

# Collatz High Cycles Do Not Exist Kevin Knight

# ▶ To cite this version:

Kevin Knight. Collatz High Cycles Do Not Exist. 2023. hal-04261183

# HAL Id: hal-04261183 https://hal.science/hal-04261183v1

Preprint submitted on 26 Oct 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Collatz High Cycles Do Not Exist

## Kevin Knight

September 23, 2023

#### Abstract

The Collatz function takes odd n to (3n + 1)/2 and even n to n/2. Under the iterated Collatz function, every positive integer is conjectured to end up in the trivial cycle 1-2-1. Two types of cycles are of special interest. Consider the set S consisting of the smallest members of all cycles containing the same number of odd terms. The *circuit* contains the smallest member of S, while the *high cycle* contains the largest. It is known that no circuits of positive integers exist (except 1-2-1); this paper shows that there are likewise no high cycles of positive integers.

## 1 Introduction

The Collatz function is

$$T(n) = \begin{cases} \frac{3n+1}{2}, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{otherwise.} \end{cases}$$

Iterating this function famously yields interesting sequences. For example:

$$\begin{array}{l} 31 \rightarrow 47 \rightarrow 71 \rightarrow 107 \rightarrow 161 \rightarrow 242 \rightarrow 121 \rightarrow 182 \rightarrow 91 \rightarrow 137 \rightarrow 206 \\ \rightarrow 103 \rightarrow 155 \rightarrow 233 \rightarrow 350 \rightarrow 175 \rightarrow 263 \rightarrow 395 \rightarrow 593 \rightarrow 890 \rightarrow 445 \\ \rightarrow 668 \rightarrow 334 \rightarrow 167 \rightarrow 251 \rightarrow 377 \rightarrow 566 \rightarrow 283 \rightarrow 425 \rightarrow 638 \rightarrow 319 \\ \rightarrow 479 \rightarrow 719 \rightarrow 1079 \rightarrow 1619 \rightarrow 2429 \rightarrow 3644 \rightarrow 1822 \rightarrow 911 \rightarrow 1367 \\ \rightarrow 2051 \rightarrow 3077 \rightarrow 4616 \rightarrow 2308 \rightarrow 1154 \rightarrow 577 \rightarrow 866 \rightarrow 433 \rightarrow 650 \\ \rightarrow 325 \rightarrow 488 \rightarrow 244 \rightarrow 122 \rightarrow 61 \rightarrow 92 \rightarrow 46 \rightarrow 23 \rightarrow 35 \rightarrow 53 \rightarrow 80 \\ \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \ldots \end{array}$$

The Collatz conjecture posits that every positive integer eventually reaches 1. The conjecture was verified in 2021 for all  $1 \le n \le 10^{21}$  [2], but it has not yet been proven or refuted.

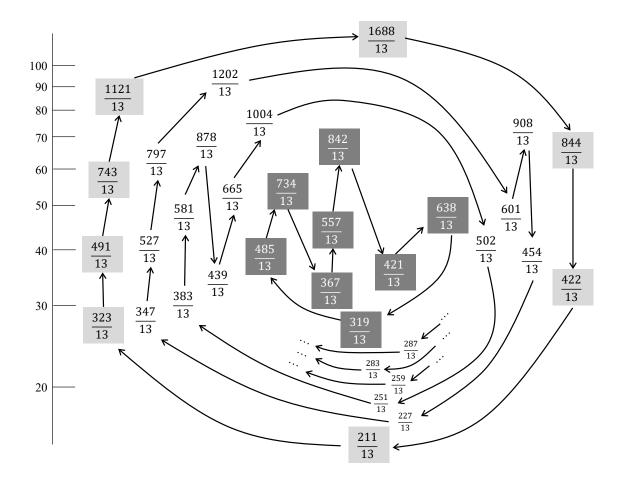


Figure 1: All of the rational Collatz cycles of length k = 8 with x = 5 odd terms. Of special note are the outermost cycle (the *circuit*), in light gray, and the innermost cycle (the *high cycle*), in dark gray.

If  $T^i(n) = n$  for some positive integer n, there is a Collatz *cycle* of length i whose first term is n. The only known T-cycle is the trivial one:  $1 \to 2 \to 1$ .

The unresolved Weak Collatz conjecture claims that there are no positive integer cycles. (It is called "weak" because it allows for the possibility of a start number that diverges to infinity, never reaching 1.)

We can also consider cycles with rational values. Figure 1 shows all cycles of length k = 8 with x = 5 odd terms. Considering the term  $\frac{211}{13}$  to be odd (by its numerator), the next term produced by the Collatz process is  $(3(\frac{211}{13}) + 1)/2 = \frac{323}{13}$  (also odd). For any k and x, the outermost and innermost cycles are of special interest. Let S contain

For any k and x, the outermost and innermost cycles are of special interest. Let S contain the smallest members of each cycle (here,  $\frac{211}{13}$ ,  $\frac{227}{13}$ , ...,  $\frac{319}{13}$ ). The outermost cycle (called the *circuit*) contains the lowest member in S, while the innermost cycle (called the *high cycle*) contains the highest member in S.

Steiner [10] showed that no positive integer circuits exist for any k, x (except k = 2, x =

1). The current paper shows likewise that no high cycles of positive integers exist.

# 2 Parity vectors

Associated with every cycle member m is a unique *parity vector* that records the even (0) and odd (1) terms obtained from k iterations of the Collatz function.

**Example 2.1** (Cycle member to parity vector). Starting with  $m = \frac{211}{13}$ , we encounter 5 odd terms followed by 3 even terms, so the corresponding parity vector is 11111000.

Likewise, given a parity vector  $\mathbf{v}$ , we can compute a cycle member m.

**Example 2.2** (Parity vector to cycle member). Applying  $\mathbf{v} = 110$  to arbitrary start number n yields  $[3\frac{3n+1}{2}+1]/2/2 = \frac{9}{8}n + \frac{5}{8}$ . To make a cycle, we set  $n = \frac{9}{8}n + \frac{5}{8}$ . Solving this equation yields n = -5, which is part of the  $-5 \rightarrow -7 \rightarrow -10 \rightarrow -5$  cycle.

Generalizing this idea, Böhm and Sontacchi [4] give the cycle member associated with arbitrary parity vector  $\mathbf{v}$  as

$$f(\mathbf{v}) = \frac{\sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} \ 3^{x-i-1}}{2^k - 3^x},$$
(2.1)

where  $d_0(\mathbf{v}) < \ldots < d_{x-1}(\mathbf{v})$  are the (zero-based) indices of 1s in  $\mathbf{v}$ . Since we are only interested in Collatz cycles with positive terms, the denominator must be positive, so  $k > x \log_2 3$ .

The Weak Collatz Conjecture can be restated: "Is there a parity vector  $\mathbf{v}$  such that  $f(\mathbf{v})$  is an integer greater than 2?"

It will sometimes be useful to refer to the numerator only:

$$g(\mathbf{v}) = \sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} 3^{x-i-1}.$$
 (2.2)

Note that rotating a parity vector produces another term in the same rational cycle.

**Example 2.3** (Rotation).  $f(100) = \frac{1}{5}$ ,  $f(001) = \frac{4}{5}$ , and  $f(010) = \frac{2}{5}$ . These are all entry points into the  $\frac{1}{5} \rightarrow \frac{4}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5}$  cycle.

Finally, we can restrict ourselves to aperiodic parity vectors. Consider  $f(1101011010) = \frac{23}{5}$ , which is part of the double cycle  $\frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5}$ . Had these been integers, the aperiodic f(11010) would also be an integer.

# 3 No integer circuits

The parity vector  $\mathbf{v_c} = 1^x 0^{k-x}$  is associated with the smallest member of the (k, x)-circuit. Following Equation 2.1,

$$f(\mathbf{v_c}) = \frac{\sum_{i=0}^{x-1} 2^i 3^{x-i-1}}{2^k - 3^x}.$$

We provide a simple proof that (non-trivial) Collatz circuits do not exist, after three lemmas.

**Lemma 3.1.**  $f(\mathbf{v_c})$  simplifies to  $\frac{3^x - 2^x}{2^k - 3^x}$ .

*Proof.* By induction, omitted.

**Lemma 3.2.** If there is a non-trivial positive integer circuit of length k with x odd terms, then  $2 \cdot (\frac{3}{2})^x \ge 2^{k-x} - 1$ .

*Proof.* This follows from the fact that the putative circuit's smallest member is at least 1.

$$\frac{3^{x} - 2^{x}}{2^{k} - 3^{x}} \ge 1$$
  

$$3^{x} - 2^{x} \ge 2^{k} - 3^{x}$$
  

$$2 \cdot 3^{x} \ge 2^{k} + 2^{x}$$
  

$$2 \cdot (\frac{3}{2})^{x} \ge 2^{k-x} - 1.$$

**Lemma 3.3** (Ellison [7]). For integers k and x, with x > 17, we have  $2^k - 3^x > 2.56^x$ .

*Proof.* We start with Ellison's original bound (for k > 27) and use the fact that  $k > x \log_2 3$  in positive cycles.

$$2^k - 3^x > \frac{2^k}{e^{k/10}} > 1.8^k > 1.8^{x \log_2 3} > 2.56^x$$
, for  $x > 17$ .

Theorem 3.4 (Steiner, 1977). Non-trivial Collatz circuits do not exist.

*Proof.* Assume  $f(\mathbf{v_c}) = \frac{3^x - 2^x}{2^k - 3^x}$  is a positive integer. Then the following are also positive integers:

$$\frac{3^{x} - 2^{x}}{2^{k} - 3^{x}} + 1,$$

$$\frac{3^{x} - 2^{x}}{2^{k} - 3^{x}} + \frac{2^{k} - 3^{x}}{2^{k} - 3^{x}},$$

$$\frac{2^{k} - 2^{x}}{2^{k} - 3^{x}},$$

$$\frac{2^{x}(2^{k-x} - 1)}{2^{k} - 3^{x}}, \text{ and } \frac{2^{k-x} - 1}{2^{k} - 3^{x}}$$

We can remove the  $2^x$  in the last step without affecting the expression's purported integrality, since the denominator is odd.

However, the last term is actually less than one; we invoke Lemmas 3.3 and 3.2 to show that the denominator outstrips the numerator. For x > 17,

$$2^k - 3^x > 2.56^x > 2 \cdot (1.5)^x \ge 2^{k-x} - 1.$$

 $2^{k-x} - 1 = 0$  only when k = 2, x = 1; setting aside this case, we obtain the contradiction  $0 < \frac{2^{k-x}-1}{2^k-3^x} < 1$ . Cases of  $x \le 17$  are handled by the fact that any non-trivial Collatz cycle must contain many millions of odd terms [6].

To date, all no-circuit proofs depend on deep lower bounds for  $2^k - 3^x$ , such as Ellison's, which are all derived from Alan Baker's pioneering transcendental number theory work [1]. As Jeffrey Lagarias [9] remarks, "The most remarkable thing about Theorem [...] is the weakness of its conclusion compared to the strength of the methods used in its proof."<sup>1</sup>

#### 4 The high cycle

In this section, characterize the Collatz high cycle.

**Definition 4.1.** Let  $\mathbf{v_h}$  be a (k, x)-parity vector with 1s indexed by  $d_i(\mathbf{v}) = \lfloor \frac{k}{x}i \rfloor$ , for  $0 \leq i \leq x - 1$ .

Example 4.2. For  $k = 8, x = 5, v_h = 11011010$ .

<sup>1</sup>Note that certain circuits can be ruled out without these deep bounds. For example, if k and x are not co-prime, then we can factor  $2^k - 3^x$ . E.g.,  $(2^{k/2} + 3^{x/2})(2^{k/2} - 3^{x/2}) > 2^{k/2} > 2^{x\log 3/2} > 1.72^x > 2^{k-x} - 1$ . Or for the case of  $\mathbf{v_c} = 1111111100000$ , it follows that if  $f(\mathbf{v_c}) = \frac{3^8 - 2^8}{2^{13} - 3^8}$  is an integer, then so is  $31f(\mathbf{v_c}) + 206$ , but it happens that  $31\frac{3^8 - 2^8}{2^{13} - 3^8} + 206\frac{2^{13} - 3^8}{2^{13} - 3^8} = \frac{3^{12}}{2^{13} - 3^8}$ , which is not integral.

Example 4.3. For  $k = 21, x = 13, v_h = 11011010110110110101010$ .

The vector  $\mathbf{v_h}$  is the well-known *(upper) Christoffel word* [3], constructed so that its 1s are roughly evenly spread among its 0s. The cycle member corresponding to  $\mathbf{v_h}$  is

$$f(\mathbf{v_h}) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x}i \rfloor}}{2^k - 3^x}.$$

Now we list a number of properties of  $\mathbf{v_h}$ .

First,  $f(\mathbf{v_h})$  is the smallest member of its cycle.

**Theorem 4.4** (Halbeisen and Hungerbühler [8]). For every (k, x) parity vector  $\mathbf{v}$  that is a rotation of  $\mathbf{v}_{\mathbf{h}}$ , we have  $f(\mathbf{v}_{\mathbf{h}}) \leq f(\mathbf{v})$ .

**Remark 4.5.** It is known that a Christoffel word  $\mathbf{v_h}$  is at a lexicographic extreme among all its rotations [3]. However, lexicographic ordering does not always coincide with f-ordering. For example,  $f(11110010) = \frac{259}{13}$ ,  $f(11100101) = \frac{395}{13}$ , and  $f(10111100) = \frac{341}{13}$ . Of course, lexicographic ordering correlates roughly with f-ordering, as left-heavy vectors tend to be associated with smaller cycle members.

**Theorem 4.6.** Among all the (k, x) upper Christoffel words,  $f(\mathbf{v_h})$  is maximized when  $k = \lceil x \log_2 3 \rceil$ .

*Proof.* Let  $\mathbf{v_h}$  be an upper Christoffel word with  $k = \lceil x \log_2 3 \rceil$ . Let  $\mathbf{u}$  be a longer (k+1, x)-high-cycle parity vector. Replacing k by k+1 increases the numerator of  $f(\mathbf{v})$  by no more than a factor of 2, while increasing the denominator by more than a factor of 2.

$$f(\mathbf{u}) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k+1}{x}i \rfloor}}{2^{k+1} - 3^x} = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x}i + \frac{i}{x} \rfloor}}{2^{k+1} - 3^x}$$
$$< \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x}i \rfloor + 1}}{2^{k+1} - 2 \cdot 3^x} = \frac{2 \cdot \sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x}i \rfloor}}{2(2^k - 3^x)} = f(\mathbf{v}).$$

Similar reasoning holds if **u** is a (k + n, x) upper Christoffel word, for any  $n \ge 1$ .

Next, we confirm the reader's suspicion that  $f(\mathbf{v_h})$  is indeed a member of the high cycle. Let S contain the smallest members of each (k, x)-cycle.

**Theorem 4.7** (Halbeisen and Hungerbühler [8]). Let  $\mathbf{v}$  be a parity vector associated with the smallest member of any (k, x)-cycle, and let  $\mathbf{v}_{\mathbf{h}}$  be an upper Christoffel word with  $k = \lceil x \log_2 3 \rceil$ . Then,  $f(\mathbf{v}_{\mathbf{h}}) \ge f(\mathbf{v})$ .

**Remark 4.8** (Tangential). This theorem is useful for proving that any Collatz cycle must be long. Just for example, it is easy to verify empirically that  $f(\mathbf{v_h}) < 10^{13}$  for all x < 10,000,000, implying that every cycle (not just the high cycle) with fewer than ten million odd terms has some member less than  $10^{13}$ . Since no Collatz counter-examples exist among the first  $10^{20}$  positive integers [2], a purported Collatz cycle must have more than ten million odd terms. Stronger results have been obtained through analytical means [8]. Of course,  $f(\mathbf{v_h})$  will outstrip any finite confirmation, after some x, because  $O(3^xx)$  exceeds  $O(2^k - 3^x)$ .

Recall the cycle member corresponding to  $\mathbf{v}_{\mathbf{h}}$ .

$$f(\mathbf{v_h}) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1} 2^{\lfloor \frac{k}{x}i \rfloor}}{2^k - 3^x},$$

Due to the floor function, we cannot simplify this expression as we did for  $f(\mathbf{v_c})$ . Its value can only be partially wrangled through upper and lower bounds.

**Theorem 4.9** (Halbeisen and Hungerbühler [8]).

$$\frac{3^x(x/20)}{2^{\lceil x \log_2 3\rceil} - 3^x} < f(\mathbf{v_h}) < \frac{3^x(7x/10)}{2^{\lceil x \log_2 3\rceil} - 3^x}.$$

These bounds can be improved from (x/20, 7x/10) to (x/6, x/2); we omit the proof for brevity.

## 5 No integer high cycles

In this section, we prove our main result that  $f(\mathbf{v_h})$  is never an integer. To do this, we need two more facts about Christoffel words. First,  $f(\mathbf{v_h}^{\mathbf{R}})$  is a member of the same cycle as  $f(\mathbf{v_h})$ .

**Theorem 5.1** (Cohn [5]). The reverse  $\mathbf{v}_{\mathbf{h}}^{\mathbf{R}}$  of an upper Christoffel word  $\mathbf{v}_{\mathbf{h}}$  is also a rotation of it.

Therefore,  $f(\mathbf{v_h})$  and  $f(\mathbf{v_h^R})$  are in the same cycle. Indeed,  $f(\mathbf{v_h^R})$  is the largest member of the cycle, though we do not need this fact; also for interest, the left-rotation distance is the multiplicative inverse of x modulo k.

**Theorem 5.2** (Berstel et al [3]). For aperiodic  $\mathbf{v}_{\mathbf{h}}$ , we have  $\mathbf{v}_{\mathbf{h}} = 1\mathbf{u}0$ , and  $\mathbf{v}_{\mathbf{h}}^{\mathbf{R}} = 0\mathbf{u}1$ .

**Example 5.3.**  $\mathbf{v_h} = 11011010$ ,  $\mathbf{v_h^R} = 01011011$ , and  $\mathbf{u} = 101101$ . Incidentally,  $\mathbf{u}$  is always a palindrome, though we do not need this fact.

Because  $\mathbf{v_h}$  and  $\mathbf{v_h^R}$  are virtually identical, differing only in their first and last components, we expect  $f(\mathbf{v_h})$  and  $f(\mathbf{v_h^R})$  to have similar summands. We express both in terms of  $\mathbf{u}$ , via Equations 2.1 and 2.2.

$$f(\mathbf{v_h}) = \frac{2g(\mathbf{u}) + 3^{x-1}}{2^k - 3^x}$$
$$f(\mathbf{v_h}^{\mathbf{R}}) = \frac{6g(\mathbf{u}) + 2^{k-1}}{2^k - 3^x}$$

In the first case, we create  $\mathbf{v}_{\mathbf{h}}$  by shifting  $\mathbf{u}$  one place to the right (doubling every summand) and adding a 1 to the left edge. In the second case, shifting  $\mathbf{u}$  requires both a doubling (for the same reason) and a tripling (from adding a 1 to the right edge).

**Theorem 5.4** (Main result). No high cycle consists of integers.

*Proof.* We know  $f(\mathbf{v_h})$  and  $f(\mathbf{v_h^R})$  are both members of the high cycle (Theorems 4.7 and 5.1). If they are both integers, then so are

$$3f(\mathbf{v_h}) - f(\mathbf{v_h}^{\mathbf{R}}) + 1,$$
  

$$3\frac{2g(\mathbf{u}) + 3^{x-1}}{2^k - 3^x} - \frac{6g(\mathbf{u}) + 2^{k-1}}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x},$$
  

$$\frac{2^k - 2^{k-1}}{2^k - 3^x}, \text{ and } \frac{2^{k-1}}{2^k - 3^x}.$$

However,  $2^{k-1}$  is not divisible by any odd number, so we have a contradiction.

**Example 5.5.** For the (8,5)-high-cycle, Figure 1 gives  $f(\mathbf{v_h}) = \frac{319}{13}$  and  $f(\mathbf{v_h}^{\mathbf{R}}) = \frac{842}{13}$ . Combining gives  $3\frac{319}{13} - \frac{842}{13} + \frac{13}{13} = \frac{2^7}{13}$ .

Note that if we left-rotate  $\mathbf{v_h}$  and  $\mathbf{v_h^R}$  by one position each, we obtain an alternate pair of high-loop members (**u01** and **u10**), whose simple difference provides the necessary contradiction. For example, cycle members  $\frac{485}{13}$  and  $\frac{421}{13}$  in Figure 1 cannot both be integers, because  $\frac{485}{13} - \frac{421}{13} = \frac{2^6}{13}$ .

#### 6 Summary

Notable features of our proof versus the no-circuit proof include:

- Instead of assuming, by way of contradiction, that  $f(\mathbf{v_c})$  is an integer, we instead assume that both  $f(\mathbf{v_h})$  and  $f(\mathbf{v_h}^{\mathbf{R}})$  are both integers.
- Unlike the no-circuit proof, we do not require deep lower bounds [1, 7] on the size of  $2^k 3^x$ .
- We require no closed-form expression for any high cycle member; by contrast, the no-circuit proof relies on the expression  $f(\mathbf{v_c}) = \frac{\mathbf{3^x} \mathbf{2^x}}{\mathbf{2^k} \mathbf{3^x}}$ .

# References

- [1] Alan Baker, Transcendental number theory, Cambridge University Press, 1975.
- [2] David Barina, *Convergence verification of the Collatz problem*, The Journal of Supercomputing **77** (2021), 2681–2688.
- [3] Jean Berstel, Aaron Lauve, Christophe Reutenauer, and Franco V. Saliola, *Combinatorics on words: Christoffel words and repetitions in words*, American Mathematical Society, Providence, RI, 2009.
- [4] Corrado Böhm and Giovanna Sontacchi, On the existence of cycles of given length in integer sequences like  $x_{n+1} = x_n/2$  when x is even and  $x_{n+1} = 3x_n + 1$  otherwise, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali **64** (1978), 260–264.
- [5] Harvey Cohn, Markoff forms and primitive words, Mathematische Annalen 196 (1972), 8–22.
- [6] Shalom Eliahou, The 3x + 1 problem: new lower bounds on nontrivial cycle lengths, Discrete Mathematics **118** (1993), 45–56.
- [7] W. J. Ellison, On a theorem of S. Sivasankaranarayana Pillai, Séminaire de théorie des nombres de Bordeaux (1971), 1–10.
- [8] Lorenz Halbeisen and Norbert Hungerbühler, Optimal bounds for the length of rational Collatz cycles, Acta Arithmetica LXXVIII (1997), 227–239.
- [9] Jeffrey C. Lagarias, The 3x+1 problem and its generalizations, The 3x+1 Problem: The Ultimate Challenge (Jeffrey C. Lagarias, ed.), American Mathematical Society, 2010, pp. 31–54.
- [10] R. P. Steiner, A theorem on the Syracuse problem, Proc. 7th Manitoba Conference on Numerical Mathematics, 1977, pp. 553–559.