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Collatz High Cycles Do Not Exist

Kevin Knight

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Abstract

The Collatz function takes odd \( n \) to \((3n + 1)/2\) and even \( n \) to \( n/2 \). Under the iterated Collatz function, every positive integer is conjectured to end up in the trivial cycle 1-2-1. Two types of cycles are of special interest. Consider the set \( S \) consisting of the smallest members of all cycles containing the same number of odd terms. The circuit contains the smallest member of \( S \), while the high cycle contains the largest. It is known that no circuits of positive integers exist (except 1-2-1); this paper shows that there are likewise no high cycles of positive integers.

1 Introduction

The Collatz function is

\[
T(n) = \begin{cases} 
\frac{3n + 1}{2}, & \text{if } n \text{ is odd;} \\
\frac{n}{2}, & \text{otherwise.}
\end{cases}
\]

Iterating this function famously yields interesting sequences. For example:

\[
\begin{align*}
31 & \rightarrow 47 \rightarrow 71 \rightarrow 107 \rightarrow 161 \rightarrow 242 \rightarrow 121 \rightarrow 182 \rightarrow 91 \rightarrow 137 \rightarrow 206 \\
& \rightarrow 103 \rightarrow 155 \rightarrow 233 \rightarrow 350 \rightarrow 175 \rightarrow 263 \rightarrow 395 \rightarrow 593 \rightarrow 890 \rightarrow 445 \\
& \rightarrow 668 \rightarrow 334 \rightarrow 167 \rightarrow 251 \rightarrow 377 \rightarrow 566 \rightarrow 283 \rightarrow 425 \rightarrow 638 \rightarrow 319 \\
& \rightarrow 479 \rightarrow 719 \rightarrow 1079 \rightarrow 1619 \rightarrow 2429 \rightarrow 3644 \rightarrow 1822 \rightarrow 911 \rightarrow 1367 \\
& \rightarrow 2051 \rightarrow 3077 \rightarrow 4616 \rightarrow 2308 \rightarrow 1154 \rightarrow 577 \rightarrow 866 \rightarrow 433 \rightarrow 650 \\
& \rightarrow 325 \rightarrow 488 \rightarrow 244 \rightarrow 122 \rightarrow 61 \rightarrow 92 \rightarrow 46 \rightarrow 23 \rightarrow 35 \rightarrow 53 \rightarrow 80 \\
& \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \ldots
\end{align*}
\]

The Collatz conjecture posits that every positive integer eventually reaches 1. The conjecture was verified in 2021 for all \( 1 \leq n \leq 10^{21} \) [2], but it has not yet been proven or refuted.
Figure 1: All of the rational Collatz cycles of length $k = 8$ with $x = 5$ odd terms. Of special note are the outermost cycle (the *circuit*), in light gray, and the innermost cycle (the *high cycle*), in dark gray.

If $T^i(n) = n$ for some positive integer $n$, there is a Collatz cycle of length $i$ whose first term is $n$. The only known $T$-cycle is the trivial one: $1 \rightarrow 2 \rightarrow 1$.

The unresolved Weak Collatz conjecture claims that there are no positive integer cycles. (It is called “weak” because it allows for the possibility of a start number that diverges to infinity, never reaching 1.)

We can also consider cycles with rational values. Figure 1 shows all cycles of length $k = 8$ with $x = 5$ odd terms. Considering the term $\frac{211}{13}$ to be odd (by its numerator), the next term produced by the Collatz process is $(3(\frac{211}{13}) + 1)/2 = \frac{323}{13}$ (also odd).

For any $k$ and $x$, the outermost and innermost cycles are of special interest. Let $S$ contain the smallest members of each cycle (here, $\frac{211}{13}, \frac{227}{13}, \ldots, \frac{319}{13}$). The outermost cycle (called the *circuit*) contains the lowest member in $S$, while the innermost cycle (called the *high cycle*) contains the highest member in $S$.

Steiner [10] showed that no positive integer circuits exist for any $k, x$ (except $k = 2, x =$
The current paper shows likewise that no high cycles of positive integers exist.

2 Parity vectors

Associated with every cycle member \( m \) is a unique parity vector that records the even (0) and odd (1) terms obtained from \( k \) iterations of the Collatz function.

Example 2.1 (Cycle member to parity vector). Starting with \( m = \frac{211}{13} \), we encounter 5 odd terms followed by 3 even terms, so the corresponding parity vector is 11111000.

Likewise, given a parity vector \( v \), we can compute a cycle member \( m \).

Example 2.2 (Parity vector to cycle member). Applying \( v = 110 \) to arbitrary start number \( n \) yields \( \frac{3^{2n+1} + 1}{2} = \frac{9}{8} n + \frac{5}{8} \). To make a cycle, we set \( n = \frac{9}{8} n + \frac{5}{8} \). Solving this equation yields \( n = -5 \), which is part of the \(-5 \rightarrow -7 \rightarrow -10 \rightarrow -5 \) cycle.

Generalizing this idea, Böhm and Sontacchi [4] give the cycle member associated with arbitrary parity vector \( v \) as

\[
f(\mathbf{v}) = \frac{\sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} 3^{x-i-1}}{2^k - 3^x}, \tag{2.1}
\]

where \( d_0(\mathbf{v}) < \ldots < d_{x-1}(\mathbf{v}) \) are the (zero-based) indices of 1s in \( \mathbf{v} \). Since we are only interested in Collatz cycles with positive terms, the denominator must be positive, so \( k > x \log_2 3 \).

The Weak Collatz Conjecture can be restated: “Is there a parity vector \( \mathbf{v} \) such that \( f(\mathbf{v}) \) is an integer greater than 2?”

It will sometimes be useful to refer to the numerator only:

\[
g(\mathbf{v}) = \sum_{i=0}^{x-1} 2^{d_i(\mathbf{v})} 3^{x-i-1}. \tag{2.2}
\]

Note that rotating a parity vector produces another term in the same rational cycle.

Example 2.3 (Rotation). \( f(100) = \frac{1}{5} \), \( f(001) = \frac{4}{5} \), and \( f(010) = \frac{2}{5} \). These are all entry points into the \( \frac{1}{5} \rightarrow \frac{4}{5} \rightarrow \frac{2}{5} \rightarrow \frac{1}{5} \) cycle.

Finally, we can restrict ourselves to aperiodic parity vectors. Consider \( f(1101011010) = \frac{23}{65} \), which is part of the double cycle \( \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \rightarrow \frac{37}{5} \rightarrow \frac{58}{5} \rightarrow \frac{29}{5} \rightarrow \frac{46}{5} \rightarrow \frac{23}{5} \). Had these been integers, the aperiodic \( f(11010) \) would also be an integer.
3 No integer circuits

The parity vector $v_c = 1^x 0^{k-x}$ is associated with the smallest member of the $(k, x)$-circuit. Following Equation 2.1,

$$f(v_c) = \frac{\sum_{i=0}^{x-1} 2^i 3^{x-i-1}}{2^k - 3^x}. $$

We provide a simple proof that (non-trivial) Collatz circuits do not exist, after three lemmas.

**Lemma 3.1.** $f(v_c)$ simplifies to $\frac{3^x - 2^x}{2^k - 3^x}$.

*Proof.* By induction, omitted. □

**Lemma 3.2.** If there is a non-trivial positive integer circuit of length $k$ with $x$ odd terms, then $2 \cdot (\frac{3}{2})^x \geq 2^{k-x} - 1$.

*Proof.* This follows from the fact that the putative circuit’s smallest member is at least 1.

$$\frac{3^x - 2^x}{2^k - 3^x} \geq 1$$

$$3^x - 2^x \geq 2^k - 3^x$$

$$2 \cdot 3^x \geq 2^k + 2^x$$

$$2 \cdot (\frac{3}{2})^x \geq 2^{k-x} - 1.$$ □

**Lemma 3.3** (Ellison [7]). For integers $k$ and $x$, with $x > 17$, we have $2^k - 3^x > 2.56^x$.

*Proof.* We start with Ellison’s original bound (for $k > 27$) and use the fact that $k > x \log_2 3$ in positive cycles.

$$2^k - 3^x > \frac{2^k}{e^{k/10}} > 1.8^k > 1.8^{x \log_2 3} > 2.56^x, \text{ for } x > 17.$$ □

**Theorem 3.4** (Steiner, 1977). Non-trivial Collatz circuits do not exist.

*Proof.* Assume $f(v_c) = \frac{3^x - 2^x}{2^k - 3^x}$ is a positive integer. Then the following are also positive integers:
\[
\frac{3^x - 2^x}{2^k - 3^x} + 1, \\
\frac{3^x - 2^x + 2^k - 3^x}{2^k - 3^x}, \\
\frac{2^k - 2^x}{2^k - 3^x}, \\
\frac{2^x(2^{k-x} - 1)}{2^k - 3^x}, \text{ and } \frac{2^{k-x} - 1}{2^k - 3^x}.
\]

We can remove the $2^x$ in the last step without affecting the expression’s purported integrality, since the denominator is odd.

However, the last term is actually less than one; we invoke Lemmas 3.3 and 3.2 to show that the denominator outstrips the numerator. For $x > 17$,

\[
2^k - 3^x > 2.56^x > 2 \cdot (1.5)^x \geq 2^{k-x} - 1.
\]

$2^{k-x} - 1 = 0$ only when $k = 2, x = 1$; setting aside this case, we obtain the contradiction $0 < \frac{2^{k-x} - 1}{2^k - 3^x} < 1$. Cases of $x \leq 17$ are handled by the fact that any non-trivial Collatz cycle must contain many millions of odd terms [6].

To date, all no-circuit proofs depend on deep lower bounds for $2^k - 3^x$, such as Ellison’s, which are all derived from Alan Baker’s pioneering transcendental number theory work [1]. As Jeffrey Lagarias [9] remarks, “The most remarkable thing about Theorem […] is the weakness of its conclusion compared to the strength of the methods used in its proof.”

\section{The high cycle}

In this section, characterize the Collatz high cycle.

\begin{definition}
Let $v_h$ be a $(k,x)$-parity vector with $1$s indexed by $d_i(v) = \lfloor \frac{k}{2}i \rfloor$, for $0 \leq i \leq x - 1$.
\end{definition}

\begin{example}
For $k = 8, x = 5, v_h = 11011010$.
\end{example}

\begin{footnote}
Note that certain circuits can be ruled out without these deep bounds. For example, if $k$ and $x$ are not co-prime, then we can factor $2^k - 3^x$. E.g., $(2^{k/2} + 3^{x/2})(2^{k/2} - 3^{x/2}) > 2^{k/2} > 2^{x \log 3/2} > 1.72^x > 2^{k-x} - 1$. Or for the case of $v_c = 1111111100000$, it follows that if $f(v_c) = \frac{3^8 - 2^8}{2^{13} - 3^8}$ is an integer, then so is $31f(v_c) + 206$, but it happens that $31 \frac{3^8 - 2^8}{2^{13} - 3^8} + 206 \frac{2^{13} - 3^8}{2^{13} - 3^8} = \frac{3^{12}}{2^{13} - 3^8}$, which is not integral.
\end{footnote}
Example 4.3. For $k = 21, x = 13$, $v_h = 110110101101011011010$. The vector $v_h$ is the well-known (upper) Christoffel word [3], constructed so that its 1s are roughly evenly spread among its 0s. The cycle member corresponding to $v_h$ is

$$f(v_h) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1}2^{\left\lfloor \frac{k}{x} i \right\rfloor}}{2^{k+1} - 3^x}.$$ 

Now we list a number of properties of $v_h$. First, $f(v_h)$ is the smallest member of its cycle.

Theorem 4.4 (Halbeisen and Hungerbührer [8]). For every $(k, x)$ parity vector $v$ that is a rotation of $v_h$, we have $f(v_h) \leq f(v)$.

Remark 4.5. It is known that a Christoffel word $v_h$ is at a lexicographic extreme among all its rotations [3]. However, lexicographic ordering does not always coincide with $f$-ordering. For example, $f(11110010) = \frac{259}{13}$, $f(11100101) = \frac{395}{13}$, and $f(10111100) = \frac{341}{13}$. Of course, lexicographic ordering correlates roughly with $f$-ordering, as left-heavy vectors tend to be associated with smaller cycle members.

Theorem 4.6. Among all the $(k, x)$ upper Christoffel words, $f(v_h)$ is maximized when $k = \lceil x \log_2 3 \rceil$.

Proof. Let $v_h$ be an upper Christoffel word with $k = \lceil x \log_2 3 \rceil$. Let $u$ be a longer $(k+1, x)$-high-cycle parity vector. Replacing $k$ by $k+1$ increases the numerator of $f(v)$ by no more than a factor of 2, while increasing the denominator by more than a factor of 2.

$$f(u) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1}2^{\left\lfloor \frac{k+1}{x} i \right\rfloor}}{2^{k+1} - 3^x} = \frac{\sum_{i=0}^{x-1} 3^{x-i-1}2^{\left\lfloor \frac{k}{x} i \right\rfloor}}{2^{k+1} - 3^x} = f(v). \quad \square$$

Similar reasoning holds if $u$ is a $(k+n, x)$ upper Christoffel word, for any $n \geq 1$.

Next, we confirm the reader’s suspicion that $f(v_h)$ is indeed a member of the high cycle. Let $S$ contain the smallest members of each $(k, x)$-cycle.

Theorem 4.7 (Halbeisen and Hungerbührer [8]). Let $v$ be a parity vector associated with the smallest member of any $(k, x)$-cycle, and let $v_h$ be an upper Christoffel word with $k = \lceil x \log_2 3 \rceil$. Then, $f(v_h) \geq f(v)$.
Remark 4.8 (Tangential). This theorem is useful for proving that any Collatz cycle must be long. Just for example, it is easy to verify empirically that $f(v_h) < 10^{13}$ for all $x < 10,000,000$, implying that every cycle (not just the high cycle) with fewer than ten million odd terms has some member less than $10^{13}$. Since no Collatz counter-examples exist among the first $10^{20}$ positive integers [2], a purported Collatz cycle must have more than ten million odd terms. Stronger results have been obtained through analytical means [8]. Of course, $f(v_h)$ will outstrip any finite confirmation, after some $x$, because $O(3^x x)$ exceeds $O(2^k - 3^x)$.

Recall the cycle member corresponding to $v_h$.

$$f(v_h) = \frac{\sum_{i=0}^{x-1} 3^{x-i-1}2^{\lfloor \frac{x}{i} \rfloor}}{2^k - 3^x}.$$ 

Due to the floor function, we cannot simplify this expression as we did for $f(v_c)$. Its value can only be partially wrangled through upper and lower bounds.

Theorem 4.9 (Halbeisen and Hungerbühler [8]).

$$\frac{3^x(x/20)}{2^\lfloor x \log_2 3 \rfloor - 3^x} \leq f(v_h) \leq \frac{3^x(7x/10)}{2^\lfloor x \log_2 3 \rfloor - 3^x}.$$ 

These bounds can be improved from $(x/20, 7x/10)$ to $(x/6, x/2)$; we omit the proof for brevity.

5 No integer high cycles

In this section, we prove our main result that $f(v_h)$ is never an integer. To do this, we need two more facts about Christoffel words. First, $f(v_R^h)$ is a member of the same cycle as $f(v_h)$.

Theorem 5.1 (Cohn [5]). The reverse $v_R^h$ of an upper Christoffel word $v_h$ is also a rotation of it.

Therefore, $f(v_h)$ and $f(v_R^h)$ are in the same cycle. Indeed, $f(v_R^h)$ is the largest member of the cycle, though we do not need this fact; also for interest, the left-rotation distance is the multiplicative inverse of $x$ modulo $k$.

Theorem 5.2 (Berstel et al [3]). For aperiodic $v_h$, we have $v_h = 1u0$, and $v_R^h = 0u1$.

Example 5.3. $v_h = 11011010$, $v_R^h = 01011011$, and $u = 101101$. Incidentally, $u$ is always a palindrome, though we do not need this fact.

Because $v_h$ and $v_R^h$ are virtually identical, differing only in their first and last components, we expect $f(v_h)$ and $f(v_R^h)$ to have similar summands. We express both in terms of $u$, via Equations 2.1 and 2.2.
\[ f(v_h) = \frac{2g(u) + 3^{x-1}}{2^k - 3^x} \]
\[ f(v^R_h) = \frac{6g(u) + 2^{k-1}}{2^k - 3^x} \]

In the first case, we create \( v_h \) by shifting \( u \) one place to the right (doubling every summand) and adding a 1 to the left edge. In the second case, shifting \( u \) requires both a doubling (for the same reason) and a tripling (from adding a 1 to the right edge).

**Theorem 5.4** (Main result). No high cycle consists of integers.

**Proof.** We know \( f(v_h) \) and \( f(v^R_h) \) are both members of the high cycle (Theorems 4.7 and 5.1). If they are both integers, then so are

\[
3f(v_h) - f(v^R_h) + 1, \\
\frac{3 \cdot 2^k - 3^x}{2^k - 3^x} - \frac{6g(u) + 2^{k-1}}{2^k - 3^x} + \frac{2^k - 3^x}{2^k - 3^x}, \\
\frac{2^k - 2^{k-1}}{2^k - 3^x}, \quad \text{and} \quad \frac{2^{k-1}}{2^k - 3^x}.
\]

However, \( 2^{k-1} \) is not divisible by any odd number, so we have a contradiction. \( \square \)

**Example 5.5.** For the \((8,5)\)-high-cycle, Figure 1 gives \( f(v_h) = \frac{319}{13} \) and \( f(v^R_h) = \frac{842}{13} \). Combining gives \( 3 \frac{319}{13} - \frac{842}{13} + \frac{13}{13} = \frac{27}{13} \).

Note that if we left-rotate \( v_h \) and \( v^R_h \) by one position each, we obtain an alternate pair of high-loop members \((u_01\text{ and } u_{10})\), whose simple difference provides the necessary contradiction. For example, cycle members \( \frac{485}{13} \) and \( \frac{421}{13} \) in Figure 1 cannot both be integers, because \( \frac{485}{13} - \frac{421}{13} = \frac{26}{13} \).

6 Summary

Notable features of our proof versus the no-circuit proof include:

- Instead of assuming, by way of contradiction, that \( f(v_c) \) is an integer, we instead assume that both \( f(v_h) \) and \( f(v^R_h) \) are both integers.

- Unlike the no-circuit proof, we do not require deep lower bounds \([1, 7]\) on the size of \( 2^k - 3^x \).

- We require no closed-form expression for any high cycle member; by contrast, the no-circuit proof relies on the expression \( f(v_c) = \frac{3^x - 2^x}{2^k - 3^x} \).
References


[4] Corrado Böhm and Giovanna Sontacchi, *On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ when $x$ is even and $x_{n+1} = 3x_n + 1$ otherwise*, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali 64 (1978), 260–264.


