the Weyl group. Denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, \mathfrak{b}^-, \mathfrak{u}^-$ and \mathfrak{h} the Lie algebras of G, B, U, B^-, U^- and T respectively. We denote by $X(T)$ and $X(T)^\vee$ the set of characters and cocharacters of T respectively. We write $\langle -, - \rangle$ for the natural pairing between $X(T)$ and $X(T)^\vee$. Let $\Phi \subseteq X(T)$ be the root system of G and $\Delta \subseteq \Phi$ be the set of simple roots determined by B . We denote by Φ^+ the positive roots and $\Phi^- = -\Phi^+$ the negative ones, so that the root space $\mathfrak{g}_\alpha \subset \mathfrak{u}$ for $\alpha \in \Phi^+$. For $\beta \in \Phi$, $\beta^\vee \in X(T)^\vee$ is the corresponding coroot. We denote by $A = (a_{\alpha,\beta})_{\alpha,\beta \in \Delta}$ the *Cartan matrix*, which is defined by $a_{\alpha,\beta} = \langle \beta, \alpha^\vee \rangle$. The character group $X(T)$ is free on the set of fundamental weights ϖ_α , indexed by the simple roots $\alpha \in \Delta$. For $\beta \in \Phi$, we denote by $s_\beta \in W$ the associated reflection. The set $\{s_\alpha : \alpha \in \Delta\}$ is a set of simple reflection of W as a Coxeter group.

Given a representation V of G and $\mu \in X(T)$, we denote the associated *weight space* by

$$V_\mu = \{v \in V : t \cdot v = \mu(t)v \text{ for } t \in T\}.$$

Recall that $V = \bigoplus_{\mu \in X(T)} V_\mu$. We consider V^* as a representation of G as follows: if $\varphi \in V^*$, $g \in G$ and $v \in V$, then $(g\varphi)(v) = \varphi(g^{-1}v)$. Let $X(T)^+$ be the set of dominant characters of T . For $\lambda \in X(T)^+$, the associated irreducible representation of G is denoted by $V(\lambda)$. If $\lambda \in X(T)$, we denote by $\lambda^* = -w_0\lambda$. This defines a linear automorphism of $X(T)$ that stabilises $X(T)^+$. In particular, $V(\lambda)^* = V(\lambda^*)$.

The following discussion holds for a general connected reductive group G . Consider the action of $G \times G$ on G defined by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. This gives to $\mathbb{C}[G]$ the structure of a $G \times G$ module. We have an isomorphism of $G \times G$ representations:

$$\begin{aligned} \bigoplus_{\lambda \in X(T)^+} V(\lambda)^* \otimes V(\lambda) &\longrightarrow \mathbb{C}[G] \\ (\varphi_\lambda \otimes v_\lambda)_\lambda &\longmapsto (g \longmapsto \sum_\lambda \varphi_\lambda(gv_\lambda)). \end{aligned} \tag{25}$$

Consider the action of U^- (resp. U) on G by left (resp. right) multiplication. Then,

$$\mathbb{C}[G]^{U^-} = \mathbb{C}[U^- \setminus G] \quad \mathbb{C}[G]^U = \mathbb{C}[G/U].$$

The two rings above inherit from G a natural structure of G -module. For any $\lambda \in X(T)^+$, we fix elements $\varphi_\lambda^- \in (V(\lambda)^*)^{U^-} = V(\lambda)_{-\lambda}^*$ and $v_\lambda^+ \in V(\lambda)^U = V(\lambda)_\lambda$ such that $\varphi_\lambda^-(v_\lambda^+) = 1$. Then, we have isomorphisms of G -modules:

$$\begin{aligned} \bigoplus_{\lambda \in X(T)^+} V(\lambda) &\longrightarrow \mathbb{C}[G]^{U^-} & \bigoplus_{\lambda \in X(T)^+} V(\lambda)^* &\longrightarrow \mathbb{C}[G]^U \\ (x_\lambda)_\lambda &\longmapsto (g \longmapsto \sum_\lambda \varphi_\lambda^-(gx_\lambda)) & (\psi_\lambda)_\lambda &\longmapsto (g \longmapsto \sum_\lambda \psi_\lambda(gv_\lambda^+)). \end{aligned} \tag{26}$$

Moreover, since T normalises U and U^- , T acts on $U^- \setminus G$ (resp. G/U) by left (resp. right) multiplication. It's easy to notice that the homogeneous components of the $X(T)$ graduations induced by these actions, on the two rings, coincide with the decomposition given above into G -modules. In particular, for $\lambda \in X(T)$,

$$(\mathbb{C}[G]^{U^-})_\lambda = V(\lambda) \quad (\mathbb{C}[G]^U)_\lambda = V(\lambda)^*, \quad (27)$$

where of course we set $V(\lambda) = V(\lambda)^* = 0$ if λ is not a dominant character. To avoid confusion, we recall that, accordingly to the notation introduced in Section 3.1

$$\begin{aligned} (\mathbb{C}[G]^{U^-})_\lambda &= \{f \in \mathbb{C}[G]^{U^-} : f(tg) = \lambda(t)f(g) \text{ for any } t \in T, g \in G\} \\ (\mathbb{C}[G]^U)_\lambda &= \{f \in \mathbb{C}[G]^U : f(gt) = \lambda(t)f(g) \text{ for any } t \in T, g \in G\}. \end{aligned}$$

5.2 Generalised minors

For any $\beta \in \Phi^+$, we chose an \mathfrak{sl}_2 -triple $(X_\beta, H_\beta, X_{-\beta})$ such that $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$, $H_\beta \in \mathfrak{h}$, and $[X_\beta, X_{-\beta}] = H_\beta$. For $t \in \mathbb{C}$ and $\beta \in \Phi$, let

$$x_\beta(t) = \exp(tX_\beta). \quad (28)$$

For $\alpha \in \Delta$, denote by $\varphi_\alpha : \mathrm{SL}_2 \rightarrow G$ the morphism determined by the \mathfrak{sl}_2 -triple associated to α , so that

$$\varphi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_\alpha(t) \quad \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \alpha^\vee(t) \quad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{-\alpha}(t).$$

We define

$$\bar{s}_\alpha := \varphi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (29)$$

A sequence $\mathbf{i} = (i_1, \dots, i_l)$, of elements of Δ , is called *reduced expression* for $w \in W$ if $s_{i_1} \dots s_{i_l} = w$ and l is minimal. In this case, $l = \ell(w)$ where ℓ is the *length* function of the Coxeter group W . We denote by $R(w)$ the set of reduced expressions of w , and by w_0 the longest element of W . Recall that, by a result of Matsumoto and Tits, any two reduced expressions of the same element can be obtained from each other by applying a sequence of braid moves. The *support* of w , denoted by $\mathrm{supp}(w)$, is the set of simple reflections appearing in a fixed (equivalently in any) reduced expression of w . Since the family $\{\bar{s}_\alpha : \alpha \in \Delta\}$ verify the braid relations, if $\mathbf{i} = (i_1, \dots, i_l) \in R(w)$, the element

$$\bar{w} = \bar{s}_{i_l} \dots \bar{s}_{i_1}$$

doesn't depend on \mathbf{i} . We will make an extensive use of the following fundamental commutation relations. If $\alpha \in \Delta$ and $t \in \mathbb{C}^*$, then

$$\begin{aligned} x_\alpha(t)\bar{s}_\alpha &= x_{-\alpha}(t^{-1})\alpha^\vee(t)x_\alpha(-t^{-1}) \\ \bar{s}_\alpha^{-1}x_{-\alpha}(t) &= x_{-\alpha}(-t^{-1})\alpha^\vee(t)x_\alpha(t^{-1}). \end{aligned} \quad (30)$$

We consider the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B^- w B$$

and denote by $G_0 = B^-B$ the open cell. Recall that the product induces an isomorphism of varieties between $U^- \times T \times B$ and B^-B . In particular, an element $x \in G_0$ can be written uniquely as $x = [x]_- [x]_0 [x]_+$ where $[x]_- \in U^-$, $[x]_0 \in T$ and $[x]_+ \in U$. Moreover, $G \setminus G_0 = \bigcup_{\alpha \in \Delta} \overline{B^-s_\alpha B}$ and the set of $\overline{B^-s_\alpha B}$ consists of pairwise distinct divisors of G .

For any $\alpha \in \Delta$, we define the *generalised minor* $\Delta_{e,e}^{\varpi_\alpha}$ as the element of $\mathbb{C}[G]$ corresponding to $\varphi_{\varpi_\alpha}^- \otimes v_{\varpi_\alpha}^+$ via (25). Namely

$$\Delta_{e,e}^{\varpi_\alpha}(g) = \varphi_{\varpi_\alpha}^-(gv_{\varpi_\alpha}^+). \quad (31)$$

The generalised minor $\Delta_{e,e}^{\varpi_\alpha}$ is the only regular function on G , such that for any $x \in G_0$, $\Delta_{e,e}^{\varpi_\alpha}(x) = \varpi_\alpha([x]_0)$. Then, if $v, w \in W$, we define the generalised minor $\Delta_{v,w}^{\varpi_\alpha}$ by $\Delta_{v,w}^{\varpi_\alpha} = (\bar{v}, \bar{w}) \cdot \Delta_{e,e}^{\varpi_\alpha}$. Explicitly, if $g \in G$, then

$$\Delta_{v,w}^{\varpi_\alpha}(g) = \Delta_{e,e}^{\varpi_\alpha}(\bar{v}^{-1}g\bar{w}). \quad (32)$$

Note that our notation is slightly different from the classical one where $\Delta_{v,w}^{\varpi_\alpha}$ is denoted by $\Delta_{v\varpi_\alpha, w\varpi_\alpha}$. We denote by \leq_R the right weak order on W : $v \leq_R w$ if and only if $\ell(w) = \ell(v) + \ell(v^{-1}w)$. We recall a bunch of well known properties of the generalised minors that are either trivial or can be found in [FZ99] ([FZ99][Propositions 2.2,2.4, Section 2.7]) and that are largely used in this text.

Lemma 5.2.1 (Basic properties of minors). *Let $\alpha \in \Delta$, $v, w, v', w' \in W$ and $h, h' \in T$, then*

1. *If $v\varpi_\alpha = v'\varpi_\alpha$ and $w\varpi_\alpha = w'\varpi_\alpha$, then $\Delta_{v,w}^{\varpi_\alpha} = \Delta_{v',w'}^{\varpi_\alpha}$.*
2. *$\Delta_{v,w}^{\varpi_\alpha}$ is vU^-v^{-1} -invariant on the left and wUw^{-1} invariant on the right.*
3. *$\Delta_{v,w}^{\varpi_\alpha}(hgh') = v\varpi_\alpha(h)w\varpi_\alpha(h')\Delta_{v,w}^{\varpi_\alpha}(g)$.*
4. *The zero locus of $\Delta_{e,e}^{\varpi_\alpha}$ is $\overline{B^-s_\alpha B}$.*
5. *$\Delta_{v,w}^{\varpi_\alpha}$ vanishes on U if and only if it vanishes on $U \cap vUv^{-1} \cap wU^-w^{-1}$.*
6. *If $v \leq_R w$, then $\Delta_{v,w}^{\varpi_\alpha}$ doesn't vanish on U .*
7. *$\Delta_{e,w}^{\varpi_\alpha}$ doesn't vanish on U .*

Fomin and Zelevinsky proved the following fundamental identity [FZ99, Theorem 1.17], which stands at the very base of the known cluster algebra structures related to G .

Theorem 5.2.2. *For any $\alpha \in \Delta$ and $v, w \in W$, such that $\ell(vs_\alpha) = \ell(v) + 1$ and $\ell(ws_\alpha) = \ell(w) + 1$, then*

$$\Delta_{v,w}^{\varpi_\alpha} \Delta_{vs_\alpha, ws_\alpha}^{\varpi_\alpha} = \Delta_{vs_\alpha, w}^{\varpi_\alpha} \Delta_{v, ws_\alpha}^{\varpi_\alpha} + \prod_{\beta \in \Delta \setminus \{\alpha\}} (\Delta_{v,w}^{\varpi_\beta})^{-a_{\beta,\alpha}}.$$

The following lemma seems well known but we add a proof because of a lack of a precise reference.

Lemma 5.2.3. *For any $\alpha \in \Delta$ and $v, w \in W$, the generalised minor $\Delta_{v,w}^{\varpi_\alpha}$ is irreducible in $\mathbb{C}[G]$. Moreover, the ideals generated by $\Delta_{e,e}^{\varpi_\alpha}$ are pairwise different.*

The proof relies on the following well known fact, which can be found in [PV94] (Theorem 3.7 and proof of Theorem 3.1) and which is frequently used in this text.

Lemma 5.2.4. *Let R be a factorial \mathbb{C} -algebra of finite type and H be a connected algebraic group acting on R . If $X(H) = \{e\}$ then*

- R^H is factorial.
- An element $f \in R^H$ is irreducible, in R^H , if and only if it is irreducible in R .
- If $f \in R^H$, its factorisations into irreducibles in R^H and in R coincide.

Proof of Lemma 5.2.3. From the definitions, for proving that $\Delta_{v,w}^{\varpi_\alpha}$ is irreducible, it's sufficient to prove that $\Delta_{e,e}^{\varpi_\alpha}$ is irreducible. Since G is semi-simple and simply connected, then $\mathbb{C}[G]$ is factorial. Consider the $U^- \times U$ -action on G defined by $(u^-, u) \cdot g = u^- g u^{-1}$. Since $U^- \times U$ is connected and has no non-trivial character, Lemma 5.2.4 applies. Let $\mathbb{C}[T_\beta]_{\beta \in \Delta}$ be a polynomial ring in the independent variables T_β . From (25), we easily deduce that the map

$$\begin{array}{ccc} \mathbb{C}[T_\beta]_{\beta} & \longrightarrow & \mathbb{C}[G]^{U^- \times U} \\ T_\beta & \longmapsto & \Delta_{e,e}^{\varpi_\beta} \end{array}$$

is an isomorphism. Since T_α is irreducible in $\mathbb{C}[T_\beta]_{\beta}$, we deduce from Lemma 5.2.4 that $\Delta_{e,e}^{\varpi_\alpha}$ is irreducible in $\mathbb{C}[G]$. The last part of the statement follows from the fact that $V(\Delta_{v,w}^{\varpi_\alpha}) = \overline{B^- s_\alpha B}$ and these divisors are pairwise distinct. \square

5.3 Cluster structure on $U(w)$

Let $w \in W$, we set $\Phi(w) = \Phi^+ \cap w^{-1}\Phi^-$ the set of *inversions* of w . Let

$$U(w) := U \cap w^{-1}U^-w.$$

This is the T -stable subgroup of U whose Lie algebra is the direct sum of the root spaces corresponding to $\Phi(w)$. We recall the cluster algebra structure constructed by Goodearl and Yakimov on $\mathbb{C}[U(w)]$, in [GY21, Theorem 7.3], which coincides with the one found by Geiss, Leclerc and Schröer, in [GLS11], in the simply laced case, for G simple. Actually, in [GY21] the construction is done in the quantum setting. The specialisation to the classical setting is made possible by [DM21, Theorem 1.6]. Note that [GY21] and [GLS11] adopt different conventions for the meaning of $N(w)$. In what follows, we slightly modify the notation of Goodearl and Yakimov, our notation is almost identical to the one used by Geiss, Leclerc and Schröer.

Fix $\mathbf{i} = (i_l, \dots, i_1) \in R(w)$. We denote

$$w_{\leq k}^{-1} = s_{i_1} \dots s_{i_k}, \quad w_{\leq k} = s_{i_k} \dots s_{i_1} = (w_{\leq k}^{-1})^{-1}. \quad (33)$$

For $k \in \{1, \dots, l\}$ we use the notation

$$k^+ = \min\{k < j \leq l : i_j = i_k\} \cup \{l+1\} \quad k^- = \max\{\{1 \leq j < k : i_j = i_k\} \cup \{0\}\}. \quad (34)$$

Moreover, for a simple root α such that $s_\alpha \in \text{supp}(w)$, we denote

$$\alpha^{\min} = \min\{1 \leq k \leq l : i_k = \alpha\} \quad \alpha^{\max} = \max\{1 \leq k \leq l : i_k = \alpha\}. \quad (35)$$

Recall that $A = (a_{\alpha,\beta})_{\alpha,\beta \in \Delta}$ is the Cartan matrix of Φ . For an integer $N \in \mathbb{N}$, we use the notation

$$[N] = \{1, \dots, N\}.$$

We consider the seed $t = t_{\mathbf{i}}$, of $\mathbb{C}(U(w))$, defined as follows:

- The vertex set is $I = [l]$. Moreover, $I_{sf} = \emptyset$ and $I_{hf} = \{\alpha^{\max} : s_\alpha \in \text{supp}(w)\}$.
- For $j \in I, k \in I_{uf}$, the coefficient $b_{j,k}$ of the generalised exchange matrix B is defined by

$$b_{j,k} = \begin{cases} 1 & \text{if } j = k^- \\ -1 & \text{if } j = k^+ \\ a_{i_j, i_k} & \text{if } j < k < j^+ < k^+ \\ -a_{i_j, i_k} & \text{if } k < j < k^+ < j^+ \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

- For $k \in I$, the cluster variable $x_k \in \mathbb{C}[U(w)]$ is the restriction at $U(w)$ of a principal minor, in particular:

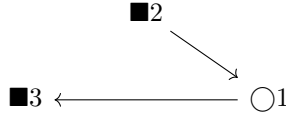
$$x_k = (\Delta_{e, w_{\leq k}^{-1}}^{\varpi_{i_k}})|_{U(w)} \quad (37)$$

We refer to the generalised minor in (37) as the *minor defining* x_k .

Example 5.3.1. Let $G = \text{SL}_3$, $\Delta = \{1, 2\}$ and $\mathbf{i} = (1, 2, 1) \in R(w_0)$. Here $U = U(w_0)$ and

$$U(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

The seed $t = t_{\mathbf{i}}$ is graphically represented by the following quiver



and its cluster variables are

$$x_1 = a \quad x_2 = ac - b \quad x_3 = b.$$

Note that $\mu_1(x_1) = c$.

The following is well known but we add a proof for completeness.

Lemma 5.3.2. *The seed t is of maximal rank.*

Proof. This is the argument of [BFZ05, Proposition 2.6]. Consider the minor of B whose rows are indexed by $I_{uf}^+ = I \setminus \{\alpha^{\min} : s_\alpha \in \text{supp}(w)\}$ and columns indexed by I_{uf} . The map $I_{uf}^+ \rightarrow I_{uf}$ sending i to i^- is a bijection. Order the rows of the minor according to the natural order, induced by \mathbb{N} , and the columns accordingly to the above bijection, that is: $i^- < j^-$ if and only if $i < j$. Then, it's immediate to check that the minor considered is upper triangular with -1 's on the diagonal. \square

Theorem 5.3.3. *We have equalities $\mathcal{A}(t) = \overline{\mathcal{A}}(t) = \mathbb{C}[U(w)]$.*

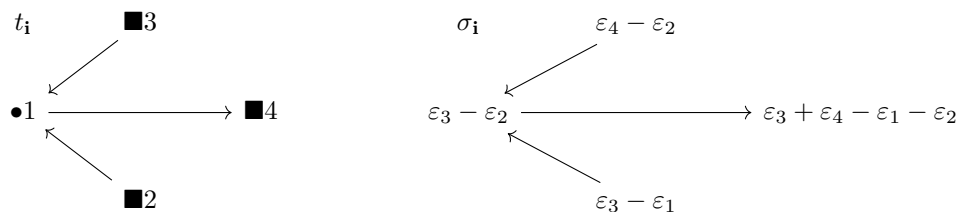
It's known that, for G simple and simply laced and for any $\mathbf{i} \in R(w)$, the seed $t_{\mathbf{i}}$ is injective reachable. See for instance [Qin22] and [GLS11]. Moreover, it's known and not difficult to verify that, in the simply laced case, the cluster structure described above doesn't depend on the reduced expression. In particular if $\mathbf{i}, \mathbf{i}' \in R(w)$, then $t_{\mathbf{i}} \sim t_{\mathbf{i}'}$. To the author's best knowledge, no similar result is known in the non simply-laced case (see [BFZ05, Remark 2.14]).

Finally, we ask a question of combinatorial nature whose importance is explained at the end of Section 7.4.

Question 5.3.4. *Let $w \in W$, $\mathbf{i} \in R(w)$ and $t = t_{\mathbf{i}}$. Moreover, let $\sigma \in X(T)^{\ell(w)}$ be the collection of weights defined by $\sigma_k = w_{\leq k}^{-1} \varpi_{i_k} - \varpi_{i_k}$. Is σ a $X(T)$ -degree configuration on t ?*

We will prove in a different work that the answer to the previous question is positive if G is simply connected and $w = w_0$ is the longest element of W .

Example 5.3.5. Let $G = \mathrm{SL}_4$. We use the notation of [Bou68]. Let $\mathbf{i} = (2, 3, 1, 2) \in R(z)$ where z is the permutation $(1, 3)(2, 4)$. We can compute that



The answer to Question 5.3.4 is positive in this case since the only mutable vertex is 1 and $\sigma_{\mathbf{i},4} = \sigma_{\mathbf{i},2} + \sigma_{\mathbf{i},3}$.

5.4 Technical properties of minors I: on vanishing and twist

This section and the next one are quite technical. The reader can skip them for a first reading, even though the results obtained are crucial for proving Theorems 8.1.1, 8.2.1.

In this section we fix $\alpha, \beta \in \Delta$ and $v, w \in W$.

Lemma 5.4.1. *The generalised minor $\Delta_{s_\alpha, w}^{\varpi_\alpha}$ vanishes on U if and only if s_α is not in the support of w .*

Proof. If s_α is not in the support of w , then $\Delta_{s_\alpha, w}^{\varpi_\alpha} = \Delta_{s_\alpha, e}^{\varpi_\alpha}$ because of Lemma 5.2.1. The zero locus of this last function is $\overline{s_\alpha B^- s_\alpha B}$, which clearly contains U .

Next, suppose that $\Delta_{s_\alpha, w}^{\varpi_\alpha}$ vanishes on U . This means that $U \subseteq \overline{s_\alpha B^- s_\alpha B w^{-1}}$. Note that $\overline{s_\alpha B^- s_\alpha B w^{-1}}$ is a closed, irreducible subvariety of G of codimension 1, and it is $s_\alpha B^- s_\alpha$ -stable by left multiplication. Since also $\overline{s_\alpha B^- s_\alpha U} = \overline{s_\alpha B^- s_\alpha B}$ is closed, irreducible and of codimension 1 in G , we have that the previous inclusion is equivalent to $\overline{s_\alpha B^- s_\alpha B} = \overline{s_\alpha B^- s_\alpha B w^{-1}}$. Then

$$\begin{aligned}
 \overline{s_\alpha B^- s_\alpha B} = \overline{s_\alpha B^- s_\alpha B w^{-1}} &\iff \overline{s_\alpha w_0 B w_0 s_\alpha B} = \overline{s_\alpha w_0 B w_0 s_\alpha B w^{-1}} \\
 &\iff \overline{B w_0 s_\alpha B} = \overline{B w_0 s_\alpha B w^{-1}} \\
 &\iff \overline{B s_\alpha w_0 B / B} = \overline{w B s_\alpha w_0 B / B}.
 \end{aligned} \tag{38}$$

The last equality is between two closed, irreducible T -stable, subvarieties of G/B . Recall that the T -fixed points of G/B are canonically identified with W , and that

$$(\overline{BvB}/B)^T = \{x \in W : x \leq v\},$$

where \leq denotes the Bruhat order on W . Note also that by the sub-word property of Coxeter groups, $s_\alpha \in \text{supp}(w)$ if and only if $s_\alpha \leq w$. Moreover, $s_\alpha \in \text{supp}(w) \iff s_\alpha \in \text{supp}(w^{-1})$. Finally, recall that right multiplication by w_0 reverses the Bruhat order. In particular, by looking at T -fixed points, we have that the last equality in (38) implies the first of the following equalities

$$\begin{aligned} \{v : v \leq s_\alpha w_0\} = \{wv : v \leq s_\alpha w_0\} &\iff \{v : vw_0 \geq s_\alpha\} = \{wv : vw_0 \geq s_\alpha\} \\ &\iff \{vw_0 : v \geq s_\alpha\} = \{wvw_0 : v \geq s_\alpha\}. \end{aligned}$$

By contradiction, if $s_\alpha \in \text{supp}(w)$, then $w^{-1} \geq s_\alpha$, hence w_0 belongs to the RHS of the last equality, looking at the LHS we deduce that $e \geq s_\alpha$, which is a contradiction. \square

The following is a well known result about the representation theory of SL_2 . See for example [FZ99, Lemma 2.8]. Recall that, if $\beta \in \Phi$ and n is a positive integer, then the *divided power* $X_\beta^{(n)}$ denotes $\frac{X_\beta^n}{n!}$, which is an element of the envelopping algebra of \mathfrak{g} .

Lemma 5.4.2. *Let V be a representation of \mathfrak{sl}_α and $v^- \in V$ (resp. $v^+ \in V$) a lowest (resp. highest) weight vector of weight $-m$ (resp. m) for $m \in \mathbb{N}$. Then*

$$X_\alpha^{(n)}v^- = \begin{cases} 0 & \text{if } n > m \\ \bar{s}_\alpha^{-1}v^- & \text{if } n = m. \end{cases} \quad X_{-\alpha}^{(n)}v^+ = \begin{cases} 0 & \text{if } n > m \\ \bar{s}_\alpha v^+ & \text{if } n = m. \end{cases}$$

Next, we do some calculations that will be crucial in the following.

Lemma 5.4.3. *Let $t \in \mathbb{C}$, $g \in G$ and $N = \langle -w\varpi_\alpha, \beta^\vee \rangle$. Then*

$$\Delta_{v,w}^{\varpi_\alpha}(gx_\beta(t)) = \begin{cases} \Delta_{v,w}^{\varpi_\alpha}(g) & \text{if } w^{-1}\beta \in \Phi^+ \\ \sum_{n=0}^N t^n Q_{N-n}(g) & \text{if } w^{-1}\beta \in \Phi^- \end{cases}$$

for some $Q_i \in \mathbb{C}[G]$, such that $Q_0 = \Delta_{v,\bar{s}_\beta w}^{\varpi_\alpha}$ and $Q_N = \Delta_{v,w}^{\varpi_\alpha}$.

Proof. If $w^{-1}\beta \in \Phi^+$, the statement follows from Lemma 5.2.1. Otherwise, we have that

$$\Delta_{v,w}^{\varpi_\alpha}(gx_\beta(t)) = \sum_{n \in \mathbb{N}} t^n \bar{v}\varphi_{\bar{\varpi}_\alpha}^-(gX_\beta^{(n)}\bar{w}v_{\bar{\varpi}_\alpha}^+).$$

Note that, if $w^{-1}\beta \in \Phi^-$, then $\bar{w}v_{\bar{\varpi}_\alpha}^+$ is a lowest weight vector for \mathfrak{sl}_β of weight $\langle w\varpi_\alpha, \beta^\vee \rangle$, hence Lemma 5.4.2 applies. In particular, for $n > N$ the corresponding factor of the above sum is zero. For $n \in \{0, \dots, N\}$ we set $Q_{N-n}(g) = \bar{v}\varphi_{\bar{\varpi}_\alpha}^-(gX_\beta^{(n)}\bar{w}v_{\bar{\varpi}_\alpha}^+)$. Clearly, $Q_N = \Delta_{v,w}^{\varpi_\alpha}$. Moreover, $X_\beta^{(N)}\bar{w}v_{\bar{\varpi}_\alpha}^+ = \bar{s}_\beta^{-1}\bar{w}v_{\bar{\varpi}_\alpha}^+$. But since $w^{-1}\beta \in \Phi^-$, then $\bar{w} = \bar{s}_\beta\bar{s}_\beta\bar{w}$. The lemma follows. \square

We have a variation of the previous lemma relative to the action of elements of U^- .

Lemma 5.4.4. *Let $t \in \mathbb{C}$, $g \in G$ and $N = \langle w\varpi_\alpha, \beta^\vee \rangle$. Then*

$$\Delta_{v,w}^{\varpi_\alpha}(gx_{-\beta}(t)) = \begin{cases} \Delta_{v,w}^{\varpi_\alpha}(g) & \text{if } w^{-1}\beta \in \Phi^- \\ \sum_{n=0}^N t^n Q_{N-n}(g) & \text{if } w^{-1}\beta \in \Phi^+ \end{cases}$$

for some $Q_i \in \mathbb{C}[G]$, such that $Q_0 = \Delta_{v,s_\beta w}^{\varpi_\alpha}$ and $Q_N = \Delta_{v,w}^{\varpi_\alpha}$.

Proof. Literally the same proof of Lemma 5.4.3. Just notice that if $w^{-1}\beta \in \Phi^+$, then $\overline{w}v_{\varpi_\alpha}^+$ is a highest weight vector for \mathfrak{sl}_β of weight $\langle w\varpi_\alpha, \beta^\vee \rangle$. \square

Lemma 5.4.5. *Let $t \in \mathbb{C}$, $g \in G$ and $N = \langle v\varpi_\alpha, \beta^\vee \rangle$. Then*

$$\Delta_{v,w}^{\varpi_\alpha}(x_\beta(t)g) = \begin{cases} \Delta_{v,w}^{\varpi_\alpha}(g) & \text{if } v^{-1}\beta \in \Phi^- \\ \sum_{n=0}^N t^n P_{N-n}(g) & \text{if } v^{-1}\beta \in \Phi^+ \end{cases}$$

for some $P_i \in \mathbb{C}[G]$, such that $P_0 = \Delta_{s_\beta v,w}^{\varpi_\alpha}$ and $P_N = \Delta_{v,w}^{\varpi_\alpha}$.

Proof. One can prove the statement by direct calculation, otherwise we can reduce to Lemma 5.4.3. Following [FZ99], let $(-)^t$ be the involutive anti-automorphism of the group G defined by

$$x_\gamma(t)^t = x_\gamma(t) \quad x_{-\gamma}(t)^t = x_{-\gamma}(t) \quad h^t = h^{-1} \quad \text{for } \gamma \in \Delta, h \in T.$$

By [FZ99, Proposition 2.7], we have that for any $\gamma \in \Delta$ and $g \in G$

$$\Delta_{v,w}^{\varpi_\gamma}(g) = \Delta_{w\overline{w}_0, v\overline{w}_0}^{\varpi_\gamma^*}(g^t) \tag{39}$$

In particular, from Lemma 5.4.3 we deduce that

$$\Delta_{v,w}^{\varpi_\alpha}(x_\beta(t)g) = \Delta_{w\overline{w}_0, v\overline{w}_0}^{\varpi_\alpha^*}(g^t x_\beta(t)) = \begin{cases} \Delta_{w\overline{w}_0, v\overline{w}_0}^{\varpi_\alpha^*}(g^t) = \Delta_{v,w}^{\varpi_\alpha}(g) & \text{if } w_0 v^{-1}\beta \in \Phi^+ \\ \sum_{n=0}^M t^n Q_{M-n}(g^t) & \text{if } w_0 v^{-1}\beta \in \Phi^- \end{cases}$$

where $M = \langle -v\overline{w}_0\varpi_\alpha^*, \beta^\vee \rangle$. Since $\varpi_\alpha^* = -w_0\varpi_\alpha$ we have that $M = N$. Then we set $P_n(g) = Q_n(g^t)$. Using again [FZ99][Proposition 2.7], we deduce the desired expression for P_0 and P_N from the expression of Q_0 and Q_N given in Lemma 5.4.3. Finally, notice that

$$w_0 v^{-1}\beta \in \Phi^+ \iff v^{-1}\beta \in \Phi^-.$$

\square

Remark 5.4.6. Note that, if $\langle v\varpi_\alpha, \beta^\vee \rangle = 0$, then $s_\beta v\varpi_\alpha = v\varpi_\alpha - \langle v\varpi_\alpha, \beta^\vee \rangle\beta = v\varpi_\alpha$. Hence P_0 and P_N agree in this case because of Lemma 5.2.1, so there's no ambiguity. For Q_0 and Q_N we have the same phenomenon. The convention about the enumeration of the coefficients P is due to the fact that we want it to be compatible with the expansion of $t^N \Delta_{v,w}^{\varpi_\alpha}(x_\beta(t^{-1})g)$, and similarly for the coefficients Q .

The next statement is a special case of the previous lemma.

Lemma 5.4.7. *Let $t \in \mathbb{C}$ and $g \in G$. We have that*

$$\Delta_{e,w}^{\varpi_\alpha}(x_\beta(t)g) = \begin{cases} \Delta_{e,w}^{\varpi_\alpha}(g) & \text{if } \alpha \neq \beta \\ \Delta_{e,w}^{\varpi_\alpha}(g) + t\Delta_{s_\alpha,w}^{\varpi_\alpha}(g) & \text{if } \alpha = \beta. \end{cases}$$

Proof. Obvious from Lemma 5.4.5. □

Remark 5.4.8 (A remark on products). We fix a quite obvious convention relative to products. Suppose that two complex, semisimple, simply connected groups G_1 and G_2 are given. Moreover, suppose that, for $i = 1, 2$ and $\alpha_i \in \Delta_i$, we have already fixed a Torus and a Borel $T_i, B_i \subseteq G_i$ and an \mathfrak{sl}_2 -triple $(X_{\alpha_i}, H_{\alpha_i}, X_{-\alpha_i})$. Then we automatically make the following choices in $G = G_1 \times G_2$: the torus $T = T_1 \times T_2$, the Borel $B = B_1 \times B_2$ and, for $\alpha \in \Delta = \Delta_1 \sqcup \Delta_2$, the \mathfrak{sl}_2 triple associated to α , in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, is the one in $\mathfrak{g}_{1/2}$ accordingly if $\alpha \in \Delta_{1/2}$. In particular, with this choice, if $\alpha = \alpha_1 \in \Delta_1$, then $\overline{s_\alpha} = (\overline{s_{\alpha_1}}, e)$, while if $\alpha = \alpha_2 \in \Delta_2$, then $\overline{s_\alpha} = (e, \overline{s_{\alpha_2}})$. Hence, for any $w = (w_1, w_2) \in W = W_1 \times W_2$, we have that

$$\overline{w} = (\overline{w_1}, \overline{w_2}).$$

The fundamental weights of $T_1 \times T_2$ are of the form $(\varpi_{\alpha_1}, 0)$ and $(0, \varpi_{\alpha_2})$ and we have

$$\Delta_{(v_1, v_2), (w_1, w_2)}^{(\varpi_{\alpha_1}, 0)} = \Delta_{v_1, w_1}^{\varpi_{\alpha_1}} \otimes 1 \quad \text{and} \quad \Delta_{(v_1, v_2), (w_1, w_2)}^{(0, \varpi_{\alpha_2})} = 1 \otimes \Delta_{v_2, w_2}^{\varpi_{\alpha_2}}. \quad (40)$$

5.5 Technical properties of minors II: on algebraic independence

We study algebraic independence of the restriction of some generalised minors to some T -stable subgroup of U .

Fix $z \in W$ and $\mathbf{i} \in R(z)$. Let $l = \ell(z)$ and $t = t_{\mathbf{i}}$. Recall that, for $1 \leq j \leq l$, $x_j \in \mathbb{C}(U(z))$ denotes the j -th cluster variable of the seed t .

Left twist: the case of $U(\mathbf{z})$

We fix $\alpha \in \Delta$ such that $s_\alpha \in \text{supp}(z)$. In this subsection, for $\beta \in \Delta$ and $v, w \in W$, we denote by $D_{v,w}^{\varpi_\beta}$ the restriction of $\Delta_{v,w}^{\varpi_\beta}$ to $U(z)$.

Lemma 5.5.1. *The following equality holds*

$$D_{s_\alpha, z_{\leq \alpha^{\min}}^{-1}}^{\varpi_\alpha} = \prod_{j=1}^{\alpha^{\min}-1} (D_{s_\alpha, z_{\leq j}^{-1}}^{\varpi_{i_j}})^{c_j}$$

for some non-negative integers c_j . Moreover, for $j < \alpha^{\min}$, we have that $D_{s_\alpha, z_{\leq j}^{-1}}^{\varpi_{i_j}} = D_{e, z_{\leq j}^{-1}}^{\varpi_{i_j}}$.

Proof. Note that $z_{\leq \alpha^{\min}}^{-1} = z' s_\alpha$ where $z' = s_{i_1} \dots s_{(i_{\alpha^{\min}-1})}$, or $z' = e$ if $\alpha^{\min} = 1$. We apply Theorem 5.2.2 with $w = z'$ and $v = e$.

Note that $s_\alpha \notin \text{supp}(z')$, hence $z'\varpi_\alpha = \varpi_\alpha$ and so $\Delta_{e,z'}^{\varpi_\alpha} = \Delta_{e,e}^{\varpi_\alpha}$ by Lemma 5.2.1. Moreover, $D_{e,e}^{\varpi_\alpha} = 1$. Similarly, from Lemma 5.4.1 we deduce that $D_{s_\alpha,z'}^{\varpi_\alpha} = D_{s_\alpha,e}^{\varpi_\alpha} = 0$. So, Theorem 5.2.2 specialises to the following identity:

$$D_{s_\alpha, z_{\leq \alpha}^{-1}}^{\varpi_\alpha} = \prod_{\beta \in \Delta \setminus \{\alpha\}} (D_{e,z'}^{\varpi_\beta})^{-a_{\beta,\alpha}}.$$

But for $\beta \neq \alpha$, $(D_{e,z'}^{\varpi_\beta})^{-a_{\beta,\alpha}} = (D_{s_\alpha,z'}^{\varpi_\beta})^{-a_{\beta,\alpha}}$ because of Lemma 5.2.1. Moreover, if $s_\beta \notin \text{supp}(z')$, $(D_{s_\alpha,z'}^{\varpi_\beta})^{-a_{\beta,\alpha}} = 1$, while if $s_\beta \in \text{supp}(z')$ and $k_\beta = \max\{j \in \{1, \dots, \alpha^{\min} - 1\} : i_j = \beta\}$, by Lemma 5.2.1 we have that

$$(D_{s_\alpha,z'}^{\varpi_\beta})^{-a_{\beta,\alpha}} = (D_{s_\alpha, z_{\leq k_\beta}^{-1}}^{\varpi_\beta})^{-a_{\beta,\alpha}}.$$

In particular, for $j < \alpha^{\min}$, setting

$$c_j = \begin{cases} -a_{\beta,\alpha} & \text{if } j = k_\beta \text{ for some } \beta \neq \alpha. \\ 0 & \text{otherwise} \end{cases}$$

we have proved the first part of the statement. The last part is obvious from Lemma 5.2.1 and the definition of α^{\min} . \square

Lemma 5.5.2. *For any $k \leq l$, the following equality of subfields of $\mathbb{C}(U(w))$ holds:*

$$\mathbb{C}(x_{\alpha^{\min}}, D_{s_\alpha, z_{\leq j}^{-1}}^{\varpi_{i_j}} : j \leq k, j \neq \alpha^{\min}) = \mathbb{C}(x_{\alpha^{\min}}, x_j : j \leq k, j \neq \alpha^{\min}).$$

In particular, the functions $\{D_{s_\alpha, z_{\leq j}^{-1}}^{\varpi_{i_j}} : j \leq l, j \neq \alpha^{\min}\}$ are algebraically independent.

Proof. Recall that $x_j = D_{e, z_{\leq j}^{-1}}^{\varpi_{i_j}}$. We prove the equality by induction on k . For $k = 0$ there is nothing to prove. Suppose that the equality has been proved for k and call

$$\mathbb{F}_k = \mathbb{C}(x_{\alpha^{\min}}, D_{s_\alpha, z_{\leq j}^{-1}}^{\varpi_{i_j}} : j \leq k, j \neq \alpha^{\min}) = \mathbb{C}(x_{\alpha^{\min}}, x_j : j \leq k, j \neq \alpha^{\min}).$$

If $k + 1 = \alpha^{\min}$, there is nothing to prove. Similarly, if $i_{k+1} \neq \alpha$, then by Lemma 5.2.1

$$D_{s_\alpha, z_{\leq k+1}^{-1}}^{\varpi_{i_{k+1}}} = D_{e, z_{\leq k+1}^{-1}}^{\varpi_{i_{k+1}}},$$

so the equality is trivial. Next, suppose that $i_{k+1} = \alpha$ and $k + 1 \neq \alpha^{\min}$. Applying Theorem 5.2.2 with $v = e$ and $w = z_{\leq k}^{-1}$, we get an equality that can be read as

$$f D_{s_\alpha, z_{\leq k+1}^{-1}}^{\varpi_{i_{k+1}}} = g D_{e, z_{\leq k+1}^{-1}}^{\varpi_{i_{k+1}}} + \varphi. \quad (41)$$

We analyse the three terms φ, f, g .

Note that

$$\varphi = \prod_{\beta \in \Delta \setminus \{\alpha\}} (D_{e, z_{\leq k}^{-1}}^{\varpi_\beta})^{-a_{\beta,\alpha}}.$$

For $\beta \in \Delta \setminus \{\alpha\}$, if $s_\beta \notin \text{supp}(z_{\leq k}^{-1})$, then $D_{e, z_{\leq k}^{-1}}^{\varpi_\beta} = 1$. Otherwise, let $k_\beta = \max\{1 \leq j \leq k : i_j = \beta\}$. Then,

$$D_{e, z \leq k}^{\varpi\beta} = D_{e, z \leq k\beta}^{\varpi\beta} = x_{k\beta} \in \mathbb{F}_k$$

Hence $\varphi \in \mathbb{F}_k$.

Next, we look at g . Let $t = (k+1)^-$, which is not zero since $k+1 \neq \alpha^{\min}$. Lemma 5.2.1 implies that:

$$g = D_{s_\alpha, z \leq k}^{\varpi i_{k+1}} = D_{s_\alpha, z \leq t}^{\varpi i_t}.$$

Since $s_\alpha \in \text{supp}(z \leq t)$, then $g \neq 0$ by Lemma 5.4.1 and statement 5 of Lemma 5.2.1. Moreover, if $t \neq \alpha^{\min}$, we clearly have $g \in \mathbb{F}_k$, while if $t = \alpha^{\min}$ we deduce that $g \in \mathbb{F}_k$ from Lemma 5.5.1.

Finally, we consider f . Notice that

$$f = D_{e, z \leq k-1}^{\varpi i_{k+1}} = D_{e, z \leq t}^{\varpi i_t} = x_t.$$

In particular, if $t = \alpha^{\min}$, then $f = x_{\alpha^{\min}} \in \mathbb{F}_k$. In the other case, it's obvious that $f \in \mathbb{F}_k$. Moreover, f is clearly non-zero since it is a cluster variable of t .

Since $f, g, \varphi \in \mathbb{F}_k$ and f, g are non-zero, the equality of fields for $k+1$ follows by induction from (41).

For the algebraic independence, notice that the functions $\{x_j : j \leq l\}$ are the cluster variables of t , which are algebraically independent. The statement follows from the well definiteness of the transcendence degree. \square

Left twist: the case of $U(z)_{\setminus \alpha}$

For a T -stable subgroup H of U and $\alpha \in \Delta$, we set

$$H_{\setminus \alpha} = H \cap s_\alpha U s_\alpha.$$

We fix $\alpha \in \Delta \cap \Phi(z)$. In particular, $s_\alpha \in \text{supp}(z)$. In this subsection, for $\beta \in \Delta$ and $v, w \in W$, we denote by $D_{v, w}^{\varpi\beta}$ the restriction of $\Delta_{v, w}^{\varpi\beta}$ to $U(z)_{\setminus \alpha}$.

Lemma 5.5.3. *The following equality holds*

$$D_{s_\alpha, z \leq \alpha^{\min}}^{\varpi\alpha} = \prod_{j=1}^{\alpha^{\min}-1} (D_{s_\alpha, z \leq j}^{\varpi i_j})^{c_j}$$

for some non-negative integers c_j . Moreover, for $j < \alpha^{\min}$, we have that $D_{s_\alpha, z \leq j}^{\varpi i_j} = D_{e, z \leq j}^{\varpi i_j}$.

Proof. This is an obvious consequence of Lemma 5.5.1. \square

Lemma 5.5.4. *The functions $\{D_{s_\alpha, z \leq j}^{\varpi i_j} : j \leq l, j \neq \alpha^{\min}\}$ are algebraically independent.*

Proof. Lemma 5.4.5 implies that $\Delta_{s_\alpha, w}^{\varpi\beta}$ is invariant by left multiplication of $U(s_\alpha)$. Since the product induces an isomorphism $U(s_\alpha) \times U(z)_{\setminus \alpha} \simeq U(z)$ by [Hum90][Proposition 28.1], we deduce the statement from Lemma 5.5.2. \square

Right twist

We fix $\alpha \in \Delta \cap \Phi(z)$. We set

$$\alpha(-) = \min\{k : z_{\leq k}\alpha \in \Phi^-\}. \quad (42)$$

Lemma 5.5.5. *We have that*

1. $\{k : z_{\leq k}\alpha \in \Phi^-\} = \{\alpha(-), \dots, l\}$.
2. $s_\alpha z_{\leq \alpha(-)}^{-1} = z_{\leq (\alpha(-)-1)}^{-1}$.
3. $\langle -z_{\leq \alpha(-)}^{-1} \varpi_{i_{\alpha(-)}}, \alpha^\vee \rangle = 1$.
4. *The expression \mathbf{i}' obtained from \mathbf{i} by deleting the term $i_{\alpha(-)}$ is a reduced expression for zs_α .*

Proof. 1. By the definition of $\alpha(-)$, we clearly have that $z_{\leq k}\alpha \in \Phi^+$ for $k < \alpha(-)$. Moreover, since \mathbf{i} is a reduced expression, from the definition of $z_{\leq k}$ it follows that $\Phi(z_{\leq k}) \subseteq \Phi(z_{\leq k+1})$. Hence, for $k \geq \alpha(-)$ we have that $z_{\leq k}\alpha \in \Phi^-$.

2. Since $z_{\leq (\alpha(-)-1)}\alpha \in \Phi^+$ and $s_{i_{\alpha(-)}}z_{\leq (\alpha(-)-1)}\alpha \in \Phi^-$, we have that

$$z_{\leq (\alpha(-)-1)}\alpha = i_{\alpha(-)}, \quad (43)$$

where we assume that $z_{\leq (\alpha(-)-1)} = e$ if $\alpha(-) = 1$. Then the identity follows from the fact that, for any $w \in W$ and $\beta \in \Phi$, we have that $ws_\beta w^{-1} = s_w \beta$.

3. Using (43), we can compute that

$$\begin{aligned} \langle -z_{\leq \alpha(-)}^{-1} \varpi_{i_{\alpha(-)}}, \alpha^\vee \rangle &= \langle -\varpi_{i_{\alpha(-)}}, (z_{\leq \alpha(-)}\alpha)^\vee \rangle \\ &= \langle -\varpi_{i_{\alpha(-)}}, (s_{i_{\alpha(-)}} i_{\alpha(-)})^\vee \rangle \\ &= \langle -\varpi_{i_{\alpha(-)}}, (-i_{\alpha(-)})^\vee \rangle \\ &= 1. \end{aligned}$$

4. Using the second statement (and paying attention if $\alpha(-) = 1$ or $\alpha(-) = l$), we have that

$$s_{i_l} \dots s_{i_{\alpha(-)+1}} s_{i_{\alpha(-)-1}} \dots s_{i_1} = s_{i_l} \dots s_{i_{\alpha(-)+1}} s_{i_{\alpha(-)}} s_{i_{\alpha(-)-1}} \dots s_{i_1} s_\alpha = zs_\alpha.$$

Hence, \mathbf{i}' is an expression for zs_α . Since $\ell(zs_\alpha) = \ell(z) - 1$, we deduce that \mathbf{i}' is reduced. \square

Here, for $\beta \in \Delta$ and $v, w \in W$, we denote by $D_{v,w}^{\varpi_\beta}$ the restriction of $\Delta_{v,w}^{\varpi_\beta}$ to $U(z)$. For $k \in [l]$, we define $f_k \in \mathbb{C}[U(z)]$ as

$$f_k = \begin{cases} D_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}} & \text{if } (z_{\leq k})\alpha \in \Phi^+ \\ D_{e, s_\alpha z_{\leq k}^{-1}}^{\varpi_{i_k}} & \text{if } (z_{\leq k})\alpha \in \Phi^- \end{cases}$$

Lemma 5.5.6. *In the above setting, $f_{\alpha(-)} = f_{\alpha(-)-}$. We use the convention that $f_0 = 1$.*

Proof. Let $k = \alpha(-)$. By statement 2 of Lemma 5.5.5 we have that

$$\begin{aligned} s_\alpha z_{\leq k}^{-1} \varpi_{i_k} &= z_{\leq k-1}^{-1} \varpi_{i_k} \\ &= s_{i_1} \cdots s_{i_{(k-1)}} \varpi_{i_k} \\ &= s_{i_1} \cdots s_{i_{k^-}} \varpi_{i_k} \\ &= z_{\leq k^-} \varpi_{i_k} \end{aligned}$$

where $z_{\leq k^-} = e$ if $k^- = 0$. The lemma follows from Lemma 5.2.1. \square

Lemma 5.5.7. *If $-z\alpha \in \Delta$, then $\{f_k : k \neq \alpha(-)\}$ are algebraically independent.*

Proof. If $z\alpha = -\beta$, with $\beta \in \Delta$, it means that $zs_\alpha = s_\beta z \leq_L z$, where \leq_L denotes the *left* weak order on W . In particular, $\Phi(zs_\alpha) \subseteq \Phi(z)$, which implies that $U(zs_\alpha) \subseteq U(z)$. For $k \in \{1, \dots, l-1\}$ let $x'_k \in \mathbb{C}[U(zs_\alpha)]$ be the k -th cluster variable of the seed $t_{\mathbf{i}'}$, where \mathbf{i}' is the reduced expression in Lemma 5.5.5. Clearly, for $k < \alpha(-)$, f_k restricts to x'_k , while using the second statement of Lemma 5.5.5 one deduces (analogously to the proof of the fourth statement of Lemma 5.5.5) that, for $k > \alpha(-)$, f_k restricts to x'_{k-1} . Since the cluster variables x'_k are algebraically independent in $\mathbb{C}[U(zs_\alpha)]$, the lemma follows. \square

6 Application to G

Let G be a semi-simple, simply connected, complex algebraic group. We show here a simple application of the minimal monomial lifting to G . This example is designed to make equality between $\mathbb{C}[G]$ and the upper cluster algebra constructed via the minimal monomial lifting to fail (see Remark 6.0.5). At the same time, this example behaves so well that we believe that it could shed a light on why equality in the second statement of Theorem 4.0.3 doesn't always hold. We use the notation of section 5.

Let $(-)^T : G \rightarrow G$ be the *transpose*, which is the involutive antihomomorphism considered in [FZ99, Section 2.1]. In particular, it is the only antihomomorphism of algebraic groups defined by

$$x_\alpha(t)^T = x_{-\alpha}(t) \quad h^T = h \quad \text{for any } \alpha \in \Delta, t \in \mathbb{C}, h \in T.$$

This restricts to an antihomomorphism between U^- and U . If $\mathbf{i} \in R(w_0)$, let $t_{\mathbf{i}^-}$ be the seed of $\mathbb{C}[U^-]$ obtained by "transposing" the seed $t_{\mathbf{i}}$ of $\mathbb{C}[U]$. Clearly, $\mathcal{A}(t_{\mathbf{i}^-}) = \overline{\mathcal{A}}(t_{\mathbf{i}^-}) = \mathbb{C}[U^-]$. Moreover, by [FZ99, Proposition 2.7], the cluster variables of $t_{\mathbf{i}^-}$ are all obtained by restricting functions of the form $\Delta_{w,e}^{\varpi_\alpha}$, with $s_\alpha \in \text{supp}(w)$, to U^- .

Fix $\mathbf{i}', \mathbf{i} \in R(w_0)$ and call

$$t_{\mathbf{i}} = (I_{uf}, I_{sf}, I_{hf}, B, x) \quad t_{\mathbf{i}'}^- = (I_{uf}^-, I_{sf}^-, I_{hf}^-, B^-, x^-) \quad t_{\mathbf{i}'}^-|t_{\mathbf{i}} = (J_{uf}, J_{sf}, J_{hf}, C, z),$$

where $t_{\mathbf{i}'}^-|t_{\mathbf{i}}$ is the disjoint union of $t_{\mathbf{i}'}^-$ and $t_{\mathbf{i}}$, as defined in Section 2.2. By Lemma 2.2.3, we have that $\overline{\mathcal{A}}(t_{\mathbf{i}'}^-|t_{\mathbf{i}}) = \mathbb{C}[U^- \times U]$, which is a factorial \mathbb{C} -algebra of finite type.

Let $D = \Delta$, $Y = U^- \times U$ and

$$\begin{aligned} \phi : T \times U^- \times U &\longrightarrow G \\ (h, u^-, u) &\longmapsto u^- hu. \end{aligned}$$

Moreover, for $\alpha \in D$, we set $X_\alpha = \Delta_{e,e}^{\varpi_\alpha}$.

Lemma 6.0.1. *The triple (G, ϕ, X) is suitable for D -lifting.*

Proof. Since G is semi-simple and simply connected, $\mathbb{C}[G]$ is a factorial domain, hence normal. Moreover, G is of finite type. Condition 1 of Definition 4.0.1 then clearly holds. Condition 2 holds because of Lemma 5.2.3. Moreover, ϕ is an open embedding with image $G_0 = B^-B$ and, for any $\alpha \in D$, $\phi^*(X_\alpha) = \varpi_\alpha \otimes 1$ by Lemma 5.2.1. Hence condition 3 of Definition 4.0.1 holds. \square

Proposition 6.0.2. *For any $\mathbf{i}', \mathbf{i} \in R(w_0)$ we have*

$$\overline{\mathcal{A}}(\mathfrak{1}(t_{\mathbf{i}'}|t_{\mathbf{i}})) = \mathbb{C}[G_0] \quad \text{and} \quad \overline{\mathcal{A}}(\mathfrak{1}(t_{\mathbf{i}'}|t_{\mathbf{i}})^D) \subseteq \mathbb{C}[G].$$

Proof. Because of Lemma 6.0.1 and Lemma 4.0.9 we can apply Theorem 4.0.3. \square

It's easy to compute the lifting configuration ν and the corresponding cluster $\mathfrak{1}z$ in this example. Let $\chi : J \rightarrow \Delta$ be the map such that, for $k \in I$ (resp. $k \in I^-$), $\chi(k) = \alpha$ if and only if the minor defining x_k (See (37)) is of the form $\Delta_{e,w}^{\varpi_\alpha}$ (resp. $\Delta_{w,e}^{\varpi_\alpha}$).

Lemma 6.0.3. *For any $j \in J$ and $\alpha \in D$, we have $\nu_{\alpha,j} = \delta_{\alpha,\chi(j)}$. Moreover, the variable $\mathfrak{1}z_j$ is the generalized minor defining z_j .*

Proof. Suppose that $j \in I$ and let $\Delta_{e,w}^{\varpi_\beta}$ be the minor defining x_j . In particular, $\chi(j) = \beta$. Using Lemma 5.2.1 and the definition of z_j , we have that, for $x = (h, u^-, u) \in T \times Y$

$$\Delta_{e,w}^{\varpi_\beta}(\phi(x)) = \varpi_\beta(h)\Delta_{e,w}^{\varpi_\beta}(u) = \Delta_{e,e}^{\varpi_\beta}(\phi(x))(1 \otimes z_j(\phi(x))).$$

In particular, identifying $1 \otimes z_j$ as a rational function on G via ϕ , we have that $\Delta_{e,w}^{\varpi_\beta} = X_\beta(1 \otimes z_j)$. But $\Delta_{e,w}^{\varpi_\beta}$ is irreducible in $\mathbb{C}[G]$ because of Lemma 5.2.3. Hence, $\mathcal{V}_\alpha(1 \otimes z_j) = -\delta_{\alpha,\chi(j)}$. The lemma follows. If $i \in I^-$ the proof is similar. \square

From the previous lemma, we deduce that for $j \in I$, $\mathfrak{1}z_j$ is a generalised minor of the form $\Delta_{e,w}^{\varpi_\alpha}$, while for $j \in I^-$, $\mathfrak{1}z_j$ is a generalised minor of the form $\Delta_{w,e}^{\varpi_\alpha}$.

Proposition 6.0.4. *Suppose that \mathbf{i} and \mathbf{i}' start with the same simple root, that is $i_1 = \alpha = i'_1$. Then, the upper cluster algebra $\overline{\mathcal{A}}(\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})^D)$ is not equal to $\mathbb{C}[G]$.*

Proof. From the definition of the seeds $t_{\mathbf{i}}$, $t_{\mathbf{i}'}^-$ and the previous lemma, we have that $\Delta_{e,s_\alpha}^{\varpi_\alpha}$ and $\Delta_{s_\alpha,e}^{\varpi_\alpha}$ are cluster variables of the initial seed $\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})$. But then, by Theorem 5.2.2, we have that

$$\Delta_{s_\alpha,s_\alpha}^{\varpi_\alpha} = \frac{\Delta_{s_\alpha,e}^{\varpi_\alpha}\Delta_{e,s_\alpha}^{\varpi_\alpha} + \prod_{\beta \in \Delta \setminus \{\alpha\}} (\Delta_{e,e}^{\varpi_\beta})^{-a_{\beta,\alpha}}}{\Delta_{e,e}^{\varpi_\alpha}} \quad (44)$$

which proves that $\Delta_{s_\alpha,s_\alpha}^{\varpi_\alpha}$ is an element of $\mathbb{C}[G] \setminus \overline{\mathcal{A}}(\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})^D)$. In fact, on the RHS of (44) we have an element of $\mathcal{L}(\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})^D)$ which is not in $\mathcal{L}(\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})^D)$. Alternatively, on the LHS of (44) we have an irreducible element of $\mathbb{C}[G]$, in particular $\mathcal{V}_\alpha(\Delta_{s_\alpha,s_\alpha}^{\varpi_\alpha}) \geq 0$ (it's easy to see that it's actually zero). But by looking at the RHS, we see that $\mathcal{CV}_\alpha(\Delta_{s_\alpha,s_\alpha}^{\varpi_\alpha}) = -1$. The statement follows from Proposition 4.1.3. \square

Remark 6.0.5. Note that, for any $\mathbf{i}, \mathbf{i}' \in R(w_0)$, the seed $\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})$ is not one of the seeds constructed in [BFZ05] for the open double Bruhat cell. Note that there is no T -action on G that makes the map ϕ equivariant. Hence, the equality between $\overline{\mathcal{A}}(\mathfrak{1}(t_{\mathbf{i}'}^-|t_{\mathbf{i}})^D)$ and $\mathbb{C}[G]$ is never reached, for any pair $\mathbf{i}, \mathbf{i}' \in R(w_0)$, because of Corollary 4.1.6. Finally, a similar construction can be done for the spherical homogeneous space G/\widehat{G} if and only if the pair $\widehat{G} \subseteq G$ is of minimal rank. This will be developed in the future.

7 Monomial lifting for branching problems

In this section we explain how to apply the minimal monomial lifting technique to study some branching problems.

7.1 The branching scheme

Let G be a semisimple, simply connected complex algebraic group and let \widehat{G} be a connected, reductive subgroup of G . Fix maximal tori and Borel subgroups T, B of G and \widehat{T}, \widehat{B} of \widehat{G} such that $T \cap \widehat{G} = \widehat{T}$ and $B \cap \widehat{G} = \widehat{B}$. Consider the $U^- \times \widehat{U} \times \widehat{G}$ action on $G \times \widehat{G}$ defined by

$$(u^-, \widehat{u}, \widehat{s}) \cdot (g, \widehat{g}) = (u^- g \widehat{s}^{-1}, \widehat{s} \widehat{g} \widehat{u}^{-1}).$$

By (25), we clearly have that

$$\mathbb{C}[G \times \widehat{G}]^{U^- \times \widehat{U} \times \widehat{G}} \simeq \bigoplus (V(\lambda) \otimes V(\widehat{\lambda})^*)^{\widehat{G}} \simeq \bigoplus \text{Hom}(V(\widehat{\lambda}), V(\lambda))^{\widehat{G}}, \quad (45)$$

where the sums run over $(\lambda, \widehat{\lambda}) \in X(T)^+ \times X(\widehat{T})^+$. Recall that, if $\lambda \in X(T) \setminus X(T)^+$ (resp. $\widehat{\lambda} \in X(\widehat{T}) \setminus X(\widehat{T})^+$), then we set $V(\lambda) = 0$ (resp. $V(\widehat{\lambda}) = 0$). Consider the $T \times \widehat{T}$ action on $G \times \widehat{G}$ defined by

$$(h, \widehat{h}) \cdot (g, \widehat{g}) = (hg, \widehat{h}\widehat{g}).$$

The induced action on $\mathbb{C}[G \times \widehat{G}]$ stabilises $\mathbb{C}[G \times \widehat{G}]^{U^- \times \widehat{U} \times \widehat{G}}$ which is then $X(T) \times X(\widehat{T})$ -graded. It's easy to verify that the homogeneous components of this graduation correspond to the decomposition given in (45). In particular, for any $(\lambda, \widehat{\lambda}) \in X(T) \times X(\widehat{T})$, we have

$$\mathbb{C}[G \times \widehat{G}]_{\lambda, \widehat{\lambda}}^{U^- \times \widehat{U} \times \widehat{G}} = (V(\lambda) \otimes V(\widehat{\lambda})^*)^{\widehat{G}}.$$

Note that, if λ and $\widehat{\lambda}$ are dominant, then $(V(\lambda) \otimes V(\widehat{\lambda})^*)^{\widehat{G}}$ is the *multiplicity space* of the representation $V(\widehat{\lambda})$ in $V(\lambda)$, seen as a representation of \widehat{G} .

Consider now the action of $U^- \times \widehat{U}$ on G given by $(u^-, \widehat{u}) \cdot g = (u^- g \widehat{u}^{-1})$. The action of $T \times \widehat{T}$ on G given by $(h, \widehat{h}) \cdot g = hg\widehat{h}$ induces a $X(T) \times X(\widehat{T})$ -graduation on $\mathbb{C}[G]^{U^- \times \widehat{U}}$.

Lemma 7.1.1. *The product $G \times \widehat{G} \rightarrow G$ induces a $T \times \widehat{T}$ -equivariant isomorphism between $\mathbb{C}[G \times \widehat{G}]^{U^- \times \widehat{U} \times \widehat{G}}$ and $\mathbb{C}[G]^{U^- \times \widehat{U}}$.*

Proof. The inverse is the map $G \rightarrow G \times \widehat{G}$ sending g to (g, e) . The proof is straightforward. \square

From now on, we identify the two algebras appearing in the previous lemma, and define the *branching algebra* $\text{Br}(G, \widehat{G})$ to be

$$\text{Br}(G, \widehat{G}) := \mathbb{C}[G \times \widehat{G}]^{U^- \times \widehat{U} \times \widehat{G}} = \mathbb{C}[G]^{U^- \times \widehat{U}}. \quad (46)$$

endowed with its $X(T) \times X(\widehat{T})$ -graduation.

Lemma 7.1.2. *The branching algebra $\text{Br}(G, \widehat{G})$ is factorial and of finite type. An element $f \in \text{Br}(G, \widehat{G})$ is irreducible if and only if it is irreducible as an element of $\mathbb{C}[G]$.*

Proof. The group $U^- \times \widehat{U}$ is connected, has no non-trivial character and $\mathbb{C}[G]$ is factorial since G is semisimple and simply connected. Then, the factoriality of $\text{Br}(G, \widehat{G})$ and the last statement follow from Lemma 5.2.4. Moreover, it is well known that $\mathbb{C}[G \times \widehat{G}]^{U^- \times \widehat{U}} = \mathbb{C}[G]^{U^-} \otimes \mathbb{C}[\widehat{G}]^{\widehat{U}}$ is of finite type. Since \widehat{G} is reductive, then $\text{Br}(G, \widehat{G})$ is of finite type by a well known theorem of Hilbert. \square

Then, we define the *branching scheme* $\mathfrak{X}(G, \widehat{G})$ as the normal, noetherian, integral, affine $T \times \widehat{T}$ -scheme

$$\mathfrak{X}(G, \widehat{G}) := \text{Spec}(\text{Br}(G, \widehat{G})). \quad (47)$$

When G and \widehat{G} are fixed, we drop the dependence on G, \widehat{G} and just write \mathfrak{X} for $\mathfrak{X}(G, \widehat{G})$ and Br for $\text{Br}(G, \widehat{G})$.

7.2 Monomial lifting for the branching scheme

In the setting of the previous section, we want to prove that $\mathfrak{X} = \mathfrak{X}(G, \widehat{G})$ is homogeneously suitable for lifting. We fix

$$D = \Delta \quad (48)$$

and for any $\alpha \in D$, we set

$$X_\alpha = \Delta_{e,e}^{\varpi_\alpha} \in \mathbb{C}[G]^{U^- \times \widehat{U}} = \text{Br}(G, \widehat{G}). \quad (49)$$

Let

$$\Omega = \mathfrak{X}_{\prod_{\alpha \in \Delta} X_\alpha} = \mathfrak{X} \setminus \bigcup_{\alpha \in D} V(X_\alpha)$$

be the principal open subscheme of X defined by the non-vanishing the X_α .

Moreover, let $\widetilde{Y} := \text{Spec}(\mathbb{C}[U]^{\widehat{U}})$, where \widehat{U} acts on U by right multiplication. The \widehat{T} action on U defined by $\widehat{h} \cdot u = \widehat{h}^{-1}u\widehat{h}$ defines a structure of \widehat{T} -scheme on \widetilde{Y} . Similarly, the action of $T \times \widehat{T}$ on $T \times U$ defined by

$$(h, \widehat{h}) \cdot (t, u) = (ht\widehat{h}, \widehat{h}^{-1}u\widehat{h}) \quad (50)$$

factors through a $T \times \widehat{T}$ action on $T \times \widetilde{Y}$. By definition, the induced T action on $T \times \widetilde{Y}$ is the left multiplication of T on the T -component.

Lemma 7.2.1. *The product map $T \times U \longrightarrow G$ induces an open embedding $\widetilde{\phi} : T \times \widetilde{Y} \longrightarrow \mathfrak{X}$, with image Ω and that satisfies*

1. For any $\alpha \in D$, $\widetilde{\phi}^*(X_\alpha) = \varpi_\alpha \otimes 1$.
2. The map $\widetilde{\phi}$ is $T \times \widehat{T}$ -equivariant.

Proof. By [FZ99, Corollary 2.5], the non vanishing locus in G of all the generalised minors $\Delta_{e,e}^{\varpi_\alpha}$ is G_0 . In particular, $\Omega = \text{Spec}(\mathbb{C}[G_0]^{U^- \times \widehat{U}})$. Recall that the product induces an isomorphism $U^- \times T \times U \simeq G_0$. Then, the proof consists of some immediate verifications. Note that the inverse of $\widetilde{\phi} : T \times \widetilde{Y} \longrightarrow \Omega$ is induced by the map $G_0 \longrightarrow T \times U$ sending x to $([x]_0, [x]_+)$. \square

Proposition 7.2.2. *The triple $(\mathfrak{X}, \widetilde{\phi}, X)$ is homogeneously-suitable for D -lifting.*

Proof. It is a direct consequence of Lemmas 7.1.2, 5.2.3 and 7.2.1. \square

The following question naturally arises.

Question 7.2.3. *For which pairs (G, \widehat{G}) the algebra $\mathbb{C}[U]^{\widehat{U}}$ has a cluster structure?*

7.3 Lifting graduation for the Branching algebra

Before describing some pairs (G, \widehat{G}) for which we can answer Question 7.2.3, we take a closer look at the information given by the lifting graduation on Br .

Note that, by Lemma 3.1.6, the $X(T)$ -graduation on Br is a lifting graduation on the pole filtration on $\mathbb{C}[U]^{\widehat{U}}$. Recall that \widehat{T} acts on \widetilde{Y} by "conjugation".

Lemma 7.3.1. *The pole filtration on $\mathbb{C}[U]^{\widehat{U}}$ is $X(\widehat{T})$ -graded.*

Proof. Identify $\mathbb{C}(\mathfrak{X})$ and $\mathbb{C}(T \times \widetilde{Y})$ using $\widetilde{\phi}$. Recall that \mathbb{Z}^D is canonically identified with $X(T)$ via the \mathbb{Z} -linear map defined by $e_\alpha \mapsto \varpi_\alpha$. Fix $\lambda \in \mathbb{Z}^D$, then

$$f \in \mathbb{C}[\widetilde{Y}]_\lambda \iff X^\lambda(1 \otimes f) \in \mathbb{C}[\mathfrak{X}], \quad (51)$$

where $\mathbb{C}[\widetilde{Y}]_\lambda$ is the λ -component of the lifting graduation. The map $\mathbb{C}(\widetilde{Y}) \rightarrow \mathbb{C}(\mathfrak{X})$ defined by $f \mapsto 1 \otimes f$ is \widehat{T} -equivariant. Since, for any $\alpha \in D$, X_α is \widehat{T} semi-invariant, we deduce from (51) that $\mathbb{C}[\widetilde{Y}]_\lambda$ is \widehat{T} -stable. \square

The $X(T) \times X(\widehat{T})$ -graduation on Br is NOT a $X(\widehat{T})$ -graded lifting graduation. Indeed, the map $\widetilde{Y} \rightarrow \mathfrak{X}$ defined by $\widetilde{y} \mapsto \widetilde{\phi}((e, \widetilde{y}))$ is not \widehat{T} -equivariant (see Lemma 3.1.5). We can fix the situation by twisting the $T \times \widehat{T}$ -action on \mathfrak{X} .

Let \mathfrak{X}^* be the scheme \mathfrak{X} endowed with the twisted $T \times \widehat{T}$ -action $*$ defined by

$$(h, \widehat{h}) * x = (h\widehat{h}^{-1}, \widehat{h}) \cdot x,$$

where \cdot denotes the standard $T \times \widehat{T}$ -action on \mathfrak{X} . We denote by $\text{Br}^* = \mathcal{O}_{\mathfrak{X}^*}(\mathfrak{X}^*)$, considered with its $X(T) \times X(\widehat{T})$ -graduation induced by the $*$ -action. As \mathbb{C} -algebras, $\text{Br}^* = \text{Br}$. Note that a subset of \mathfrak{X} is stable for the \cdot action if and only if it is stable for the $*$ action.

Corollary 7.3.2. *The triple $(\mathfrak{X}^*, \widetilde{\phi}, X)$ is homogeneously suitable for D -lifting. Moreover, the $X(T) \times X(\widehat{T})$ -graduation on Br^* is a $X(\widehat{T})$ -graded lifting graduation.*

Proof. The first part of the statement is obvious from Proposition 7.2.2. Moreover, using (50), we easily deduce that, for any $h, t \in T$, $\widehat{h} \in \widehat{T}$ and $\widetilde{y} \in \widetilde{Y}$, then

$$(h, \widehat{h}) * (t, \widetilde{y}) = (ht, \widehat{h} \cdot \widetilde{y})$$

where $h \cdot \widetilde{y}$ denotes the conjugation action. In particular, the map $\iota : \widetilde{Y} \rightarrow \mathfrak{X}^*$ defined by $\widetilde{y} \mapsto \widetilde{\phi}((e, \widetilde{y}))$ is \widehat{T} -equivariant. We conclude by applying Lemma 3.1.5. \square

Let $\iota : \widetilde{Y} \rightarrow \mathfrak{X}^*$ be as in the proof of the previous corollary.

Proposition 7.3.3. *For any $(\lambda, \widehat{\lambda}) \in X(T)^+ \times X(\widehat{T})^+$, ι^* induces an isomorphism*

$$\text{Hom}(V(\lambda), V(\widehat{\lambda}))^{\widehat{G}} \longrightarrow \mathbb{C}[U]_{\lambda, \widehat{\lambda} - \rho(\lambda)}^{\widehat{U}}$$

where

$$\mathbb{C}[U]_{\lambda, \widehat{\lambda} - \rho(\lambda)}^{\widehat{U}} = \{f \in \mathbb{C}[U]^{\widehat{U}} : \begin{array}{ll} f(\widehat{h}^{-1}u\widehat{h}) = (\widehat{\lambda} - \lambda)(\widehat{h})f(u) & \text{for } \widehat{h} \in \widehat{T}, u \in U, \text{ and} \\ \mathcal{V}_\alpha(1 \otimes f) \geq -\langle \lambda, \alpha^\vee \rangle & \text{for } \alpha \in D \end{array} \}.$$

Proof. It's a simple computation that $\text{Br}_{\lambda, \widehat{\lambda}} = \text{Br}_{\lambda, \widehat{\lambda} - \rho(\lambda)}^*$. Then the proposition follows from Corollary 7.3.2 and Definition 3.1.4. \square

Remark 7.3.4. The above proposition gives a geometric realisation of the multiplicity spaces for the branching problem of \widehat{G} in G . For the tensor product case, hence when $G = \widehat{G} \times \widehat{G}$ and \widehat{G} is diagonally embedded in G , by identifying $U/\widehat{U} = (\widehat{U} \times \widehat{U})/\widehat{U}$ with \widehat{U} via the map $(\widehat{u}_1, \widehat{u}_2) \mapsto \widehat{u}_1 \widehat{u}_2^{-1}$, we can easily deduce Proposition 1.4 of [Zel02]. Actually, [Zel99][Proposition 1.4] is the starting point for the interest in perfect bases, which have classically been one of the most used items to study tensor product decomposition. See [Kam22] for a beautiful survey on perfect bases. We wonder if Proposition 7.3.3 can lead to the development of a similar theory for other branching problems.

7.4 On question 7.2.3

We discuss some cases in which the answer to Question 7.2.3 is positive because we can naturally identify \widetilde{Y} with $U(z)$ for a certain $z \in W$. In general, Question 7.2.3 being quite vague, the author feels the following question more interesting.

Question 7.4.1. *For which pairs (G, \widehat{G}) there exist a $z \in W$ such that the product $U(z) \times \widehat{U} \rightarrow U$ is an isomorphism?*

In the following, we exhibit two families for which we can answer positively to Question 7.4.1. In general, the answer to the previous question is positive in many other interesting cases, as we will see in some forthcoming works.

7.4.1 Levi subgroups

Let $I \subsetneq \Delta$ be a subset, strictly contained in Δ , and $\Phi_I = \Phi \cap \mathbb{Z}I$. This is a root system with I as simple set of roots and Weyl group $W_I = \langle s_\alpha : \alpha \in I \rangle \subseteq W$. Consider $\widehat{G} = G_I$ the *Levi subgroup* of G , which is a reductive group with root system $\widehat{\Phi} = \Phi_I$. Let $\widehat{T} = T = \widehat{G} \cap T$ and let $\widehat{B} = B \cap \widehat{G}$. Note that $\widehat{W} = W_I$. Let $w_{0,I}$ be the longest element of \widehat{W} and $z = w_{0,I}^\vee = w_0 w_{0,I}$. Note that $\widehat{U} = U(w_{0,I})$. Then we define

$$Y = U(z) = U(w_{0,I}^\vee).$$

As a special case, when $I = \emptyset$, then $\widehat{G} = T$.

7.4.2 Products

Let \widehat{G} and H be semisimple, simply connected, complex algebraic groups such that $\widehat{G} \subseteq H$. Let T_H, B_H (resp. \widehat{T}, \widehat{B}) be a maximal torus and a Borel subgroup of H (resp. \widehat{G}) such that $T_H \subseteq B_H$ (resp. $\widehat{T} \subseteq \widehat{B}$) and denote by U_H (resp. \widehat{U}) the unipotent radical of B_H resp. \widehat{B} . Assume furthermore that $\widehat{T} \subseteq T_H$ and $\widehat{B} \subseteq B_H$.

Let $G = H \times \widehat{G}$, and consider \widehat{G} embedded in G diagonally. Then, $T = T_H \times \widehat{T}$, $B = B_H \times \widehat{B}$ and $U = U_H \times \widehat{U}$ (recall our convention on products 5.4.8).

Let $z = (w_{0,H}, e) \in W = W_H \times \widehat{W}$, where $w_{0,H}$ is the longest element of W_H . Then we set

$$Y = U(z) = U_H \times \{e\}.$$

Note that, as a special case of this situation, we have the case of \widehat{G} diagonally embedded into \widehat{G}^n which is the product of n copies of \widehat{G} . The corresponding branching problem consists in decomposing the tensor product of n irreducible representations of \widehat{G} , under the diagonal action.

Remark 7.4.2. Note that the reduced expressions of $z \in W$ are in obvious bijection with the reduced expressions of $w_{0,H} \in W_H$. The reduced expression $\mathbf{i} \in R(z)$ gives a seed $t_{\mathbf{i}} \in \mathbb{C}[U(z)]$ while, thinking about the same reduced expression as an expression for $w_{0,H}$, we obtain a seed $t'_{\mathbf{i}} \in \mathbb{C}[U_H]$. It's trivial to verify, using the last identity in Remark 5.4.8, that $t_{\mathbf{i}} = t'_{\mathbf{i}}$ under the obvious identification between $U(z)$ and U_H . So, we make no difference between these two seeds.

From now on we suppose that the pair $\widehat{G} \subseteq G$ belongs to one of the two cases discussed above.

Lemma 7.4.3. *The natural inclusion $Y = U(z) \rightarrow U$ induces a \widehat{T} -equivariant isomorphism between Y and $\widetilde{Y} = \text{Spec}(\mathbb{C}[U]^{\widehat{U}})$.*

Proof. We claim that in the two cases, the product gives a \widehat{T} -equivariant isomorphism $Y \times \widehat{U} \rightarrow U$. In fact: for the product case, the above map is given explicitly by

$$\begin{aligned} U_H \times \widehat{U} &\longrightarrow U_H \times \widehat{U} \\ (u_h, \widehat{u}) &\longmapsto (u_h \widehat{u}, \widehat{u}) \end{aligned}$$

which is obviously an isomorphism. For the Levi case, the claim follows from [Hum75][Proposition 28.1]. \square

Let $\phi : T \times Y \rightarrow \mathfrak{X}$ be the map obtained from $\widetilde{\phi}$, by identifying Y and \widetilde{Y} as in Lemma 7.4.3.

Corollary 7.4.4. *The triple (\mathfrak{X}, ϕ, X) is homogeneously suitable for D -lifting.*

Proof. It is obvious from Corollary 7.3.2 and Lemma 7.4.3. \square

Theorem 7.4.5. *For any $\mathbf{i} \in R(z)$, we have*

$$\overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}) = \text{Br}_{\prod_{\alpha \in D} X_{\alpha}} \quad \text{and} \quad \overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}^D) \subseteq \text{Br}.$$

Moreover $\overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}^D)$, with the canonical graduation induced by $\uparrow 0$, is a graded subalgebra of Br .

Proof. Because of Corollary 7.4.4 and Lemma 4.0.9 we can apply Theorem 4.0.3. The part on the graduation is a consequence of Lemma 4.1.7. \square

From now on, fix $\mathbf{i} \in R(z)$ as above. Let $t = t_{\mathbf{i}}$ and $\nu \in \mathbb{Z}^{D \times I}$ be the minimal lifting matrix of the seed t associated to \mathfrak{X} . Note that, for any $k \in I$, $\nu_{\bullet, k}$ is canonically an element of $X(T)$.

Lemma 7.4.6. *For any $k \in I$, $\uparrow x_k$ is a $X(T) \times X(\widehat{T})$ -homogeneous element of Br of degree*

$$(\nu_{\bullet, k}, \rho(\nu_{\bullet, k}) + \rho(z_{\leq k}^{-1} \varpi_{i_k} - \varpi_{i_k})).$$

Proof. Because of (50) and Lemma 5.2.1, we easily deduce that $1 \otimes x_k$ is a $X(T) \times X(\widehat{T})$ -homogeneous rational function of weight $(0, \rho(z_{\leq k}^{-1} \varpi_{i_k} - \varpi_{i_k}))$. Moreover, for any $\alpha \in D$, X_{α} is homogeneous of degree $(\varpi_{\alpha}, \rho(\varpi_{\alpha}))$. The lemma follows from the definition of $\uparrow x_k$. \square

Note that the previous lemma gives some explicit components of the branching from G to \widehat{G} .

We want to stress that the graduation induced by $\uparrow 0$, only accounts for the $X(T)$ -part of the graduation on Br . It's natural to ask if the whole $X(T) \times X(\widehat{T})$ -graduation on Br can be understood using cluster theory. In particular, because of Proposition 3.1.9 and Corollary 7.3.2, if the $X(\widehat{T})$ -grading on $\mathbb{C}[U(z)]$ corresponding to the conjugation action of \widehat{T} , is induced by a $X(\widehat{T})$ -degree configuration σ on t , then $\overline{\mathcal{A}}(\uparrow t^D)$ has a $X(T) \times X(\widehat{T})$ -graduation induced by $\uparrow \sigma$ and $\overline{\mathcal{A}}(\uparrow t^D)$ is a $X(T) \times X(\widehat{T})$ -graded subalgebra of Br^* . Note that, if such a degree configuration exists, then for any $k \leq \ell(z)$, σ_k has to be the weight of x_k with respect to the conjugation action of \widehat{T} on $U(z)$. By Lemma 5.2.1, then

$$\sigma_k = \rho(z_{\leq k}^{-1} \varpi_{i_k} - \varpi_{i_k}). \quad (52)$$

This explains the importance of Question 5.3.4.

If the answer to Question 5.3.4 turns out to be negative in general, it's natural to ask the following.

Question 7.4.7. *Let G and \widehat{G} be as in Section 7.1. Moreover let $z \in W$, $\mathbf{i} \in R(z)$, $t = t_{\mathbf{i}}$ and $\bar{\sigma} = \rho(\sigma) \in X(\widehat{T})^{\ell(z)}$, where $\sigma \in X(T)^{\ell(z)}$ is defined by (52). When is $\bar{\sigma}$ a $X(\widehat{T})$ -degree configuration on t ?*

Note that, a positive answer to Question 5.3.4 or 7.4.7 would allow, using Lemma 7.4.6 and the mutation formula for degree configurations, to obtain an algorithmic way to identify a (a priori infinite) class of non-zero components for the branching from G to \widehat{G} .

8 Study of equality

In this section we study whether there is equality, between $\overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}^D)$ and Br , in Theorem 7.4.5. By Proposition 4.1.3, for any $\alpha \in D$, we need to compare the valuations \mathcal{V}_{α} and the cluster valuations $\mathcal{C}\mathcal{V}_{\alpha}$. To compute the valuations \mathcal{V}_{α} , the strategy is to produce suitable open charts intersecting the corresponding divisor. Actually, this procedure is performed in G where we have "more charts" than in \mathfrak{X} . This is made possible by Lemma 8.0.1.

Recall that $\mathbb{C}(\mathfrak{X})$ is naturally identified with a subfield of $\mathbb{C}(G)$. If a rational function $f \in \mathbb{C}(G)$ belongs to $\mathbb{C}(\mathfrak{X})$, then f is invariant for left multiplication by U^- and for right multiplication by \widehat{U} .

Lemma 8.0.1. *For any $f \in \mathbb{C}(\mathfrak{X})$ and $\alpha \in D$, $\mathcal{V}_{\alpha}(f) = \mathcal{V}_{\overline{B^{-s_{\alpha}}B}}(f)$.*

Proof. It's sufficient to prove the case of $f \in \mathbb{C}[\mathfrak{X}]$. Then $\mathcal{V}_{\alpha}(f)$, is the multiplicity of X_{α} , in the decomposition of f into irreducible factors in $\mathbb{C}[\mathfrak{X}]$. By Lemma 5.2.4, this is the same as the multiplicity of X_{α} in the decomposition of f into irreducible factors in $\mathbb{C}[G]$. The lemma follows by Lemma 5.2.1, since $V(X_{\alpha}) = \overline{B^{-s_{\alpha}}B}$. \square

8.1 The Levi case

The notation of this section is as in Subsection 7.4.1. We want to prove the following theorem.

Theorem 8.1.1. *Suppose that G is simple and $\widehat{G} = G_I$ is the Levi subgroup corresponding to $I \subsetneq \Delta$. For any $\mathbf{i} \in R(z)$, there's equality in Theorem 7.4.5. That is $\overline{\mathcal{A}}(t_{\mathbf{i}}^D) = \text{Br}(G, G_I)$.*

Note that the branching problem from a semisimple group to a Levi subgroup, can always be reduced to the study of a finite number of branchings from a simple group, to a Levi subgroup.

Corollary 8.1.2. *If G is simply laced and $\mathbf{i}, \mathbf{i}' \in R(z)$, then $|t_{\mathbf{i}}^D| \simeq |t_{\mathbf{i}'}^D|$.*

Proof. Since G is simply laced, $t_{\mathbf{i}} \simeq t_{\mathbf{i}'}$. Then $|t_{\mathbf{i}}^D| \simeq |t_{\mathbf{i}'}^D|$ by Theorem 8.1.1, Corollary 4.0.11 and Lemma 3.0.8. \square

From now on, \widehat{G} and G are as in the hypothesis of Theorem 8.1.1 and $\mathbf{i} \in R(z)$ is fixed. Recall that $z = w_{0,I}^\vee$. We denote $t = t_{\mathbf{i}}$ and $\text{Br} = \text{Br}(G, \widehat{G})$. Here $D = \Delta$ and we have a cartesian square

$$\begin{array}{ccc} U^- \times T \times Y \times \widehat{U} & \xrightarrow{\psi} & G \\ \downarrow & \lrcorner & \downarrow \pi \\ T \times Y & \xrightarrow{\phi} & X \end{array} \quad (53)$$

where the leftmost vertical map is defined by $(u^-, h, y, \widehat{u}) \mapsto (h, y)$, ψ by $(u^-, h, y, \widehat{u}) \mapsto u^- h y \widehat{u}$ and π is the natural projection. Note that the image of ψ is G_0 .

Recall that, for a T -stable subgroup H of U and $\alpha \in \Delta$, $H_{\setminus \alpha} = H \cap s_\alpha U s_\alpha$.

Lemma 8.1.3. *For $\alpha \in D$, let ι_α be the map defined by*

1. *If $\alpha \in I$*

$$\begin{array}{ccc} \iota_\alpha : U^- \times T \times \mathbb{C} \times Y \times \widehat{U}_{\setminus \alpha} & \longrightarrow & G \\ x = (u^-, h, t, y, \widehat{u}) & \longmapsto & \iota_\alpha(x) = u^- h \overline{s}_\alpha^{-1} x_{-\alpha}(t) y \widehat{u}. \end{array}$$

2. *If $\alpha \notin I$*

$$\begin{array}{ccc} \iota_\alpha : U^- \times T \times \mathbb{C} \times Y_{\setminus \alpha} \times \widehat{U} & \longrightarrow & G \\ x = (u^-, h, t, y, \widehat{u}) & \longmapsto & \iota_\alpha(x) = u^- h \overline{s}_\alpha^{-1} x_{-\alpha}(t) y \widehat{u}. \end{array}$$

Then, ι_α is an open embedding whose image intersects $\overline{B^- s_\alpha B}$. Moreover, $\iota_\alpha(x) \in \overline{B^- s_\alpha B}$ if and only if $t = 0$, otherwise $\iota_\alpha(x) \in G_0$.

Proof. It's clear that, for any $\alpha \in \Delta$, the map

$$\begin{array}{ccc} \tilde{\iota}_\alpha : U^- \times T \times U & \longrightarrow & G \\ (u^-, h, u) & \longmapsto & u^- h u \overline{s}_\alpha^{-1} \end{array} \quad (54)$$

is an open embedding whose image intersects $\overline{B^- s_\alpha B}$. Moreover, the map

$$\begin{array}{ccc} \mathbb{C} \times U_{\setminus \alpha} & \longrightarrow & U \\ (t, u) & \longmapsto & x_\alpha(-t)u \end{array}$$

is an isomorphism and conjugation by \overline{s}_α induces an automorphism of $U_{\setminus \alpha}$. To conclude that ι_α is an open embedding, note first that the product induces an isomorphism between $Y \times \widehat{U}_{\setminus \alpha}$ (resp.

$Y_{\setminus\alpha} \times \widehat{U}$) and $U_{\setminus\alpha}$, if $\alpha \in I$ (resp $\alpha \notin I$), because of [Hum90, Proposition 28.1]. Finally, we use that for any $t \in \mathbb{C}$,

$$x_{\alpha}(-t)\overline{s_{\alpha}^{-1}} = \overline{s_{\alpha}^{-1}}x_{-\alpha}(t),$$

which is an easy identity that can be checked in SL_2 . It's clear from the definition of ι_{α} that $\iota_{\alpha}(x) \in \overline{B^{-s_{\alpha}}B}$ if the t -coordinate of x is zero. If $t \neq 0$, $\iota_{\alpha}(x) \in G_0$ from the second identity in (30). \square

By the previous lemma, the divisor $\overline{B^{-s_{\alpha}}B}$, in the chart given by ι_{α} , corresponds to $\{t = 0\}$. Then we make a very important computation.

Lemma 8.1.4. *Let $\lambda \in X(T)$, $\alpha \in D$ and $f \in \mathbb{C}(\mathfrak{X})_{\lambda, \bullet}$. For a generic x as in the notation of Lemma 8.1.3, such that $t \neq 0$, we have*

$$f(\iota_{\alpha}(x)) = \lambda(h)t^{\langle \lambda, \alpha^{\vee} \rangle} f(x_{\alpha}(t^{-1})y).$$

Moreover, if $\beta \in D$, then

$$X_{\beta}(\iota_{\alpha}(x)) = \varpi_{\beta}(h)t^{\langle \varpi_{\beta}, \alpha^{\vee} \rangle}.$$

Remark 8.1.5. With little abuse, for $\lambda \in X(T)$, we denote by $\mathbb{C}(\mathfrak{X})_{\lambda, \bullet}$ the set of T -equivariant rational functions of weight λ .

Proof. If $x = (u^{-}, h, t, y, \widehat{u})$, using the second formula in (30) and the fact that f is $U^{-} \times \widehat{U}$ -invariant, we have that

$$\begin{aligned} f(\iota_{\alpha}(x)) &= f(h\overline{s_{\alpha}^{-1}}x_{-\alpha}(t)y) \\ &= \lambda(h)f(x_{-\alpha}(-t^{-1})\alpha^{\vee}(t)x_{\alpha}(t^{-1})y) \\ &= \lambda(h)t^{\langle \lambda, \alpha^{\vee} \rangle} f(x_{\alpha}(t^{-1})y). \end{aligned}$$

The statement on $X_{\beta} = \Delta_{e, e}^{\varpi_{\beta}}$ follows immediately from Lemma 5.2.1. \square

We use the convention that, whenever $\alpha \in D$ is fixed, for any $v, w \in W$ and $\beta \in \Delta$, we denote

$$D_{v, w}^{\varpi_{\beta}} = \begin{cases} (\Delta_{v, w}^{\varpi_{\beta}})|_Y & \text{if } \alpha \in I \\ (\Delta_{v, w}^{\varpi_{\beta}})|_{Y_{\setminus\alpha}} & \text{if } \alpha \notin I. \end{cases} \quad (55)$$

Proposition 8.1.6. *Let $k \in I$ and $\alpha \in D$. Then $\nu_{\alpha, k} = \delta_{\alpha, i_k}$. Moreover for x as in Lemma 8.1.3*

$$\upharpoonright_{x_k}(\iota_{\alpha}(x)) = \varpi_{i_k}(h)(p_{k,0}(y) + tp_{k,1}(y)) \quad \text{where} \quad p_{k,0} = D_{s_{\alpha}, z_{\leq k}^{-1}}^{\varpi_{i_k}}.$$

Moreover:

$$p_{k,1} = \begin{cases} 0 & \text{if } \alpha \neq i_k \\ D_{e, z_{\leq k}}^{\varpi_{i_k}} & \text{otherwise.} \end{cases}$$

Remark 8.1.7. Note that $p_{k,0}$ and $p_{k,1}$ depend on α . So, when we use such a notation, we assume that it refers to an α which is sufficiently clear from the context.

Proof. First, we compute ν .

Suppose that $\alpha \in I$ and that x is such that $t \neq 0$. Since $1 \otimes x_k \in \mathbb{C}(\mathfrak{X})_{0,\bullet}$, from Lemma 8.1.4 we deduce that $x_k(\iota_\alpha(x)) = 1 \otimes x_k(x_\alpha(t^{-1})y)$. Recall that, in this case, $y \in Y = U(z)$. Note that conjugation by $x_\alpha(t^{-1})$ stabilises Y . In fact, if $\delta \in \Phi(z) = \Phi^+ \setminus \Phi_I^+$ and $\delta + \alpha \in \Phi$, then $\delta + \alpha \in \Phi(z)$. Hence

$$\begin{aligned} 1 \otimes x_k(x_\alpha(t^{-1})y) &= 1 \otimes x_k(x_\alpha(t^{-1})yx_\alpha(-t^{-1})x_\alpha(t^{-1})) \\ &= 1 \otimes x_k(x_\alpha(t^{-1})yx_\alpha(-t^{-1})) \end{aligned}$$

where the last equality follows from the fact that, if $\alpha \in I$, then $x_\alpha(t^{-1}) \in \widehat{U}$. Note that (see (53))

$$\phi\left((e, x_\alpha(t^{-1})yx_\alpha(-t^{-1}))\right) = \pi(x_\alpha(t^{-1})yx_\alpha(-t^{-1})).$$

In particular, from the definition of $1 \otimes x_k$ we deduce the first of the following equalities

$$\begin{aligned} 1 \otimes x_k(x_\alpha(t^{-1})yx_\alpha(-t^{-1})) &= x_k(x_\alpha(t^{-1})yx_\alpha(-t^{-1})) \\ &= \Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}((x_\alpha(t^{-1})yx_\alpha(-t^{-1})) \\ &= \Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}((x_\alpha(t^{-1})y). \end{aligned}$$

The second equality is the definition of x_k and the last one follows from Lemma 5.4.3 and the fact that $z_{\leq k}\alpha \in \Phi^+$. In fact, from the definition of $z_{\leq k}$, we have that $z_{\leq k} \leq_L z$, where \leq_L denotes the left weak order. Hence, $\Phi(z_{\leq k}) \subseteq \Phi(z)$ and $\alpha \notin \Phi(z)$. Now we apply Lemma 5.4.7.

If $i_k \neq \alpha$, we deduce that

$$\Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}((x_\alpha(t^{-1})y) = \Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y) = \Delta_{s_\alpha, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y).$$

The last equality follows from Lemma 5.2.1, since $s_\alpha \varpi_{i_k} = \varpi_{i_k}$. In particular, we deduce that $\nu_{\alpha, k} = 0$.

If $i_k = \alpha$, we have that

$$\Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}((x_\alpha(t^{-1})y) = \Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y) + t^{-1} \Delta_{s_\alpha, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y).$$

By Lemma 5.4.1, we have that $\Delta_{s_\alpha, z_{\leq k}^{-1}}^{\varpi_{i_k}}$ doesn't vanish on U , in fact, $s_{i_k} = s_\alpha \in \text{supp}(z_{\leq k}^{-1})$. Then, by Lemma 5.2.1, $\Delta_{s_\alpha, z_{\leq k}^{-1}}^{\varpi_{i_k}}$ doesn't vanish on

$$U \cap s_\alpha U s_\alpha \cap z_{\leq k}^{-1} U^- z_{\leq k} \subseteq U \cap z^{-1} U^- z = Y.$$

We deduce that $\nu_{\alpha, k} = 1$.

We assume now that $\alpha \notin I$. In this case, $x_\alpha(t^{-1}) \in Y$, so for $y \in Y_{\setminus \alpha}$ we have that $x_\alpha(t^{-1})y \in Y$. In particular, by a very similar but simpler argument than the previous one, we have that

$$1 \otimes x(k)(\iota_\alpha(x)) = 1 \otimes x_k(x_\alpha(t^{-1})y) = \Delta_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}(x_\alpha(t^{-1})y).$$

Then, literally the same proof of the previous case, up to replacing Y with $Y_{\setminus \alpha}$, implies that $\nu_{\alpha, k} = \delta_{\alpha, i_k}$.

From the explicit expression of ν , it follows that $\downarrow x_k = X_{i_k}(1 \otimes x_k)$. Then, the last part of the statement follows from the previous calculations and Lemma 8.1.4. \square

Corollary 8.1.8. *For any $k \in [l]$, $\rho(z_{\leq k}^{-1} \varpi_{i_k}) \in X(\widehat{T})^+$ and the \widehat{G} -representation $V(\rho(z_{\leq k}^{-1} \varpi_{i_k}))$ is a subrepresentation of $V(\varpi_{i_k})$.*

Proof. By Lemma 7.4.6 and Proposition 8.1.6, $\uparrow x_k$ is homogeneous of degree $(\varpi_{i_k}, \rho(z_{\leq k}^{-1} \varpi_{i_k}))$. \square

Lemma 8.1.9. *For any $I \subsetneq \Delta$, $\text{supp}(z) = \Delta$.*

Remark 8.1.10. This is the only lemma in which we use that G is simple.

Proof. Let $\alpha \in \Delta$ and $\Phi_\alpha^+ = \{\sum_{\beta \in \Delta} n_\beta \beta \in \Phi^+ : n_\alpha > 0\}$. If by contradiction $s_\alpha \notin \text{supp}(z)$, then $z\Phi_\alpha^+ \subseteq \Phi^+$. In fact, for any $\delta \in \Phi$, the α -coefficients of δ and $z\delta$ (in the base Δ) are the same.

Since G is simple, we can consider $\gamma \in \Phi^+$ the longest root of Φ . Since $\gamma \in \Phi_\beta^+$ for any $\beta \in \Delta$, we deduce that $\gamma \notin \Phi(z)$. But then $\gamma \in \Phi(w_{0,I})$. This is a contradiction since $\gamma \notin \Phi_I$. \square

By the previous lemma, for any $\alpha \in \Delta$, α^{\min} is well defined.

Lemma 8.1.11. *In the notation of Proposition 8.1.6*

$$p_{\alpha^{\min}, 0} = \prod_{j=1}^{\alpha^{\min}-1} p_{j,0}^{c_j}$$

for some non-negative integers c_j .

Again, the exponents c_j depend on α (as the functions $p_{j,0}$), so we use such a notation when it refers to an α which is sufficiently clear from the context.

Proof. Accordingly to if $\alpha \in I$ or not, this is a reformulation of Lemma 5.5.1 or of Lemma 5.5.3 respectively. \square

Lemma 8.1.12. *The functions $p_{k,0}$, for $k \neq \alpha^{\min}$ and $k \in [l]$, are algebraically independent.*

Proof. Again this is a reformulation of Lemma 5.5.2 or Lemma 5.5.4. \square

We are ready to prove Theorem 8.1.1.

Proof of Theorem 8.1.1. By contradiction suppose that $\overline{\mathcal{A}}(\uparrow t^D)$ is strictly contained in Br . By Proposition 4.1.3, there exists an $\alpha \in D$ and $f \in \text{Br}$ such that $\mathcal{CV}_\alpha(f) < 0$. Since Br is $X(T)$ -graded, we can suppose f homogeneous. Recall that $\overline{\mathcal{A}}(\uparrow t^D)$ is a graded subalgebra of Br , which in turn is a graded subalgebra of $\overline{\mathcal{A}}(\uparrow t)$. In particular, up to multiplying for a monomial in the X_β , $\beta \in D$, and in the unfrozen variables of $\uparrow x$, we can suppose that there exist $\lambda \in X(T)$ and $f \in \text{Br}_{\lambda, \bullet}$ such that

$$f = \frac{P}{\uparrow x_\alpha} \quad \text{where} \quad P = \sum_{n \in \mathbb{N}^I} a_n \uparrow x^n : \uparrow x_\alpha \nmid P. \quad (56)$$

Hence, P is a polynomial in the variables of the cluster $\uparrow x$, which is not divisible by $\uparrow x_\alpha$, here divisibility is intended in the polynomial ring. Up to changing f , we can assume that if $n_\alpha > 0$, then $a_n = 0$, so that the sum defining P runs over $\mathbb{N}^{I \setminus \{\alpha\}}$. We use the convention that, if $\beta \in D$, then $i_\beta = \beta$ so that i_j makes sense for any $j \in I$. The fact that f is homogeneous of degree λ imposes (using Proposition 8.1.6) that, for any n such that $a_n \neq 0$

$$\sum_{j \in I \setminus \{\alpha\}} n_j \varpi_{i_j} = \lambda + \varpi_\alpha. \quad (57)$$

Next, let x as in Lemma 8.1.3. Using Proposition 8.1.6 and Lemma 8.1.4 (recall that $\imath x_\beta = X_\beta$) we deduce that

$$f(\iota_\alpha(x)) = \lambda(h)t^{-1} \sum_{k \geq 0} f_k(y)$$

for certain $f_k \in \mathbb{C}[Y]$ (resp. $\mathbb{C}[Y \setminus \alpha]$) if $\alpha \in I$ (resp. $\alpha \notin I$). Moreover, for $\beta \in \Delta$, we set $p_{\beta,0} = 1$ so that, for $\beta \neq \alpha$,

$$\imath x_\beta(\iota_\alpha(x)) = \varpi_\beta(h) = \varpi_\beta(h)p_{\beta,0}.$$

Using the last expression, the fact that $n_\alpha = 0$ if $a_n \neq 0$ and Proposition 8.1.6, we get that

$$f_0 = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha\}}} a_n p_{\bullet,0}^n \quad \text{where} \quad p_{\bullet,0}^n = \prod_{j \in I \setminus \{\alpha\}} p_{j,0}^{n_j}.$$

But since $f \in \text{Br}$, $\iota_\alpha^*(f)$ is regular, hence $f_0 = 0$.

Consider the linear map $\tilde{\pi} : \mathbb{Z}^{I \setminus \{\alpha\}} \rightarrow \mathbb{Z}^{(I \setminus \alpha^{\min})}$ defined in the following way. For $\beta \in D \setminus \{\alpha\}$, $\tilde{\pi}(e_\beta) = 0$, for $j \in I \setminus \{\alpha^{\min}\}$ $\tilde{\pi}(e_j) = e_j$ and $\tilde{\pi}(e_{\alpha^{\min}}) = \sum_{j < \alpha^{\min}} c_j e_j$, where the coefficients c_j are the ones of Lemma 8.1.11. Since $p_{\beta,0} = 1$ for $\beta \in D$ and because of Lemma 8.1.11, we have that for any $n \in \mathbb{N}^{I \setminus \{\alpha\}}$

$$p_{\bullet,0}^n = p_{\bullet,0}^{\tilde{\pi}(n)}.$$

In particular

$$f_0 = \sum_{m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}} b_m p_{\bullet,0}^m \quad \text{where} \quad b_m = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha\}} : \tilde{\pi}(n)=m} a_n.$$

By Lemma 8.1.12, the functions $p_{j,0}$ with $j \in [l] \setminus \{\alpha^{\min}\}$ are algebraically independent. Since $f_0 = 0$, then for any $m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}$, $b_m = 0$. We claim that, for any $m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}$, there exists at most one $n \in \mathbb{N}^{I \setminus \{\alpha\}}$ such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$. This implies at ones that, for any $n \in \mathbb{N}^{I \setminus \{\alpha\}}$, $a_n = 0$, which is a contradiction.

To prove the claim, just notice that any n such that $a_n \neq 0$ satisfies the weight condition (57). In particular, consider the linear map $\pi : \mathbb{Z}^{I \setminus \{\alpha\}} \rightarrow X(T) \times \mathbb{Z}^{I \setminus \{\alpha^{\min}\}}$ defined by $\pi(e_j) = (\varpi_{i_j}, \tilde{\pi}(e_j))$. Remember that, for $\beta \in \Delta$, $i_\beta = \beta$. If n is such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$, then $\pi(n) = (\lambda + \varpi_\alpha, m)$. But the map π is injective. To prove this, consider π as a matrix with columns indexed by $I \setminus \{\alpha\}$ and rows indexed by $I \setminus \{\alpha^{\min}\}$. Recall that $X(T)$ is identified with \mathbb{Z}^D by means of the fundamental weights. Enumerate the roots of $\Delta \setminus \{\alpha\}$ from 1 to $r-1$. Then order the columns of the matrix representing π accordingly to the order

$$\beta_1 < \cdots < \beta_{r-1} < 1 < \cdots < l$$

and the rows according to the order

$$\beta_1 < \cdots < \beta_{r-1} < 1 < \cdots < \alpha^{\min} - 1 < \alpha < \alpha^{\min} + 1 < \cdots < l.$$

Then, the matrix representing π is upper triangular with ones on the diagonal. In particular π is invertible. \square

8.2 The product case: tensor product

To simplify the notation, in this section we switch the role of G and \widehat{G} . In particular, G is a subgroup of \widehat{G} . We assume that G is semisimple, simply connected and we consider G as a subgroup of $\widehat{G} = G \times G$ by the diagonal embedding. This is the simplest case of Subsection 7.4.2: when $H = G$. In this section we want to prove the following

Theorem 8.2.1. *For any $\mathbf{i} \in R(w_0)$, there's equality in Theorem 7.4.5. That is:*

$$\overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}^D) = \text{Br}(G \times G, G).$$

Corollary 8.2.2. *If G is simply laced and $\mathbf{i}, \mathbf{i}' \in R(w_0)$, then $\uparrow t_{\mathbf{i}}^D \simeq \uparrow t_{\mathbf{i}'}^D$.*

Proof. Since G is simply laced, then $t_{\mathbf{i}} \simeq t_{\mathbf{i}'}$. Then $\uparrow t_{\mathbf{i}}^D \simeq \uparrow t_{\mathbf{i}'}^D$ by Theorem 8.2.1, Corollary 4.0.11 and Lemma 3.0.8. \square

Here we chose a maximal torus T contained in a Borel B , of G , and we consider $\widehat{T} = T \times T$ and $\widehat{B} = B \times B$. Because of Remark 7.4.2, we make no difference between $z = (w_0, e)$ and w_0 . Moreover, we identify $\widehat{U}(z)$ and $U = U(w_0)$ in the obvious way. Recall that, for any $\mathbf{i} \in R(w_0) = R(z)$, the seed $t_{\mathbf{i}}$ gives a cluster structure to $Y = U$. From now on, we fix $\mathbf{i} = (i_l, \dots, i_1) \in R(w_0)$ and set $t = t_{\mathbf{i}}$. Here, $D = D_l \sqcup D_r$ is the disjoint union of two copies of Δ , namely: the *left copy* is $D_l = \{\alpha_l : \alpha \in \Delta\}$ and the *right copy* is $D_r = \{\alpha_r : \alpha \in \Delta\}$. For $\alpha \in \Delta$ (see Remark 5.4.8)

$$X_{\alpha_l} = \Delta_{e,e}^{\varpi_\alpha} \otimes 1 \quad X_{\alpha_r} = 1 \otimes \Delta_{e,e}^{\varpi_\alpha}. \quad (58)$$

We denote $\text{Br} = \text{Br}(G \times G, G)$. Note that $X_{\alpha_l} \in \text{Br}_{(\varpi_\alpha, 0), \varpi_\alpha}$ and $X_{\alpha_r} \in \text{Br}_{(0, \varpi_\alpha), \varpi_\alpha}$. The map

$$\begin{aligned} T \times T \times U &\longrightarrow G \times G \\ (h_l, h_r, u) &\longmapsto (h_l u, h_r) \end{aligned}$$

induces the $T \times T$ -equivariant open embedding $\phi : T \times T \times Y \longrightarrow \mathfrak{X} := \mathfrak{X}(G \times G, G)$. We have a cartesian square

$$\begin{array}{ccc} U^- \times U^- \times T \times T \times U \times U & \xrightarrow{\psi} & G \times G \\ \downarrow & \lrcorner & \downarrow \pi \\ T \times T \times U & \xrightarrow{\phi} & X \end{array} \quad (59)$$

where the leftmost vertical map is defined by $(u_l^-, u_r^-, h_l, h_r, u_l, u_r) \longmapsto (h_l, h_r, u_l)$, π is the natural projection and ψ is defined by $(u_l^-, u_r^-, h_l, h_r, u_l, u_r) \longmapsto (u_l^- h_l u_l u_r, u_r^- h_r u_r)$. By Lemma 8.0.1, for any $\alpha \in \Delta$, computing the valuation \mathcal{V}_{α_l} (resp. \mathcal{V}_{α_r}) of a rational function on X is the same as computing its valuation along the divisor $\overline{B^- s_\alpha B} \times G$ (resp. $G \times \overline{B^- s_\alpha B}$).

Lemma 8.2.3. *For $\alpha \in \Delta$, let*

$$\begin{aligned} \iota_\alpha : T \times T \times \mathbb{C}^* \times Y &\longrightarrow G \times G \\ x = (h_l, h_r, t, y) &\longmapsto \iota_\alpha(x) = (h_l \alpha^\vee(t) x_\alpha (t^{-1}) y, h_r) \end{aligned}$$

and

$$\begin{aligned} j_\alpha : T \times T \times \mathbb{C}^* \times Y &\longrightarrow G \times G \\ x = (h_l, h_r, t, y) &\longmapsto j_\alpha(x) = (h_l y x_\alpha (t^{-1}), h_r \alpha^\vee(t)). \end{aligned}$$

Then, for any $f \in \mathbb{C}(\mathfrak{X})$,

$$\mathcal{V}_{\alpha_l}(f) = \mathcal{V}_{\{t=0\}}(\iota_\alpha^*(f)) \quad \text{and} \quad \mathcal{V}_{\alpha_r}(f) = \mathcal{V}_{\{t=0\}}(j_\alpha^*(f)).$$

Proof. First, note that the maps obtained by composing ι_α and j_α with the natural projection $G \times G \rightarrow \mathfrak{X}$ are dominant. In fact, at $t \in \mathbb{C}^*$ fixed, both maps are obtained from ϕ by an appropriate "multiplication twist" by $\alpha^\vee(t)$ and $x_\alpha(t^{-1})$, so they're dominant at t fixed. Hence ι_α^* and j_α^* make sense. Since f is generically defined on \mathfrak{X} , we make the assumption, throughout the proof, that the points on which we evaluate f are sufficiently generic so that the evaluation makes sense.

We start from the "left side", so we prove the statement about ι_α . First, we prove that the map ϕ_α defined by

$$\begin{aligned} U^- \times U^- \times T \times T \times \mathbb{C} \times \mathbb{C} \times U_{\setminus\alpha} \times U_{\setminus\alpha} &\longrightarrow G \times G \\ x = (u_l^-, u_r^-, h_l, h_r, t, r, u, \tilde{u}) &\longmapsto \phi_\alpha(x) = (u_l^- h_l \bar{s}_\alpha^{-1} x_{-\alpha}(t) u \tilde{u}, u_r^- h_r x_\alpha(r) \tilde{u}) \end{aligned}$$

is an open embedding whose image intersects $\overline{B^- s_\alpha B} \times G$. Note that, for x as in the definition of ϕ_α , we have that $\phi_\alpha(x) \in \overline{B^- s_\alpha B} \times G$ if and only if $t = 0$ and, by (30), $\phi_\alpha(x) \in G_0 \times G_0$ otherwise.

The map

$$\begin{aligned} \tilde{\phi}_\alpha : U^- \times U^- \times T \times T \times U \times U &\longrightarrow G \times G \\ (u_l^-, u_r^-, h_l, h_r, u_l, u_r) &\longmapsto (u_l^- h_l u_l \bar{s}_\alpha^{-1}, u_r^- h_r u_r) \end{aligned}$$

is clearly an open embedding. Moreover, the map

$$\begin{aligned} \mathbb{C} \times U_{\setminus\alpha} &\longrightarrow U \\ (t, u) &\longmapsto x_\alpha(t)u \end{aligned}$$

is an isomorphism and conjugation by \bar{s}_α is an automorphism of $U_{\setminus\alpha}$. Furthermore, the map

$$\begin{aligned} U_{\setminus\alpha} \times U_{\setminus\alpha} &\longrightarrow U_{\setminus\alpha} \times U_{\setminus\alpha} \\ u, \tilde{u} &\longmapsto u \tilde{u}, \tilde{u} \end{aligned}$$

is an isomorphism. In particular, looking in the definition of $\tilde{\phi}_\alpha$, we see that we can write uniquely $(u_l \bar{s}_\alpha^{-1}, u_r) = (x_\alpha(-t) \bar{s}_\alpha^{-1} u \tilde{u}, x_\alpha(r) \tilde{u})$. Using that

$$x_\alpha(-t) \bar{s}_\alpha^{-1} = \bar{s}_\alpha^{-1} x_{-\alpha}(t)$$

we deduce that ϕ_α is an open embedding. Now, let x as in the definition of ϕ_α , such that $t \neq 0$. Then, by the invariance properties of f and (30), we have that

$$f(\phi_\alpha(x)) = f(h_l \bar{s}_\alpha^{-1} x_{-\alpha}(t) u, h_r x_\alpha(r)) = f(h_l \alpha^\vee(t) x_\alpha(t^{-1}) u, h_r x_\alpha(r)).$$

In particular, it's clear that if

$$\begin{aligned} \tilde{\iota}_\alpha : T \times T \times \mathbb{C}^* \times \mathbb{C} \times U_{\setminus\alpha} &\longrightarrow G \times G \\ (h_l, h_r, t, r, u) &\longmapsto (h_l \alpha^\vee(t) x_\alpha(t^{-1}) u, h_r x_\alpha(r)) \end{aligned}$$

then

$$\mathcal{V}_{\alpha_l}(f) = \mathcal{V}_{\{t=0\}} \tilde{\iota}_\alpha^*(f) = \mathcal{V}_{\{t=0\}} \tilde{\iota}_\alpha^*(f).$$

But

$$\begin{aligned} \mathbb{C} \times U_{\setminus \alpha} &\longrightarrow Y \\ (r, u) &\longmapsto ux_{\alpha}(-r) \end{aligned} \quad (60)$$

is an isomorphism and, since f is invariant by the right action of diagonal multiplication by U , we have that

$$f(\tilde{\iota}_{\alpha}(x)) = f(h_l \alpha^{\vee}(t) x_{\alpha}(t^{-1}) u x_{\alpha}(-r), h_r).$$

In particular, under the identification induced by (60), we have that

$$\tilde{\iota}_{\alpha}^*(f) = \iota_{\alpha}^*(f).$$

We conclude thanks to the following very simple statement.

Lemma 8.2.4. *Let $\psi : Z \rightarrow W$ be an isomorphism of normal, complex, algebraic varieties and $\phi : \mathbb{C}^* \times Z \rightarrow \mathbb{C}^* \times W$ be defined by $(t, z) \mapsto (t, \psi(z))$. Then, for any $f \in \mathbb{C}(\mathbb{C}^* \times W)$,*

$$\mathcal{V}_{\{t=0\}}(f) = \mathcal{V}_{\{t=0\}}\phi^*(f).$$

Next, we pass to the "right side", that is to j_{α} . The argument is very similar, we sketch through it. First, one proves that the map

$$\begin{aligned} \psi_{\alpha} : U^{-} \times U^{-} \times T \times T \times \mathbb{C} \times Y \times U_{\setminus \alpha} &\longrightarrow G \times G \\ x = (u_l^{-}, u_r^{-}, h_l, h_r, t, y, u) &\longmapsto \psi_{\alpha}(x) = (u_l^{-} h_l y u, u_r^{-} h_r x_{\alpha}(t) \bar{s}_{\alpha} u) \end{aligned}$$

is an open embedding. Moreover, $\psi_{\alpha}(x) \in G \times \overline{B^{-} s_{\alpha} B}$ if $t = 0$ and $\psi_{\alpha}(x) \in G_0 \times G_0$ otherwise. But then, for $t \neq 0$ we compute that

$$f(\psi_{\alpha}(x)) = f(h_l y, h_r \alpha^{\vee}(t) x_{\alpha}(-t^{-1})).$$

In particular, considering

$$\begin{aligned} \tilde{j}_{\alpha} : T \times T \times \mathbb{C}^* \times Y &\longrightarrow G \times G \\ (h_l, h_r, t, y) &\longmapsto (h_l y, h_r \alpha^{\vee}(t) x_{\alpha}(-t^{-1})) \end{aligned}$$

we clearly have that

$$\mathcal{V}_{\alpha_r}(f) = \mathcal{V}_{\{t=0\}}\psi_{\alpha}^*(f) = \mathcal{V}_{\{t=0\}}\tilde{j}_{\alpha}^*(f).$$

But then, by invariance of f , we see that $\tilde{j}_{\alpha}^*(f) = j_{\alpha}^*(f)$. □

Then, we make this important but very easy calculation.

Lemma 8.2.5. *Let $(\lambda, \mu) \in X(T) \times X(T) = X(T \times T)$, $f \in \mathbb{C}(\mathfrak{X})_{(\lambda, \mu), \bullet}$ and x generic as in the notation of Lemma 8.2.3. Then, for any $\alpha \in \Delta$,*

$$\begin{aligned} f(\iota_{\alpha}(x)) &= \lambda(h_l) \mu(h_r) t^{\langle \lambda, \alpha^{\vee} \rangle} f((x_{\alpha}(t^{-1}) y, e)) \quad \text{and} \\ f(j_{\alpha}(x)) &= \lambda(h_l) \mu(h_r) t^{\langle \mu, \alpha^{\vee} \rangle} f((y x_{\alpha}(t^{-1}), e)). \end{aligned}$$

Moreover, for any $\beta \in \Delta$

$$\begin{aligned} X_{\beta_l}(\iota_{\alpha}(x)) &= \varpi_{\beta}(h_l) t^{\langle \varpi_{\beta}, \alpha^{\vee} \rangle} & X_{\beta_r}(\iota_{\alpha}(x)) &= \varpi_{\beta}(h_r) \\ X_{\beta_l}(j_{\alpha}(x)) &= \varpi_{\beta}(h_l) & X_{\beta_r}(j_{\alpha}(x)) &= \varpi_{\beta}(h_r) t^{\langle \varpi_{\beta}, \alpha^{\vee} \rangle} \end{aligned}$$

Remark 8.2.6. By little abuse, $\mathbb{C}(\mathfrak{X})_{(\lambda, \mu), \bullet}$ denotes the set of $T \times T$ semi-invariant rational functions of weight (λ, μ) .

Proof. The proof is trivial. □

Recall that, an element $\mu \in \mathbb{Z}^D$, is identified with the character

$$\sum_{\alpha \in \Delta} (\mu_{\alpha_l} \varpi_{\alpha}, \mu_{\alpha_r} \varpi_{\alpha}) \in X(T \times T).$$

For any $\alpha \in \Delta$ and $v, w \in W$, we denote by $D_{v, w}^{\varpi_{\alpha}}$ the restriction of $\Delta_{v, w}^{\varpi_{\alpha}}$ to $Y = U$. Here, we describe explicitly the cluster variables $\uparrow x$ and how they behave in the charts given by ι_{α} and j_{α} .

Lemma 8.2.7. *For any $k \in [l]$ and $\alpha \in \Delta$*

$$\nu_{\alpha_l, k} = \delta_{i_k, \alpha} \quad \text{and} \quad \nu_{\alpha_r, k} = \max\{0, \langle -z_{\leq k}^{-1} \varpi_{i_k}, \alpha^{\vee} \rangle\}.$$

Moreover,

$$\uparrow x_k(\iota_{\alpha}(x)) = \nu_{\bullet, k}((h_l, h_r))(p_{k,0}(y) + tp_{k,1}(y)) \quad \text{where} \quad p_{k,0} = D_{s_{\alpha}, z_{\leq k}^{-1}}^{\varpi_{i_k}}$$

and

$$\uparrow x_k(j_{\alpha}(x)) = \nu_{\bullet, k}((h_l, h_r)) \left[\sum_{n=0}^{\nu_{\alpha_r, k}} t^n q_{k,n}(y) \right] \quad \text{where} \quad q_{k,0} = \begin{cases} D_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}} & \text{if } (z_{\leq k})\alpha \in \Phi^+ \\ D_{e, s_{\alpha} z_{\leq k}^{-1}}^{\varpi_{i_k}} & \text{if } (z_{\leq k})\alpha \in \Phi^-. \end{cases}$$

Proof. Note that $1 \otimes x_k \in \mathbb{C}(\mathfrak{X})_{(0,0), \bullet}$. In particular, by Lemma 8.2.5 we have that

$$1 \otimes x_k(\iota_{\alpha}(x)) = 1 \otimes x_k((x_{\alpha}(t^{-1})y, e)).$$

But (see diagram (59))

$$\phi\left((e, e, x_{\alpha}(t^{-1})y)\right) = \pi\left((x_{\alpha}(t^{-1})y, e)\right).$$

Then, by the definition of $1 \otimes x_k$, we have that

$$1 \otimes x_k(\iota_{\alpha}(x)) = 1 \otimes x_k((x_{\alpha}(t^{-1})y, e)) = x_k(x_{\alpha}(t^{-1})y).$$

Now we apply Lemma 5.4.7.

If $i_k \neq \alpha$, then

$$x_k(x_{\alpha}(t^{-1})y) = D_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y) = D_{s_{\alpha}, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y) = p_{k,0}(y). \quad (61)$$

The central equality follows from Lemma 5.2.1. In particular, by Lemma 8.2.3, we deduce that $\nu_{\alpha_l, k} = 0$.

If $i_k = \alpha$, then

$$x_k(x_{\alpha}(t^{-1})y) = p_{k,1}(y) + t^{-1}p_{k,0}(y) \quad (62)$$

where $p_{k,0}$ is as described in the statement and $p_{k,1}$ can be deduced from Lemma 5.4.7. By Lemma 5.4.1, $p_{k,0}$ doesn't vanish on Y . Hence, from Lemma 8.2.3, we deduce that $\nu_{\alpha_l, k} = \delta_{i_k, \alpha}$.

We switch to j_α . The argument is very similar. We set $N_k(\alpha) = \max\{0, \langle -z_{\leq k}^{-1} \varpi_{i_k}, \alpha^\vee \rangle\}$. First, observe that if $(z_{\leq k})\alpha \in \Phi^+$, then $N_k(\alpha) = 0$ and

$$1 \otimes x_k(j_\alpha(x)) = D_{e, z_{\leq k}^{-1}}^{\varpi_{i_k}}(y) \quad (63)$$

by Lemmas 8.2.5 and 5.4.3. If $(z_{\leq k})\alpha \in \Phi^-$, then $N_k(\alpha) = \langle -z_{\leq k}^{-1} \varpi_{i_k}, \alpha^\vee \rangle$. Again, Lemmas 8.2.5 and 5.4.3 imply that

$$1 \otimes x_k(j_\alpha(x)) = \sum_{n=0}^{N_k(\alpha)} t^{-n} Q_{(N_k(\alpha)-n)}(y). \quad (64)$$

Then, we set $q_{k,m} = Q_m$. By Lemma 5.4.3, $q_{k,0}$ agrees with the expression given in the statement. Since the minors of the form $\Delta_{e,w}^{\varpi_\beta}$ don't vanish on $Y = U$, we have that $q_{k,0} \neq 0$. Then, we deduce from expressions (64), (63) and Lemma 8.2.3 that $\nu_{\alpha_r, k} = N_k(\alpha)$. Now, since

$$\uparrow x_k = X^{\nu_{\bullet, k}}(1 \otimes x_k),$$

the desired expression for $\iota_\alpha^*(\uparrow x_k)$ and $j_\alpha^*(\uparrow x_k)$ is obtained from (61), (62), (63), (64) and Lemma 8.2.5. \square

Before going on, we characterise the $X(T)^3$ -weight of the cluster variables of the seeds $\uparrow t_{i'}^D$, for the various $i' \in R(w_0)$. Recall the following theorem.

Theorem 8.2.8 ([PRV67]). *For any $\lambda, \mu, \nu \in X(T)^+$, if there exist $v \in W$ such that*

$$w_0\lambda + v\mu = v\nu \quad (65)$$

then, for any $k \in \mathbb{N}$,

$$\dim\left(\text{Hom}(V(k\lambda) \otimes V(k\mu), V(k\nu))^G\right) = 1.$$

We say that the triple of dominant weights (λ, μ, ν) satisfies the PRV condition, for $v \in W$, if condition (65) holds. If $\lambda = \sum n_\alpha \varpi_\alpha \in X(T)$, we denote $\lambda^+ = \sum n_\alpha^+ \varpi_\alpha$ and $\lambda^- = \sum n_\alpha^- \varpi_\alpha$.

Proposition 8.2.9. *For any $k \in [l]$, $\uparrow x_k$ is $X(T)^3$ -homogeneous of weight*

$$(\varpi_{i_k}, (z_{\leq k}^{-1} \varpi_{i_k})^-, (z_{\leq k}^{-1} \varpi_{i_k})^+).$$

This triple of dominant weights satisfies the PRV condition for $w_0 z_{\leq k}$.

Proof. By Lemma 8.2.7, $\nu_{\bullet, k} = (\varpi_{i_k}, (z_{\leq k}^{-1} \varpi_{i_k})^-)$. Moreover, Lemma 7.4.6 implies that (recall that we modified our notation for this section) $\uparrow x_k$ is $X(T)^3$ -homogeneous of weight

$$(\varpi_{i_k}, (z_{\leq k}^{-1} \varpi_{i_k})^-, \varpi_{i_k} + (z_{\leq k}^{-1} \varpi_{i_k})^- + z_{\leq k}^{-1} \varpi_{i_k} - \varpi_{i_k}).$$

Recalling that, for any $b \in \mathbb{R}$, $b = b^+ - b^-$, we immediately have that the triple of weights above coincides with the one of the statement. The same formula allows to verify, easily, the PRV condition. \square

Proposition 8.2.10. *Let $w \in W$ and $\alpha \in \Delta$.*

1. *There exist $\mathbf{i}' \in R(w_0)$ and a cluster variable $\uparrow x_{\alpha,w}$ of $\uparrow t_{\mathbf{i}'}^D$, which is $X(T)^3$ -homogeneous of weight $(\varpi_\alpha, (w\varpi_\alpha)^-, (w\varpi_\alpha)^+)$.*
2. *If G is simply laced and \tilde{x} is a cluster variable of a seed $\tilde{t} \simeq \uparrow t_{\mathbf{i}'}^D$, for a certain $\mathbf{i}' \in R(w_0)$, such that \tilde{x} is $X(T)^3$ -homogeneous of weight $(\varpi_\alpha, (w\varpi_\alpha)^-, (w\varpi_\alpha)^+)$, then $\tilde{x} = \uparrow x_{\alpha,w}$.*

Proof. For statement one, let $\mathbf{j} = (j_1, \dots, j_m) \in R(w)$ (pay attention to the unusual enumeration). Let $k = \alpha^{\{\max\}}$ and $v = \prod_{n=1}^k s_{j_n}$. Since $w\varpi_\alpha = v\varpi_\alpha$, it's sufficient to prove the statement for v . If $k = 0$, hence $v = e$, then $\uparrow x_{\alpha,e} = \uparrow x_{\alpha_l}$ works. Note that $\uparrow x_{\alpha_l} = X_{\alpha_l}$ is by definition a cluster variable of $\uparrow t_{\mathbf{i}'}^D$ for any \mathbf{i}' . If $k > 0$, take $\mathbf{i}' \in R(w_0)$ such that $i'_n = j_n$ for $n \leq k$. By Proposition 8.2.9, setting $\uparrow x_{\alpha,v} = \uparrow x_{\mathbf{i}',k}$ works.

For statement two, by Corollary 8.2.2 we have that $\tilde{t} \simeq \uparrow t_{\mathbf{i}'}^D$. By Theorem 8.2.8 and Proposition 8.2.9, $\dim \text{Br}_{(\varpi_\alpha, (w\varpi_\alpha)^-, (w\varpi_\alpha)^+)} = 1$. Hence, there exist $c \in \mathbb{C}^*$ such that $\uparrow x_{\alpha,w} = c\tilde{x}$. The statement follows from Theorem 2.1.8. \square

Note that, for any $\alpha \in \Delta$, α^{\min} and $\alpha(-) = \min\{k : (z_{\leq k})\alpha \in \Phi^-\}$ are defined.

Lemma 8.2.11. *For any $\alpha \in \Delta$ we have*

$$p_{\alpha^{\min},0} = \prod_{j=1}^{\alpha^{\min}-1} p_{j,0}^{c_j} \quad \text{and} \quad q_{\alpha(-),0} = \prod_{j=1}^{\alpha(-)-1} q_{j,0}^{d_j}$$

for some non-negative integers c_j and d_j .

Proof. For the functions p , this is a reformulation of Lemma 5.5.1. For the functions q it is a reformulation of Lemma 5.5.6. \square

Lemma 8.2.12. *The functions $p_{k,0}$ for $k \neq \alpha^{\min}$ (resp. $q_{r,0}$ for $r \neq \alpha(-)$) are algebraically independent.*

Proof. This is a reformulation of Lemma 5.5.2 for the p and of Lemma 5.5.7 for the q . \square

We are ready to prove Theorem 8.2.1.

Proof of Theorem 8.2.1. By contradiction, suppose that $\overline{\mathcal{A}}(\uparrow t^D)$ is strictly contained in Br . By Proposition 4.1.3, there exists a $d \in D$ and $f \in \text{Br}$ such that $\mathcal{CV}_d(f) < 0$. We can suppose that f is homogeneous for the $T \times T$ -action. In particular, up to multiplying for a monomial in the X_s , $s \in D$, and in the unfrozen variables of $\uparrow x$, we can suppose that there exist $(\lambda, \mu) \in X(T) \times X(T)$ and $f \in \text{Br}_{(\lambda, \mu), \bullet}$ such that

$$f = \frac{P}{\uparrow x_d} \quad \text{where} \quad P = \sum_{n \in \mathbb{N}^{|I|}} a_n \uparrow x^n : \uparrow x_d \nmid P. \quad (66)$$

Hence, P is a polynomial in the variables of the cluster $\uparrow x$, which is not divisible by $\uparrow x_d$. Here, divisibility is intended in the polynomial ring. Up to changing f , we can assume that, if $n_d > 0$, then $a_n = 0$, so that the sum defining P runs over $\mathbb{N}^{|I| \setminus \{d\}}$. We use the convention that, if $\alpha \in \Delta$,

then $\nu_{\bullet, \alpha_l} = (\varpi_\alpha, 0)$ and $\nu_{\bullet, \alpha_r} = (0, \varpi_\alpha)$. The fact that f is homogeneous of degree (λ, μ) imposes that, for any n such that $a_n \neq 0$,

$$\sum_{j \in I} n_j \nu_{\bullet, j} = (\lambda, \mu) + \nu_d. \quad (67)$$

For $s \in D$, we set $p_{s,0} = 1 = q_{s,0}$. We distinguish two cases.

Suppose first that $d = \alpha_l$ for a certain $\alpha \in \Delta$. Let x as in Lemma 8.2.3. Using Proposition 8.2.7 and Lemma 8.2.5 we deduce that

$$f(\iota_\alpha(x)) = (\lambda, \mu)((h_l, h_r)) \left(t^{-1} \sum_{k \geq 0} f_k(y) \right)$$

for certain $f_k \in \mathbb{C}[Y]$. Using Lemma 8.2.5, the fact that $n_{\alpha_l} = 0$ if $a_n \neq 0$ and Proposition 8.2.7, we get that

$$f_0 = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha_l\}}} a_n p_{\bullet,0}^n \quad \text{where} \quad p_{\bullet,0}^n = \prod_{j \in I \setminus \{\alpha_l\}} p_{j,0}^{n_j}.$$

But since $f \in \text{Br}$, $\mathcal{V}_{\{t=0\}} \iota_\alpha^*(f) \geq 0$, in particular $f_0 = 0$.

Consider the linear map $\tilde{\pi} : \mathbb{Z}^{I \setminus \{\alpha_l\}} \longrightarrow \mathbb{Z}^{I \setminus \alpha^{\min}}$ defined in the following way. For $s \in D \setminus \{\alpha_l\}$, $\tilde{\pi}(e_s) = 0$, for $j \in I \setminus \{\alpha^{\min}\}$ $\tilde{\pi}(e_j) = e_j$ and $\tilde{\pi}(e_{\alpha^{\min}}) = \sum_{j < \alpha^{\min}} c_j e_j$, where the coefficients c_j are the ones of Lemma 8.2.11. Since $p_{s,0} = 1$ for $s \in D$ and because of Lemma 8.2.11 we have that, for any $n \in \mathbb{N}^{I \setminus \{\alpha_l\}}$,

$$p_{\bullet,0}^n = p_{\bullet,0}^{\tilde{\pi}(n)}.$$

In particular,

$$f_0 = \sum_{m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}} b_m p_{\bullet,0}^m \quad \text{where} \quad b_m = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha_l\}} : \tilde{\pi}(n) = m} a_n.$$

Since, by Lemma 8.2.12, the functions $p_{j,0}$ with $j \in [l] \setminus \{\alpha^{\min}\}$ are algebraically independent and $f_0 = 0$, then for any $m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}$, $b_m = 0$. We claim that for any $m \in \mathbb{N}^{I \setminus \{\alpha^{\min}\}}$ there exists at most one $n \in \mathbb{N}^{I \setminus \{\alpha_l\}}$ such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$. This implies at ones that, for any $n \in \mathbb{N}^{I \setminus \{\alpha_l\}}$, $a_n = 0$, which is a contradiction.

To prove the claim, just notice that any n such that $a_n \neq 0$ satisfies the weight condition (67). In particular, consider the linear map $\pi : \mathbb{Z}^{I \setminus \{\alpha_l\}} \longrightarrow X(T \times T) \times \mathbb{Z}^{I \setminus \{\alpha^{\min}\}}$ defined by $\pi(e_j) = (\nu_{\bullet, j}, \tilde{\pi}(e_j))$. Remember that, for $s \in D$, $\nu_{\bullet, s}$ has been previously defined. If n is such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$, then $\pi(n) = ((\lambda + \varpi_\alpha, \mu), m)$. But the map π is injective. To prove this, we can consider π as a matrix with columns indexed by $I \setminus \{\alpha_l\}$ and rows indexed by $I \setminus \{\alpha^{\min}\}$. Recall that $X(T \times T)$ is identified with \mathbb{Z}^D by means of the fundamental weights. Enumerate the roots of Δ from 1 to k , such that $\alpha = \beta_k$. Then, order the columns of the matrix representing π accordingly to the order

$$\beta_{1,l} < \cdots < \beta_{k-1,l} < \beta_{1,r} < \cdots < \beta_{k,r} < 1 < \cdots < l$$

and the rows according to the order

$$\beta_{1,l} < \cdots < \beta_{k-1,l} < \beta_{1,r} < \cdots < \beta_{k,r} < 1 < \cdots < \alpha^{\min} - 1 < \alpha_l < \alpha^{\min} + 1 < \cdots < l.$$

Then, the matrix representing π is upper triangular with ones on the diagonal (note that $\nu_{\alpha_l, \alpha^{\min}} = 1$ by Lemma 8.2.7). In particular, π is invertible.

Next suppose that $d = \alpha_r$, for a certain $\alpha \in \Delta$. We can compute $j_\alpha^*(f)$, similarly as before, and by the same argument we deduce that the function $f_0 \in \mathbb{C}[Y]$ defined by

$$f_0 = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha_r\}}} a_n q_{\bullet,0}^n \quad \text{where} \quad q_{\bullet,0}^n = \prod_{j \in I \setminus \{\alpha_r\}} q_{j,0}^{n_j}$$

is zero.

We introduce the linear map $\tilde{\pi} : \mathbb{Z}^{I \setminus \{\alpha_r\}} \longrightarrow \mathbb{Z}^{I \setminus \alpha(-)}$ defined in the following way. For $s \in D \setminus \{\alpha_r\}$, $\tilde{\pi}(e_s) = 0$, for $j \in I \setminus \{\alpha(-)\}$, $\tilde{\pi}(e_j) = e_j$ and $\tilde{\pi}(e_{\alpha(-)}) = \sum_{j < \alpha^{\min}} d_j e_j$, where the coefficients d_j are the ones of Lemma 8.2.11. Since $q_{s,0} = 1$ for $s \in D$ and because of Lemma 8.2.11, we have that for any $n \in \mathbb{N}^{I \setminus \{\alpha_r\}}$,

$$q_{\bullet,0}^n = q_{\bullet,0}^{\tilde{\pi}(n)}.$$

In particular,

$$f_0 = \sum_{m \in \mathbb{N}^{I \setminus \{\alpha(-)\}}} b_m q_{\bullet,0}^m \quad \text{where} \quad b_m = \sum_{n \in \mathbb{N}^{I \setminus \{\alpha_r\}} : \tilde{\pi}(n) = m} a_n.$$

Since, by Lemma 8.2.12, the functions $q_{j,0}$ with $j \in [l] \setminus \{\alpha(-)\}$ are algebraically independent and $f_0 = 0$, then for any $m \in \mathbb{N}^{I \setminus \{\alpha(-)\}}$, $b_m = 0$. We claim that, for any $m \in \mathbb{N}^{I \setminus \{\alpha(-)\}}$, there exists at most one $n \in \mathbb{N}^{I \setminus \{\alpha_r\}}$ such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$. This implies at ones that for any $n \in \mathbb{N}^{I \setminus \{\alpha_r\}}$, $a_n = 0$, which is a contradiction.

As before, consider the linear map $\pi : \mathbb{Z}^{I \setminus \{\alpha_r\}} \longrightarrow X(T \times T) \times \mathbb{Z}^{I \setminus \{\alpha(-)\}}$ defined by $\pi(e_j) = (\nu_{\bullet,j}, \tilde{\pi}(e_j))$. If n is such that $a_n \neq 0$ and $\tilde{\pi}(n) = m$, then $\pi(n) = ((\lambda, \mu + \varpi_\alpha), m)$. But the map π is injective. To prove this, consider π as a matrix with columns indexed by $I \setminus \{\alpha_r\}$ and rows indexed by $I \setminus \{\alpha(-)\}$. Enumerate the roots of Δ from 1 to k such that $\alpha = \beta_k$. Then, order the columns of the matrix representing π accordingly to the order

$$\beta_{1,l} < \cdots < \beta_{k,l} < \beta_{1,r} < \cdots < \beta_{k-1,r} < 1 < \cdots < l$$

and the rows according to the order

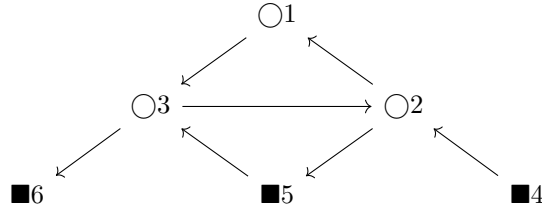
$$\beta_{1,l} < \cdots < \beta_{k,l} < \beta_{1,r} < \cdots < \beta_{k-1,r} < 1 < \cdots < \alpha(-) - 1 < \alpha_r < \alpha(-) + 1 < \cdots < l.$$

Then, the matrix representing π is upper triangular with ones on the diagonal. Here we used that $\nu_{\alpha_r, \alpha(-)} = 1$ by Lemma 8.2.7 and the third statement in Lemma 5.5.5. In particular, π is invertible. \square

Example 8.2.13. Let $G = \mathrm{SL}_4$, then $\Delta = \{1, 2, 3\}$ with standard notation. Consider the reduced expression of w_0 : $\mathbf{i} = (1, 2, 3, 1, 2, 1)$ and let $t = t_{\mathbf{i}}$. Then

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

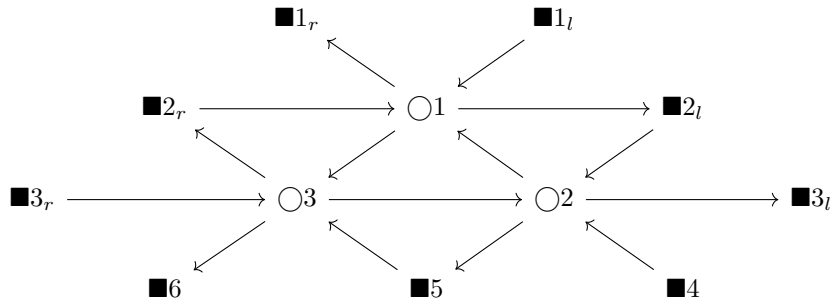
where the rows and columns of B are ordered in the obvious way. Graphically, the valued quiver of t is the following:



Here $D = \{1_l, 2_l, 3_l, 1_r, 2_r, 3_r\}$. Reading the previous list from left to right allows to identify the minimal lifting matrix ν with a six by six matrix. Using Lemma 8.2.7, we have that

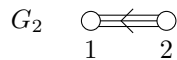
$$\nu = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A direct computation of $-\nu B$ allows to deduce that, the seed $t_{\mathbf{i}}^D$, corresponds to the following valued quiver



which is the *ice hive quiver* defined in [Fei17].

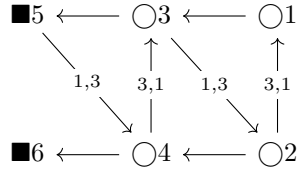
Example 8.2.14. Let $G = G_2$. Label the two simple roots as prescribed by the following picture.



Let $\mathbf{i} = (1, 2, 1, 2, 1, 2) \in R(w_0)$ and $t = t_{\mathbf{i}}$. Then

$$B = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 3 & 0 & -3 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

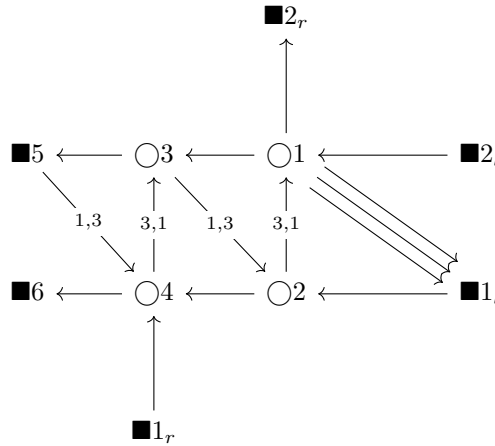
where the rows and columns of B are ordered in the obvious way. Graphically, the valued quiver of t is the following:



Here $D = \{1_l, 2_l, 1_r, 2_r\}$. Reading the previous list from left to right allows to identify the minimal monomial lifting matrix ν with a four by six matrix. A direct computation using Lemma 8.2.7, implies that

$$\nu = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

A direct computation of $-\nu B$ allows to deduce that, the seed $t_{\mathbf{i}}^D$, corresponds to the following valued quiver



which, in the notation of [Fei21], up to some arrows between frozen vertices, is the iART-quiver Δ_Q^2 corresponding to G_2 .

We end this section comparing our construction to [Fei21], which has been of great inspiration for this work. Indeed, in [Fei21], the author constructs cluster structures on $\mathfrak{X}(G \times G, G)$ assuming

G to be simple and simply laced. The proof highly relies on the theory of quiver with potentials [DWZ08] [DWZ10]. We believe our construction to be simpler, especially because it only relies on the geometric properties of the branching scheme and on the minimal monomial lifting, which is a versatile technique. As an evidence of this fact, in the present text we drop the hypothesis that G is simple and simply laced. Moreover, in the following we prove that Fei's cluster structures are obtained through minimal monomial lifting. It should also be true the the cluster structures constructed here agree with Fei's ones, we return on this aspect later. Finally, in [Fei21], the proof that the branching scheme has cluster structure is simultaneously achieved with the existence of a good basis for the corresponding upper cluster algebra. Nevertheless, the two statements can't be proved independently. Even if the construction of a good basis for the upper cluster algebra is a remarkable achievement, we believe that the present approach, in which the construction of cluster structures on branching schemes are independent on the existence of good bases is more in the spirit of studying branching problems through cluster theory. In a forthcoming work, we will see how to construct good bases for some of the upper cluster algebras constructed in the present text, only relying on cluster theory.

Suppose from now on that G is simple. In [Fei21], the author constructs a seed

$$\tilde{t} = (\tilde{I}_{uf}, \tilde{I}_{sf} = \emptyset, \tilde{I}_{hf}, B^2, S^2)$$

of $\mathbb{C}(\mathfrak{X})$, which is $X(T)^3$ -graded by a degree configuration σ^2 , such that [Fei21][Theorem 8.1]

Theorem 8.2.15. *If G is simply laced*

1. $\overline{\mathcal{A}}(\tilde{t}) = \mathbb{C}[\mathfrak{X}]$.
2. *The graduation induced by σ^2 , on $\overline{\mathcal{A}}(\tilde{t})$, coincides with the natural graduation on $\mathbb{C}[\mathfrak{X}]$.*
3. *The generic cluster character C_{W^2} , associated to a certain potential W^2 on the seed \tilde{t} , gives a good (in the sense of Qin [Qin22]) basis for $\overline{\mathcal{A}}(\tilde{t})$.*

Suppose from now on that G is simply laced and fix an orientation of the Dynkin diagram of G . This choice allows to construct a maximal rank seed t , of $\mathbb{C}(U)$, such that $\overline{\mathcal{A}}(t) = \mathbb{C}[U]$ (see [Fei21][Section 6.3]). Then, let $\tilde{\iota} : T \times T \times U \rightarrow G \times G$ be the map defined by $(h_l, h_r, u) \mapsto (h_l, h_r u)$ and $\iota : T \times T \times U \rightarrow \mathfrak{X}$ be the map obtained by composing $\tilde{\iota}$ with the natural projection $G \times G \rightarrow \mathfrak{X}$. As for Corollary 7.4.4, one can easily prove that the triple (\mathfrak{X}, ι, X) is homogeneously suitable for D -lifting. We write \mathfrak{X}_ι , for the scheme \mathfrak{X} , endowed with the suitable for D -lifting scheme structure described above, and $\uparrow t_\iota$ for the minimal monomial lifting of t , with respect to \mathfrak{X}_ι .

Proposition 8.2.16. *We have that $\tilde{t} = \uparrow t_\iota^D$.*

Proof. We want to apply Theorem 4.0.10. First, $S_D^2 = X$ because of [Fei21][Corollary 8.8, Corollary 7.9]. By [Fei21][Proposition 3.6], we have that $\tilde{I}_{uf} = I_{uf}$, $\tilde{I}_{hf} = I_{hf} \sqcup D$ and $B_{I \times I_{uf}}^2 = B$. Hence, \tilde{t} is a D -pointed seed extension of t . Moreover, condition 2 and 3 of Theorem 4.0.10 hold because of [Fei21][Lemma 7.8, Corollary 7.11] and the discussion at the beginning of [Fei21][Section 6.3]. Since $\overline{\mathcal{A}}(\tilde{t}) = \mathbb{C}[\mathfrak{X}_\iota]$ by Theorem 8.2.15, we have that $\tilde{t} = \uparrow t_\iota^D$ by Theorem 4.0.10. \square

It should be true that the seed t equals $t_{\mathbf{i}}$, for a certain $\mathbf{i} \in R(w_0)$ adapted, in the sense of [GLS07][Section 1.3], to the chosen orientation of the Dynkin diagram, but this is not clear to the

author, in general. Nevertheless, it is easy to verify this statement on small examples.

Finally, if G is not simply laced, the inclusion $\overline{\mathcal{A}}(\tilde{t}) \subseteq \mathbb{C}[\mathfrak{X}]$ still holds and the first part of [Fei21][Conjecture 8.2] states that this inclusion is actually an equality. Proceeding as in the proof of Proposition 8.2.16, if one could identify $\mathbf{i} \in R(w_0)$ such that $t = t_{\mathbf{i}}$ and prove that $B^2 = 1B_{\mathbf{i}}$, then Theorem 8.2.1 would imply the first part of [Fei21][Conjecture 8.2]. For example, in the G_2 case, since everything is explicit by Example 8.2.14 and [Fei21][Appendix A], we have that the conjecture holds for G_2 .

Remark 8.2.17. As a final remark, note that the cluster algebra $\overline{\mathcal{A}}(t)$ is categorified and the work of Fei produces a categorification for $1t_{\iota}$ which is compatible with the one on $\overline{\mathcal{A}}(t)$. Moreover, [Fei21][Corollary 7.8] gives an homological interpretation of the minimal lifting matrix ν_{ι} of the seed t , with respect to \mathfrak{X}_{ι} , in terms of this categorification. A more explicit understanding of the relation between the minimal lifting matrix in term of this categorification seems to the author the fundamental tool needed to prove [Fei21][Conjecture 8.2] via Theorem 8.2.1. Actually, comparing the minimal monomial lifting to the extension construction of [Fei23][Section 2.3], could provide a possible lead to accomplish this task.

8.3 The base affine space, or branching to a maximal Torus

We use the previous result to construct a cluster structure on $\mathfrak{X} := \text{Spec}(\mathbb{C}[G]^{U^-})$. Note that $\mathfrak{X} = \mathfrak{X}(G, T)$ and T is a Levi subgroup of G , so the results of this section, for G simple, can be deduced from the ones of Section 8.1.1. One can also observe that, the results of Section 8.1.1 hold in the case of $G_T = T$, even if G is not necessarily simple, because the same proofs work. See Remark 8.1.10. In this section, we reinterpret this cluster structure on \mathfrak{X} , as a quotient of the one on $\mathfrak{X}(G \times G, G)$. We slightly modify the notation of Subsection 7.4.1.

Let $D_{\iota} = \{\alpha_{\iota} : \alpha \in \Delta\}$ and $\tilde{X}_{\alpha_{\iota}} = \Delta_{e, e}^{\varpi, \alpha}$. We have a cartesian square

$$\begin{array}{ccc} G_0 & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow \\ T \times U & \xrightarrow{\tilde{\phi}} & \mathfrak{X} \end{array}$$

where $\tilde{\phi} : T \times U \rightarrow \mathfrak{X}$ is the T -equivariant open embedding induced by the product $T \times U \rightarrow G$ and the leftmost vertical map is defined by $x \mapsto ([x]_0, [x]_+)$. Then $(\mathfrak{X}, \tilde{\phi}, \tilde{X})$ is homogeneously suitable for D_{ι} -lifting by Corollary 7.4.4.

Note also that we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} \times T & \xrightarrow{\iota} & \mathfrak{X}(G \times G, G) \\ (\tilde{\phi}, id) \uparrow & & \uparrow \phi \\ T \times U \times T & \longrightarrow & T \times T \times U \end{array} \quad (68)$$

where the map $T \times U \times T \rightarrow T \times T \times U$ is defined by $(h_{\iota}, u, h_r) \mapsto (h_{\iota}, h_r, u)$ and ι is the open embedding induced by the inclusion $G \times T \subseteq G \times G$.

Let $\mathbf{i} \in R(w_0)$. We denote by $t_{\mathbf{i}}$ (resp $\tilde{t}_{\mathbf{i}}$) the associated seed of $\mathbb{C}(U)$ when $\mathbb{C}(U)$ is considered as a subfield of $\mathbb{C}(\mathfrak{X}(G \times G, G))$ (resp. $\mathbb{C}(\mathfrak{X})$). We call $I_{\mathbf{i}}$ the vertex set of both $\tilde{t}_{\mathbf{i}}$ and $t_{\mathbf{i}}$. Let $\tilde{\nu}_{\mathbf{i}} \in \mathbb{Z}^{D_{\iota} \times I}$ and $\nu_{\mathbf{i}} \in \mathbb{Z}^{(D_{\iota} \sqcup D_r) \times I}$ the minimal lifting matrix corresponding to $\tilde{t}_{\mathbf{i}}$ and $t_{\mathbf{i}}$ respectively.

Lemma 8.3.1. *We have that $(\nu_{\mathbf{i}})_{D_{\mathbf{i}}, \bullet} = \tilde{\nu}_{\mathbf{i}}$*

Proof. Note that, for any $\alpha \in \Delta$, $\iota^*(X_{\alpha_l}) = \tilde{X}_{\alpha_l} \otimes 1$. The statement follows at ones from the fact that ι is an open embedding and the diagram (68) is commutative. \square

Theorem 8.3.2. *We have equality $\overline{\mathcal{A}}(\tilde{t}_{\mathbf{i}}^{D_{\mathbf{i}}}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$.*

Proof. It's easy to verify that $\text{Br}(G \times G, G)/(X_{\alpha_r} - 1 : \alpha \in \Delta) \simeq \mathbb{C}[G]^{U^-}$. Geometrically, the isomorphism corresponds to the closed embedding $\mathfrak{X} \rightarrow \mathfrak{X}(G \times G, G)$ defined by $p \mapsto \iota(p, e)$. By Theorem 8.2.1, $\mathbb{C}[G]^{U^-}$ is the quotient of $\overline{\mathcal{A}}(\uparrow t_{\mathbf{i}}^{D_{\mathbf{i}} \sqcup D_r})$ by the ideal $(\uparrow x_{\mathbf{i}, \alpha_r} - 1 : \alpha_r \in D_r)$. We conclude using Corollary 3.1.11 and Lemma 8.3.1. \square

Corollary 8.3.3. *If G is simply laced, the mutation equivalence class of $\tilde{t}_{\mathbf{i}}$ doesn't depend on \mathbf{i} .*

Proof. It's a direct consequence of Corollary 8.2.2. \square

For $\alpha \in \Delta$ and $w \in W$, we denote by $D_{e,w}^{\varpi_{\alpha}}$ the restriction of $\Delta_{e,w}^{\varpi_{\alpha}}$ to U .

Lemma 8.3.4. *For $k \in [l]$, $\uparrow \tilde{x}_{\mathbf{i},k} = \Delta_{e,(w_0)_{\leq k}}^{\varpi_{i_k}}$.*

Here $(w_0)_{\leq k}$ is computed with respect to \mathbf{i} .

Proof. By Lemmas 8.3.1 and 8.2.7 we have that $\uparrow \tilde{x}_{\mathbf{i},k} = \tilde{X}_{(i_k)_l}(1 \otimes \tilde{x}_{\mathbf{i},k})$. Hence

$$\uparrow \tilde{x}_{\mathbf{i},k}((h, u)) = \varpi_{i_k}(h) D_{e,(w_0)_{\leq k}}^{\varpi_{i_k}}(u) = \Delta_{e,(w_0)_{\leq k}}^{\varpi_{i_k}}((t, u)).$$

Since $T \times U$ is open in \mathfrak{X} , which is irreducible, this proves the lemma. \square

Lemma 8.3.5. *Let $w \in W$ and $\alpha \in \Delta$.*

1. *There exist $\mathbf{i}' \in R(w_0)$ such that $\Delta_{e,w}^{\varpi_{\alpha}}$ is a cluster variable of $\uparrow \tilde{t}_{\mathbf{i}'}^{D_{\mathbf{i}'}}$.*
2. *If G is simply laced, then $\Delta_{e,w}^{\varpi_{\alpha}}$ is a cluster variable of $\overline{\mathcal{A}}(\uparrow \tilde{t}_{\mathbf{i}}^{D_{\mathbf{i}}})$, for any \mathbf{i} .*

Proof. For the first statement, using Lemma 8.3.4, the proof is literally the same of Proposition 8.2.10. The second statement is an obvious consequence of the first one and of Corollary 8.3.3. \square

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