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# Collatz High Cycles Do Not Exist

Kevin Knight

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## Abstract

The Collatz function takes odd  $n$  to  $(3n + 1)/2$  and even  $n$  to  $n/2$ . Under the iterated Collatz function, every positive integer is conjectured to end up in the trivial cycle 1-2-1. Two types of cycles are of special interest. Consider the set  $S$  consisting of the smallest members of all cycles containing the same number of odd terms. The *circuit* contains the smallest member of  $S$ , while the *high cycle* contains the largest. It is known that no circuits of positive integers exist (except 1-2-1); this paper shows that there are likewise no high cycles of positive integers.

## 1 Introduction

The Collatz function is

$$T(n) = \begin{cases} \frac{3n + 1}{2}, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{otherwise.} \end{cases}$$

Iterating this function famously yields interesting sequences. For example:

31 → 47 → 71 → 107 → 161 → 242 → 121 → 182 → 91 → 137 → 206  
→ 103 → 155 → 233 → 350 → 175 → 263 → 395 → 593 → 890 → 445  
→ 668 → 334 → 167 → 251 → 377 → 566 → 283 → 425 → 638 → 319  
→ 479 → 719 → 1079 → 1619 → 2429 → 3644 → 1822 → 911 → 1367  
→ 2051 → 3077 → 4616 → 2308 → 1154 → 577 → 866 → 433 → 650  
→ 325 → 488 → 244 → 122 → 61 → 92 → 46 → 23 → 35 → 53 → 80  
→ 40 → 20 → 10 → 5 → 8 → 4 → 2 → 1 → 2 → 1 → ...

The Collatz conjecture posits that every positive integer eventually reaches 1. The conjecture was verified in 2021 for all  $1 \leq n \leq 10^{21}$  [2], but it has not yet been proven or refuted.

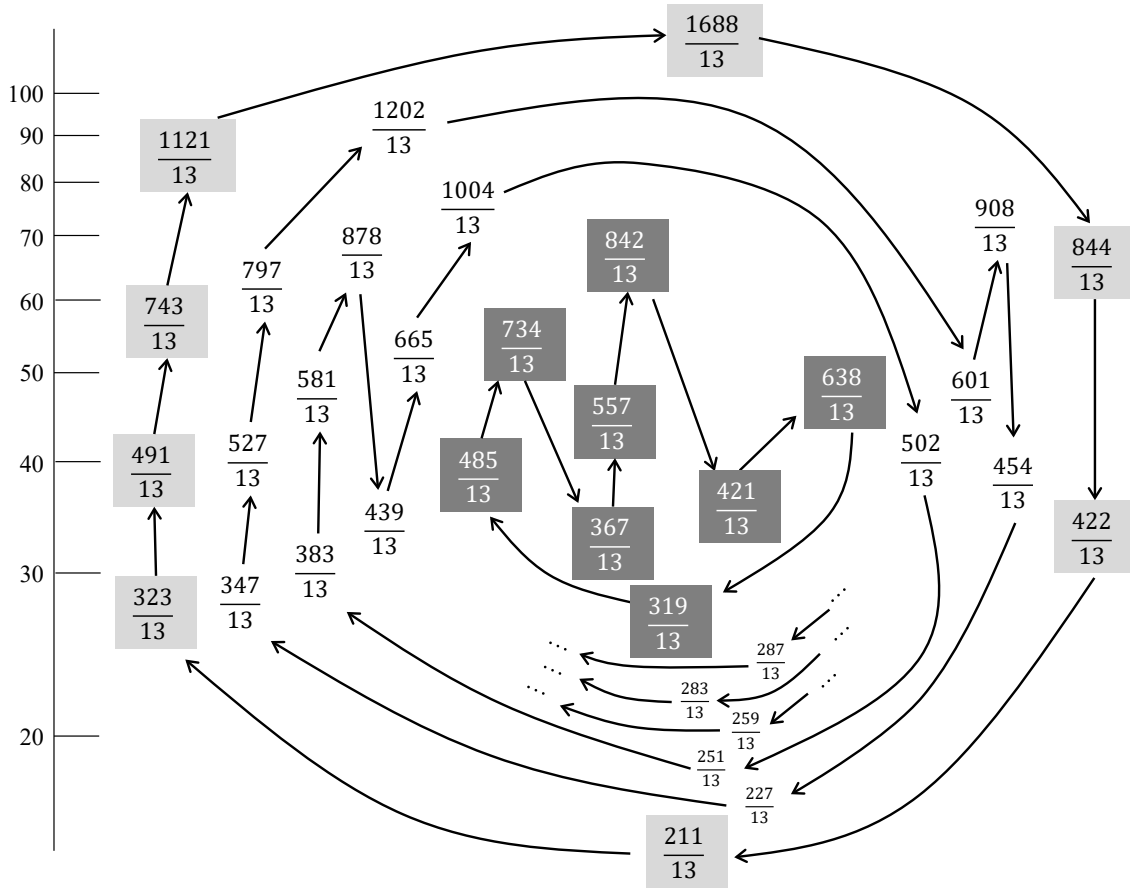


Figure 1: All of the rational Collatz cycles of length  $k = 8$  with  $x = 5$  odd terms. Of special note are the outermost cycle (the *circuit*), in light gray, and the innermost cycle (the *high cycle*), in dark gray.

If  $T^i(n) = n$  for some positive integer  $n$ , there is a Collatz *cycle* of length  $i$  whose first term is  $n$ . The only known  $T$ -cycle is the trivial one:  $1 \rightarrow 2 \rightarrow 1$ .

The unresolved Weak Collatz conjecture claims that there are no positive integer cycles. (It is called “weak” because it allows for the possibility of a start number that diverges to infinity, never reaching 1.)

We can also consider cycles with rational values. Figure 1 shows all cycles of length  $k = 8$  with  $x = 5$  odd terms. Considering the term  $\frac{211}{13}$  to be odd (by its numerator), the next term produced by the Collatz process is  $(3(\frac{211}{13}) + 1)/2 = \frac{323}{13}$  (also odd).

For any  $k$  and  $x$ , the outermost and innermost cycles are of special interest. Let  $S$  contain the smallest members of each cycle (here,  $\frac{211}{13}, \frac{227}{13}, \dots, \frac{319}{13}$ ). The outermost cycle (called the *circuit*) contains the lowest member in  $S$ , while the innermost cycle (called the *high cycle*) contains the highest member in  $S$ .

Steiner [10] showed that no positive integer circuits exist for any  $k, x$  (except  $k = 2, x =$

1). The current paper shows likewise that no high cycles of positive integers exist.

## 2 Parity vectors

Associated with every cycle member  $m$  is a unique *parity vector* that records the even (0) and odd (1) terms obtained from  $k$  iterations of the Collatz function.

**Example 2.1** (Cycle member to parity vector). *Starting with  $m = \frac{211}{13}$ , we encounter 5 odd terms followed by 3 even terms, so the corresponding parity vector is 11111000.*

Likewise, given a parity vector  $\mathbf{v}$ , we can compute a cycle member  $m$ .

**Example 2.2** (Parity vector to cycle member). *Applying  $\mathbf{v} = 110$  to arbitrary start number  $n$  yields  $[3\frac{3n+1}{2} + 1]/2/2 = \frac{9}{8}n + \frac{5}{8}$ . To make a cycle, we set  $n = \frac{9}{8}n + \frac{5}{8}$ . Solving this equation yields  $n = -5$ , which is part of the  $-5 \rightarrow -7 \rightarrow -10 \rightarrow -5$  cycle.*

Generalizing this idea, Böhm and Sontacchi [4] give the cycle member associated with arbitrary parity vector  $\mathbf{v}$  as

$$f(\mathbf{v}) = \frac{\sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} 3^{x-i-1}}{2^k - 3^x}, \quad (2.1)$$

where  $d_0(\mathbf{v}) < \dots < d_{x-1}(\mathbf{v})$  are the (zero-based) indices of 1s in  $\mathbf{v}$ . Since we are only interested in Collatz cycles with positive terms, the denominator must be positive, so  $k > x \log_2 3$ .

The Weak Collatz Conjecture can be restated: “Is there a parity vector  $\mathbf{v}$  such that  $f(\mathbf{v})$  is an integer greater than 2?”

It will sometimes be useful to refer to the numerator only:

$$g(\mathbf{v}) = \sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} 3^{x-i-1}. \quad (2.2)$$

Note that rotating a parity vector produces another term in the same rational cycle.

**Example 2.3** (Rotation).  $f(100) = \frac{1}{5}$ ,  $f(001) = \frac{4}{5}$ , and  $f(010) = \frac{2}{5}$ . *These are all entry points into the  $\frac{1}{5} \rightarrow \frac{4}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5}$  cycle.*

Finally, we can restrict ourselves to aperiodic parity vectors. Consider  $f(1101011010) = \frac{23}{5}$ , which is part of the double cycle  $\frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5}$ . Had these been integers, the aperiodic  $f(11010)$  would also be an integer.

### 3 No integer circuits

The parity vector  $\mathbf{v}_c = 1^x 0^{k-x}$  is associated with the smallest member of the  $(k, x)$ -circuit. Following Equation 2.1,

$$f(\mathbf{v}_c) = \frac{\sum_{i=0}^{x-1} 2^i 3^{x-i-1}}{2^k - 3^x}.$$

We provide a simple proof that (non-trivial) Collatz circuits do not exist, after three lemmas.

**Lemma 3.1.**  $f(\mathbf{v}_c)$  simplifies to  $\frac{3^x - 2^x}{2^k - 3^x}$ .

*Proof.* By induction, omitted. □

**Lemma 3.2.** *If there is a non-trivial positive integer circuit of length  $k$  with  $x$  odd terms, then  $2 \cdot (\frac{3}{2})^x \geq 2^{k-x} - 1$ .*

*Proof.* This follows from the fact that the putative circuit's smallest member is at least 1.

$$\begin{aligned} \frac{3^x - 2^x}{2^k - 3^x} &\geq 1 \\ 3^x - 2^x &\geq 2^k - 3^x \\ 2 \cdot 3^x &\geq 2^k + 2^x \\ 2 \cdot (\frac{3}{2})^x &\geq 2^{k-x} - 1. \end{aligned}$$

□

**Lemma 3.3** (Ellison [7]). *For integers  $k$  and  $x$ , with  $x > 17$ , we have  $2^k - 3^x > 2.56^x$ .*

*Proof.* We start with Ellison's original bound (for  $k > 27$ ) and use the fact that  $k > x \log_2 3$  in positive cycles.

$$2^k - 3^x > \frac{2^k}{e^{k/10}} > 1.8^k > 1.8^{x \log_2 3} > 2.56^x, \text{ for } x > 17. \quad \square$$

**Theorem 3.4** (Steiner, 1977). *Non-trivial Collatz circuits do not exist.*

*Proof.* Assume  $f(\mathbf{v}_c) = \frac{3^x - 2^x}{2^k - 3^x}$  is a positive integer. Then the following are also positive integers:

$$\begin{aligned} & \frac{3^x - 2^x}{2^k - 3^x} + 1, \\ & \frac{3^x - 2^x}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x}, \\ & \frac{2^k - 2^x}{2^k - 3^x}, \\ & \frac{2^x(2^{k-x} - 1)}{2^k - 3^x}, \text{ and } \frac{2^{k-x} - 1}{2^k - 3^x}. \end{aligned}$$

We can remove the  $2^x$  in the last step without affecting the expression's purported integrality, since the denominator is odd.

However, the last term is actually less than one; we invoke Lemmas 3.3 and 3.2 to show that the denominator outstrips the numerator. For  $x > 17$ ,

$$2^k - 3^x > 2.56^x > 2 \cdot (1.5)^x \geq 2^{k-x} - 1.$$

$2^{k-x} - 1 = 0$  only when  $k = 2, x = 1$ ; setting aside this case, we obtain the contradiction  $0 < \frac{2^{k-x}-1}{2^k-3^x} < 1$ . Cases of  $x \leq 17$  are handled by the fact that any non-trivial Collatz cycle must contain many millions of odd terms [6].  $\square$

To date, all no-circuit proofs depend on deep lower bounds for  $2^k - 3^x$ , such as Ellison's, which are all derived from Alan Baker's pioneering transcendental number theory work [1]. As Jeffrey Lagarias [9] remarks, "The most remarkable thing about Theorem [...] is the weakness of its conclusion compared to the strength of the methods used in its proof."<sup>1</sup>

## 4 The high cycle

In this section, characterize the Collatz high cycle.

**Definition 4.1.** Let  $\mathbf{v}_h$  be a  $(k, x)$ -parity vector with 1s indexed by  $d_i(\mathbf{v}) = \lfloor \frac{k}{x}i \rfloor$ , for  $0 \leq i \leq x - 1$ .

**Example 4.2.** For  $k = 8, x = 5$ ,  $\mathbf{v}_h = 11011010$ .

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<sup>1</sup>Note that certain circuits can be ruled out without these deep bounds. For example, if  $k$  and  $x$  are not co-prime, then we can factor  $2^k - 3^x$ . E.g.,  $(2^{k/2} + 3^{x/2})(2^{k/2} - 3^{x/2}) > 2^{k/2} > 2^{x \log 3/2} > 1.72^x > 2^{k-x} - 1$ . Or for the case of  $\mathbf{v}_c = 1111111100000$ , it follows that if  $f(\mathbf{v}_c) = \frac{3^8 - 2^8}{2^{13} - 3^8}$  is an integer, then so is  $31f(\mathbf{v}_c) + 206$ , but it happens that  $31 \frac{3^8 - 2^8}{2^{13} - 3^8} + 206 \frac{2^{13} - 3^8}{2^{13} - 3^8} = \frac{3^{12}}{2^{13} - 3^8}$ , which is not integral.

**Example 4.3.** For  $k = 21, x = 13, \mathbf{v}_h = 110110101101101010$ .

The vector  $\mathbf{v}_h$  is the well-known (*upper*) *Christoffel word* [3], constructed so that its 1s are roughly evenly spread among its 0s. The cycle member corresponding to  $\mathbf{v}_h$  is

$$f(\mathbf{v}_h) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x} i \rfloor}}{2^k - 3^x}.$$

Now we list a number of properties of  $\mathbf{v}_h$ .

First,  $f(\mathbf{v}_h)$  is the smallest member of its cycle.

**Theorem 4.4** (Halbeisen and Hungerbühler [8]). *For every  $(k, x)$  parity vector  $\mathbf{v}$  that is a rotation of  $\mathbf{v}_h$ , we have  $f(\mathbf{v}_h) \leq f(\mathbf{v})$ .*

**Remark 4.5.** *It is known that a Christoffel word  $\mathbf{v}_h$  is at a lexicographic extreme among all its rotations [3]. However, lexicographic ordering does not always coincide with  $f$ -ordering. For example,  $f(11110010) = \frac{259}{13}$ ,  $f(11100101) = \frac{395}{13}$ , and  $f(10111100) = \frac{341}{13}$ . Of course, lexicographic ordering correlates roughly with  $f$ -ordering, as left-heavy vectors tend to be associated with smaller cycle members.*

**Theorem 4.6.** *Among all the  $(k, x)$  upper Christoffel words,  $f(\mathbf{v}_h)$  is maximized when  $k = \lceil x \log_2 3 \rceil$ .*

*Proof.* Let  $\mathbf{v}_h$  be an upper Christoffel word with  $k = \lceil x \log_2 3 \rceil$ . Let  $\mathbf{u}$  be a longer  $(k+1, x)$ -high-cycle parity vector. Replacing  $k$  by  $k+1$  increases the numerator of  $f(\mathbf{v})$  by no more than a factor of 2, while increasing the denominator by more than a factor of 2.

$$\begin{aligned} f(\mathbf{u}) &= \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k+1}{x} i \rfloor}}{2^{k+1} - 3^x} = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x} i + \frac{i}{x} \rfloor}}{2^{k+1} - 3^x} \\ &< \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x} i \rfloor + 1}}{2^{k+1} - 2 \cdot 3^x} = \frac{2 \cdot \sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x} i \rfloor}}{2(2^k - 3^x)} = f(\mathbf{v}). \quad \square \end{aligned}$$

Similar reasoning holds if  $\mathbf{u}$  is a  $(k+n, x)$  upper Christoffel word, for any  $n \geq 1$ .

Next, we confirm the reader's suspicion that  $f(\mathbf{v}_h)$  is indeed a member of the high cycle. Let  $S$  contain the smallest members of each  $(k, x)$ -cycle.

**Theorem 4.7** (Halbeisen and Hungerbühler [8]). *Let  $\mathbf{v}$  be a parity vector associated with the smallest member of any  $(k, x)$ -cycle, and let  $\mathbf{v}_h$  be an upper Christoffel word with  $k = \lceil x \log_2 3 \rceil$ . Then,  $f(\mathbf{v}_h) \geq f(\mathbf{v})$ .*

**Remark 4.8** (Tangential). *This theorem is useful for proving that any Collatz cycle must be long. Just for example, it is easy to verify empirically that  $f(\mathbf{v}_h) < 10^{13}$  for all  $x < 10,000,000$ , implying that every cycle (not just the high cycle) with fewer than ten million odd terms has some member less than  $10^{13}$ . Since no Collatz counter-examples exist among the first  $10^{20}$  positive integers [2], a purported Collatz cycle must have more than ten million odd terms. Stronger results have been obtained through analytical means [8]. Of course,  $f(\mathbf{v}_h)$  will outstrip any finite confirmation, after some  $x$ , because  $O(3^x x)$  exceeds  $O(2^k - 3^x)$ .*

Recall the cycle member corresponding to  $\mathbf{v}_h$ .

$$f(\mathbf{v}_h) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x} i \rfloor}}{2^k - 3^x},$$

Due to the floor function, we cannot simplify this expression as we did for  $f(\mathbf{v}_c)$ . Its value can only be partially wrangled through upper and lower bounds.

**Theorem 4.9** (Halbeisen and Hungerbühler [8]).

$$\frac{3^x(x/20)}{2^{\lceil x \log_2 3 \rceil} - 3^x} < f(\mathbf{v}_h) < \frac{3^x(7x/10)}{2^{\lceil x \log_2 3 \rceil} - 3^x}.$$

These bounds can be improved from  $(x/20, 7x/10)$  to  $(x/6, x/2)$ ; we omit the proof for brevity.

## 5 No integer high cycles

In this section, we prove our main result that  $f(\mathbf{v}_h)$  is never an integer. To do this, we need two more facts about Christoffel words. First,  $f(\mathbf{v}_h^R)$  is a member of the same cycle as  $f(\mathbf{v}_h)$ .

**Theorem 5.1** (Cohn [5]). *The reverse  $\mathbf{v}_h^R$  of an upper Christoffel word  $\mathbf{v}_h$  is also a rotation of it.*

Therefore,  $f(\mathbf{v}_h)$  and  $f(\mathbf{v}_h^R)$  are in the same cycle. Indeed,  $f(\mathbf{v}_h^R)$  is the largest member of the cycle, though we do not need this fact; also for interest, the left-rotation distance is the multiplicative inverse of  $x$  modulo  $k$ .

**Theorem 5.2** (Berstel et al [3]). *For aperiodic  $\mathbf{v}_h$ , we have  $\mathbf{v}_h = 1\mathbf{u}0$ , and  $\mathbf{v}_h^R = 0\mathbf{u}1$ .*

**Example 5.3.**  $\mathbf{v}_h = 11011010$ ,  $\mathbf{v}_h^R = 01011011$ , and  $\mathbf{u} = 101101$ . *Incidentally,  $\mathbf{u}$  is always a palindrome, though we do not need this fact.*

Because  $\mathbf{v}_h$  and  $\mathbf{v}_h^R$  are virtually identical, differing only in their first and last components, we expect  $f(\mathbf{v}_h)$  and  $f(\mathbf{v}_h^R)$  to have similar summands. We express both in terms of  $\mathbf{u}$ , via Equations 2.1 and 2.2.



$$f(\mathbf{v}_h) = \frac{2g(\mathbf{u}) + 3^{x-1}}{2^k - 3^x}$$

$$f(\mathbf{v}_h^R) = \frac{6g(\mathbf{u}) + 2^{k-1}}{2^k - 3^x}$$

In the first case, we create  $\mathbf{v}_h$  by shifting  $\mathbf{u}$  one place to the right (doubling every summand) and adding a 1 to the left edge. In the second case, shifting  $\mathbf{u}$  requires both a doubling (for the same reason) and a tripling (from adding a 1 to the right edge).

**Theorem 5.4** (Main result). *No high cycle consists of integers.*

*Proof.* We know  $f(\mathbf{v}_h)$  and  $f(\mathbf{v}_h^R)$  are both members of the high cycle (Theorems 4.7 and 5.1). If they are both integers, then so are

$$3f(\mathbf{v}_h) - f(\mathbf{v}_h^R) + 1,$$

$$3 \frac{2g(\mathbf{u}) + 3^{x-1}}{2^k - 3^x} - \frac{6g(\mathbf{u}) + 2^{k-1}}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x},$$

$$\frac{2^k - 2^{k-1}}{2^k - 3^x}, \quad \text{and} \quad \frac{2^{k-1}}{2^k - 3^x}.$$

However,  $2^{k-1}$  is not divisible by any odd number, so we have a contradiction.  $\square$

**Example 5.5.** *For the (8, 5)-high-cycle, Figure 1 gives  $f(\mathbf{v}_h) = \frac{319}{13}$  and  $f(\mathbf{v}_h^R) = \frac{842}{13}$ . Combining gives  $3\frac{319}{13} - \frac{842}{13} + \frac{13}{13} = \frac{2^7}{13}$ .*

Note that if we left-rotate  $\mathbf{v}_h$  and  $\mathbf{v}_h^R$  by one position each, we obtain an alternate pair of high-loop members ( $\mathbf{u01}$  and  $\mathbf{u10}$ ), whose simple difference provides the necessary contradiction. For example, cycle members  $\frac{485}{13}$  and  $\frac{421}{13}$  in Figure 1 cannot both be integers, because  $\frac{485}{13} - \frac{421}{13} = \frac{2^6}{13}$ .

## 6 Summary

Notable features of our proof versus the no-circuit proof include:

- Instead of assuming, by way of contradiction, that  $f(\mathbf{v}_c)$  is an integer, we instead assume that both  $f(\mathbf{v}_h)$  and  $f(\mathbf{v}_h^R)$  are both integers.
- Unlike the no-circuit proof, we do not require deep lower bounds [1, 7] on the size of  $2^k - 3^x$ .
- We require no closed-form expression for any high cycle member; by contrast, the no-circuit proof relies on the expression  $f(\mathbf{v}_c) = \frac{3^x - 2^x}{2^k - 3^x}$ .

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