Collatz High Cycles Do Not Exist

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September 23, 2023

Abstract

The Collatz function takes odd $n$ to $(3n + 1)/2$ and even $n$ to $n/2$. Under the iterated Collatz function, every positive integer is conjectured to end up in the trivial cycle 1-2-1. Two types of cycles are of special interest. Consider the set $S$ consisting of the smallest members of all cycles containing the same number of odd terms. The circuit contains the smallest member of $S$, while the high cycle contains the largest. It is known that no circuits of positive integers exist (except 1-2-1); this paper shows that there are likewise no high cycles of positive integers.

1 Introduction

The Collatz function is

$$T(n) = \begin{cases} 
\frac{3n+1}{2}, & \text{if } n \text{ is odd;} \\
\frac{n}{2}, & \text{otherwise.}
\end{cases}$$

Iterating this function famously yields interesting sequences. For example:

$31 \rightarrow 47 \rightarrow 71 \rightarrow 107 \rightarrow 161 \rightarrow 242 \rightarrow 121 \rightarrow 182 \rightarrow 91 \rightarrow 137 \rightarrow 206$

$\rightarrow 103 \rightarrow 155 \rightarrow 233 \rightarrow 350 \rightarrow 175 \rightarrow 263 \rightarrow 395 \rightarrow 593 \rightarrow 890 \rightarrow 445$

$\rightarrow 668 \rightarrow 334 \rightarrow 167 \rightarrow 251 \rightarrow 377 \rightarrow 566 \rightarrow 283 \rightarrow 425 \rightarrow 638 \rightarrow 319$

$\rightarrow 479 \rightarrow 719 \rightarrow 1079 \rightarrow 1619 \rightarrow 2429 \rightarrow 3644 \rightarrow 1822 \rightarrow 911 \rightarrow 1367$

$\rightarrow 2051 \rightarrow 3077 \rightarrow 4616 \rightarrow 2308 \rightarrow 1154 \rightarrow 577 \rightarrow 866 \rightarrow 433 \rightarrow 650$

$\rightarrow 325 \rightarrow 488 \rightarrow 244 \rightarrow 122 \rightarrow 61 \rightarrow 92 \rightarrow 46 \rightarrow 23 \rightarrow 35 \rightarrow 53 \rightarrow 80$

$\rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \ldots$

The Collatz conjecture posits that every positive integer eventually reaches 1. The conjecture was verified in 2021 for all $1 \leq n \leq 10^{21}$ [2], but it has not yet been proven or refuted.
Figure 1: All of the rational Collatz cycles of length $k = 8$ with $x = 5$ odd terms. Of special note are the outermost cycle (the circuit), in light gray, and the innermost cycle (the high cycle), in dark gray.

If $T^i(n) = n$ for some positive integer $n$, there is a Collatz cycle of length $i$ whose first term is $n$. The only known $T$-cycle is the trivial one: $1 \rightarrow 2 \rightarrow 1$.

The unresolved Weak Collatz conjecture claims that there are no positive integer cycles. (It is called “weak” because it allows for the possibility of a start number that diverges to infinity, never reaching 1.)

We can also consider cycles with rational values. Figure 1 shows all cycles of length $k = 8$ with $x = 5$ odd terms. Considering the term $\frac{211}{13}$ to be odd (by its numerator), the next term produced by the Collatz process is $(3(\frac{211}{13}) + 1)/2 = \frac{323}{13}$ (also odd).

For any $k$ and $x$, the outermost and innermost cycles are of special interest. Let $S$ contain the smallest members of each cycle (here, $\frac{211}{13}, \frac{227}{13}, \ldots, \frac{319}{13}$). The outermost cycle (called the circuit) contains the lowest member in $S$, while the innermost cycle (called the high cycle) contains the highest member in $S$.

Steiner [10] showed that no positive integer circuits exist for any $k, x$ (except $k = 2, x =$...
1). The current paper shows likewise that no high cycles of positive integers exist.

2 Parity vectors

Associated with every cycle member \( m \) is a unique parity vector that records the even (0) and odd (1) terms obtained from \( k \) iterations of the Collatz function.

**Example 2.1** (Cycle member to parity vector). Starting with \( m = \frac{211}{13} \), we encounter 5 odd terms followed by 3 even terms, so the corresponding parity vector is 11111000.

Likewise, given a parity vector \( v \), we can compute a cycle member \( m \).

**Example 2.2** (Parity vector to cycle member). Applying \( v = 110 \) to arbitrary start number \( n \) yields \( \frac{3^{3n+1} + 1}{2} = \frac{9}{8} n + \frac{5}{8} \). To make a cycle, we set \( n = \frac{9}{8} n + \frac{5}{8} \). Solving this equation yields \( n = -5 \), which is part of the \(-5 \rightarrow -7 \rightarrow -10 \rightarrow -5 \) cycle.

Generalizing this idea, Böhm and Sontacchi [4] give the cycle member associated with arbitrary parity vector \( v \) as

\[
f(v) = \frac{\sum_{i=0}^{x-1} 2^{d_i(v)} \cdot 3^{x-i-1}}{2^k - 3^x},
\]

(2.1)

where \( d_0(v) < \ldots < d_{x-1}(v) \) are the (zero-based) indices of 1s in \( v \). Since we are only interested in Collatz cycles with positive terms, the denominator must be positive, so \( k > x \text{ log}_2 3 \).

The Weak Collatz Conjecture can be restated: “Is there a parity vector \( v \) such that \( f(v) \) is an integer greater than 2?”

It will sometimes be useful to refer to the numerator only:

\[
g(v) = \sum_{i=0}^{x-1} 2^{d_i(v)} \cdot 3^{x-i-1}.
\]

(2.2)

Note that rotating a parity vector produces another term in the same rational cycle.

**Example 2.3** (Rotation). \( f(100) = \frac{1}{5}, f(001) = \frac{4}{5}, \) and \( f(010) = \frac{2}{5} \). These are all entry points into the \( \frac{1}{5} \rightarrow \frac{4}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5} \) cycle.

Finally, we can restrict ourselves to aperiodic parity vectors. Consider \( f(1101011010) = \frac{23}{6}, \) which is part of the double cycle \( \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5}. \) Had these been integers, the aperiodic \( f(11010) \) would also be an integer.
3 No integer circuits

The parity vector $v_c = 1^x0^{k-x}$ is associated with the smallest member of the $(k,x)$-circuit. Following Equation 2.1,

$$f(v_c) = \frac{\sum_{i=0}^{x-1} 2^i 3^{x-i-1}}{2^k - 3^x}.$$  

We provide a simple proof that (non-trivial) Collatz circuits do not exist, after three lemmas.

**Lemma 3.1.** $f(v_c)$ simplifies to $\frac{3^x - 2^x}{2^k - 3^x}$.

*Proof. By induction, omitted.*

**Lemma 3.2.** If there is a non-trivial positive integer circuit of length $k$ with $x$ odd terms, then $2 \cdot \left(\frac{3}{2}\right)^x \geq 2^{k-x} - 1$.

*Proof. This follows from the fact that the putative circuit’s smallest member is at least 1.

$$\frac{3^x - 2^x}{2^k - 3^x} \geq 1$$

$$3^x - 2^x \geq 2^k - 3^x$$

$$2 \cdot 3^x \geq 2^k + 2^x$$

$$2 \cdot \left(\frac{3}{2}\right)^x \geq 2^{k-x} - 1.$$  

**Lemma 3.3** (Ellison [7]). For integers $k$ and $x$, with $x > 17$, we have $2^k - 3^x > 2.56^x$.

*Proof. We start with Ellison’s original bound (for $k > 27$) and use the fact that $k > x \log_2 3$ in positive cycles.

$$2^k - 3^x > \frac{2^k}{e^{k/10}} > 1.8^k > 1.8^x \log_2 3 > 2.56^x,$$ for $x > 17$.  

**Theorem 3.4** (Steiner, 1977). Non-trivial Collatz circuits do not exist.

*Proof. Assume $f(v_c) = \frac{3^x - 2^x}{2^k - 3^x}$ is a positive integer. Then the following are also positive integers:
\[ \frac{3^x - 2^x}{2^k - 3^x} + 1, \]
\[ \frac{3^x - 2^x}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x}, \]
\[ \frac{2^k - 2^x}{2^k - 3^x}, \]
\[ \frac{2^x(2^{k-x} - 1)}{2^k - 3^x}, \text{ and } \frac{2^{k-x} - 1}{2^k - 3^x}. \]

We can remove the \(2^x\) in the last step without affecting the expression’s purported integrality, since the denominator is odd.

However, the last term is actually less than one; we invoke Lemmas 3.3 and 3.2 to show that the denominator outstrips the numerator. For \(x > 17\),

\[ 2^k - 3^x > 2.56^x > 2 \cdot (1.5)^x \geq 2^{k-x} - 1. \]

\(2^{k-x} - 1 = 0\) only when \(k = 2, x = 1\); setting aside this case, we obtain the contradiction \(0 < \frac{2^{k-x} - 1}{2^k - 3^x} < 1\). Cases of \(x \leq 17\) are handled by the fact that any non-trivial Collatz cycle must contain many millions of odd terms \([6]\).

To date, all no-circuit proofs depend on deep lower bounds for \(2^k - 3^x\), such as Ellison’s, which are all derived from Alan Baker’s pioneering transcendental number theory work \([1]\). As Jeffrey Lagarias \([9]\) remarks, ”The most remarkable thing about Theorem […] is the weakness of its conclusion compared to the strength of the methods used in its proof.”

4 The high cycle

In this section, characterize the Collatz high cycle.

Definition 4.1. Let \(v_h\) be a \((k, x)\)-parity vector with 1s indexed by \(d_i(v) = \left\lfloor \frac{k}{2} i \right\rfloor\), for \(0 \leq i \leq x - 1\).

Example 4.2. For \(k = 8, x = 5\), \(v_h = 11011010\).

\(^1\)Note that certain circuits can be ruled out without these deep bounds. For example, if \(k\) and \(x\) are not co-prime, then we can factor \(2^k - 3^x\). E.g., \((2^{k/2} + 3^{x/2})(2^{k/2} - 3^{x/2}) > 2^{k/2} > 2 \cdot \log 3/2 > 1.72^x > 2^{k-x} - 1\). Or for the case of \(v_c = 111111110000\), it follows that if \(f(v_c) = \frac{3^8 - 2^8}{2^{13} - 3^8}\) is an integer, then so is \(31f(v_c) + 206\), but it happens that \(31 \cdot \frac{3^8 - 2^8}{2^{13} - 3^8} + 206 \cdot \frac{2^{13} - 3^8}{2^{13} - 3^8} = \frac{3^{12}}{2^{13} - 3^8}\), which is not integral.
Example 4.3. For $k = 21, x = 13$, $v_h = 110110101101101011010$.

The vector $v_h$ is the well-known (upper) Christoffel word [3], constructed so that its 1s are roughly evenly spread among its 0s. The cycle member corresponding to $v_h$ is

$$f(v_h) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} \cdot 2^\lfloor \frac{k}{x} i \rfloor}{2^k - 3^x}.$$

Now we list a number of properties of $v_h$.

First, $f(v_h)$ is the smallest member of its cycle.

Theorem 4.4 (Halbeisen and Hungerbühler [8]). For every $(k, x)$ parity vector $v$ that is a rotation of $v_h$, we have $f(v_h) \leq f(v)$.

Remark 4.5. It is known that a Christoffel word $v_h$ is at a lexicographic extreme among all its rotations [3]. However, lexicographic ordering does not always coincide with $f$-ordering. For example, $f(11110010) = \frac{259}{13}$, $f(11100101) = \frac{395}{13}$, and $f(10111100) = \frac{341}{13}$. Of course, lexicographic ordering correlates roughly with $f$-ordering, as left-heavy vectors tend to be associated with smaller cycle members.

Theorem 4.6. Among all the $(k, x)$ upper Christoffel words, $f(v_h)$ is maximized when $k = \lceil x \log_2 3 \rceil$.

Proof. Let $v_h$ be an upper Christoffel word with $k = \lceil x \log_2 3 \rceil$. Let $u$ be a longer $(k + 1, x)$-high-cycle parity vector. Replacing $k$ by $k + 1$ increases the numerator of $f(v)$ by no more than a factor of 2, while increasing the denominator by more than a factor of 2.

$$f(u) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} \cdot 2^\lfloor \frac{k+1}{x} i \rfloor}{2^{k+1} - 3^x} = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} \cdot 2^\lfloor \frac{k}{x} i \rfloor + 1}{2^{k+1} - 3^x} = \frac{2 \cdot \sum_{i=0}^{x-1} 3^{x-i-1} \cdot 2^\lfloor \frac{k}{x} i \rfloor}{2(2^k - 3^x)} = f(v).$$

Similar reasoning holds if $u$ is a $(k + n, x)$ upper Christoffel word, for any $n \geq 1$.

Next, we confirm the reader’s suspicion that $f(v_h)$ is indeed a member of the high cycle. Let $S$ contain the smallest members of each $(k, x)$-cycle.

Theorem 4.7 (Halbeisen and Hungerbühler [8]). Let $v$ be a parity vector associated with the smallest member of any $(k, x)$-cycle, and let $v_h$ be an upper Christoffel word with $k = \lceil x \log_2 3 \rceil$. Then, $f(v_h) \geq f(v)$. 

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Remark 4.8 (Tangential). This theorem is useful for proving that any Collatz cycle must be long. Just for example, it is easy to verify empirically that \( f(v_h) < 10^{13} \) for all \( x < 10,000,000 \), implying that every cycle (not just the high cycle) with fewer than ten million odd terms has some member less than \( 10^{13} \). Since no Collatz counter-examples exist among the first \( 10^{20} \) positive integers \([2]\), a purported Collatz cycle must have more than ten million odd terms. Stronger results have been obtained through analytical means \([8]\). Of course, \( f(v_h) \) will outstrip any finite confirmation, after some \( x \), because \( O(3^x x) \) exceeds \( O(2^k - 3^x) \).

Recall the cycle member corresponding to \( v_h \).

\[
f(v_h) = \sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{x}{i} \rfloor} \frac{x-i-1}{2^k - 3^x},
\]

Due to the floor function, we cannot simplify this expression as we did for \( f(v_c) \). Its value can only be partially wrangled through upper and lower bounds.

Theorem 4.9 (Halbeisen and Hungerbühler \([8]\)).

\[
\frac{3^x(x/20)}{2^{\lfloor x \log_2 3 \rfloor} - 3^x} < f(v_h) < \frac{3^x(7x/10)}{2^{\lfloor x \log_2 3 \rfloor} - 3^x}.
\]

These bounds can be improved from \((x/20, 7x/10)\) to \((x/6, x/2)\); we omit the proof for brevity.

5 No integer high cycles

In this section, we prove our main result that \( f(v_h) \) is never an integer. To do this, we need two more facts about Christoffel words. First, \( f(v_h^R) \) is a member of the same cycle as \( f(v_h) \).

Theorem 5.1 (Cohn \([5]\)). The reverse \( v_h^R \) of an upper Christoffel word \( v_h \) is also a rotation of it.

Therefore, \( f(v_h) \) and \( f(v_h^R) \) are in the same cycle. Indeed, \( f(v_h^R) \) is the largest member of the cycle, though we do not need this fact; also for interest, the left-rotation distance is the multiplicative inverse of \( x \) modulo \( k \).

Theorem 5.2 (Berstel et al \([3]\)). For aperiodic \( v_h \), we have \( v_h = 1u0 \), and \( v_h^R = 0u1 \).

Example 5.3. \( v_h = 11011010 \), \( v_h^R = 01011011 \), and \( u = 101101 \). Incidentally, \( u \) is always a palindrome, though we do not need this fact.

Because \( v_h \) and \( v_h^R \) are virtually identical, differing only in their first and last components, we expect \( f(v_h) \) and \( f(v_h^R) \) to have similar summands. We express both in terms of \( u \), via Equations 2.1 and 2.2.
\[ f(v_h) = \frac{2g(u) + 3^{x-1}}{2^k - 3^x}, \]
\[ f(v_R^h) = \frac{6g(u) + 2^{k-1}}{2^k - 3^x}. \]

In the first case, we create \( v_h \) by shifting \( u \) one place to the right (doubling every summand) and adding a 1 to the left edge. In the second case, shifting \( u \) requires both a doubling (for the same reason) and a tripling (from adding a 1 to the right edge).

**Theorem 5.4** (Main result). No high cycle consists of integers.

Proof. We know \( f(v_h) \) and \( f(v_R^h) \) are both members of the high cycle (Theorems 4.7 and 5.1). If they are both integers, then so are

\[ 3f(v_h) - f(v_R^h) + 1, \]
\[ \frac{3}{2^k - 3^x} \left( 2g(u) + 3^{x-1} \right) - \frac{6g(u) + 2^{k-1}}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x}, \]
\[ \frac{2^k - 2^{k-1}}{2^k - 3^x}, \quad \text{and} \quad \frac{2^{k-1}}{2^k - 3^x}. \]

However, \( 2^{k-1} \) is not divisible by any odd number, so we have a contradiction. \( \square \)

**Example 5.5.** For the \((8,5)\)-high-cycle, Figure 1 gives \( f(v_h) = \frac{319}{13} \) and \( f(v_R^h) = \frac{842}{13}. \) Combining gives \( \frac{3 \cdot 319}{13} - \frac{3 \cdot 842}{13} + \frac{13}{13} = \frac{27}{13}. \)

Note that if we left-rotate \( v_h \) and \( v_R^h \) by one position each, we obtain an alternate pair of high-loop members (\( u_01 \) and \( u_{10} \)), whose simple difference provides the necessary contradiction. For example, cycle members \( \frac{485}{13} \) and \( \frac{421}{13} \) in Figure 1 cannot both be integers, because \( \frac{485}{13} - \frac{421}{13} = \frac{2^6}{13}. \)

6 Summary

Notable features of our proof versus the no-circuit proof include:

- Instead of assuming, by way of contradiction, that \( f(v_c) \) is an integer, we instead assume that both \( f(v_h) \) and \( f(v_R^h) \) are both integers.

- Unlike the no-circuit proof, we do not require deep lower bounds \([1, 7]\) on the size of \( 2^k - 3^x \).

- We require no closed-form expression for any high cycle member; by contrast, the no-circuit proof relies on the expression \( f(v_c) = \frac{3^x - 2^x}{2^k - 3^x} \).
References


[4] Corrado Böhm and Giovanna Sontacchi, *On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ when $x$ is even and $x_{n+1} = 3x_n + 1$ otherwise*, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali 64 (1978), 260–264.


